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Numerical computation of connecting orbits in planar piecewise smooth dynamical system ☆

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A B S T R A C T

In this paper, a numerical algorithm for computing the connecting orbits in piecewise smooth dynamical systems is derived and is analyzed. A nondegenerate condition for the connecting orbit with respect to its bifurcation parameter is presented to ensure the defining equation being well posed, which is a generalization of the Melnikov condition for smooth systems. The error caused by the truncation of time interval is also analyzed. Some numerical calculations are carried out to illustrate the theoretical analysis.

1. Introduction

In recent years, there are growing interests in piecewise smooth dynamical systems for their wide applications in applied science and engineering, such as the stick-slip mechanical systems [2,16,32], the mechanical systems with clearances or elastic constraints [34,33,36,37], the earthquake engineering [6,18,20], the power electronic converters [8,9,13], the suspension bridges [10] and so on. The discontinuity of the system is a special form of nonlinearity, which causes rich complicated new phenomena.

Our research interests in this work emanate from a model of a free-standing rigid block subjected to harmonic forcing, see Fig. 1 for a sketch. The model is often used to describe the behavior of man-made structures undergoing earthquakes.

The mathematical modeling of this rocking rigid block can be formulated as follows (see [21,18]),

$$\alpha\ddot{u} + \sin[\alpha(1-u)] = -\alpha\beta\cos[\alpha(1-u)]\cos\omega t, \qquad u > 0, \tag{1}$$

$$\alpha\ddot{u} - \sin[\alpha(1+u)] = -\alpha\beta\cos[\alpha(1+u)]\cos\omega t, \qquad u < 0, \tag{2}$$

$$\dot{u}(t^A) = r\dot{u}(t^B), \tag{3}$$

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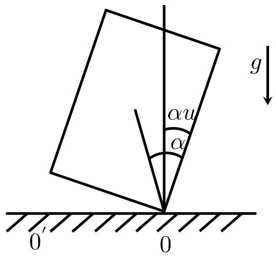

Fig. 1. Sketch of a rocking rigid block.

where α is the block shape parameter, αu presents the angle between one edge of the block and the droop, $\ddot{u}$ is the second derivative of u with respect to the time variable t, β and ω are amplitude and frequency parameters of the excitation, respectively. $0 \leqslant r \leqslant 1$ is the coefficient of restitution characterizing the energy loss at impact, t^A is the time just after impact and t^B is the time just before impact. If there is no external excitation the coefficient $\beta = 0$ and if the impact is completely elastic the parameter $r = 1$.

Much work has been carried out for the case $\alpha \ll 1$ (the slender block). In this situation, system (1)–(3) is reduced to a piecewise-linear system and its solutions can be obtained analytically. Hogan [19] shows that heteroclinic bifurcations appear in this piecewise smooth system. Bruhn and Koch [6] calculate a heteroclinic bifurcation condition without using perturbation methods and also use the Melnikov method in the case of small excitation and damping, etc. In many papers, they have mentioned that their methods also apply to the nonlinear case when α is an arbitrary angle which is related to the ordinary man-made structures.

One research interest on this problem is to study the structure stability of this rocking block. The terms in the right-hand side of equations (1)–(2) represent the external force added to the block undergoing earthquakes and equation (3) is the impact equation which reflects the ability of the block reverting to its original state during earthquakes. These two factors play an important role while studying the structure stability of the block. Besides these two external influences, the block's shape characterized by parameter α is also a key aspect to determine the stability of the block undergoing earthquakes. It is worthwhile to study the structure properties of the block without external influences, which correspond to studying the dynamics of equations (1)–(3) with parameters $\beta = 0$ and $r = 1$. It follows from numerical simulations that there exists a heteroclinic loop in the phase space $(u, \dot{u})$, inside of which are piecewise smooth periodic solutions corresponding to bounded oscillations of the block around the rest situation $u = \dot{u} = 0$, outside of which are orbits of large scale motions leading to overturning. This heteroclinic loop is the separatrix to distinct the stable and unstable motions of the block. In other words, it characterizes the critical situation of the block changing from stable motions to unstable motions. The main purpose of this paper is to construct a numerical method for computing the connecting orbits including homoclinic and heteroclinic orbits in planar piecewise smooth dynamical systems.

The numerical methods for computing connecting orbits in smooth dynamical systems are well studied by many authors, see [4,14,15,26] and the references therein. But these well posed methods are unable to be directly applied to piecewise smooth dynamical systems due to the influences by the discontinuity of the system. In this paper we study the numerical method for approximating a connecting orbit which transversally intersects the line of discontinuity. We define a nondegenerate condition for the piecewise smooth connecting orbit together with its bifurcation parameter, which is the generalization of the counterpart in smooth dynamical systems. In the geometric point of view, this nondegenerate condition is interpreted as that the stable and unstable manifolds pass each other along the line of discontinuity with non-vanishing velocity with respect to the bifurcation parameter. This nondegenerate condition ensures the regularity of an extending equation for computing a piecewise smooth connecting orbit together with its bifurcation parameter.

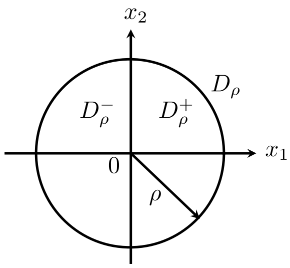

Fig. 2. $D_\rho = \{x \in \mathbb{R}^2 : \|x\| < \rho\}$, $D_\rho^\pm = D_\rho \cap \{(x_1, x_2)^T : \pm x_1 > 0\}$.

For the connecting orbits in piecewise smooth dynamical systems, the onset of chaos near the homoclinic orbits has been treated. Awrejcewicz and Holicke [1] study a homoclinic orbit in a smooth system perturbed by non-smooth components involving stick-slip motions. They use the standard Melnikov function to study the appearance of chaos if the Melnikov function has a simple zero. Fechan [12] follows a different approach based on some kind of Lyapunov–Schmidt reduction and a topological method for multivalued problems. Zou and Küpper [30] study a non-smooth homoclinic orbit under small time-periodic perturbations by deriving an extended Melnikov function which contains an extra term reflecting the change of the vector fields at the discontinuity. Du and Wang [11] extend the Melnikov function to nonlinear impact systems and also give a method to compute the Melnikov functions up to the n-th order. Medrano et al. [25] present a general numerical method to demonstrate the existence of a connecting orbit in a piecewise linear three dimensional system. Kuznetsov et al. [23] study several cases of codimension 1 bifurcation of a sliding homoclinic orbit (having a sliding segment on the line of discontinuity). In each case they propose a defining system and then trace the bifurcation curve using standard continuation techniques.

This paper is organized as follows. In section 2, we introduce and discuss our basis assumptions for this work. In section 3, we propose a nondegenerate condition for a piecewise smooth connecting orbit together with its bifurcation parameter. We also construct a defining system for computing the piecewise smooth connecting orbit and its bifurcation parameter simultaneously. In section 4 we prove that the piecewise smooth connecting orbit is a regular solution to our defining system under the nondegenerate condition. In section 5 we set up a numerical computation method for the piecewise smooth connecting orbit by truncating the defining equation on a finite interval using the projection boundary conditions. We also prove the existence of solutions of the truncated system and estimate the truncation errors. In section 6 we apply the method to compute homoclinic orbits and heteroclinic orbits, respectively, to illustrate the theoretical analysis on the truncated errors.

2. Basic assumptions

In this section we describe and discuss the basic assumptions used in this paper.

Let D_ρ be an open disk with radius $\rho > 0$ centered about the origin. Define a semi-disk $D_\rho^\pm = D_\rho \cap \{(x_1, x_2)^T : \pm x_1 > 0\}$ and denote its closure by $\overline{D_\rho^\pm}$, see Fig. 2.

Consider the following parameterized piecewise smooth dynamical system

$$\frac{dx}{dt} = f(x, \lambda), \quad x \in D_\rho, \quad \lambda \in \Lambda, \tag{4}$$

where Λ is an open interval in $\mathbb{R}$ and f is piecewisely defined by

$$f(x, \lambda) = \begin{cases} f^+(x, \lambda), & x \in D_\rho^+, \\ f^-(x, \lambda), & x \in D_\rho^-. \end{cases}$$

Our assumptions are:

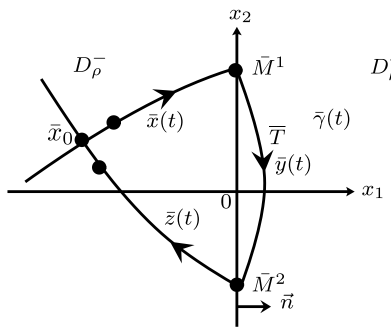

Fig. 3. A homoclinic orbit transversally intersects the line of discontinuity at two points $\bar{M}^1$ and $\bar{M}^2$.

(H1) The function $f^{\pm}$ is C^r $(r \geq 2)$ smooth in $\overline{D_\rho} \times \Lambda$, i.e., $f^{\pm}$ and all its derivatives up to the r-th order are continuous.

Remark 2.1. The function f is piecewise smooth and may lose smoothness only along the x_2-axis which is the line of discontinuity of system (4).

(H2) There exists a saddle point $\bar{x}_0 \in D_\rho^-$ at $\lambda = \bar{\lambda} \in \Lambda$, i.e.,

$$f^-(\bar{x}_0, \bar{\lambda}) = 0, \quad \det f_x^-(\bar{x}_0, \bar{\lambda}) < 0.$$

Remark 2.2. Saying $\bar{x}_0$ is a saddle point means that at the equilibrium $\bar{x}_0$, the Jacobian matrix $f_x^-(\bar{x}_0, \bar{\lambda})$ has one negative and one positive eigenvalues, respectively.

(H3) System (4) possesses a homoclinic orbit $\bar{\gamma}(t)$ with endpoint $\bar{x}_0$ at $\lambda = \bar{\lambda}$.

(H4) The homoclinic orbit $\bar{\gamma}(t)$ transversally intersects the line of discontinuity at $\bar{M}^1$ and $\bar{M}^2$, respectively, see Fig. 3. Without loss of generality, we assume

$$f^{\pm}(\bar{M}^1, \bar{\lambda})^T \cdot \vec{n} \ > \ 0 \qquad \text{and} \qquad f^{\pm}(\bar{M}^2, \bar{\lambda})^T \cdot \vec{n} \ < \ 0, \tag{5}$$

where $\vec{n} = (1, 0)^T$ represents the normal vector to the line of discontinuity directed from D_ρ^- to D_ρ^+.

Remark 2.3. A connecting orbit is either a homoclinic orbit or a heteroclinic orbit. For convenience, we only discuss homoclinic orbits in this paper and all the results can be easily generalized to heteroclinic orbits.

Remark 2.4. At the point $\bar{M}^1$ there is a pair of vectors $f^-(\bar{M}^1, \bar{\lambda})$ and $f^+(\bar{M}^1, \bar{\lambda})$. In the case of a smooth system, they coincide with each other, i.e.,

$$f^-(\bar{M}^1, \bar{\lambda}) = f^+(\bar{M}^1, \bar{\lambda}).$$

In the case of a piecewise smooth system, they might be different. These two vectors are either linearly dependent or linearly independent. Similar phenomena occur at the point $\bar{M}^2$. Precisely, we have three cases,

Case (I) Both the two pairs of vectors at point $\bar{M}^1$ and $\bar{M}^2$ are respectively linearly dependent, i.e., there exist two constants $\bar{\theta}^1 \neq 0$ and $\bar{\theta}^2 \neq 0$ such that

$$f^-(\bar{M}^1, \bar{\lambda}) \ = \ \bar{\theta}^1 \ f^+(\bar{M}^1, \bar{\lambda}), \quad f^-(\bar{M}^2, \bar{\lambda}) \ = \ \bar{\theta}^2 \ f^+(\bar{M}^2, \bar{\lambda}) \tag{6}$$

(this includes the smooth case when $\bar{\theta}^1 = \bar{\theta}^2 = 1$).

Case (II) The pair of vectors at $\bar{M}^1(\bar{M}^2)$ is linearly dependent while the pair at $\bar{M}^2$ $(\bar{M}^1)$ is linearly independent.

Case (III) Both the two pairs of vectors are respectively linearly independent.

3. Nondegenerate connecting orbits

In this section, we first introduce a defining system for computing the piecewise smooth homoclinic orbit $\bar{\gamma}$, then define and discuss a nondegenerate condition which ensures the regularity of the defining system.

Since system (4) is autonomous, we assume without loss of generality that $\bar{\gamma}(0) = \bar{M}^1$ and $\bar{\gamma}(\overline{T}) = \bar{M}^2$, where $\overline{T}$ is the flight time of the homoclinic orbit $\bar{\gamma}$ from point $\bar{M}^1$ to point $\bar{M}^2$ inside the region D_ρ^+, cf. Fig. 3. For convenience, we denote the three pieces of the homoclinic orbit $\bar{\gamma}$ by

$$\bar{x}(t) \; = \; \bar{\gamma}|_{(-\infty,0]}\,(t), \qquad \bar{y}(t) \; = \; \bar{\gamma}(\overline{T}t)|_{[0,1]}\,(t), \qquad \bar{z}(t) \; = \; \bar{\gamma}|_{[0,\infty)}\,(t+\overline{T}).$$

We call $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot), \overline{T}, \bar{\lambda})$ a homoclinic pair (HOP for short).

Let $\mathbb{R}^\pm = \{t : \pm t \geqslant 0,\; t \in \mathbb{R}\}$. Define Banach spaces,

$$B^{k,\pm} = \left\{ x(\cdot) \in C^k(\mathbb{R}^\pm, \mathbb{R}^2) : \lim_{t\to\pm\infty} x^{(j)}(t) \text{ exists},\; j = 0, \cdots, k \right\},\; k = 0, 1,$$

with the norm

$$\|x\|_{k,\pm} \;=\; \sum_{j=0}^{k} \sup_{t\in\mathbb{R}^\pm} \|x^{(j)}(t)\|,$$

where $\|\cdot\|$ denotes Euclidian norm in $\mathbb{R}^2$. In order to simplify the notations, denoted by $B^k = B^{k,-} \times C^k([0,1], \mathbb{R}^2) \times B^{k,+},\; k = 0, 1$. Define a function

$$F : \begin{array}{l} B^1 \times \mathbb{R} \times \mathbb{R} \to B^0 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \\ (x, y, z, T, \lambda) \to \Big(\dot{x} - f^-(x,\lambda),\; \dot{y} - Tf^+(y,\lambda),\; \dot{z} - f^-(z,\lambda), \\ \qquad\qquad x(0) - y(0),\; y(1) - z(0),\; \vec{n}^T x(0),\; \vec{n}^T y(1) \Big)^T. \end{array}$$

Obviously, the HOP $(\bar{x}, \bar{y}, \bar{z}, \overline{T}, \bar{\lambda}) \in B^1 \times \mathbb{R} \times \mathbb{R}$ is a solution of equation $F(x, y, z, T, \lambda) = 0$.

Definition 3.1. A homoclinic orbit $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{z}(\cdot))$ of system (4) is called nondegenerate with respect to parameter λ if the Jacobian matrix $f_x^-(\bar{x}_0, \bar{\lambda}) = \lim_{t\to-\infty} f_x^-(\bar{x}(t), \bar{\lambda})$ is hyperbolic, and

$$\int_{-\infty}^{0} \bar{u}(t)^T f_\lambda^-(\bar{x}(t), \bar{\lambda})dt + \int_{0}^{1} \bar{v}(t)^T \overline{T} f_\lambda^+(\bar{y}(t), \bar{\lambda})dt + \int_{0}^{+\infty} \bar{w}(t)^T f_\lambda^-(\bar{z}(t), \bar{\lambda})dt \neq 0 \tag{7}$$

for all bounded nontrivial solutions $(\bar{u}, \bar{v}, \bar{w})$ of the following equations

$$\dot{u}(t) \;+\; (f_x^-(\bar{x}(t), \bar{\lambda}))^T\, u(t) \;=\; 0, \tag{8}$$

$$\dot{v}(t) \;+\; \overline{T}(f_x^+(\bar{y}(t), \bar{\lambda}))^T\, v(t) \;=\; 0, \tag{9}$$

$$\int_0^1 v(t)^T \; f^+(\bar{y}(t),\bar{\lambda}) \; dt \;=\; 0, \tag{10}$$

$$\dot{w}(t) \;+\; (f_x^-(\bar{z}(t),\bar{\lambda}))^T \; w(t) \;=\; 0, \tag{11}$$

$$(u(0)-v(0))^T \; \vec{n}^{\perp} \;=\; 0, \tag{12}$$

$$(v(1)-w(0))^T \; \vec{n}^{\perp} \;=\; 0, \tag{13}$$

where $\vec{n}^{\perp} = (-n_2, n_1)^T$ for $\vec{n} = (n_1, n_2)^T$.

Remark 3.2. In the next section we will prove that equations (8)–(13) always possess nontrivial bounded solutions (cf. Lemma 4.11).

Remark 3.3. In smooth systems, the nondegenerate condition for a connecting orbit $\bar{\gamma}$ in terms of a variational equation is

$$\dot{y} - f_x(\bar{\gamma}(t),\bar{\lambda})y = f_\lambda(\bar{\gamma}(t),\bar{\lambda})\mu \quad \Rightarrow \quad \mu = 0 \text{ and } y = c\dot{\bar{\gamma}}(t) \text{ for some } c \in \mathbb{R}. \tag{14}$$

This condition was proposed by Beyn (see Definition 2.1 in [4]) and is consequently used later by other authors (cf. [3,5,14,15,24,26,27]). Beyn [4] also proves that the nondegenerate condition (14) is equivalent to the following Melnikov condition,

$$\int_{-\infty}^{\infty} \bar{u}(t)^T f_\lambda(\bar{\gamma}(t),\bar{\lambda})dt \;\neq\; 0, \tag{15}$$

for any nontrivial solution $\bar{u}$ of the equation

$$\dot{u}(\cdot) + f_x(\bar{\gamma}(t),\bar{\lambda})^T u(\cdot) \;=\; 0. \tag{16}$$

In this paper we generalize the Melnikov condition (15) and (16) to piecewise smooth systems with the form of (7)–(13). It is difficult to find an equivalent version of the form (14), see Corollary 4.19 for details.

Our final assumption is

(H5) The homoclinic orbit $(\bar{x}, \bar{y}, \bar{z}, \overline{T})$ is nondegenerate with respect to λ.

The next theorem is a main result of this paper and its proof is left to the next section.

Theorem 3.4. *Assume* (H1)–(H5)*. Then $(\bar{x}, \bar{y}, \bar{z}, \overline{T}, \bar{\lambda})$ is a regular solution of equation $F = 0$.*

By regular we mean that the total derivative $D\overline{F} \triangleq DF(\bar{x}, \bar{y}, \bar{z}, \overline{T}, \bar{\lambda})$ of operator F at $(\bar{x}, \bar{y}, \bar{z}, \overline{T}, \bar{\lambda})$ is a linear homeomorphism from $B^1 \times \mathbb{R} \times \mathbb{R}$ onto $B^0 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$. Direct computation gives

$$D\overline{F} = \begin{pmatrix} & & & 0 & -f_\lambda^-(\bar{x},\bar{\lambda}) \\ & & & -f^+(\bar{y},\bar{\lambda}) & -\overline{T}\; f_\lambda^+(\bar{y},\bar{\lambda}) \\ & \overline{L} & & 0 & -f_\lambda^-(\bar{z},\bar{\lambda}) \\ & & & 0 & 0 \\ & & & 0 & 0 \\ \vec{n}^T l_1 & 0 & 0 & 0 & 0 \\ 0 & \vec{n}^T l_2 & 0 & 0 & 0 \end{pmatrix}$$

where $\overline{L}: B^1 \to B^0 \times \mathbb{R}^2 \times \mathbb{R}^2$ is defined by

$$\overline{L}\begin{pmatrix} x(\cdot) \\ y(\cdot) \\ z(\cdot) \end{pmatrix} = \begin{pmatrix} \dot{x} - f_x^-(\bar{x}(\cdot),\bar{\lambda})\, x \\ \dot{y} - \overline{T} f_x^+(\bar{y}(\cdot),\bar{\lambda})\, y \\ \dot{z} - f_x^-(\bar{z}(\cdot),\bar{\lambda})\, z \\ x(0) - y(0) \\ y(1) - z(0) \end{pmatrix} \triangleq \begin{pmatrix} \overline{L}_1\, x(\cdot) \\ \overline{L}_2\, y(\cdot) \\ \overline{L}_3\, z(\cdot) \\ x(0) - y(0) \\ y(1) - z(0) \end{pmatrix}, \tag{17}$$

$$l_1: \quad X^{1,-} \to \mathbb{R}^2, \quad l_1(x) = x(0) \quad \text{and} \quad l_2: \quad C^1 \to \mathbb{R}^2, \quad l_2(y) = y(1).$$

The next lemma plays an important role in the proof of the main results of this work.

Lemma 3.5 *(Bordering lemma, [4], Lemma 2.3). Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and consider the operator*

$$S = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in L\ [\ \mathcal{X} \times \mathbb{R}^p,\ \mathcal{Y} \times \mathbb{R}^q\ ],$$

with bounded linear operators $A_{11} \in L\ [\mathcal{X},\ \mathcal{Y}]$, $A_{12} \in L\ [\mathbb{R}^p, \mathcal{Y}]$, $A_{21} \in L\ [\mathcal{X}, \mathbb{R}^q]$, and $A_{22} \in L\ [\mathbb{R}^p, \mathbb{R}^q]$. If A_{11} is Fredholm of index n then S is Fredholm of index $n + p - q$.

The next section is devoted to proving that $\overline{L}$ is Fredholm of index 0, then from the Bordering Lemma 3.5, the operator $D\overline{F}$ is Fredholm of index 0. Hence we only need to prove that $D\overline{F}(x, y, z, T, \lambda) = 0$ has only trivial solution which ensures that $D\bar{F}$ is a homeomorphism.

4. Linearization along connecting orbits

In this section, we first prove that the linear operator $\overline{L}$ defined in (17) is Fredholm of index 0, which is a conclusion from studying the properties of operators $\overline{L}_1, \overline{L}_2$ and $\overline{L}_3$, respectively. Then we investigate the dimensions of the kernel space $\mathcal{N}(\overline{L})$ and the range space $\mathcal{R}(\overline{L})$ of the linear operator $\overline{L}$, respectively. At the end of this section we complete the proof of Theorem 3.4.

Denote the fundamental matrix for $\overline{L}_1$ by $\bar{X}(t)$ satisfying $\bar{X}(0) = I$. According to [7,28], $\overline{L}_1$ has an exponential dichotomy on $\mathbb{R}^-$ with properties summed up in the following Lemma 4.1 and Lemma 4.3.

Lemma 4.1. *Assume* (H1)–(H3). *Then $\overline{L}_1$ has an exponential dichotomy on $\mathbb{R}^-$ with data $(\bar{Q}(\cdot), \bar{K}_1^-, \bar{K}_2^-, \bar{\alpha}_1^-, \bar{\alpha}_2^-)$ such that*

$$\begin{aligned} \bar{X}(t)\bar{X}^{-1}(s)\bar{Q}(s) &= \bar{Q}(t)\bar{X}(t)\bar{X}^{-1}(s), \quad && \text{for } t, s \in \mathbb{R}^-, \\ \|\bar{X}(t)\bar{X}^{-1}(s)\bar{Q}(s)\| &\leqslant \bar{K}_1^- e^{-\bar{\alpha}_1^-(t-s)}, && \text{for } t, s \in \mathbb{R}^- \text{ with } t \geq s, \\ \|\bar{X}(t)\bar{X}^{-1}(s)(I - \bar{Q}(s))\| &\leqslant \bar{K}_2^- e^{-\bar{\alpha}_2^-(s-t)}, && \text{for } t, s \in \mathbb{R}^- \text{ with } t \leq s, \end{aligned}$$

$$\mathcal{N}(\bar{Q}(0)) = \left\{ \xi \in \mathbb{R}^2 : \quad \sup_{t \in \mathbb{R}^-} \|\bar{X}(t)\xi\| \ < \ \infty \right\}$$

and

$$\lim_{t \to -\infty} \bar{X}(t)\ (I - \bar{Q}(t))\ \bar{X}(t)^{-1} = \bar{Q}_u.$$

Here $\bar{Q}_u$ is the projector onto the unstable subspace along the stable subspace of the matrix $f_x^-(\bar{x}_0, \bar{\lambda})$.

Corollary 4.2. *Assume* (H1)–(H3)*. Then*

- **(I)** *If $x(t)$ is a bounded solution of $\overline{L}_1 x = 0$, there holds $x(0) \in \mathcal{N}(\bar{Q}(0))$.*
- **(II)** *Since* $\dim\ \mathcal{N}(\bar{Q}_u) = 1$*, we have* $\dim \mathcal{N}(\bar{Q}(t)) = 1$ *for all $t \in \mathbb{R}^-$.*
- **(III)** $\dot{\bar{x}}(t) = \dfrac{d\bar{x}}{dt}(t)$ *is a bounded solution of equation $\overline{L}_1 x(t) = 0$ and*

$$\mathcal{N}(\overline{L}_1) = \mathrm{span}\{\dot{\bar{x}}(\cdot)\}, \qquad \mathcal{N}(\bar{Q}(0)) = \mathrm{span}\{\dot{\bar{x}}(0)\}.$$

Define the adjoint operator of $\overline{L}_1$ as

$$\overline{L}_1^*:\ B^{1,-} \to B^{0,-}, \qquad \overline{L}_1^*\, u(\cdot) \ = \ -\dot{u}(\cdot) - f_x^-(\bar{x}(\cdot), \bar{\lambda})^T u(\cdot).$$

Then $\overline{L}_1^*$ has a fundamental matrix $\bar{X}^{-1}(t)^T$ with $\bar{X}^{-1}(0)^T = I$.

Lemma 4.3. *Assume* (H1)–(H3)*. Then the adjoint operator $\overline{L}_1^*$ has an exponential dichotomy on $\mathbb{R}^-$ with data $(I - \bar{Q}(\cdot)^T, \bar{K}_2^-, \bar{K}_1^-, \bar{\alpha}_2^-, \bar{\alpha}_1^-)$. Moreover,*

$$\mathcal{N}(I - \bar{Q}(0)^T) \ = \ \{\xi \in \mathbb{R}^2 :\ \sup_{t \in \mathbb{R}^-} \|\bar{X}^{-1}(t)^T \xi\| < \infty\} \ = \ \mathcal{N}(\bar{Q}(0))^{\perp}.$$

From Corollary 4.2 (I) and Lemma 4.3, it follows that

Corollary 4.4. *Assume* (H1)–(H3)*. Then for any bounded solution $x(t)$ of $\overline{L}_1 x = 0$ and any bounded solution $u(t)$ of $\overline{L}_1^* u = 0$, there holds*

$$(x(t),\ u(t)) \ = \ (\bar{X}(t)x(0),\ \bar{X}^{-1}(t)^T u(0)) \ = \ (x(0),\ u(0)) \ = \ 0,$$

where $(\cdot,\cdot)$ denotes the Euclidian inner product. Moreover,

$$\mathcal{N}(L_1^*) = \mathit{span}\{\bar{X}^{-1}(t)^T \dot{\bar{x}}(0)^{\perp}\}.$$

Remark 4.5. From Corollary 4.2 (III) and Corollary 4.4, it follows that for any bounded solution $u(t)$ of $\overline{L}_1^* u = 0$, we have,

$$\int_{-\infty}^{0} u(t)^T f^-(\bar{x}(t), \bar{\lambda}) dt = \int_{-\infty}^{0} u(t)^T \dot{\bar{x}}(t) dt = 0. \tag{18}$$

In the definition of the Melnikov condition (15) for smooth systems, any nontrivial bounded solution $\bar{u}$ of equation (16) automatically satisfies equation (10). In the case of piecewise smoothness, only $u(t)$ satisfies (18) and $w(t)$ satisfies a similar equation to (18). In our Definition 3.1 we require the piece $v(t)$ to satisfy the integral equation (10).

The properties of the operator $\overline{L}_3$ are similar to those of operator $\overline{L}_1$. Denoted the fundamental matrix for $\overline{L}_3$ by $\bar{Z}(t)$ satisfying $\bar{Z}(0) = I$. Similarly, we obtain the following properties.

Lemma 4.6. *Assume* (H1)–(H3)*. Then $\overline{L}_3$ has an exponential dichotomy on $\mathbb{R}^+$ with data $(\bar{P}(\cdot), \bar{K}_1^+, \bar{K}_2^+, \bar{\alpha}_1^+, \bar{\alpha}_2^+)$ such that*

$$\bar{Z}(t)\bar{Z}^{-1}(s)\bar{P}(s) = \bar{P}(t)\bar{Z}(t)\bar{Z}^{-1}(s), \quad \textit{for } t,\ s \in \mathbb{R}^-,$$
$$\|\bar{Z}(t)\bar{Z}^{-1}(s)\bar{P}(s)\| \leqslant \bar{K}_1^+ e^{-\bar{\alpha}_1^+(t-s)}, \quad \textit{for } t,\ s \in \mathbb{R}^- \textit{ with } t \geq s,$$
$$\|\bar{Z}(t)\bar{Z}^{-1}(s)(I - \bar{P}(s))\| \leqslant \bar{K}_2^+ e^{-\bar{\alpha}_2^+(s-t)}, \quad \textit{for } t,\ s \in \mathbb{R}^- \textit{ with } t \leq s,$$

$$\mathcal{R}(\bar{P}(0)) = \left\{\xi \in \mathbb{R}^2 : \sup_{t\in\mathbb{R}^+} \|\bar{Z}(t)\xi\| < \infty\right\}$$

and

$$\lim_{t\to\infty} \bar{Z}(t)\ \bar{P}(t)\ \bar{Z}(t)^{-1} = \bar{P}_s.$$

Here $\bar{P}_s$ is the projector onto the stable subspace along the unstable subspace of the matrix $f_x^-(\bar{x}_0, \bar{\lambda})$.

Corollary 4.7. *Assume* (H1)–(H3). *Then*

- **(I)** *If $z(t)$ is a bounded solution of $\overline{L}_3 z = 0$, there holds $z(0) \in \mathcal{R}(\bar{P}(0))$.*
- **(II)** *Since* $\dim \mathcal{R}(\bar{P}_s) = 1$, *we have* $\dim \mathcal{R}(\bar{P}(t)) = 1$ *for all* $t \in \mathbb{R}^+$.
- **(III)** $\dot{\bar{z}}(t) = \dfrac{d\bar{z}(t)}{dt}$ *is a bounded solution of equation $\overline{L}_3 z(t) = 0$ and*

$$\mathcal{N}(\overline{L}_3) = \operatorname{span}\{\dot{\bar{z}}(\cdot)\}, \qquad \mathcal{R}(\bar{P}(0)) = \operatorname{span}\{\dot{\bar{z}}(0)\}.$$

Define the adjoint operator of $\overline{L}_3$ as

$$\overline{L}_3^* : B^{1,+} \to B^{0,+}, \qquad \overline{L}_3^*\, w(\cdot) = -\dot{w}(\cdot) - f_x^-(\bar{z}(\cdot), \bar{\lambda})^T w(\cdot)$$

Lemma 4.8. *Assume* (H1)–(H3). *Then the adjoint operator $\overline{L}_3^*$ has a fundamental matrix $\bar{Z}^{-1}(t)^T$ and has an exponential dichotomy on $\mathbb{R}^+$ with data $(I - \bar{P}(\cdot)^T, \bar{K}_2^+, \bar{K}_1^+, \bar{\alpha}_2^+, \bar{\alpha}_1^+)$. Moreover,*

$$\mathcal{R}(I - \bar{P}(0)^T) = \{\xi \in \mathbb{R}^2 : \sup_{t\in\mathbb{R}^+} \|\bar{Z}^{-1}(t)^T \xi\| < \infty\} = \mathcal{R}(\bar{P}(0))^\perp.$$

Corollary 4.9. *Assume* (H1)–(H3). *Then for any bounded solution $z(t)$ of $\overline{L}_3 z = 0$ and any bounded solution $w(t)$ of $\overline{L}_3^* w = 0$, we have*

$$(z(t),\ w(t)) = (\bar{Z}(t)z(0),\ \bar{Z}^{-1}(t)^T w(0)) = (z(0),\ w(0)) = 0.$$

Moreover, $\mathcal{N}(L_3^*) = span\{\bar{Z}^{-1}(t)^T \dot{\bar{z}}(0)^\perp\}$.

Next, we study the properties of the operator $\overline{L}_2$. Let $\bar{Y}(t)$ be the fundamental matrix of $\overline{L}_2$ satisfying $\bar{Y}(0) = I$, then the adjoint operator of $\overline{L}_2$ is defined as

$$\overline{L}_2^* : C^1([0,1], \mathbb{R}^2) \to C^0([0,1], \mathbb{R}^2), \quad \overline{L}_2^*\, y(\cdot) = -\dot{y}(\cdot) - \overline{T} f_x^+(\bar{y}(\cdot), \bar{\lambda})^T y(\cdot),$$

with a fundamental matrix $\bar{Y}^{-1}(t)^T$.

Remark 4.10. Notice that $\dot{\bar{y}}(t) = \dfrac{d\bar{y}}{dt}(t)$ is a bounded solution of equation $\overline{L}_2(y) = 0$, thus $\dot{\bar{y}}(t) = \bar{Y}(t)\dot{\bar{y}}(0)$. Moreover, $\dim \mathcal{N}(\overline{L}_2) = 2$.

Lemma 4.11. *Assume* (H1)–(H4). *Then the solution space of equations* (8)–(13) *is always one dimensional.*

Proof. Define

$$(\bar{u}(t), \bar{v}(t), \bar{w}(t)) = (c_1 \bar{X}^{-1}(t)^T \dot{\bar{x}}(0)^\perp,\ Y^{-1}(t)^T \dot{\bar{y}}(0)^\perp, c_2 \bar{Z}^{-1}(t)^T \dot{\bar{z}}(0)^\perp),$$

then from Corollaries 4.4 and 4.9, it follows that $(\bar{u}, \bar{v}, \bar{w})$ is a nontrivial solution of equations (8), (9) and (11) for any $c_1, c_2 \in \mathbb{R}$.

By Remark 4.10, we have

$$\int_0^1 \bar{v}(t)^T f^+(\bar{y}(t), \bar{\lambda}) dt = \frac{1}{\overline{\overline{T}}} \int_0^1 \bar{v}(t)^T \dot{\bar{y}}(t) dt$$

$$= \frac{1}{\overline{\overline{T}}} \int_0^1 (\bar{Y}^{-1}(t)^T \dot{\bar{y}}(0)^\perp)^T \bar{Y}(t) \dot{\bar{y}}(0) dt = 0,$$

which leads to equation (10).

Now we only need to find appropriate constants c_1 and c_2 such that equations (12) and (13) hold.

Denoted by $\dot{\bar{x}}(0) = (\bar{x}_1, \bar{x}_2)^T$ and $\dot{\bar{y}}(0) = (\widetilde{x}_1, \widetilde{x}_2)^T$. Then equation (12) turns out to be

$$(c_1 \dot{\bar{x}}(0)^\perp - \dot{\bar{y}}(0)^\perp)^T \vec{n}^\perp = (c_1(-\bar{x}_2, \bar{x}_1)^T - (-\widetilde{x}_2, \widetilde{x}_1)^T)^T (0,1)^T = c_1 \bar{x}_1 - \widetilde{x}_1 = 0.$$

From the transversality condition (H4), $\bar{x}_1 \neq 0$ and $\widetilde{x}_1 \neq 0$, thus $c_1 = \widetilde{x}_1 / \bar{x}_1 \neq 0$. And c_2 is obtained in a similar way. □

Next, we investigate the properties of the Null space $\mathcal{N}(L)$.

Theorem 4.12. *Assume* (H1)–(H4). *Then*

in Case (I), dim $\mathcal{N}(\overline{L}) = 1$ *and* $\bar{Y}(1)\ \mathcal{N}(\bar{Q}(0)) = \mathcal{R}(\bar{P}(0))$. *Precisely,*

$$\mathcal{N}(\overline{L}) \;=\; \text{span}\{(\dot{\bar{x}},\ (\bar{\theta}^1/\overline{T})\dot{\bar{y}},\ (\bar{\theta}^1/\bar{\theta}^2)\dot{\bar{z}})\}.$$

In Case (II), dim $\mathcal{N}(\overline{L}) = 0$ *and* $\bar{Y}(1)\ \mathcal{N}(\bar{Q}(0)) \neq \mathcal{R}(\bar{P}(0))$.

In Case (III), *if* $\bar{Y}(1)\ \mathcal{N}(\bar{Q}(0)) = \mathcal{R}(\bar{P}(0))$, *there holds* dim $\mathcal{N}(\overline{L}) = 1$, *otherwise* dim $\mathcal{N}(\overline{L}) = 0$.

Proof. We first prove Case (I). It follows from Corollary 4.2 (III) that

$$\dim\ \mathcal{N}(\overline{L}) \;\leqslant\; \dim\ \mathcal{N}(\overline{L}_1) \;=\; 1.$$

Assume $(x^*(\cdot), y^*(\cdot), z^*(\cdot)) \in \mathcal{N}(\overline{L})$. Then from Corollary 4.2 (III) and Corollary 4.7 (III), there exist constants c_1 and c_2 such that

$$x^*(t) \;=\; c_1 \dot{\bar{x}}(t), \qquad\qquad z^*(t) \;=\; c_2 \dot{\bar{z}}(t).$$

From $x^*(0) = y^*(0)$, we have $y^*(t) = \bar{Y}(t) y^*(0) = \bar{Y}(t) c_1 \dot{\bar{x}}(0)$. Therefore, dim $\mathcal{N}(\overline{L}) = 1$ if and only if the last equation in (17) holds for nonzero c_1 and c_2, i.e.,

$$c_1 \bar{Y}(1)\ \dot{\bar{x}}(0) \;=\; c_2 \dot{\bar{z}}(0).$$

Thus if $\bar{Y}(1)\dot{\bar{x}}(0)$ and $\dot{\bar{z}}(0)$ are linearly dependent, dim $\mathcal{N}(\overline{L}) = 1$, otherwise dim $\mathcal{N}(\overline{L}) = 0$.

Notice that $\dot{\bar{x}}(0) = f^-(\bar{M}^1, \bar{\lambda})$, $\dot{\bar{y}}(0) = \overline{T} f^+(\bar{M}^1, \bar{\lambda})$, $\dot{\bar{y}}(1) = \overline{T} f^+(\bar{M}^2, \bar{\lambda})$ and $\dot{\bar{z}}(0) = f^-(\bar{M}^2, \bar{\lambda})$, then from (6), we have

$$\begin{aligned}\bar{Y}(1)\dot{\bar{x}}(0) &= \bar{Y}(1) f^-(\bar{M}^1, \bar{\lambda}) = \bar{Y}(1)\bar{\theta}^1 f^+(\bar{M}^1, \bar{\lambda}) = \bar{Y}(1)(\bar{\theta}^1/\overline{T})\dot{\bar{y}}(0)\\ &= (\bar{\theta}^1/\overline{T})\dot{\bar{y}}(1) = \bar{\theta}^1 f^+(\bar{M}^2, \bar{\lambda}) = (\bar{\theta}^1/\bar{\theta}^2) f^-(\bar{M}^2, \bar{\lambda}) = (\bar{\theta}^1/\bar{\theta}^2)\dot{\bar{z}}(0).\end{aligned}$$

Let $c_1 = 1$ and $c_2 = \bar{\theta}^1/\bar{\theta}^2$, then $\mathcal{N}(\overline{L}) = \operatorname{span}\{(\dot{\bar{x}},\ (\bar{\theta}^1/\overline{T})\dot{\bar{y}},\ (\bar{\theta}^1/\bar{\theta}^2)\dot{\bar{z}})\}$. And from Corollary 4.2 (III) and Corollary 4.7 (III),

$$\operatorname{span}\{\bar{Y}(1)\ \dot{\bar{x}}(0)\} = \bar{Y}(1)\mathcal{N}(\bar{Q}(0)), \qquad \operatorname{span}\{\dot{\bar{z}}(0)\} = \mathcal{R}(\bar{P}(0)),$$

thus $\bar{Y}(1)\ \mathcal{N}(\bar{Q}(0)) = \mathcal{R}(\bar{P}(0))$.

The other two cases can be proved in a similar way. □

Corollary 4.13. *Assume* (H1)–(H4). *If* $\bar{Y}(1)\ \mathcal{N}(\bar{Q}(0)) = \mathcal{R}(\bar{P}(0))$ *then* $\dim\ \mathcal{N}(\overline{L}) = 1$, *otherwise* $\dim\ \mathcal{N}(\overline{L}) = 0$.

Now, we present the properties of the space $\mathcal{R}(\overline{L})$ via the following Lemma 4.14 and Lemma 4.17.

Lemma 4.14. *Assume* (H1)–(H4). *Then for any* $(x, y, z) \in B^1$ *and for any* $(u, v, w, \mu, \omega) \in B^1 \times \mathbb{R}^2 \times \mathbb{R}^2$, *there holds*

$$\begin{aligned}\langle \overline{L}(x,y,z),\ (u,v,w,\mu,\omega)\rangle &= \langle (x,y,z,x(0),y(0),y(1),z(0)),\ \overline{L}^*(u,v,w,\mu,\omega)\rangle\\ &\quad + \lim_{t\to-\infty} x(t)^T u(t) + \lim_{t\to\infty} z(t)^T w(t),\end{aligned} \tag{19}$$

where the adjoint operator $\overline{L}^*$ *is defined as*

$$\overline{L}^* \begin{pmatrix} u(\cdot)\\ v(\cdot)\\ w(\cdot)\\ \mu\\ \omega\end{pmatrix} = \begin{pmatrix} -\dot{u} - f_x^-(\bar{x},\bar{\lambda})^T\ u\\ -\dot{v} - \overline{T}\ f_x^+(\bar{y},\bar{\lambda})^T\ v\\ -\dot{w} - f_x^-(\bar{z},\bar{\lambda})^T\ w\\ u(0) + \mu\\ -v(0) - \mu\\ v(1) + \omega\\ -w(0) - \omega\end{pmatrix} = \begin{pmatrix} \overline{L}_1^*\ u(\cdot)\\ \overline{L}_2^*\ v(\cdot)\\ \overline{L}_3^*\ w(\cdot)\\ u(0) + \mu\\ -v(0) - \mu\\ v(1) + \omega\\ -w(0) - \omega\end{pmatrix}.$$

Proof. Integrating by part, we obtain

$$\begin{aligned}&\langle \overline{L}(x,y,z),\ (u,v,w,\mu,\omega)\rangle\\ &= \int_{-\infty}^{0} (\overline{L}_1 x)^T\ udt + \int_0^1 (\overline{L}_2 y)^T\ vdt + \int_0^\infty (\overline{L}_3 z)^T\ wdt + (x(0) - y(0))^T\mu\\ &\quad + (y(1) - z(0))^T\omega\\ &= x^T u|_{-\infty}^0 - \int_{-\infty}^0 x^T(\dot{u} + f_x^-(\bar{x},\bar{\lambda})^T u)dt + y^T v|_0^1 - \int_0^1 y^T(\dot{v} + \overline{T} f_x^+(\bar{y},\bar{\lambda})^T v)dt\\ &\quad + z^T w|_0^\infty - \int_0^\infty z^T(\dot{w} + f_x^-(\bar{z},\bar{\lambda})^T w)dt + (x(0) - y(0))^T\mu + (y(1) - z(0))^T\omega\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} x^T(\overline{L}_1^* u)\, dt + \int_0^1 y^T(\overline{L}_2^* v)\, dt + \int_0^{\infty} z^T(\overline{L}_3^* w)\, dt \\
&\quad + x(0)^T(u(0)+\mu) - y(0)^T(v(0)+\mu) + y(1)^T(v(1)+\omega) - z(0)^T(w(0)+\omega) \\
&\quad + \lim_{t\to-\infty} x(t)^T u(t) + \lim_{t\to\infty} z(t)^T w(t) \\
&= \langle (x,y,z,x(0),y(0),y(1),z(0)),\ \overline{L}^*(u,v,w,\mu,\omega)\rangle \\
&\quad + \lim_{t\to-\infty} x(t)^T u(t) + \lim_{t\to\infty} z(t)^T w(t). \qquad \square
\end{aligned}
$$

Similar to Theorem 4.12, we have

Theorem 4.15. *Assume* (H1)–(H4). *Then*

$$
\begin{aligned}
&\textit{in}\ \text{Case (I)}, && \dim\ \mathcal{N}(\overline{L}^*) = 1, \\
&\textit{in}\ \text{Case (II)}, && \dim\ \mathcal{N}(\overline{L}^*) = 0, \\
&\textit{in}\ \text{Case (III)}, && \textit{if}\ \bar{Y}(1)\mathcal{N}(\bar{Q}(0)) = \mathcal{R}(\bar{P}(0)),\ \dim\ \mathcal{N}(\overline{L}^*) = 1, \\
& && \textit{otherwise}\ \dim\ \mathcal{N}(\overline{L}^*) = 0.
\end{aligned}
$$

Remark 4.16. This theorem shows that if we replace the equations (12) and (13) by $u(0)-v(0)=0$ and $v(1)-w(0)=0$, respectively, equations (8)–(13) will have only trivial solution in the case $\dim\mathcal{N}(\overline{L}^*)=0$.

Lemma 4.17. *Assume* (H1)–(H4). *Then* $(f_1(\cdot), f_2(\cdot), f_3(\cdot), f_4, f_5) \in \mathcal{R}(\overline{L})$, *if and only if*

$$
\langle (f_1,f_2,f_3,f_4,f_5),(u,v,w,\mu,\omega)\rangle = 0, \quad \forall (u(\cdot),v(\cdot),w(\cdot),\mu,\omega)\in\mathcal{N}(\overline{L}^*) \tag{20}
$$

Proof. The necessity follows from equation (19) and the fact $u(t), w(t) \to 0$ exponentially fast as $t\to\mp\infty$.

The sufficiency means that there exists $(x^*,y^*,z^*)\in B^1$, such that

$$
\overline{L}\ (x^*,y^*,z^*) = (f_1(\cdot), f_2(\cdot), f_3(\cdot), f_4, f_5)^T,
$$

if $(f_1,f_2,f_3,f_4,f_5)\in B^0\times\mathbb{R}^2\times\mathbb{R}^2$ satisfies condition (20).

For any given $\xi,\ \eta\in\mathbb{R}^2$, define three functions as follows

$$
\begin{aligned}
x^*(t) =&\ \bar{X}(t)(I-\bar{Q}(0))\xi + \textstyle\int_{-\infty}^t \bar{Q}(t)\bar{X}(t)\bar{X}^{-1}(s)f_1(s)ds \\
&- \textstyle\int_t^0 (I-\bar{Q}(t))\bar{X}(t)\bar{X}^{-1}(s)f_1(s)ds, && t\leqslant 0, \\
y^*(t) =&\ \bar{Y}(t)y^*(0) + \textstyle\int_0^t \bar{Y}(t)\bar{Y}^{-1}(s)f_2(s)ds, && t\in[0,1], \\
z^*(t) =&\ \bar{Z}(t)\bar{P}(0)\eta + \textstyle\int_0^t \bar{P}(t)\bar{Z}(t)\bar{Z}^{-1}(s)f_3(s)ds \\
&- \textstyle\int_t^\infty (I-\bar{P}(t))\bar{Z}(t)\bar{Z}^{-1}(s)f_3(s)ds, && t\geqslant 0.
\end{aligned}
$$

Then, (x^*,y^*,z^*) is a bounded solution of equation

$$
\overline{L}(x,y,z) = (f_1,f_2,f_3,f_4,f_5),
$$

if and only if there exist ξ and η, such that $x^*(0)-y^*(0)=f_4$ and $y^*(1)-z^*(0)=f_5$, which read respectively

$$(I-\bar{Q}(0))\xi + \int\limits_{-\infty}^{0} \bar{Q}(0)\bar{X}^{-1}(s)f_1(s)ds - y^*(0) = f_4,$$

$$\bar{Y}(1)y^*(0) - \bar{P}(0)\eta + \int\limits_{0}^{1} \bar{Y}(1)\bar{Y}^{-1}(s)f_2(s)ds + \int\limits_{0}^{\infty}(I-\bar{P}(0))\bar{Z}^{-1}(s)f_3(s)ds = f_5.$$

Eliminating $y^*(0)$ from these two equations we obtain

$$\begin{aligned} & \bar{P}(0)\eta - \bar{Y}(1)\ (I-\bar{Q}(0))\xi \\ = \ & \bar{Y}(1)\int_{-\infty}^{0} \bar{Q}(0)\bar{X}^{-1}(s)f_1(s)ds + \int_0^1 \bar{Y}(1)\bar{Y}^{-1}(s)f_2(s)ds \\ & + \int_0^{\infty}(I-\bar{P}(0))\bar{Z}^{-1}(s)f_3(s)ds - \bar{Y}(1)f_4 - f_5. \end{aligned} \tag{21}$$

To simplify the notations, we write the total terms in the right-hand side of (21) as h. Then the equation (21) is solvable if

$$h \ \in \ \mathcal{R}(\bar{P}(0)) \ + \ \mathcal{R}(\bar{Y}(1)(I-\bar{Q}(0))). \tag{22}$$

If $\mathcal{R}(\bar{P}(0)) \neq \mathcal{R}(\bar{Y}(1)(I-\bar{Q}(0))$, we have

$$\mathcal{R}(\bar{P}(0)) + \mathcal{R}(\bar{Y}(1)(I-\bar{Q}(0))) \ = \ \mathbb{R}^2,$$

therefore equation (22) holds. Otherwise, we only need to prove $\kappa^T h = 0$ for any $\kappa \in \mathcal{N}(\bar{P}(0)^T)$. Direct calculation gives

$$\begin{aligned} \kappa^T h &= \int_{-\infty}^{0}[\bar{X}^{-1}(s)^T\bar{Q}(0)^T\bar{Y}(1)^T\kappa]^T f_1(s)ds + \int_0^1[(\bar{Y}^{-1}(s)^T\bar{Y}(1)^T\kappa]^T f_2(s)ds \\ &\quad + \int_0^{\infty}[(\bar{Z}^{-1}(s)^T(I-\bar{P}(0)^T)\kappa]^T f_3(s)ds - \kappa^T\bar{Y}(1)f_4 - \kappa^T f_5 \\ &= \langle (u^*(\cdot), v^*(\cdot), w^*(\cdot), \mu^*, \omega^*), (f_1(\cdot), f_2(\cdot), f_3(\cdot), f_4, f_5)\rangle, \end{aligned}$$

where

$$\begin{array}{rcll} u^*(t) &=& \bar{X}^{-1}(t)^T\ \bar{Q}(0)^T\ \bar{Y}(1)^T\kappa, & t \leqslant 0, \\ v^*(t) &=& \bar{Y}^{-1}(t)^T\ \bar{Y}(1)^T\kappa, & t \in [0,1], \\ w^*(t) &=& \bar{Z}^{-1}(t)^T\ (I-\bar{P}(0)^T)\kappa, & t \geqslant 0, \\ \mu^* &=& -\bar{Y}(1)^T\kappa, & \\ \omega^* &=& -\kappa. & \end{array} \tag{23}$$

Clearly, $\kappa^T h = 0$ if $(u^*, v^*, w^*, \mu^*, \omega^*) \in \mathcal{N}(\overline{L}^*)$. It follows from Lemmas 4.3 and 4.8 that

$$\overline{L}_1^* u^* \ = \ 0, \quad \overline{L}_2^* v^* \ = \ 0 \quad \text{and} \quad \overline{L}_3^* w^* \ = \ 0.$$

Since $\kappa \in \mathcal{N}(\bar{P}(0)^T)$, we have $\bar{P}(0)^T\kappa = 0$, hence $\kappa = (I-\bar{P}(0)^T)\kappa$, therefore $v^*(1) = w^*(0)$. On the other hand,

$$\mathcal{N}(\bar{P}(0)^T) = (\mathcal{R}(\bar{P}(0)))^{\perp} = (\mathcal{R}(\bar{Y}(1)(I-\bar{Q}(0)))^{\perp} = \mathcal{N}((I-\bar{Q}(0)^T)\bar{Y}(1)^T),$$

therefore $\kappa \in \mathcal{N}(\bar{P}(0)^T)$ implies that

$$(I-\bar{Q}(0))^T\ \bar{Y}(1)^T\kappa = 0.$$

Hence $\bar{Q}(0)^T\ \bar{Y}(1)^T\kappa = \bar{Y}(1)^T\kappa$ which leads to $u^*(0) = v^*(0)$. $\square$

By now, we have proved the following theorem which states the Fredholm properties of the linear operator $\overline{L}$.

Theorem 4.18. *Assume* (H1)–(H4). *Then the linear operator* $\overline{L}$ *is Fredholm of index 0.*

Corollary 4.19. *Assume* (H1)–(H4). *If* $\bar{Y}(1)\ \mathcal{N}(\bar{Q}(0)) \;=\; \mathcal{R}(\bar{P}(0))$, $\operatorname{codim}\mathcal{R}(\overline{L}) = 1$, *otherwise* $\overline{L}$ *is onto.*

Now, we are ready to complete the proof of Theorem 3.4.

Proof of Theorem 3.4. We only need to prove that equation

$$D\overline{F}(x, y, z, T, \lambda) = 0 \tag{24}$$

has only trivial solution. Let $(x, y, z, T, \lambda) \in B^1 \times \mathbb{R} \times \mathbb{R}$ satisfy equation (24), which reads

$$\overline{L}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \overline{L}_1\ x \\ \overline{L}_2\ y \\ \overline{L}_3\ z \\ x(0) \;-\; y(0) \\ y(1) \;-\; z(0) \end{pmatrix} = \begin{pmatrix} f_\lambda^-(\bar{x}, \bar{\lambda})\lambda \\ \overline{T} f_\lambda^+(\bar{y}, \bar{\lambda})\lambda + f^+(\bar{y}, \bar{\lambda})T \\ f_\lambda^-(\bar{z}, \bar{\lambda})\lambda \\ 0 \\ 0 \end{pmatrix} \tag{25}$$

and

$$\vec{n}^T x(0) \;=\; 0, \qquad \vec{n}^T y(1) \;=\; 0. \tag{26}$$

We need to prove that $(x, y, z, T, \lambda) = 0$.

Let $(\bar{u}, \bar{v}, \bar{w}) \in B^1$ be any nontrivial solution of equations (8)–(13). Multiplying equation (25) by $(\bar{u}, \bar{v}, \bar{w}, 0, 0)$ and by virtue of equation (10) we get,

$$\langle(\bar{u}, \bar{v}, \bar{w}, 0, 0), \overline{L}(x, y, z)\rangle = \left(\int_{-\infty}^{0} \bar{u}^T f_\lambda^-\ dt + \int_0^1 \overline{T}\bar{v}^T f_\lambda^+\ dt + \int_0^\infty \bar{w}^T f_\lambda^-\ dt \right) \lambda. \tag{27}$$

Then, the nondegeneracy (7) implies that $\lambda = 0$ if and only if the left hand side of equation (27) vanishes. By Lemma 4.14, we have

$$\begin{aligned}
&\langle(\bar{u}, \bar{v}, \bar{w}, 0, 0), \overline{L}(x, y, z)\rangle \\
&= \langle \overline{L}^*(\bar{u}, \bar{v}, \bar{w}, 0, 0), (x, y, z, x(0), y(0), y(1), z(0))\rangle \\
&\quad + \lim_{t\to-\infty} x(t)^T \bar{u}(t) + \lim_{t\to\infty} z(t)^T \bar{w}(t) \\
&= \langle(0, 0, 0, \bar{u}(0), -\bar{v}(0), \bar{v}(1), -\bar{w}(0)), (x, y, z, x(0), y(0), y(1), z(0))\rangle \\
&= \bar{u}(0)^T x(0) - \bar{v}(0)^T y(0) - \bar{v}(0)^T y(1) - \bar{w}(0)^T z(0),
\end{aligned}$$

where the limits vanish because that $\bar{u}$ ($\bar{\omega}$) tends to zero exponentially fast as $t \to -\infty$ (∞) (see Lemma 4.1). From equation (12) it follows that $\bar{u}(0) - \bar{v}(0) = c\vec{n}$ for some constant $c \in \mathbb{R}$. Then by the fourth equation of (25) and (26), we obtain

$$\bar{u}(0)^T x(0) - \bar{v}(0)^T y(0) \;=\; (\bar{u}(0) - \bar{v}(0))^T x(0) \;=\; c\vec{n}^T x(0) = 0.$$

Similarly $-\bar{v}(0)^T y(1) - \bar{w}(0)^T z(0) = 0$.

Now equation (25) becomes,

$$(\overline{L}_1 x, \overline{L}_2 y, \overline{L}_3 z, x(0) - y(0), y(1) - z(0))^T = (0, f^+(\bar{y}, \bar{\lambda})T, 0, 0, 0)^T. \tag{28}$$

From Corollary 4.2 (III), $x = c_1 \dot{\bar{x}}$ for some $c_1 \in \mathbb{R}$. Then by (26) and the assumption (H4), $c_1 = 0$, thus $x = 0$. Similarly $z = 0$.

Finally, y and T satisfy the following boundary value problem due to equation (28)

$$\overline{L}_2 y \;=\; f^+(\bar{y}, \bar{\lambda})\, T, \quad y(0) \;=\; 0, \quad y(1) \;=\; 0.$$

Then $y(t) \;=\; \bar{Y}(t)y(0) \;+\; T\int_0^t \bar{Y}(t)\; \bar{Y}^{-1}(s)\; f^+(\bar{y}(s), \bar{\lambda})ds$. By Remark 4.10,

$$\begin{aligned} 0 \;= y(1) \;&=\; T\int_0^1 \bar{Y}(1)\bar{Y}^{-1}(s) f^+(\bar{y}(s), \bar{\lambda})ds \;=\; T/\overline{T}\int_0^1 \bar{Y}(1)\bar{Y}^{-1}(s)\dot{\bar{y}}(s)ds \\ &= T/\overline{T}\int_0^1 \bar{Y}(1)\bar{Y}^{-1}(s)\bar{Y}(s)\dot{\bar{y}}(0)ds = \; T/\overline{T}\int_0^1 \dot{\bar{y}}(1)ds \;=\; T/\overline{T}\;\dot{\bar{y}}(1), \end{aligned}$$

therefore $T = 0$, which implies $y = 0$. □

5. Truncation to a finite interval

In order to obtain a numerical HOP, we truncate the orbit pieces $x(t)$ and $z(t)$ on a finite interval using the projection boundary conditions, for more details see [4,14,15,24,29]. Precisely, we consider the following boundary value problem on a finite interval $[-\mathcal{T}_-, \mathcal{T}_+]$ with $\mathcal{T}_\pm > 0$.

$$\dot{x}(t) = f^-(x(t), \lambda), \qquad t \in (-\mathcal{T}_-, 0), \tag{29}$$

$$\dot{y}(t) = Tf^+(y(t), \lambda), \qquad t \in (0, 1), \tag{30}$$

$$\dot{z}(t) = f^-(z(t), \lambda), \qquad t \in (0, \mathcal{T}_+), \tag{31}$$

$$b^-(x(-\mathcal{T}_-), \lambda) = 0, \tag{32}$$

$$b^+(z(\mathcal{T}_+), \lambda) = 0, \tag{33}$$

$$x(0) = y(0), \tag{34}$$

$$y(1) = z(0), \tag{35}$$

$$\vec{n}^T x(0) = 0, \tag{36}$$

$$\vec{n}^T y(1) = 0, \tag{37}$$

where $b^\pm(x, \lambda) = 0$ represents a standard projection boundary condition such that its zeroes are on some computable approximation of unstable (stable) manifold of the hyperbolic equilibrium $\bar{x}_\lambda$ at parameter λ.

We will investigate the existence of solutions to equations (29)–(37) and their error estimates. We use a well-known linearization technique (see for examples, Keller [22]; Vainikko [31]; Beyn [4]). The basic tool for analyzing the nonlinear problem (29)–(37) is the following perturbation lemma. See Vainikko (§3, [31]) for a proof.

Lemma 5.1. *Let $\mathcal{F} : U_\delta(\mathcal{X}_\mathbb{J}) \to \mathcal{B}_2$ be a C^1-mapping from some ball $U_\delta(\mathcal{X}_\mathbb{J})$ of radius δ in a Banach space $\mathcal{B}_1$ into some Banach space $\mathcal{B}_2$. Assume that $D\mathcal{F}(\mathcal{X}_\mathbb{J})$ is a homeomorphism and that for some constants κ, σ we have*

$$\|D\mathcal{F}(\mathcal{X}) - D\mathcal{F}(\mathcal{X}_\mathbb{J})\| \leqslant \kappa < \sigma \leqslant \|D\mathcal{F}(\mathcal{X}_\mathbb{J})^{-1}\|^{-1}, \quad \forall \mathcal{X} \in U_\delta(\mathcal{X}_\mathbb{J}), \tag{38}$$

$$\|\mathcal{F}(\mathcal{X}_\mathbb{J})\| \leqslant (\sigma - \kappa)\ \delta. \tag{39}$$

Then $\mathcal{F}$ has a unique zero $\mathcal{X}_0$ in $U_\delta(\mathcal{X}_\mathbb{J})$ and the following estimates hold

$$\|\mathcal{X}_0 - \mathcal{X}_\mathbb{J}\| \leqslant (\sigma - \kappa)^{-1}\|\mathcal{F}(\mathcal{X}_\mathbb{J})\|, \tag{40}$$

$$\|\mathcal{X}_1 - \mathcal{X}_2\| \leqslant (\sigma - \kappa)^{-1}\|\mathcal{F}(\mathcal{X}_1) - \mathcal{F}(\mathcal{X}_2)\|, \qquad \forall \mathcal{X}_1, \mathcal{X}_2 \in U_\delta(\mathcal{X}_\mathbb{J}).$$

For simplicity, denoted by $\mathbb{J}^- = [-\mathcal{T}_-, 0]$, $\mathbb{J}^+ = [0, \mathcal{T}_+]$ and $\mathbb{J} = [-\mathcal{T}_-, \mathcal{T}_+]$. By saying that $\mathbb{J}$ is sufficiently large, we mean that $\mathbb{J}^-$ and $\mathbb{J}^+$ are sufficiently large, respectively. We use the spaces $C^k(\mathbb{J}^\pm, \mathbb{R}^2)$ $(k = 0, 1)$ equipped with the standard C^k-norm denoted by $\|\cdot\|_k$. The restriction of a function $x \in B^{k,\pm}$ to the interval $\mathbb{J}^\pm$ is written as $x|_{\mathbb{J}^\pm}$, which belongs to $C^k(\mathbb{J}^\pm, \mathbb{R}^2)$.

The following theorem ensures the existence of truncated connecting orbits and provides a way to approximate the HOP using numerical methods.

Theorem 5.2. *Assume* (H1)–(H5) *and that $b^\pm(\cdot) \in C^1(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R})$ satisfies*

$$b^\pm(\bar{x}_0, \bar{\lambda}) = 0, \qquad b_x^-(\bar{x}_0, \bar{\lambda})E_s(\bar{\lambda}) \neq 0, \qquad b_x^+(\bar{x}_0, \bar{\lambda})E_u(\bar{\lambda}) \neq 0, \tag{41}$$

where $E_{s,u}(\bar{\lambda})$ are the stable and unstable eigenspaces of $f_x^-(\bar{x}_0, \bar{\lambda})$, respectively.

Then there exist constants $\delta > 0$ and $C > 0$, such that for sufficiently large $\mathbb{J}^-$ and $\mathbb{J}^+$, the finite boundary-value problem (29)–(37) *has a unique solution $(x_\mathbb{J}, y_\mathbb{J}, z_\mathbb{J}, T_\mathbb{J}, \lambda_\mathbb{J})$ in the tube*

$$\begin{aligned} K_\delta \;=\; & \{(x, y, z, T, \lambda) \in C^1(\mathbb{J}^-, \mathbb{R}^2) \times C^1([0,1], \mathbb{R}^2) \times C^1(\mathbb{J}^+, \mathbb{R}^2) \times \mathbb{R} \times \mathbb{R} : \\ & \|x - \bar{x}|_{\mathbb{J}^-}\|_1 + \|y - \bar{y}\|_1 + \|z - \bar{z}|_{\mathbb{J}^+}\|_1 + |T - \overline{T}| + |\lambda - \bar{\lambda}| \leqslant \delta\} \end{aligned}$$

and the following estimate holds

$$\begin{aligned} & \|x_\mathbb{J} - \bar{x}|_{\mathbb{J}^-}\|_1 + \|y_\mathbb{J} - \bar{y}\|_1 + \|z_\mathbb{J} - \bar{z}|_{\mathbb{J}^+}\|_1 + |T_\mathbb{J} - \overline{T}| + |\lambda_\mathbb{J} - \bar{\lambda}| \\ & \qquad \leqslant C\ (|b^-(\bar{x}(-\mathcal{T}_-), \bar{\lambda})| + |b^+(\bar{z}(\mathcal{T}_+), \bar{\lambda})|). \end{aligned} \tag{42}$$

Proof. We apply Lemma 5.1 with the settings

$$\begin{aligned} \mathcal{B}_1 = \;& C^1(\mathbb{J}^-, \mathbb{R}^2) \times C^1([0,1], \mathbb{R}^2) \times C^1(\mathbb{J}^+, \mathbb{R}^2) \times \mathbb{R} \times \mathbb{R}, \\ \mathcal{B}_2 = \;& C^0(\mathbb{J}^-, \mathbb{R}^2) \times C^0([0,1], \mathbb{R}^2) \times C^0(\mathbb{J}^+, \mathbb{R}^2) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}, \\ \mathcal{X}_\mathbb{J} = \;& (\bar{x}|_{\mathbb{J}^-},\ \bar{y},\ \bar{z}|_{\mathbb{J}^+},\ \overline{T},\ \bar{\lambda}), \end{aligned}$$

and

$$\begin{aligned} & \mathcal{F}(x, y, z, T, \lambda) \\ = \;& (\dot{x} - f^-(x, \lambda),\ \dot{y} - Tf^+(y, \lambda),\ \dot{z} - f^-(z, \lambda), b^-(x(-\mathcal{T}_-), \lambda), \\ & b^+(z(\mathcal{T}_+), \lambda),\ x(0) - y(0),\ y(1) - z(0),\ \vec{n}^T x(0),\ \vec{n}^T y(1)). \end{aligned}$$

Suppose and we will prove later that $\|D\mathcal{F}(\mathcal{X}_{\mathbb{J}})^{-1}\|$ has a uniform bound σ^{-1} for all sufficiently large $\mathbb{J}$, i.e., the following estimate holds for any given $(\phi(\cdot), \psi(\cdot), \varphi(\cdot), \kappa_1, \kappa_2, \varrho_1, \varrho_2, \epsilon_1, \epsilon_2) \in \mathcal{B}_2$ and for all sufficiently large $\mathbb{J}$,

$$\begin{aligned} &\|x\|_1 + \|y\|_1 + \|z\|_1 + |T| + |\lambda| \\ &\leqslant \sigma^{-1}(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + |\kappa_1| + |\kappa_2| + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2|), \end{aligned} \tag{43}$$

where (x, y, z, T, λ) is the solution of the following variational equation,

$$D\mathcal{F}(\mathcal{X}_{\mathbb{J}})\ (x, y, z, T, \lambda) \ = \ (\phi, \psi, \varphi, \kappa_1, \kappa_2, \varrho_1, \varrho_2, \epsilon_1, \epsilon_2). \tag{44}$$

Then, we find a $\delta > 0$ by the smoothness properties of $f^{\pm}$ and $b^{\pm}$, such that condition (38) holds with $\kappa = \frac{1}{2}\sigma$ for all sufficiently large $\mathbb{J}$.

Condition (39) holds for sufficiently large $\mathbb{J}$ by virtue of assumption (41) and the fact

$$\begin{aligned} \|\mathcal{F}(\mathcal{X}_{\mathbb{J}})\| =&\ \|\mathcal{F}(\bar{x}|_{\mathbb{J}^-}, \bar{y}, \bar{z}|_{\mathbb{J}^+}, \overline{T}, \bar{\lambda})\| \\ =&\ |b^-(\bar{x}(-\mathcal{T}_-), \bar{\lambda})| + |b^+(\bar{z}(\mathcal{T}_+), \bar{\lambda})| \to 0, \quad \text{as } \mathcal{T}_-, \mathcal{T}_+ \to \infty. \end{aligned} \tag{45}$$

Therefore by Lemma 5.1, $\mathcal{F}$ has a unique zero $\mathcal{X}_0 = (x_{\mathbb{J}}, y_{\mathbb{J}}, z_{\mathbb{J}}, T_{\mathbb{J}}, \lambda_{\mathbb{J}})$ in K_δ and the estimate (42) follows from (40).

Now, we prove the estimate (43). Equation (44) turns out to be the following variational equation

$$\dot{x}(t) \ - \ f_x^-(\bar{x}(t), \bar{\lambda})x(t) \ - \ f_\lambda^-(\bar{x}(t), \bar{\lambda})\lambda = \phi(t), \tag{46}$$

$$\dot{y}(t) \ - \ \overline{T} f_x^+(\bar{y}(t), \bar{\lambda})y(t) \ - \ f^+(\bar{y}(t), \bar{\lambda})T \ - \ \overline{T} f_\lambda^+(\bar{y}(t), \bar{\lambda})\lambda = \psi(t), \tag{47}$$

$$\dot{z}(t) \ - \ f_x^-(\bar{z}(t), \bar{\lambda})z(t) \ - \ f_\lambda^-(\bar{z}(t), \bar{\lambda})\lambda = \varphi(t), \tag{48}$$

$$b_x^-(\bar{x}(-\mathcal{T}_-), \bar{\lambda})\ x(-\mathcal{T}_-) \ + \ b_\lambda^-(\bar{x}(-\mathcal{T}_-), \bar{\lambda})\lambda = \kappa_1, \tag{49}$$

$$b_x^+(\bar{z}(\mathcal{T}_+), \bar{\lambda})\ z(\mathcal{T}_+) \ + \ b_\lambda^+(\bar{z}(\mathcal{T}_+), \bar{\lambda})\lambda = \kappa_2, \tag{50}$$

$$x(0) \ - \ y(0) = \varrho_1, \tag{51}$$

$$y(1) \ - \ z(0) = \varrho_2, \tag{52}$$

$$\vec{n}^T\ x(0) = \epsilon_1, \tag{53}$$

$$\vec{n}^T\ y(1) = \epsilon_2. \tag{54}$$

First, we will find an estimate for parameter λ. Let $(\bar{u}, \bar{v}, \bar{w}) \in B^1$ be any nontrivial solution of equations (8)–(13), then there exist two constants $c_1, c_2 \in \mathbb{R}$, such that

$$\bar{u}(0) \ - \ \bar{v}(0) \ = \ c_1\vec{n}, \qquad \bar{v}(1) \ - \ \bar{w}(0) \ = \ c_2\vec{n}. \tag{55}$$

Multiply equations (46), (47) and (48) by the functions $\bar{u}(t)|_{\mathbb{J}^-}, \bar{v}(t)$ and $\bar{w}(t)|_{\mathbb{J}^+}$, respectively, then integrate by part and sum them up to get an equation for λ

$$\begin{aligned} &\left(\int_{-\mathcal{T}_-}^0 \bar{u}^T f_\lambda^- dt \ + \int_0^1 \bar{v}^T f_\lambda^+ dt \ + \ \int_0^{\mathcal{T}_+} \bar{w}^T f_\lambda^- dt \right) \lambda \\ &= \bar{u}(0)^T x(0) - \bar{v}(0)^T y(0) + \bar{v}(1)^T y(1) - \bar{w}(0)^T z(0) - \bar{u}(-\mathcal{T}_-)^T x(-\mathcal{T}_-) \\ &\quad + \bar{w}(\mathcal{T}_+)^T z(\mathcal{T}_+) \ - \ \int_{-\mathcal{T}_-}^0 \bar{u}^T \phi dt \ - \ \int_0^1 \bar{v}^T \psi dt \ - \ \int_0^{\mathcal{T}_+} \bar{w}^T \varphi dt. \end{aligned}$$

The coefficient of λ tends to a nonzero constant as $\mathcal{T}_-, \mathcal{T}_+ \to \infty$ by the nondegeneracy (7). And by equations (55), (51) and (53)

$$\begin{aligned}\bar{u}(0)^T x(0) \; - \; \bar{v}(0)^T y(0) &= (\bar{u}(0) - \bar{v}(0))^T x(0) \; + \; \bar{v}(0)^T (x(0) - y(0)) \\ &= c_1 \vec{n}^T x(0) \; + \; \bar{v}(0)^T \varrho_1 \quad = \quad c_1 \epsilon_1 \; + \; \bar{v}(0)^T \varrho_1,\end{aligned}$$

similarly

$$\bar{v}(1)^T y(1) \; - \; \bar{w}(0)^T z(0) \quad = \quad c_2 \epsilon_2 \; + \; \bar{w}(0)^T \varrho_2.$$

Since $\bar{u}(-\mathcal{T}_-)$ $(\bar{w}(\mathcal{T}_+)) \to 0$ exponentially fast as $\mathcal{T}_-(\mathcal{T}_+) \to \infty$, then we get an initial estimate for λ

$$|\lambda| \leqslant c_{\mathbb{J}-} \|x\|_0 + c_{\mathbb{J}+} \|z\|_0 + c_3(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2|),$$

where $c_{\mathbb{J}-}$, $c_{\mathbb{J}+}$ and c_3 are all independent of $x,\ z,\ \phi,\ \psi,\ \varphi,\ \varrho_1,\ \varrho_2,\ \epsilon_1,\ \epsilon_2$, and c_3 is also independent of $\mathbb{J}$, but $c_{\mathbb{J}\pm} \to 0$ as $\mathcal{T}_-(\mathcal{T}_+) \to \infty$. By a slight abuse of notation, we write that as

$$\lambda = o(\|x\|_0) + o(\|z\|_0) + O(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2|). \tag{56}$$

Using the same argument as Appendix D in [4], we obtain

$$\begin{aligned}\|x\|_0 &= o(\|z\|_0) + O(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2| + |\kappa_1|), \\ \|z\|_0 &= o(\|x\|_0) + O(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2| + |\kappa_2|),\end{aligned}$$

therefore,

$$\begin{aligned}\|x\|_0 &= O(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2| + |\kappa_1| + |\kappa_2|), \\ \|z\|_0 &= O(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2| + |\kappa_1| + |\kappa_2|).\end{aligned} \tag{57}$$

Thus from (56), we get an estimate for λ,

$$\lambda = O(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2| + |\kappa_1| + |\kappa_2|).$$

And from (57) and (46), we obtain an estimate for x,

$$\|x\|_1 = O(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2| + |\kappa_1| + |\kappa_2|).$$

Similarly,

$$\|z\|_1 = O(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2| + |\kappa_1| + |\kappa_2|).$$

Next we estimate T. Let $\widetilde{v}(t) = \bar{Y}^{-1}(t)^T \dot{\bar{y}}(0)$, then $\widetilde{v} \in \mathcal{N}(\overline{L}_2^*)$. We multiply equation (47) by $\widetilde{v}(t)$ and integrate over $[0, 1]$ which leads to an equation for T,

$$\begin{aligned}&T \int_0^1 \widetilde{v}(t)^T f^+(\bar{y}(t), \bar{\lambda}) dt \\ &= \widetilde{v}(1)^T y(1) - \widetilde{v}(0)^T y(0) - \lambda \int_0^1 \widetilde{v}(t)^T \overline{T} f_\lambda^+(\bar{y}(t), \bar{\lambda}) dt - \int_0^1 \widetilde{v}(t)^T \psi(t) dt.\end{aligned}$$

Since,

$$\int_0^1 \widetilde{v}(t)^T\ f^+(\bar{y}(t),\bar{\lambda})\ dt\ =\ \frac{1}{\overline{\overline{T}}}\int_0^1 \widetilde{v}(t)^T\ \dot{\bar{y}}(t)\ dt$$
$$=\frac{1}{\overline{\overline{T}}}\int_0^1 (\bar{Y}^{-1}(t)^T\dot{\bar{y}}(0))^T\ \bar{Y}(t)\dot{\bar{y}}(0)\ dt\ =\ \frac{1}{\overline{\overline{T}}}\ \|\dot{\bar{y}}(0)\|^2\ \neq\ 0, \tag{58}$$

there exists a constant $c_4 \in \mathbb{R}$, such that

$$|T| \leqslant c_4(\|y(1)\| + \|y(0)\| + |\lambda| + \|\psi\|).$$

Then, an estimate for T is obtained by (51), (52) and the estimates for x, z, λ,

$$T = O(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2| + |\kappa_1| + |\kappa_2|).$$

Finally we estimate y. The solution of equation (47) can be written as

$$y(t) = \bar{Y}(t)y(0) + \int_0^t \bar{Y}(t)\bar{Y}^{-1}(s)\Big(\overline{T}f^+_\lambda(\bar{y}(s),\bar{\lambda})\lambda + f^+(\bar{y}(s),\bar{\lambda})T + \psi(s)\Big)ds.$$

Then there exists a constant $c_5 \in \mathbb{R}$, such that

$$\|y\|_0\ \leqslant\ c_5(\|y(0)\| + |\lambda| + |T| + \|\psi\|_0).$$

From equation (51) and the estimates for x, λ, T we get an estimate for $\|y\|_0$. Then by (47), we obtain

$$\|y\|_1 = O(\|\phi\|_0 + \|\psi\|_0 + \|\varphi\|_0 + |\kappa_1| + |\kappa_2| + \|\varrho_1\| + \|\varrho_2\| + |\epsilon_1| + |\epsilon_2|).$$

Hence we complete the proof. $\square$

Corollary 5.3. *Under the assumptions of Theorem 5.2 we have, for sufficiently large* $\mathbb{J} = [-\mathcal{T}_-, \mathcal{T}_+]$,

$$\begin{aligned}\|x_{\mathbb{J}} - \bar{x}|_{\mathbb{J}^-}\|_1 + \|y_{\mathbb{J}} - \bar{y}\|_1 + \|z_{\mathbb{J}} - \bar{z}|_{\mathbb{J}^+}\|_1 + |T_{\mathbb{J}} - \overline{T}| + |\lambda_{\mathbb{J}} - \bar{\lambda}| \\ \leqslant\ C\ \exp(-\min\{\mu_+\mathcal{T}_+, \mu_-\mathcal{T}_-\}).\end{aligned}$$

If, in addition, $b^-_x(\bar{x}_0,\bar{\lambda})E_u(\bar{\lambda}) = 0$, $b^+_x(\bar{x}_0,\bar{\lambda})E_s(\bar{\lambda}) = 0$, *we have*

$$\begin{aligned}\|x_{\mathbb{J}} - \bar{x}|_{\mathbb{J}^-}\|_1 + \|y_{\mathbb{J}} - \bar{y}\|_1 + \|z_{\mathbb{J}} - \bar{z}|_{\mathbb{J}^+}\|_1 + |T_{\mathbb{J}} - \overline{T}| + |\lambda_{\mathbb{J}} - \bar{\lambda}| \\ \leqslant\ C\ \exp(-2\min\{\mu_+\mathcal{T}_+, \mu_-\mathcal{T}_-\}),\end{aligned}$$

where $-\mu_-$ and μ_+ are the negative and positive eigenvalues of $f^-_x(\bar{x}_0,\bar{\lambda})$.

6. Numerical implementation and applications

In this section we apply the numerical method described in (29)–(37) to compute homoclinic orbits and heteroclinic orbits, respectively in two examples. In each example, we show some numerical computations and illustrate the truncation errors, respectively.

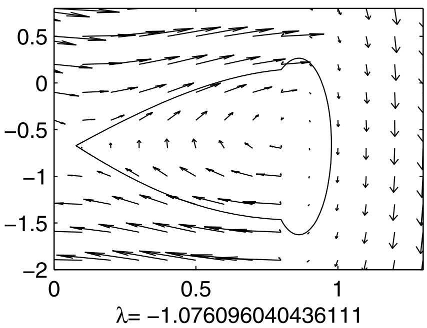

Fig. 4. A numerical piecewise smooth homoclinic orbit.

Fix $h > 0$ to be a stepsize with $Mh = 1$ for a suitable integer M. Let $\mathcal{T}_- = M_-h$ and $\mathcal{T}_+ = M_+h$ for large integers $M_\pm > 0$. In order to implement equations (29)–(37), we use the midpoint Euler scheme to discretize equation (29)–(31), then equations (29)–(37) become nonlinear equations

$$\begin{aligned}
x_{n+1} - x_n &= hf^-(\frac{x_{n+1}+x_n}{2}, \lambda), && n = -M_-, \cdots, -1,\\
y_{m+1} - y_m &= Thf^+(\frac{y_{m+1}+y_m}{2}, \lambda), && m = 0, 1, \cdots, M-1,\\
z_{l+1} - z_l &= hf^-(\frac{z_{l+1}+z_l}{2}, \lambda), && l = 0, 1, \cdots, M_+ - 1,\\
b^-(x_{-M_-}, \lambda) &= 0,\\
b^+(z_{M_+}, \lambda) &= 0,\\
x_0 &= y_0,\\
y_M &= z_0,\\
\vec{n}^T x_0 &= 0,\\
\vec{n}^T y_M &= 0.
\end{aligned}$$

We apply the standard Newton's method to solve these equations to obtain a discrete approximation $(x_{[-M_-,0]}, y_{[0,M]}, z_{[0,M_+]}, T, \lambda)$ of the HOP.

In the next two numerical examples we fix the stepsize $h = 0.01$ for carrying out the calculation.

Example 1. Consider an equation

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 - (x_1 - \sigma^\pm \phi(x_1)) \\ -p^\pm(x_1 - \kappa^\pm \phi(x_1) - \lambda) \end{pmatrix}, \qquad \pm(x_1 - 0.8) > 0, \tag{59}$$

where $\sigma^\pm$, $\kappa^\pm$, $\lambda \in \mathbb{R}$, $p^\pm > 0$ and $\phi(z) = (1 + \exp(-4z))^{-1}$.

If $\sigma^- = \sigma^+$, $\kappa^- = \kappa^+$ and $p^- = p^+$, system (59) is smooth and is a special case of the generalized Lienard system [35]. The homoclinic bifurcation properties in smooth case are analyzed in [17] by numerical computation. Now we also apply numerical method to study a piecewise smooth system of (59) with the following settings $\kappa^- = \kappa^+ = 2$, $\sigma^- = 1.3$, $\sigma^+ = 1.67$, $p^- = 3.35$ and $p^+ = 85.5$. Applying the extending equations (29)–(37) by taking λ as the bifurcation parameter, we approximately compute a piecewise smooth homoclinic orbit $\bar{\gamma}(t)$ on $[-10, 10]$, which is shown in Fig. 4.

Denoted by $(\bar{x}, \bar{y}, \bar{z}, \overline{T}, \bar{\lambda})$ the exact solution and by $(x_\mathbb{J}, y_\mathbb{J}, z_\mathbb{J}, T_\mathbb{J}, \lambda_\mathbb{J})$ the truncated solution on $\mathbb{J} = [-\mathcal{T}_-, \ \mathcal{T}_+]$. We define a local error as,

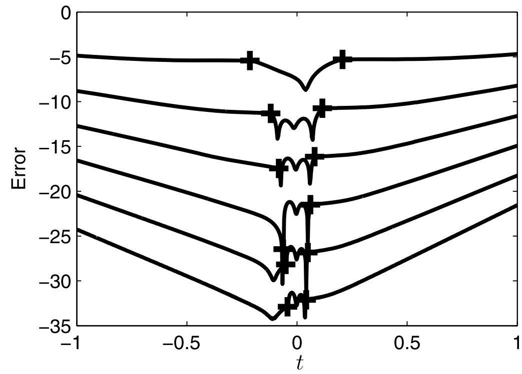

Fig. 5. Errors $\ln(e(-\mathcal{T},\mathcal{T},t\ ))$ by varying $\mathcal{T}$ from 1 to 6 with step 1, t is scaled to $[-1,1]$. '+' represents the errors at the two points of the homoclinic orbit at which the orbit crosses the line of discontinuity.

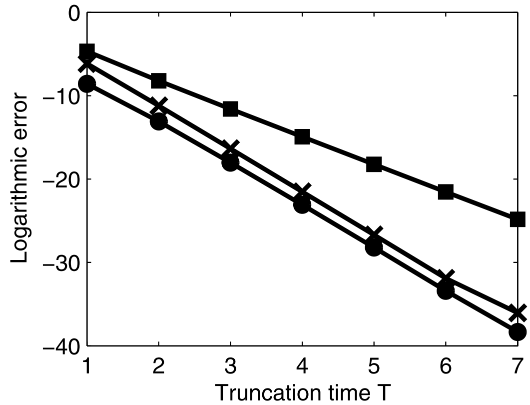

Fig. 6. Global errors versus the truncation time $\mathcal{T}$, ■ − ■: $\ln(e_x(\mathcal{T}))$, $\times-\times$: $\ln(e_T(\mathcal{T}))$, • − •: $\ln(e_\lambda(\mathcal{T}))$.

$$e\ (-\mathcal{T}_-,\mathcal{T}_+,t\ )\ =\ \begin{cases} \|\bar{x}(t)\ -\ x_{\mathbb{J}}(t)\|, & -\mathcal{T}_-\leqslant t\leqslant 0,\\ \|\bar{y}(t)\ -\ y_{\mathbb{J}}(t)\|, & 0\leqslant t\leqslant 1,\\ \|\bar{z}(t-1)\ -\ z_{\mathbb{J}}(t-1)\|, & 1\leqslant t\leqslant \mathcal{T}_++1. \end{cases}$$

Define a global error as

$$e_x(\mathcal{T})=\max_{-\mathcal{T}\leqslant t\leqslant\mathcal{T}+1}\{e(-\mathcal{T},\mathcal{T},t)\},\ e_T(\mathcal{T})=|\overline{T}-T_{\mathbb{J}}|,\ e_\lambda(\mathcal{T})=|\bar{\lambda}-\lambda_{\mathbb{J}}|.$$

Since we can not find the exact piecewise smooth homoclinic orbit, we regard a truncated homoclinic orbit on a large interval as the standard homoclinic orbit and calculate all the error estimates with respect to this standard homoclinic orbit. Here we choose the large interval as $[-15,15]$. All finite boundary-value problems (29)–(37) are solved at high accuracy ($\sim 10^{-13}$) so that discretization errors do not spoil the error arising from truncation.

Take $\mathcal{T}_-=\mathcal{T}_+=\mathcal{T}$, we calculate the local errors when $\mathcal{T}=1,2,\cdots,6$, respectively, the numerical results are shown in Fig. 5. Then we vary $\mathcal{T}$ from 1 to 7 with step 1 and calculate the global errors, respectively which are shown in Fig. 6. The error $e_x(\mathcal{T})$ shows the expected slope of $-\min\{2\mu_-,2\mu_+\}$ which is predicted in Corollary 5.3. While $e_\lambda(\mathcal{T})$ and $e_T(\mathcal{T})$ show the superconvergence with a slope approximately $-\min\{(2\mu_-+\mu_+),\ (2\mu_++\mu_-)\}$ which is beyond the reach of Corollary 5.3.

Next, we discuss the influence on the errors of the choice of the truncation interval $[-\mathcal{T}_-,\mathcal{T}_+]$. As a consequence of our previous calculations we only consider the local error $e(-\mathcal{T}_-,\mathcal{T}_+,t)$. Fix $\mathcal{T}_+=5$ and vary $\mathcal{T}_-$ from 1 to 9 with step 1, then the local error estimates are shown in Fig. 7. Clearly, as $\mathcal{T}_-$ increases,

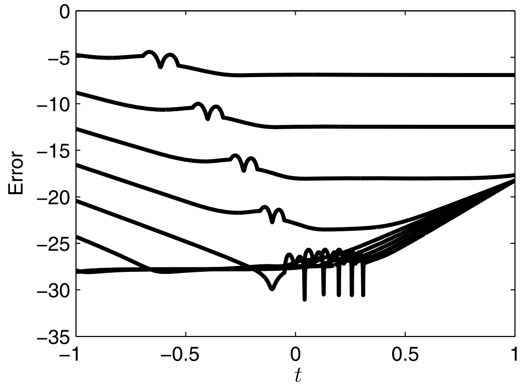

Fig. 7. Errors $\ln[e(-\mathcal{T}_-, 5, t)]$ by fixing $\mathcal{T}_+ = 5$ and varying $\mathcal{T}_-$ from 1 to 9 with step 1, t was scaled to $[-1, 1]$.

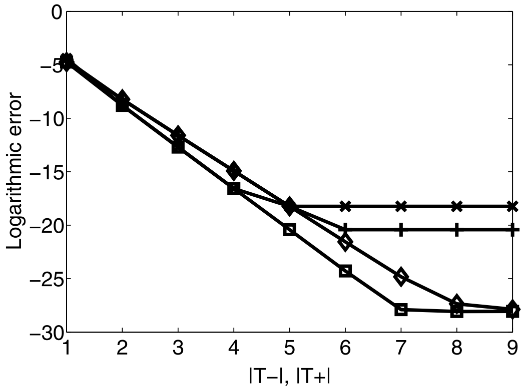

Fig. 8. Local and global errors obtained by varying one endpoint of the interval $[-\mathcal{T}_-, \mathcal{T}_+]$ only.

the error cannot pass below a certain error function whose maximum is at $\mathcal{T}_+$. But the error at $t = -\mathcal{T}_-$ decreases over a certain range. This is summarized in Fig. 8 which compares the local and global errors.

$$\begin{aligned} &\Box - \Box: \ e\left(-\mathcal{T}_-, 5, -\mathcal{T}_-\right), && \Diamond - \Diamond: \ e\left(-5, \mathcal{T}_+, \mathcal{T}_+\right), \\ &\times - \times: \max_{-\mathcal{T}_- \leqslant t \leqslant 5+1} e(-\mathcal{T}_-, 5, t), && + - +: \max_{-5 \leqslant t \leqslant \mathcal{T}_+ + 1} e(-5, \mathcal{T}_+, t). \end{aligned}$$

Example 2. Consider the unforced and undamped rocking block

$$\begin{aligned} \alpha\ddot{u} + \sin[\alpha(1-u)] &= 0, \qquad u > 0, \\ \alpha\ddot{u} - \sin[\alpha(1+u)] &= 0, \qquad u < 0. \end{aligned} \tag{60}$$

This is a piecewise-defined Hamiltonian system which contains a piecewise smooth heteroclinic loop. In order to approximate the heteroclinic loop, we rewrite this second order equation as the following first order equations by setting $x_1 = u$ and $x_2 = \dot{u}$,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \mp \sin(\alpha \mp x_1 \mp \lambda) \end{pmatrix}, \qquad \pm\, x_1 > 0. \tag{61}$$

The artificial parameter λ introduced here is to ensure the nondegenerate property. Obviously, at $\lambda = 0$, equation (61) coincides with equation (60).

At $\alpha = 0.3$, we obtain a numerical heteroclinic orbit on $[-10, 10]$, see Fig. 9, and a heteroclinic loop which is shown in Fig. 10.

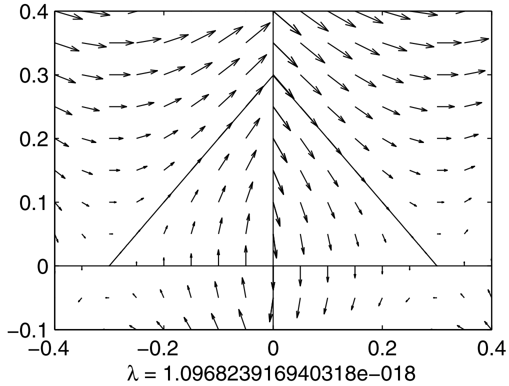

Fig. 9. The numerical heteroclinic orbit.

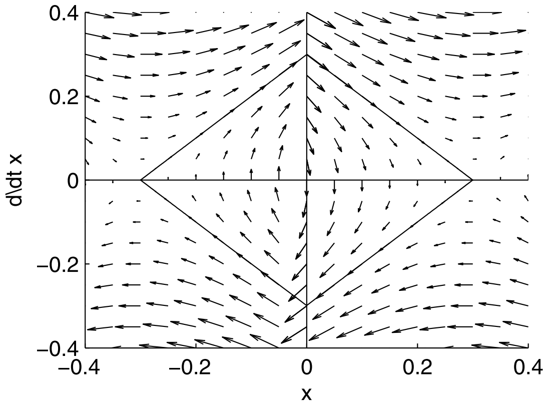

Fig. 10. The numerical heteroclinic loop.

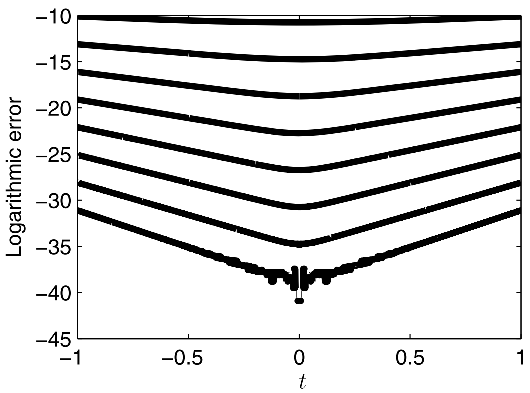

Fig. 11. Local errors $\ln(e(-\mathcal{T},\mathcal{T},t))$ by varying $\mathcal{T}$ from 1 to 8 with step 1, t was scaled to $[-1,1]$.

The next four figures (Figs. 11–14) show the error estimates similar to the previous example. Here, the heteroclinic orbit approximated on $[-15,15]$ is taken as the standard solution $(\bar{x},\bar{y},\bar{\lambda})$.

In Fig. 11 we present the logarithmic error of the approximation. In Fig. 12, $e_x(T)$ shows the expected slope of

$$-\min\{\min\{2\mu_-^-,2\mu_+^-\},\min\{2\mu_-^+,2\mu_+^+\}\}$$

as predicted in Corollary 5.3. In Fig. 13, we exhibit the logarithmic error of the approximation by varying

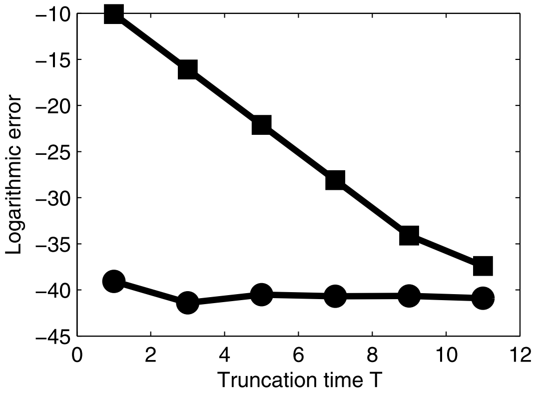

Fig. 12. Global errors by varying $\mathcal{T} = 1, 3, 5, \cdots, 11$. ■ − ■: $\ln(e_x(\mathcal{T}))$, • − •: $\ln(e_\lambda(\mathcal{T}))$.

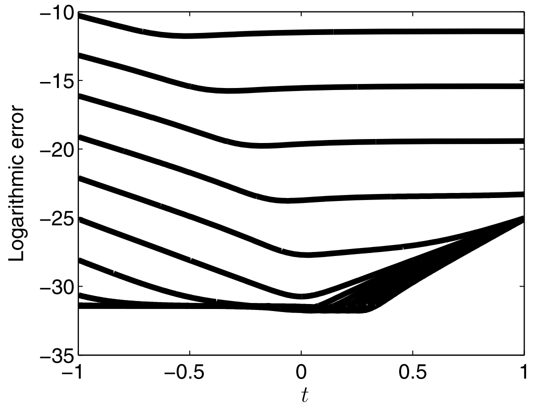

Fig. 13. Errors $\ln[e(-\mathcal{T}_-, 6, t)]$ by fixing $\mathcal{T}_+ = 6$ and varying $\mathcal{T}_-$ from 1 to 12 with step 1, t is scaled to $[-1, 1]$.

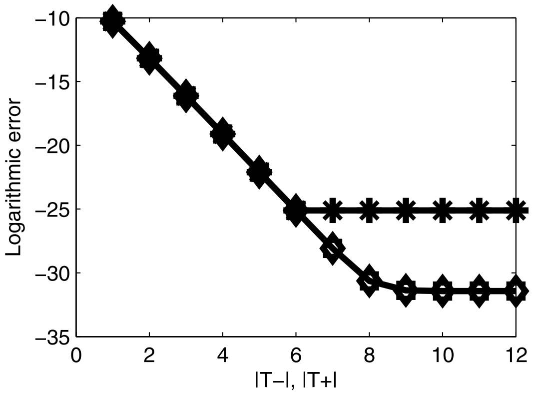

Fig. 14. Local and global errors obtained by varying one endpoint of the interval $[-\mathcal{T}_-, \mathcal{T}_+]$ only.

the left endpoint $\mathcal{T}_-$ and fixing the right endpoint $\mathcal{T}_+$. In Fig. 14, we depict the local and global errors with four different settings of the endpoint $\mathcal{T}_-$ and $\mathcal{T}_+$.

$$\square - \square: e(-\mathcal{T}_-, 6, -\mathcal{T}_-); \qquad \times - \times: \max_{-\mathcal{T}_- \leqslant t \leqslant 6+1} e(-\mathcal{T}_-, 6, t);$$
$$\diamond - \diamond: \quad e(-6, \mathcal{T}_+, \mathcal{T}_+); \qquad + - +: \max_{-6 \leqslant t \leqslant \mathcal{T}_+ + 1} e(-6, \mathcal{T}_+, t).$$

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