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On the convergence of cardinal interpolation by parameterized radial basis functions

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A R T I C L E I N F O

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We consider cardinal interpolation on gridded data by using various radial basis functions associated with one or two parameters, one of which leads asymptotically to so-called flat-limits. Previously it had been shown that the classical Paley–Wiener functions can be recovered by such cardinal interpolations as the parameter tends to infinity. In this article, we extend the results by relaxing the requirements on the approximand functions from several points of view. The radial basis functions that we are concerned with and which are of special interest contain the celebrated multiquadrics, inverse multiquadrics and shifted thin-plate spline radial basis functions for instance. We also generalise the classes of admitted approximands as well as the radial basis functions to generalised multiquadrics in place of the well-known ordinary or for example inverse multiquadrics. An interesting analytical aspect of this work is that – unlike the classical Whittaker–Shannon theorem – functions (approximands) may be reproduced for the parameter $c \to \infty$ in the generalised multiquadrics cardinal approximands, where the usual Shannon series does not converge with theses approximands due to the slow decay of the sinc-function which does not allow e.g. polynomials as approximands. In contrast to the latter, the generalised multiquadrics cardinal functions employed here decay sufficiently fast for each fixed parameter c that even polynomials may be admitted as approximands and are reproduced when then the parameter tends to infinity.

1. Introduction

Approximation and interpolation in multiple dimensions of d-variable functions and data by computationally simpler expressions is a task that is often addressed for instance by using linear combinations of shifts of a single kernel function. This is because the computation of the aforementioned approximant or interpolant is greatly simplified in this way especially when the said kernel function has certain symmetries

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for example. Especially in high dimensions $d \gg 1$, one type of symmetry is resulting from using a radially symmetric kernel $\varphi(\|\cdot\|) : \mathbb{R}^d \to \mathbb{R}$. Here and anywhere else the norm $\|\cdot\|$ is Euclidean and the radial part $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is called the *radial basis function.*

Various different approaches to approximate the approximand f may be taken. When going back to the radial basis functions, for instance one may work by varying on the positions of the shifts – here called centres because of the radial symmetry about them – and among them we wish to study cardinal interpolation on equally spaced data. Indeed, the problem of interpolating to a multivariate function on an integer grid using the radial basis function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is formulated classically in the following way: given the continuous function $f : \mathbb{R}^d \to \mathbb{R}$ (the approximand), find a set of real coefficients $\{d_k\}_{k\in\mathbb{Z}^d}$ such that

$$If(x) = \sum_{k\in\mathbb{Z}^d} d_k \varphi(\|x-k\|), \qquad x \in \mathbb{R}^d,$$

is well-defined (the sum converges at a minimum quadratically, that is, in the ℓ^2-sense and not, e.g., uniform, thus we may not in certain cases evaluate pointwise everywhere) and agrees with f everywhere on $\mathbb{Z}^d$. Alternatively, and this is our approach here, we may initially try to find coefficients $\{c_k\}_{k\in\mathbb{Z}^d}$ such that the so-called cardinal function

$$\chi(x) = \sum_{k\in\mathbb{Z}^d} c_k \varphi(\|x-k\|), \qquad x \in \mathbb{R}^d, \tag{1.1}$$

is an absolutely pointwise with respect to x convergent sum with the cardinality conditions $\chi(j) = \delta_{0,j}$ for all multi-integers $j \in \mathbb{Z}^d$, where δ is the Dirac functional, that is, $\delta_{s,t} = 1$ if $s = t$ and $\delta_{s,t} = 0$ if $s \neq t$. We then set

$$If(x) = \sum_{k\in\mathbb{Z}^d} f(k)\chi(x-k), \qquad x \in \mathbb{R}^d, \tag{1.2}$$

whenever the approximant's sum (1.2) converges absolutely with respect to x or at a minimum in an L^2-sense. In the latter case we may be unable to evaluate pointwise but may consider the error

$$\|f - If\|_2$$

nonetheless.

This approach provides a useful and flexible family of approximants for many choices of φ. For instance, the famous multiquadric radial basis function (MQ) $\varphi(r) = \varphi_c(r) = \sqrt{r^2+c^2}$, further inverse multiquadrics (IM)

$$\varphi(r) = \frac{1}{\sqrt{r^2+c^2}},$$

inverse quadratics (IQ)

$$\varphi(r) = \frac{1}{r^2+c^2},$$

which all unify and generalise in

$$\varphi_{c\gamma}(r) = \left(r^2+c^2\right)^{\gamma}, \qquad \gamma \notin \mathbb{Z}_+;$$

nonnegative integers are forbidden because they force the radial function composed with the Euclidean norm to be simply a polynomial of degree 2γ in d unknowns. Finally, among the most often used radial basis functions are the popular Gaussians (GA) $\varphi(r) = \exp(-(cr)^2)$, the Poisson kernel $\varphi(r) = \exp(-cr)$ and shifted thin-plate spline radial basis function $\varphi(r) = (r^2 + c^2)\log(r^2 + c^2)$.

However, in this article we will focus mostly on the multiquadrics $\varphi_c(r) = \sqrt{r^2 + c^2}$ with real parameter c and its aforementioned generalisation for γ not a nonnegative integer

$$\varphi_{c\gamma}(r) = \left(r^2 + c^2\right)^{\gamma}.$$

In this case, the existence of the cardinal function $\chi = \chi_c$ defined by (1.1) was confirmed for example by the first author [3], where it is furthermore proved that for instance beginning in one dimension and for the multiquadrics proper it is true that at a minimum

$$|\chi_c(x)| = O(\|x\|^{-5}) = O_c(\|x\|^{-5}) \qquad \text{as} \quad \|x\| \to \infty,$$

with the constant absorbed in $O = O_c$ being dependent on c but not on x. This is a first indication that the convergence of the infinite series for the cardinal interpolants may also be hoped for in the context of some polynomially increasing approximands f or indeed polynomials $p = f$ of certain degrees themselves.

Continuing now, from the broad theory in Chapter 4 in [4], and when c is not zero, it follows that for the generalised multiquadrics function we get further decay estimates of

$$|\chi_c(x)| = O_c(\|x\|^{-4\gamma-3d}), \qquad \text{as} \quad \|x\| \to \infty, \tag{1.3}$$

for $x \in \mathbb{R}^d$ so long as $2\gamma + d$ is an even positive integer, and in all other cases

$$|\chi_c(x)| = O_c(\|x\|^{-2\gamma-2d}), \qquad \text{as} \quad \|x\| \to \infty. \tag{1.4}$$

Then, a frequently occurring question is whether the limits of interpolants (1.2) will recover the original function on the whole space either immediately or indeed asymptotically when the parameter c tends to infinity – which makes the radial basis functions “increasingly flat” in a term coined by Fornberg and Larsson [6]. This aspect of radial basis function interpolation and its numerical solution is useful because it also concerns the numerical problem with ill-conditioned matrices when solving the mentioned interpolation problems for extreme parameters and how to solve the interpolation problems for the interpolation coefficients efficiently in the face of this ill-conditioning. An interesting feature of the so-called “flat limits” is that they are often as simple functions as polynomials. The reproduction of polynomials happens also in our context, see for instance the last theorem of this section or the polynomial reproduction on infinite grids by interpolation or quasi-interpolation, see for instance the standard references (but there are many others) [4] or [3] or [5].

An earlier paper [2] by Baxter gave out certain sufficient conditions on functions f such that (1.2) uniformly converges to f on $\mathbb{R}^d$ when the parameter c tends to infinity. More precisely, the result is stated in the following theorem.

Theorem 1.1. [2] *Given a continuous function $f \in L^2(\mathbb{R}^d)$, whose square-integrable Fourier transform $\hat{f}$ is compactly supported in $[-\pi, \pi]^d$, so that it is band-limited, then the interpolant*

$$(I_c f)(x) = \sum_{k\in\mathbb{Z}^d} f(k)\chi_c(x-k), \quad x \in \mathbb{R}^d, \tag{1.5}$$

is well-defined in $L^2(\mathbb{R}^d)$, where χ_c denotes the cardinal function for the integer grid using the classical multiquadric radial function ($\gamma = 1/2$) with parameter c. Furthermore, it is true that

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$$\lim_{c\to\infty}(I_c f)(x) = f(x) \tag{1.6}$$

uniformly for all arguments on $\mathbb{R}^d$.

In another recent article [9] by Ledford, the author established the result (see [9, Theorem 2]) with respect to a relatively general family of basis functions. But in [9] it is still required all approximand functions satisfying the same conditions. However, Powell [12, Section 5] had pointed out that (1.6) holds for $f(x) = x^2$, which, obviously, as an approximand does not in fact satisfy the conditions of Theorem 1.1. Therefore, the central purpose of this paper is to extend the uniform approximation property (1.6) by relaxing the requirements on the approximands much further.

When the approximand is again the sinc-function, this theorem implies that the limit of the cardinal functions with respect to the c parameter (the "flat limit") as the parameter c tends to ∞ is the sinc-function pointwise, for this and other radial basis functions as in [9]. Our first main result establishes the uniform convergence of (1.6) for L^p-integrable functions, $1 < p < \infty$, with limited support of Fourier transforms. Also, it is shown that such approximation is true under the corresponding derivatives.

Theorem 1.2. *Let* $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, *with a Fourier transform* $\hat{f}$ *in the distributional sense. If the radial basis function in use is the generalised multiquadric function and* $\hat{f}$ *is supported in* $[-\pi, \pi]^d$, *we have that*

$$\lim_{c\to\infty}(I_c f)(x) = f(x) \tag{1.7}$$

uniformly on $\mathbb{R}^d$. *More generally, for any* $\alpha \in \mathbb{Z}_+^d$,

$$\lim_{c\to\infty}\partial^\alpha(I_c f)(x) = \partial^\alpha f(x) \tag{1.8}$$

uniformly on $\mathbb{R}^d$, *where* ∂^α *is a short notation for the partial derivative*

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_d^{\alpha_d}}$$

of order $\alpha \in \mathbb{Z}_+^d$.

Remarks.

(i) In this sense, χ_c can be seen as a generalisation of the sinc function which provided the famous sampling theorem (see [7]). However, the sinc function decays far too slowly, so it is not very well localised, and it has to be used employing the tensor product form in the high dimensional case.

(ii) By Paley–Wiener's theorem, the functions satisfying the conditions in Theorem 1.2 can be extended to entire functions of exponential type at most π. For details, one can refer to [15] and [11].

(iii) The conclusions of Theorem 1.2 are still justified for any radial basis function with its Fourier transform using the modified Bessel functions K_{v_j} in the form of

$$\hat{\phi}_c(r) = \sum_{j=1}^{m} g_j(r)c^{s_j}\frac{K_{v_j}(cr)}{r^{v_j}},$$

where for each $j = 1, \ldots, m$, v_j being always positive, $s_j \in \mathbb{R}_+$, and g_j are univariate functions which have continuous derivatives with g_j and g_j' possessing at most polynomial growth.

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Notice that when $p = \infty$, (1.6) may not be true. To see this, one can consider $f(x) = \sin \pi x$ as an example, which is nonzero but vanishes at every integer. In this view, we turn to establish (1.6) as well for approximand functions, which are in some special forms-Fourier transform of Borel measure, Fourier–Stieltjes integral and multivariate polynomials, respectively.

Theorem 1.3. *Let f be a multivariate function on $\mathbb{R}^d$ which is band-limited and defined by a Fourier transform of any Borel measure, that is*

$$f(x) = \int\limits_{[-\pi,\pi]^d} \exp(ix \cdot u)\, d\mu(u), \tag{1.9}$$

where μ is a Borel measure on $\mathbb{R}^d$ with $\mu([-\pi, \pi]^d) < \infty$. The $\cdot$ denotes the usual inner product. Then we still have for the generalised multiquadric radial basis function

$$\lim_{c\to\infty} (I_c f)(x) = f(x)$$

uniformly for all $x \in \mathbb{R}^d$.

Theorem 1.4. *Let f be a multivariate function on $\mathbb{R}^d$ defined by a Fourier–Stieltjes integral, that is*

$$f(x) = \int\limits_{[-\pi,\pi]^d} \exp(ix \cdot u)\, d\alpha_1(u_1) \cdots d\alpha_d(u_d), \qquad x \in \mathbb{R}^d,\ u = (u_1, \ldots, u_d), \tag{1.10}$$

where each $\alpha_j(u_j)$, $j = 1, \ldots, d$, is of bounded variation in $[-\pi, \pi]$ with $\alpha_j(-\pi + 0) - \alpha_j(-\pi) = \alpha_j(\pi) - \alpha_j(\pi - 0)$. The cardinal interpolation in multiple dimensions using the aforementioned cardinal function χ_c with radial basis functions $\varphi_{c\gamma} = (r^2 + c^2)^\gamma$ will then in fact satisfy for all γ that are not non-negative integers

$$\lim_{c\to\infty} (I_c f)(x) = f(x)$$

uniformly for all $x \in \mathbb{R}^d$.

Theorem 1.5. *If f is a multivariate polynomial on $\mathbb{R}^d$ of degree componentwise less than $4\gamma + 3d - 1$ when $2\gamma + d$ is even or $2\gamma + 2d - 1$ for all other cases, it enjoys for the generalised multiquadric function the identity (1.6) pointwise with an absolutely convergent infinite sum. For a polynomial of degree componentwise less than $4\gamma + 3d - 1/2$ when $2\gamma + d$ is even or $2\gamma + 2d - 1/2$ for all other cases, the same is true in the sense of L^2 with a square summable series. So the L^2-error of the difference between approximand and approximant vanishes.*

We remark that the generalisation also could be seen easily by applying Theorem 1.4 to the example $f(x) = \cos \pi x$ as approximand for which therefore Theorem 1.1, Theorem 1.3 and Theorem 1.5 are not applicable. Also the observation of Powell [12, Section 5] about $f(x) = x^2$ is justified by Theorem 1.5.

In the papers [2] and [9], the authors essentially accomplished their proofs by applying the limit behaviour of $\widehat{\chi}_c$, the Fourier transform of cardinal function χ_c. However, in our cases it is no longer enough for the proofs. Hence, in the next section after recalling some well known facts we will first give an estimate of χ_c. Then, in particular, taking into account special properties of the modified Bessel functions we gave an estimate of a sum of χ_c and its derivatives, which are crucial for the proofs of our main results. Finally, we will complete that section by proving Theorem 1.2, Theorem 1.3, Theorem 1.4 and Theorem 1.5.

2. Limit for the parameter of cardinal interpolation with RBF

Again, the radial basis function we consider is called the generalised multiquadric with a parameter $c > 0$ and a nonzero parameter γ, not a positive integer, where incidentally for positive exponent γ also $c = 0$ is explicitly allowed,

$$\varphi_{c\gamma}(r) = \left(r^2 + c^2\right)^{\gamma}, \quad r > 0.$$

So we now have two parameters in our radial basis function.

As it is well known, the Fourier transform preserves the radial symmetry property; that is, if f is a radial function on $\mathbb{R}^d$, its Fourier transform satisfies that

$$\widehat{f}(\xi) = \widehat{f}(\eta), \qquad \text{if } \|\xi\| = \|\eta\|, \ \xi, \eta \in \mathbb{R}^d.$$

So for convenience, given a fixed dimension d, we define

$$\widehat{\varphi_{c\gamma}}(r) := \widehat{\Phi_{c\gamma}}(x), \qquad r = \|x\|, \ \ x \in \mathbb{R}^d,$$

with $\Phi_{c\gamma}(x) = \varphi_{c\gamma}(\|x\|)$. Here and in what follows, we specify the Fourier transform normalised incidentally as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \exp(-ix \cdot \xi)\, dx, \qquad \xi \in \mathbb{R}^d. \tag{2.1}$$

So long as we have the classical case $\gamma = \frac{1}{2}$, $\widehat{\varphi_{c\gamma}}$ can be formulated as

$$\widehat{\varphi_{c\gamma}}(r) = \widehat{\varphi_{c,1/2}}(r) = -\frac{(2\pi c)^{(d+1)/2} K_{(d+1)/2}(cr)}{\pi r^{(d+1)/2}},$$

where $K_{(d+1)/2}$ is modified Bessel function with degree $(d+1)/2$. In particular, for the one-dimensional case with $\gamma = 1/2$, we have the simple expression

$$\widehat{\varphi_{c\gamma}}(\|x\|) = \widehat{\varphi_{c,1/2}}(\|x\|) = -\frac{2cK_1(c\|x\|)}{\|x\|} = -2\int_1^{\infty} \exp(-c\|x\|t)(t^2 - 1)^{\frac{1}{2}}\, dt. \tag{2.2}$$

Now, in the general d-dimensional case for γ not a nonnegative integer and $c > 0$,

$$\widehat{\varphi_{c\gamma}}(r) = -2\Gamma(\gamma + 1)\pi^{d/2-1}(2c/r)^{\gamma+d/2} \sin(\pi\gamma) K_{\gamma+d/2}(cr)$$

which has an integral representation as

$$-2\pi^{(d-1)/2} c^{2\gamma+d} \frac{\Gamma(\gamma + 1)\sin(\pi\gamma)}{\Gamma\left(\gamma + \frac{d+1}{2}\right)} \int_1^{\infty} \exp(-crt)(t^2 - 1)^{\gamma + \frac{d-1}{2}}\, dt, \tag{2.3}$$

and for the case $c = 0$, $\gamma > 0$, not integral,

$$\widehat{\varphi_{0\gamma}}(r) = -\Gamma\left(\gamma + \frac{d}{2}\right)\Gamma(1+\gamma)\sin(\pi\gamma) 2^{2\gamma+d} \pi^{d/2-1} r^{-2\gamma-d}.$$

For further details of above formulae, one can refer to [8] for instance.

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Especially the exponential decay of $\widehat{\varphi_{c\gamma}}$ for large argument is essential for our proofs, that is, for $0 < \|\xi\| < \|\eta\|$ in particular

$$|\widehat{\varphi_{c\gamma}}(\|\eta\|)| \leq \exp\Big[-c(\|\eta\| - \|\xi\|)\Big]\, |\widehat{\varphi_{c\gamma}}(\|\xi\|)|\,, \tag{2.4}$$

which is in fact a slight generalisation of Lemma 2.1 in [2] and is guaranteed by an asymptotic behaviour of modified Bessel functions (see [1, 9.7.2]); that is, for any degree $v \in \mathbb{R}_+$,

$$K_v(x) \sim \frac{e^{-x}}{\sqrt{x}}, \quad x \to +\infty, \tag{2.5}$$

where $A \sim B$ means there is a positive constant θ independent of x such that $\theta^{-1}A \leq B \leq \theta A$. Apart from this, we need two more facts on modified Bessel functions. Namely,

$$K_v(x) \geq \sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}}, \quad x > 0, \;\; |v| \geq \frac{1}{2}, \tag{2.6}$$

and the formulas for derivatives (see for instance [1, 9.6.28]), that is

$$\frac{d}{dz}\frac{K_v(z)}{z^v} = -\frac{K_{v+1}(z)}{z^v}, \;\; z \in \mathbb{C}. \tag{2.7}$$

Furthermore, due to [3], with respect to the generalised multiquadric radial function again, the cardinal function defined by (1.1) in $\mathbb{R}^d$ exists, and its Fourier transform is given by

$$\widehat{\chi}_c(x) = \frac{\widehat{\varphi_{c\gamma}}(\|x\|)}{\sum_\ell \widehat{\varphi_{c\gamma}}(\|x + 2\pi\ell\|)}, \tag{2.8}$$

where the sum is taken over all d-dimensional multi-integers ℓ. Based on this, the following two lemmas provide us with further details about the cardinal function χ_c and its Fourier transform.

Lemma 2.1. *For any $u \in (-\pi, \pi)$,*

$$|1 - \widehat{\chi}_c(u)| \leq e^{-c|\pi - u|}, \tag{2.9}$$

and for $u \in \mathbb{R} \setminus [-\pi, \pi]$, say $u = \zeta + 2\pi k$ with $k \geq 1$ and $\zeta \in (-\pi, \pi)$,

$$|\widehat{\chi}_c(u)| \leq e^{-c\pi k} + e^{-c|\pi - \zeta|}. \tag{2.10}$$

Remarks.

(i) Lemma 2.1 can be seen as a deeper characterisation of Proposition 2.2 in [2]. For the clarity of presentation, it is convenient to rewrite it as a lemma.

(ii) This result can be easily extended to any high dimensional case $\mathbb{R}^d$ by replacing $|\pi - \zeta|$ and k in (2.9), (2.10) by $\sigma_d(\zeta)$ and $|\kappa|_\infty$ respectively, where $|k|_\infty = \max |k_j|$ and

$$\sigma_d(\zeta) = \min\{|\pi\underline{\varepsilon} - \zeta| : \underline{\varepsilon} \in \{-1, 0, 1\}^d, \underline{\varepsilon} \neq 0\}, \quad \zeta \in (-\pi, \pi)^d. \tag{2.11}$$

(iii) By employing the inverse Fourier transform, this lemma immediately implies that

$$|\partial^\alpha \chi_c(x)| \leq A, \qquad \alpha \in \mathbb{Z}_+^d, \;\; x \in \mathbb{R}^d, \tag{2.12}$$

where A is a constant independent of c and x.

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(iv) It is a particular consequence of this result that the pointwise convergence of the cardinal function with respect to c as $c \to \infty$ to the sinc-function is exponential.

Proof. By using (2.8), we get

$$|\widehat{\chi}_c(2\pi - u)| \leq \frac{\widehat{\varphi_{c\gamma}}(|2\pi - u|)}{\widehat{\varphi_{c\gamma}}(|u|)}.$$

Since $|2\pi - u| - |u| \geq |\pi - u|$ for $u \in [0, \pi)$, by (2.4) we have (2.9) immediately.

Similarly, when $k \geq 2$, notice that

$$|2k\pi - u| = |(2k-1)\pi + \pi - u| \geq (2k-1)\pi,$$

which means $|2k\pi - u| - |u| \geq (2k-2)\pi \geq k\pi$, and therefore (2.10) holds by (2.8) and (2.4). □

Lemma 2.2. *For any $\varepsilon > 0$,*

$$\sum_{j \in \mathbb{Z}^d} |\chi_c(x+j)|^{1+\varepsilon} < A < \infty, \tag{2.13}$$

where A is a constant independent of c and x. Furthermore, for any $\alpha \in \mathbb{Z}_+^d$,

$$\sum_{j \in \mathbb{Z}^d} |\partial^\alpha \chi_c(x+j)|^{1+\varepsilon} < A' < \infty, \tag{2.14}$$

where A' is a constant independent of c and $x \in \mathbb{R}^d$.

Proof. It will be instructive to consider first the one dimensional case where the arguments can be transferred to the higher dimensional situation easily under a slight change.

By combining (2.8), (2.2) and (2.7), after a straightforward calculation, we have that

$$\begin{aligned}\widehat{\chi}_c'(\xi) &= \frac{c\left[\frac{K_1(c|\xi|)}{|\xi|}\sum_\ell \frac{K_2(c|\xi+2\pi\ell|)H(\xi+2\pi\ell)}{|\xi+2\pi\ell|} - \frac{K_2(c|\xi|)}{|\xi|}\sum_\ell \frac{K_1(c|\xi+2\pi\ell|)}{|\xi+2\pi\ell|}\right]}{\left[\sum_\ell \frac{K_1(c|\xi+2\pi\ell|)}{|\xi+2\pi\ell|}\right]^2} \\ &= \frac{c\left[\frac{K_1(c|\xi|)}{|\xi|}\sum_{\ell\neq 0} \frac{K_2(c|\xi+2\pi\ell|)H(\xi+2\pi\ell)}{|\xi+2\pi\ell|} - \frac{K_2(c|\xi|)}{|\xi|}\sum_{\ell\neq 0} \frac{K_1(c|\xi+2\pi\ell|)}{|\xi+2\pi\ell|}\right]}{\left[\sum_\ell \frac{K_1(c|\xi+2\pi\ell|)}{|\xi+2\pi\ell|}\right]^2},\end{aligned}$$

where $H(x) = 1$ for $x \geq 0$ and $H(x) = -1$ otherwise. Now, suppose that the parameter c is sufficient large, by using (2.5) and (2.6), we have that

$$|\widehat{\chi}_c'(\xi)| \lesssim \begin{cases} ce^{-c|\pi-\zeta|}, & |k| \leq 1; \\ cke^{-c\pi k}, & |k| > 1, \end{cases} \tag{2.15}$$

for $\xi = \zeta + 2\pi k$ with $\zeta \in (-\pi, \pi)$ and $k \in \mathbb{Z}$ and $|\xi| > \varepsilon$. Here and in what follows we use $\lesssim$ to denote that there is an extra constant independent of c in the proposed upper bound.

Note that in case of choosing $\xi = \pi$ for example, the first infinite sum in the numerator in the pen-ultimate display cancels, that is

$$\sum_{\ell} \frac{K_2(c|\xi + 2\pi\ell|)H(\xi + 2\pi\ell)}{|\xi + 2\pi\ell|}$$

vanishes, which results in a nonzero numerator, because the two series no longer cancel each other asymptotically, and explains the c factor for $\pm\pi$ as arguments in $\widehat{\chi}_c'(\xi)$.

Therefore, by symmetry,

$$\int_{-\infty}^{\infty} |\widehat{\chi}_c'(\xi)|d\xi = 2\sum_{k=0}^{\infty} \int_{[-\pi,\pi]} |\widehat{\chi}_c'(\zeta + 2\pi k)|d\zeta < B < \infty, \tag{2.16}$$

where $B > 0$ is independent of the parameter c. It turns out that

$$|\chi_c(x)| \le \frac{1}{2\pi|x|} \left| \int_{-\infty}^{\infty} e^{ix\xi} \widehat{\chi}_c'(\xi)d\xi \right| \le \frac{B}{|x|} \tag{2.17}$$

which, by combining with (2.12), implies the desired (2.13) for $d = 1$.

Then, for the general d-dimensional case, using the same argument we can obtain that for $\xi = \zeta + 2\pi k$ with $\zeta \in (-\pi,\pi)^d$ and $k \in \mathbb{Z}^d$,

$$|\partial_\xi^{\mathbb{1}} \widehat{\chi}_c(\xi)| \lesssim \begin{cases} ce^{-c\sigma_d(\zeta)}, & |k|_\infty \le 1; \\ ce^{-c\pi k}, & |k|_\infty > 1, \end{cases}$$

where $\partial_\xi^{\mathbb{1}} = \frac{\partial^d}{\partial\xi_1 \cdots \partial\xi_d}$ and σ_d is as defined in the remarks after Lemma 2.1. This immediately implies that

$$|\chi_c(x)| \le \frac{1}{2\pi \prod_{j=1}^d |x_j|} \left| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \partial^{\mathbb{1}} \widehat{\chi}_c(\xi)d\xi \right| \le \frac{B'}{\prod_{j=1}^d |x_j|}$$

with B' independent of c and x – and thus (2.13) is justified.

Finally, to prove (2.14), when $d = 1$ we notice that (2.15) implies the analogues of (2.15), (2.16) and (2.17); that is, for any $a \in \mathbb{Z}_+$, we have that

$$|\xi^a \widehat{\chi}_c'(\xi)| \lesssim \begin{cases} ce^{-c|\pi-\zeta|}, & |k| \le 1; \\ ck^a e^{-c\pi k}, & |k| > 1, \end{cases}$$

for $\xi = \zeta + 2\pi k$ with $\zeta \in (-\pi,\pi)$, $k \in \mathbb{Z}$, and there is a constant B'' independent of c such that

$$\int_{-\infty}^{\infty} |\xi^a \widehat{\chi}_c'(\xi)|d\xi < B'' \tag{2.18}$$

and

$$\left| \frac{d^a}{dx^a} \chi_c(x) \right| \le \frac{1}{2\pi|x|} \left| \int_{-\infty}^{\infty} (i\xi)^a e^{ix\xi} \widehat{\chi}_c'(\xi)d\xi \right| \le \frac{B''}{2\pi|x|}, \quad |x| > 0. \tag{2.19}$$

Consequently, with (2.12) we can conclude (2.14) for one dimension and indeed for any higher dimension. □

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Now we are in the position to prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, with its Fourier transform supported in $[-\pi, \pi]^d$. Let $f_n \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that supp $\hat{f} \subset [-\pi, \pi]$ and $f_n \to f$ in $L^p(\mathbb{R}^d)$. Here $n \in \mathbb{N}$. For instance, one can set

$$f_n(x) = f * g_n(x)$$

with $g_n(x) = (n/\pi)^{d/2} e^{-n\|x\|^2}$. Here, the star denotes the classical convolution by integrals on $\mathbb{R}^d$. Noticing indeed the Nikolskii type inequality for exponential type obtained by Nessel and Wilmes [10, Theorem 3], f_n also converges to f uniformly as $n \to \infty$.

Then by Hölder's inequality, for any $x \in \mathbb{R}^d$ and $p > 1$ with $p' = p/(p-1)$, we have

$$|I_c(f_n)(x) - I_c(f)(x)| \leq \left(\sum_j |f_n(j) - f(j)|^p \right)^{1/p} \left(\sum_j |\chi_c(x-j)|^{p'} \right)^{1/p'},$$

which, with Lemma 2.2 and Plancherel–Pólya's theorem (see, for instance [11]), implies that

$$|I_c(f_n)(x) - I_c(f)(x)| \leq C\|f_n - f\|_p,$$

where C is a constant dependent on p but not dependent on x, n and neither on c. Therefore, since

$$|I_c(f)(x) - f(x)| \leq |I_c(f)(x) - I_c(f_n)(x)| + |I_c(f_n)(x) - f_n(x)| + |f_n(x) - f(x)|,$$

by applying Theorem 1.1 to f_n, we conclude (1.7), and the same is true for (1.8) by a similar argument using the statement about the sum of partial derivatives from the previous lemma. □

Next we turn to prove Theorem 1.3, Theorem 1.4 and Theorem 1.5, which will be essentially relying on the following Lemma 2.4. However, for more clarity of the presentation, before that we state a corollary of Lemma 2.1 since it will be used many times in the proof of Lemma 2.4.

Corollary 2.3. *Let m_1, m_2 be any two nonnegative integers with $m_1 + m_2 = d$. Then for $u \in (-\pi, \pi)^d$, the series*

$$\sum_{k_1 \in \mathcal{A}_1 \text{ or } k_2 \in \mathcal{A}_2} \widehat{\chi_c}\big(2\pi k_1 + u, \pi k_2\big) \lesssim e^{-c\sigma_d(u)}, \quad as \quad c \to \infty, \tag{2.20}$$

where $\mathcal{A}_1 = \mathbb{Z}^{m_1} \setminus \{0\}$ and $\mathcal{A}_2 = \mathbb{Z}^{m_2} \setminus \{0, -1, 1\}^{m_2}$.

Lemma 2.4. *Let χ_c be the cardinal interpolation function as above, employing the said generalised multiquadric function $\varphi_{c\gamma}$. Then for any $x = (\underline{x}_1, \underline{x}_2) \in \mathbb{R}^d$ with $\underline{x}_1 \in \mathbb{R}^{m_1}$, $\underline{x}_2 \in \mathbb{R}^{m_2}$, and $m_1 + m_2 = d$, m_1, m_2 being nonnegative integers, if $u \in (-\pi, \pi)^{m_1}$,*

$$\left| \sum_{j_1 \in \mathbb{Z}^{m_1}} \sum_{j_2 \in \mathbb{Z}^{m_2}} e^{ij_1 \cdot u} (-1)^{j_2} \chi_c(\underline{x}_1 - j_1, \underline{x}_2 - j_2) - e^{i\underline{x}_1 \cdot u} \cos(\pi \underline{x}_2) \right| \lesssim e^{-c\sigma_d(u)}, \tag{2.21}$$

as $c \to \infty$, where $\sigma_d(u)$ is as defined in (2.11). Here and anywhere else we adopt the convention that for some nonnegative integer m we have $(-1)^\alpha := (-1)^{\alpha_1} \cdots (-1)^{\alpha_m}$ if $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$ and $x \cdot y$, for

$x, y \in \mathbb{R}^m$, is the inner product as before, and $\cos x$ denotes the componentwise product $\cos x_1 \cdots \cos x_{\hat{d}}$ if $x = (x_1, \ldots, x_{\hat{d}}) \in \mathbb{R}^{\hat{d}}$, any $\hat{d} \in \mathbb{N}$.

In particular, when m_2 vanishes, one can simplify (2.21) as the estimate

$$\left| \sum_{j \in \mathbb{Z}^d} e^{ij\cdot u} \chi_c(\underline{x} - j) - e^{i\underline{x}\cdot u} \right| \lesssim e^{-c\sigma_d(u)}, \quad u \in (-\pi, \pi)^d, \ \underline{x} \in \mathbb{R}^d. \tag{2.22}$$

Proof. Recall that with the specification of the Fourier transform (2.1), the Poisson summation formula states that, if for example – see e.g. [16] also for weaker requirements –

$$|f(x)| + \left|\widehat{f}(x)\right| = O\left(1 + \|x\|^{-d-\epsilon}\right) \text{ with some } \epsilon > 0, \tag{2.23}$$

then it is true that

$$\sum_{j \in \mathbb{Z}^d} f(x - j) = \sum_{k \in \mathbb{Z}^d} \widehat{f}(2\pi k) e^{-2\pi i x\cdot k}, \qquad x \in \mathbb{R}^d. \tag{2.24}$$

Now, the proof is the same in all dimensions, but the description is simpler for $\mathbb{R}^2$, so first our proof is carried out for $\mathbb{R}^2$, and next we indicate the necessary changes to the desired generalisation to higher dimensions.

Obviously, the decay properties (1.3), (1.4) and (2.4) guarantee the requirement (2.23). Therefore, for fixed $x_1, x_2 \in \mathbb{R}$, since

$$\begin{aligned}
&\sum_{j_1 \in \mathbb{Z}} \sum_{j_2 \in \mathbb{Z}} e^{ij_1 u} (-1)^{j_2} \chi_c(x_1 - j_1, x_2 - j_2) = \\
&e^{ix_1 u} \sum_{j_1 \in \mathbb{Z}} \sum_{j_2 \in \mathbb{Z}} e^{-i(x_1 - j_1)u} \left[\chi_c\left(x_1 - j_1, 2\left(\frac{x_2}{2} - j_2\right)\right) - \chi_c\left(x_1 - j_1, 2\left(\frac{x_2 - 1}{2} - j_2\right)\right)\right]
\end{aligned}$$

and by using the stated Poisson summation formula (2.24) with

$$f(x) = e^{-ix_1 u} \chi_c(x_1, 2x_2),$$

we have that

$$\begin{aligned}
&\sum_{j_1 \in \mathbb{Z}} \sum_{j_2 \in \mathbb{Z}} e^{ij_1 u} (-1)^{j_2} \chi_c(x_1 - j_1, x_2 - j_2) \\
&= \frac{e^{ix_1 u}}{2} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \widehat{\chi}_c(2\pi k_1 + u, \pi k_2) e^{-2i\pi k_1 x_1} \left[e^{-i\pi k_2 x_2} - e^{-i\pi k_2 (x_2 - 1)} \right] \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 = e^{ix_1 u} \left[\widehat{\chi}_c(u, \pi) e^{-i\pi x_2} + \widehat{\chi}_c(u, -\pi) e^{i\pi x_2} \right]$$

and

$$I_2 = \frac{e^{ix_1 u}}{2} \sum_{k_1 \neq 0 \text{ or } |k_2| > 1} \widehat{\chi}_c(2\pi k_1 + u, \pi k_2) e^{-2i\pi k_1 x_1} \left[e^{-i\pi k_2 x_2} - e^{-i\pi k_2 (x_2 - 1)} \right].$$

Notice that by using the symmetry of (2.8),

$$\begin{aligned}
&\left|\widehat{\chi}_c(u,\pi)-\frac{1}{2}\right| \\
&=\left|\widehat{\chi}_c(u,-\pi)-\frac{1}{2}\right| \\
&=\left|\frac{\widehat{\varphi_{c\gamma}}\big(\|(u,\pi)\|\big)}{\sum_{\ell_1,\ell_2\in\mathbb{Z}}\widehat{\varphi_{c\gamma}}\big(\|(u+2\pi\ell_1,\pi+2\pi\ell_2)\|\big)}-\frac{1}{2}\right| \qquad (2.25)\\
&=\left|\left(2+\sum_{\substack{|\ell_1|\geq 1\\ \ell_2\neq 0,-1}}\frac{\widehat{\varphi_{c\gamma}}\big(\|(u+2\pi\ell_1,\pi+2\pi\ell_2)\|\big)}{\widehat{\varphi_{c\gamma}}\big(\|(u,\pi)\|\big)}\right)^{-1}-\frac{1}{2}\right| \\
&\leq\left|\left(2+\sum_{\substack{|\ell_1|\geq 1\\ \ell_2\neq 0,\pm 1}}\frac{\widehat{\varphi_{c\gamma}}\big(\|(u+2\pi\ell_1,\pi\ell_2)\|\big)}{\widehat{\varphi_{c\gamma}}\big(\|(u,\pi)\|\big)}\right)^{-1}-\frac{1}{2}\right| \\
&=o(1), \qquad (2.26)
\end{aligned}$$

which uniformly approaches zero as $c\to\infty$ after a straightforward calculation by using (2.4).

For I_2, using (2.8) again, we have that

$$|I_2|\leq\sum_{k_1\neq 0\ \text{or}\ |k_2|>1}\widehat{\chi}_c\big(2\pi k_1+u,\pi k_2\big)\lesssim e^{-c\sigma_1(u)}, \qquad (2.27)$$

where the last step follows from Corollary 2.3.

Then, with a slight modification, the proof works equally well for the remaining cases when for instance $m_1=2$ and $m_1=0$, and therefore we have completed the proof now for 2-dimensional case.

In the general $\mathbb{R}^d$ case, with $m_1+m_2=d$, m_1,m_2 being both nonnegative integers, and $u\in(-\pi,\pi)^{m_1}$,

$$\begin{aligned}
&\sum_{\substack{j_1\in\mathbb{Z}^{m_1}\\ j_2\in\mathbb{Z}^{m_2}}}e^{ij_1\cdot u}(-1)^{j_2}\chi_c(\underline{x}_1-j_1,\underline{x}_2-j_2) \\
&=e^{i\underline{x}_1\cdot u}\sum_{s_2\in\{0,1\}^{m_2}}\sum_{\substack{j_1\in\mathbb{Z}^{m_1}\\ j_2\in\mathbb{Z}^{m_2}}}e^{-i(\underline{x}_1-j_1)\cdot u}(-1)^{s_2}\chi_c(\underline{x}_1-j_1,\underline{x}_2-s_2-2j_2) \\
&=\frac{e^{i\underline{x}_1\cdot u}}{2^{m_2}}\sum_{\substack{k_1\in\mathbb{Z}^{m_1}\\ k_2\in\mathbb{Z}^{m_2}}}\widehat{\chi}_c(2\pi k_1+u,\pi k_2)e^{i2\pi\underline{x}_1\cdot k_1}e^{i\pi\underline{x}_2\cdot k_2}\sum_{s_2\in\{0,1\}^{m_2}}(-1)^{s_2}e^{-i\pi s_2\cdot k_2}. \qquad (2.28)
\end{aligned}$$

Then one can check that for any nonnegative integer m, if $k\in\{1,-1\}^m$,

$$\sum_{s\in\{0,1\}^m}(-1)^s e^{i\pi k\cdot s}=\sum_{s\in\{0,1\}^m}(-1)^{2s}=2^m, \qquad (2.29)$$

and if $k\in\{0,1,-1\}^m\setminus\{1,-1\}^m$, say $k_{i_1}=k_{i_2}=\cdots=k_{i_t}=0$, $1\leq i_1<\cdots<i_t\leq m$ with a positive integer $0<t\leq m$,

$$\sum_{s\in\{0,1\}^m}(-1)^s e^{i\pi k\cdot s}=\sum_{s_{i_1},\ldots,s_{i_t}\in\{0,1\}}(-1)^{s_{i_1}+\cdots+s_{i_t}}=0.$$

By applying these equalities to (2.28), it implies that

$$\sum_{j_1\in\mathbb{Z}^{m_1},j_2\in\mathbb{Z}^{m_2}} e^{ij_1\cdot u}(-1)^{j_2}\chi_c(\underline{x}_1-j_1,\underline{x}_2-j_2)=e^{i\underline{x}_1\cdot u}\widehat{\chi}_c(u,\pi\underline{e})\sum_{k_2\in\{-1,1\}^{m_2}} e^{i\pi\underline{x}_2\cdot k_2}+J_2 \tag{2.30}$$

$$=2^{m_2}e^{i\underline{x}_1\cdot u}\cos(\pi\underline{x}_2)\widehat{\chi}_c(u,\pi\underline{e})+J_2. \tag{2.31}$$

Here recall that $\underline{e}=(1,1,\dots,1)\in\mathbb{R}^{m_2}$ and J_2 equals

$$\frac{e^{i\underline{x}_1\cdot u}}{2^{m_2}}\sum_{k_1\in\mathcal{A}_1\text{or }k_2\in\mathcal{A}_2}\left[\widehat{\chi}_c(2\pi k_1+u,\pi k_2)e^{i2\pi\underline{x}_1\cdot k_1}e^{i\pi\underline{x}_2\cdot k_2}\sum_{s_2\in\{0,1\}^{m_2}}(-1)^{s_2}e^{-i\pi s_2\cdot k_2}\right]$$

satisfying that, with $\mathcal{A}_1=\mathbb{Z}^{m_1}\setminus\{0\}$ and $\mathcal{A}_2=\mathbb{Z}^{m_2}\setminus\{0,1,-1\}^{m_2}$, as before

$$|J_2|\le\sum_{k_1\in\mathcal{A}_1\quad\text{or}\quad k_2\in\mathcal{A}_2}\widehat{\chi_c}\big(2\pi k_1+u,\pi k_2\big)\lesssim e^{-c\sigma_d(u)},\ \ c\to\infty,$$

by using Corollary 2.3. Moreover, in a similar way as in (2.26), we obtain that uniformly in u,

$$\lim_{c\to\infty}\widehat{\chi}_c(u,\pi\underline{e})=2^{-m_2}.$$

Then consequently (2.31) yields the desired (2.21). □

Now we are in the position to prove Theorem 1.3, Theorem 1.4 and Theorem 1.5. Beginning by using Lemma 2.4 and the dominated convergence theorem, we can obtain Theorem 1.3 directly, basically in the same way as in the following

Proof of Theorem 1.4. For the sake of convenience and being concise, we shall carry out the proof only for $d=2$, while the general case follows in a most similar way.

For each $j=1,2$, let

$$\alpha_{j,0}(u)=\begin{cases}\alpha_j(-\pi+0), & \text{if } u=-\pi,\\ \alpha_j(u), & \text{if } -\pi<u<\pi,\\ \alpha_j(\pi-0), & \text{if } u=\pi;\end{cases}$$

and define furthermore

$$A_j=\alpha_j(-\pi+0)-\alpha_j(-\pi)=B_j=\alpha_j(\pi)-\alpha_j(\pi-0)\quad\text{and}\quad C_j=A_j+B_j=2A_j.$$

Then using that $A_j=B_j$, $j=1,2$,

$$f(x)=\prod_{j=1}^{2}\left[\int_{-\pi}^{\pi}e^{ix_ju_j}\,d\alpha_{j,0}(u_j)+C_j\cos(\pi x_j)\right]$$

where $u=(u_1,u_2)$, $x=(x_1,x_2)$.

Then, by expanding $f(k)$ for each $k = (k_1, k_2) \in \mathbb{Z}^2$, we have

$$
\begin{aligned}
&|(I_c f)(x) - f(x)| \\
&= |\sum_{k\in\mathbb{Z}^2} f(k)\chi_c(x-k) - f(x)| \\
&\le \int\limits_{[-\pi,\pi]^2} \left|\sum_{k\in\mathbb{Z}^2} e^{ik\cdot u}\chi_c(x-k) - e^{ix\cdot u}\right| d\alpha_{1,0}(u_1)d\alpha_{2,0}(u_2) \\
&\quad + |C_1| \int\limits_{-\pi}^{\pi} \left|\sum_{k\in\mathbb{Z}^2} (-1)^{k_1} e^{ik_2u_2}\chi_c(x-k) - e^{ix_2\cdot u_2}\cos\pi x_1\right| d\alpha_{2,0}(u_2) + \\
&\quad + |C_2| \int\limits_{-\pi}^{\pi} \left|\sum_{k\in\mathbb{Z}^2} (-1)^{k_2} e^{ik_1u_1}\chi_c(x-k) - e^{ix_1u_1}\cos\pi x_2\right| d\alpha_{1,0}(u_1) + \\
&\quad + |C_1C_2|| \sum_{k\in\mathbb{Z}^2} (-1)^k \chi_c(x-k) - \cos(\pi x)|,
\end{aligned}
$$

which, by applying Lemma 2.4 and the continuity of $\alpha_{1,0}, \alpha_{2,0}$, allows us to claim that

$$\lim_{c\to\infty} |f(x) - (I_c f)(x)| = 0$$

uniformly on $x \in \mathbb{R}^2$ and therefore – using the analogous arguments in the general multivariate case – conclude the proof of Theorem 1.4. □

In order to prove Theorem 1.5, it is sufficient to notice that for example

$$x^2 = 2\lim_{u\to 0^+} \frac{1-\cos(xu)}{u^2} \tag{2.32}$$

and

$$x^3 = \lim_{u\to 0^+} \frac{2\sin(xu) - \sin(2xu)}{u^3}.$$

Moreover,

$$x^4 = \lim_{u\to 0^+} \frac{6 - 8\cos(xu) + 2\cos(2xu)}{u^4},$$

and similarly for all other powers. Then, by applying Lemma 2.4 again and choosing certain linear combinations, we arrive directly at Theorem 1.5. □

We remark that the idea of Theorem 1.4 using Fourier–Stieltjes integrals follows from the work of I.J. Schoenberg in [14], where it concerns the spline interpolation. Moreover, in [13], he also proved the necessity of the condition (1.10). Nonetheless, this problem is still open for our case.

We also remark a straightforward generalisation of the Corollary 2.3, where the decay property and the existence of the Lagrange functions are needed and guaranteed by the work in Chapter 4 in [4], and the remaining part of the proof follows the same lines as above.

Corollary 2.5. *Let $\underline{\varphi}_c$ be any radial basis function, depending on a positive parameter c, that possesses a generalised Fourier transform $\widehat{\underline{\varphi}_c}$ which is positive, decays exponentially with*

$$\widehat{\underline{\varphi}_c}(r) = O\Big(\exp(-\overline{\alpha}cr)\Big), \qquad c, r \to \infty, \tag{2.33}$$

and

$$1/\widehat{\underline{\varphi}_c}(r) = O\Big(\exp(\underline{\alpha}cr)\Big), \qquad c, r \to \infty,$$

for some positive $\underline{\alpha}, \overline{\alpha}$, and has a singularity of positive order $\overline{\mu}$ at the origin. Then the identities of the previous Lemmas 2.1 and 2.2 hold. If moreover, the standard conditions in [4], p. 59, are satisfied, namely for $M > d + \overline{\mu}$ that $\widehat{\underline{\varphi}_c} \in C^M(\mathbb{R}_+)$ with all its derivatives satisfying (2.33) and having singularities

$$\widehat{\underline{\varphi}^{(\ell)}}(r) \sim r^{-\overline{\mu}-\ell}$$

at the origin, $\ell = 0, 1, \ldots, M$, then the cardinal function satisfies the decay estimate that at a minimum

$$|\chi_c(x)| = O(\|x\|^{-d-\overline{\mu}})$$

for large argument. Therefore in particular

$$\sum_{j \in \mathbb{Z}^d} \Big|\chi_c(x - j)\Big|$$

is uniformly convergent and bounded for all arguments.

Note that the proof of Theorem 1.4 essentially only relies on the decay property of radial basis function $\underline{\varphi}_c$ given in Corollary 2.5. Naturally we extend our results to this more general class of radial basis functions. A typical example is the generalised shifted thin-plate spline radial basis function

$$\underline{\varphi}_c(r) = (r^2 + c^2)\log(r^2 + c^2)$$

with Fourier transform

$$2(2\pi)^{d/2}\frac{d}{d\beta}2^{\beta/2}/\Gamma(\beta/2)\Big|_{\beta=2} K_{d/2+1}(cr)(c/r)^{d/2+1},$$

see for example [5] Example 2.7.

Corollary 2.6. *Let f be an entire multivariate function on $\mathbb{C}^d$ defined by a Fourier–Stieltjes integral, that is*

$$f(x) = \int_{[-\pi,\pi]^d} \exp(ix \cdot u)\, d\alpha_1(u_1) \cdots d\alpha_d(u_d), \qquad x \in \mathbb{R}^d,\ u = (u_1, \ldots, u_d),$$

where each $\alpha_j(u_j)$, $j = 1, \ldots, d$, is of bounded variation in $[-\pi, \pi]$ with $\alpha_j(-\pi + 0) - \alpha_j(-\pi) = \alpha_j(\pi) - \alpha_j(\pi - 0)$. The cardinal interpolation in d dimensions using the aforementioned cardinal function χ_c with radial basis functions $\underline{\varphi}_c$ as given in Corollary 2.5 will then in fact satisfy

$$\lim_{c \to \infty} (I_c f)(x) = f(x)$$

uniformly for all $x \in \mathbb{R}^d$.

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