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Lyapunov Functional for Multiple Delay General Lurie Systems with Multiple Non-linearities[1]

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In this paper, the absolute stability problem of a multiple delay general Lurie control system with multiple non-linearities is considered. Necessary and sufficient conditions are obtained for the existence of the Lyapunov functional of extended Lurie form with negative definite derivative. From those conditions, a very general algebraic criterion for absolute stability is obtained, which extends the previous results. A concise example illustrates the effectiveness of the present results.

Key Words: general Lurie system; delay; multiple non-linearities; absolute stability; Lyapunov functional.

1. INTRODUCTION

The theory of absolute stability occupies an important place among exact mathematical methods being used in the design and analysis of control systems. To date, the number of works on the absolute stability exceeds hundreds of books and papers; the absolute stability of a control system with multiple non-linearities was extensively studied. Some algebraic criteria and frequency criteria were obtained by means of the

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Lyapunov function with $\dot{V} < 0$ (i.e., negative definite derivative) [4, 5]. Some necessary and sufficient conditions for existence of the Lyapunov function with a quadratic form plus an integral form with $\dot{V} < 0$ were obtained by Zhao [7]. The results of [7] not only overcame the difficulty of checking the Popov frequency criterion for absolute stability of the control system with multiple non-linearities, but also the stability region of [7] was larger than those of [1, 3, 5]. Xiong [6] generalized the results of [7] to a time lag control system with multiple non-linearities. However, the books and papers about the Lyapunov function for a general Lurie system are few [2, 5]. The absolute Lyapunov function candidates were studied by Narendra and Taylor [5]. Some necessary and sufficient conditions for existence of a G-type Lyapunov function were obtained by Grujic [2]. But, the verification of the criteria is very difficult.

However, so far, we have never seen a necessary and sufficient condition for the existence of the Lyapunov functional of Lurie form for multiple delay general Lurie systems with $\dot{V} < 0$. In this paper, by the methods of Zhao [7], we shall investigate the absolute stability of multiple delay general Luire control systems with multiple non-linearities, and some necessary and sufficient conditions for the existence of Lyapunov functional of extended Lurie form with $\dot{V} < 0$ will be obtained, which contain the results of [6, 7], as a particular case, when $D = 0$ or $E = 0$. From those conditions, a very general algebraic criterion for absolute stability is obtained.

Consider the multiple delay general Lurie control system with multiple non-linearities

$$\begin{cases} \dot{x}_i(t) = \sum\limits_{j=1}^{n} a_{ij} x_j(t) + \sum\limits_{j=1}^{n} e_{ij} x_j(t - \tau_j) + \sum\limits_{j=1}^{m} b_{ij} f_j\big(\sigma_j(t)\big) \\ \sigma_j(t) = \sum\limits_{i=1}^{n} c_{ji} x(t) - d_j f_j\big(\sigma_j(t)\big) \\ i = 1, 2, \dots, n, \qquad j = 1, 2, \dots, m, \end{cases} \tag{1.1}$$

where a_{ij}, e_{ij}, b_{ij}, c_{ij}, d_j, τ_i are real constants, and $A = (a_{ij})_{n\times n}$, $E = (e_{ij})_{n\times n}$, $\operatorname{Re} \lambda(A) < 0$, $\operatorname{Re} \lambda(E) < 0$, $x(t) = \operatorname{col}(x_1(t), x_2(t), \dots, x_n(t))$, $x(t-\tau) = \operatorname{col}(x_1(t-\tau_1), x_2(t-\tau_2), \dots, x_n(t-\tau_n))$, $\tau_i > 0$, $B = (b_{ij})_{n\times m}$, $C = (c_{ij})_{n\times m}$, $D = \operatorname{diag}(d_1, d_2, \dots, d_m)$, $d_i \geq 0$, $\sigma(t) = \operatorname{col}(\sigma_1(t), \sigma_2(t), \dots, \sigma_m(t))$, $\sigma_i(t) \in R$, $f(\sigma(t)) = \operatorname{col}(f_1(\sigma_1(t)), f_2(\sigma_2(t)), \dots, f_m(\sigma_m(t)))$, $f_i(\cdot) \in \mathscr{F}_{\mu_i}$,

$$\mathscr{F}_{\mu_i} \triangleq \big\{ f_i(\cdot) : f_i(0) = 0, 0 \leq \sigma_i f_i(\sigma_i) \leq \mu_i \sigma_i^2, \quad \sigma_i \neq 0, \quad \text{and} \quad f_i(\cdot) \in C(-\infty, +\infty) \big\} \tag{1.2}$$

with $0 < \mu_i < +\infty$, $i = 1, 2, \dots, m$.

For system (1.1), we assume that there exist $n \times n$ positive definite matrices P and N such that the matrix

$$G = \begin{bmatrix} -(A^T P + PA) - N & -PE \\ -E^T P & N \end{bmatrix} \tag{1.3}$$

is a positive definite matrix.

Let $\psi = \mathrm{col}(\psi_1, \psi_2, \ldots, \psi_n)$, $\psi_i \in C([-\tau_i, 0], R)$, $\|\psi_i(\cdot)\| = \sup_{-\tau_i \le u \le 0} |\psi_i(u)|$, and $\psi_i(0) = x_i(t)$, $\psi_i(-\tau_i) = x_i(t - \tau_i)$, $\psi(0) = x(t)$, $\psi(-\tau) = x(t - r)$. $C = C([-\tau_i, 0], R)$ is the Banach space of continuous functions mapping the interval $[-\tau_i, 0]$ into the set of real numbers R, $i = 1, 2, \ldots, n$.

Taking the Lyapunov functional of extended Lurie form,

$$\begin{aligned} V(\psi) = {} & \psi^T(0) P \psi(0) + \sum_{i=1}^{n} \int_{-\tau_i}^{0} \psi_i(u) \sum_{j=1}^{n} n_{ij} \psi_j(u)\, du \\ & + \sum_{i=1}^{m} \theta_i \int_0^{\sigma_i} f_i(\xi)\, d\xi + \frac{1}{2} f^T(\sigma) \Theta D f(\sigma), \end{aligned} \tag{1.4}$$

where $\Theta = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_m)$, $\theta_i \ge 0 (i = 1, 2, \ldots, m)$, $N = (n_{ij})_{n \times n}$.

Calculating the derivative of $V(\psi)$ along the solutions of system (1.1), we obtain

$$\begin{aligned} \dot{V}(\psi) |_{(1.1)} = {} & [A\psi(0) + E\psi(-\tau) + Bf(\sigma)]^T P\psi(0) + \psi^T(0) P \\ & \times [A\psi^T(0) + E\psi(-\tau) + Bf(\sigma)] \\ & + \psi^T(0) N \psi(0) - \psi^T(-\tau) N \psi(-\tau) \\ & + \sum_{i=1}^{m} \theta_i f_i(\sigma_i) c_i^T [A\psi(0) + E\psi(-\tau) + Bf(\sigma)] \\ = {} & \psi^T(0)[A^T P + PA + N]\psi(0) + 2\psi^T(-\tau) E^T P \psi(0) \\ & + \psi^T(0)(2PB + A^T C\Theta) f(\sigma) + \psi^T(-\tau) E^T C \Theta f(\sigma) \\ & + \frac{1}{2} f^T(\sigma)(\Theta C^T B + B^T C \Theta) f(\sigma) - \psi^T(-\tau) N \psi(-\tau) \\ = {} & -\begin{bmatrix} \psi(0) \\ \psi(-\tau) \end{bmatrix}^T G \begin{bmatrix} \psi(0) \\ \psi(-\tau) \end{bmatrix} + 2 \begin{bmatrix} \psi(0) \\ \psi(-\tau) \end{bmatrix}^T U f(\sigma) \\ & + f^T(\sigma) W f(\sigma), \end{aligned}$$

where

$$U = \begin{bmatrix} PB + \frac{1}{2}A^T C\Theta \\ \frac{1}{2}E^T C\Theta \end{bmatrix}_{2n\times m}, \qquad W = \tfrac{1}{2}(\Theta C^T B + B^T C\Theta)_{m\times m}.$$

Thus, we have

$$-\dot{V}(\psi)\mid_{(1.1)} = \begin{bmatrix} f \\ \psi(0) \\ \psi(-\tau) \end{bmatrix}^T \begin{bmatrix} -W & -U^T \\ -U & G \end{bmatrix} \begin{bmatrix} f \\ \psi(0) \\ \psi(-\tau) \end{bmatrix}. \tag{1.5}$$

DEFINITION 1. The functional $V(\psi)$ of (1.4) is said to be a Lyapunov functional of system (1.1) with negative definite derivative, that is,

$$\dot{V}(\psi)\mid_{(1.1)} < 0 \quad \text{on } \mu = \operatorname{diag}(\mu_1, \mu_2, \dots, \mu_m), \quad \text{if for any } f_i(\cdot) \in \mathscr{F}_{\mu_i}$$
$$\text{and} \quad \sum_{j=1}^{n} |\psi_j(0)| + \sum_{j=1}^{n} |\psi_j(-\tau_j)| \neq 0 \ (i = 1, 2, \dots, m). \tag{1.6}$$

DEFINITION 2. The system (1.1) is said to be absolutely stable, if for any $f_i(\cdot) \in \mathscr{F}_{\mu_i}$ and $\tau_i > 0$ $(i = 1, 2, \dots, m)$, the system (1.1) is globally asymptotically stable.

2. MAIN RESULTS

THEOREM 1. *Assume that $m \geq 2$. Then necessary and sufficient conditions for the condition* (1.6) *are that*

(i) $\dot{V}\mid_{(1.1)} < 0$ *for* $f_1(\sigma_1) = \alpha_1\sigma_1(\alpha_1 = 0, \mu_1)$ *and any* $f_i(\sigma_i) \in \mathscr{F}_{\mu_i}(i = 2, \dots, m)$; $\sum_{j=1}^{n}|\psi_j(0)| \neq 0$ *and* $\sum_{j=1}^{n}|\psi_j(-\tau_j)| \neq 0$.

(ii) $\dot{V}\mid_{(1.1)} < 0$ *for* $f_1(\sigma_1) \in \mathscr{F}_{\mu_i}$ *and* $f_i(\sigma_i) = 0(i = 2, \dots, m)$; $\sum_{j=1}^{n}|\psi_j(0)| \neq 0$ *and* $\sum_{j=1}^{n}|\psi_j(-\tau_j)| \neq 0$.

THEOREM 2. *If* $0 < \mu_i < +\infty(i = 1, \dots, m)$ *and* $m \geq 1$, *then, necessary and sufficient conditions for absolute stability of system* (1.1) *by means of the*

Lyapunov functional of (1.4) *are*

(i) *There exist $n \times n$ positive definite matrices P and N such that*

$$G = \begin{bmatrix} -(A^T P + PA) - N & -PE \\ -E^T P & N \end{bmatrix}$$

is a $2n \times 2n$ positive definite matrix.

(ii) *There exist θ_i $(i = 1, 2, \ldots, m)$ such that*

$$\frac{1}{\mu_i} > \max_{\alpha \in D_i} Z_i[\alpha] \qquad (i = 1, 2, \ldots, m), \tag{2.1}$$

where $D_i = \{\alpha = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m) \mid \alpha_j = 0$ for $j \geq i$ and $\alpha_j = 0, \mu_j$ for $j \leq i\}$ with 2^{i-1} elements,

$$A[\alpha] = A + B\alpha L^* C^T, \qquad P[\alpha] = P + \tfrac{1}{2}\Theta\alpha L^* C^T,$$

$$G[\alpha] = \begin{bmatrix} -\{A^T[\alpha]P[\alpha] + P[\alpha]A[\alpha]\} - N & -P[\alpha]E \\ -E^T P[\alpha] & N \end{bmatrix},$$

$$2u[\alpha] = \begin{bmatrix} 2P[\alpha]B + A^T[\alpha]C\Theta \\ E^T C\Theta \end{bmatrix}_{2n \times m},$$

$$u[\alpha] = (u_1[\alpha], \ldots, u_m[\alpha]),$$

$$Z_i[\alpha] = \left\{\left[\theta_i c_i^T b_i + u_i^T[\alpha]G^{-1}[\alpha]u_i[\alpha]\right]\left[\bar{c}_i^T G^{-1}[\alpha]\bar{c}_i\right]\right\}^{1/2} + \bar{c}_i^T G^{-1}[\alpha]u_i[\alpha].$$

where $L^* = \operatorname{diag}((1 + d_1\alpha_1)^{-1}, (1 + d_2\alpha_2)^{-1}, \ldots, (1 + d_m\alpha_m)^{-1})$, $\bar{c}_i^T = (c_i^T, 0, \ldots, 0)$.

THEOREM 3. *A necessary and sufficient condition for the existence of Lyapunov functional $V(x)$ of* (1.4) *satisfying the condition* (1.6), *which ensures system* (1.1) *being absolutely stable in a finite sector $\mu = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_m)$, is that there exist a matrix $G = G^T > 0$ (i.e., positive definite) and real number $\theta_i \geq 0$ $(i = 1, 2, \ldots, m)$ such that inequality* (2.1) *holds.*

This is a very general criterion for absolute stability described by inequality (2.1). For each choice of $G > 0$ and $\theta_i \geq 0$ $(i = 1, 2, \ldots, m)$, the inequality (2.1) only contains the parameters of system (1.1) and μ_i $(i = 1, 2, \ldots, m)$. With the help of a computer, we can easily check inequality (2.1) by calculating $G[\alpha]$, $u_i[\alpha]$ and $Z_i[\alpha]$. Hence, the criterion is simple and convenient.

3. THE PROOF OF THE MAIN RESULTS

To prove Theorem 1, we need the following work.

We represent W, U, G as follows respectively [7],

$$W = \begin{pmatrix} w_{11} & W_{21}^T \\ W_{21} & W_{22} \end{pmatrix}_{m\times m}, \qquad U = \begin{pmatrix} u_{11} & U_{12} \\ U_{21} & U_{22} \\ U_{31} & U_{32} \end{pmatrix}_{2n\times m},$$

$$G = \begin{pmatrix} g_{11} & G_{21}^T & G_{31}^T \\ G_{21} & G_{22} & G_{32}^T \\ G_{31} & G_{32} & G_{33} \end{pmatrix}_{2n\times 2n},$$

where w_{11}, u_{11}, $g_{11} \in R$, W_{21}, U_{21}, $G_{21} \in R^{(m-1)\times 1}$, $U_{12} \in R^{1\times(m-1)}$, W_{22}, $U_{22}, G_{22} \in R^{(m-1)\times(m-1)}$, U_{31}, $G_{31} \in R^{(2n-m)\times 1}$, $U_{32}, G_{32} \in R^{(2n-m)\times(m-1)}$, $G_{33} \in R^{(2n-m)\times(2n-m)}$.

Case 1. Suppose that the vectors $c_1, c_2, \ldots, c_m$ are linearly independent. Without loss of generality, we may assume that $C = \left[\begin{smallmatrix} E_{m\times m} \\ 0_{(n-m)\times m}\end{smallmatrix}\right]$; thus, we have

$$-\dot{V}|_{(1.1)} = \begin{bmatrix} f_1 \\ \bar{f} \\ \bar{x}_1 \\ y \\ z \end{bmatrix}^T \begin{bmatrix} -w_{11} & -W_{21}^T & -u_{11}^T & -U_{21}^T & -U_{31}^T \\ -W_{21} & -W_{22} & -U_{12}^T & -U_{22}^T & -U_{32}^T \\ -u_{11} & -U_{12} & g_{11}^T & G_{21}^T & G_{31}^T \\ -U_{21} & -U_{22} & G_{21} & G_{22} & G_{32}^T \\ -U_{31} & -U_{32} & G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} f_1 \\ \bar{f} \\ \bar{x}_1 \\ y \\ z \end{bmatrix} \tag{3.1}$$

where

$$\begin{bmatrix} \psi(0) \\ \psi(-\tau) \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ y \\ z \end{bmatrix}, \qquad f = \begin{bmatrix} f_1 \\ \bar{f} \end{bmatrix},$$

$$\bar{x}_1 = \psi_1(0), \quad f_1 = f_1(\sigma_1), \quad \bar{f} = \mathrm{col}(f_2, f_3, \ldots, f_m),$$

$$y = \mathrm{col}(\psi_2(0), \psi_3(0), \ldots, \psi_m(0)),$$

$$z = \mathrm{col}(\psi_{m+1}(0), \psi_{m+2}(0), \ldots, \psi_n(0), \psi_1(-\tau_1), \psi_2(-\tau_2), \ldots, \psi_n(-\tau_n)).$$

For any $f_1 \in A_{\mu_1}$, $f_j \in A_{\mu_j}$ $(j = 2, \dots, m)$, we assume that

$$k_1 = \begin{cases} \dfrac{f_1(\sigma_1)}{\sigma_1} & \text{if } \sigma_1 \neq 0 \\ 0 & \text{if } \sigma_1 = 0 \end{cases},$$

$$k_j = \begin{cases} \dfrac{f_j(\sigma_j)}{\sigma_j} & \text{if } \sigma_j \neq 0 \\ 0 & \text{if } \sigma_j = 0 \end{cases} \quad (j = 2, 3, \dots, m) \text{ and } \overline{K} = \operatorname{diag}(k_2, \dots, k_m).$$

Then $0 \le k_j \le \mu_j$, $f_1(\sigma_1) = k_1 l_1 \psi_1(0)$, and $f_j(\sigma_j) = k_j l_j \psi_j(0)$ $(j = 2, \dots, m)$, where $l_i = (1 + d_i k_i)^{-1}$, $\sigma_i = l_i \psi_i(0)$, $i = 1, 2, \dots, m$.

Thus, we have

$$-\dot{V}\,|_{(1.1)} = \begin{bmatrix} \bar{x}_1 \\ y \\ z \end{bmatrix} I(k_1, k_2, \dots, k_m) \begin{bmatrix} \bar{x}_1 \\ y \\ z \end{bmatrix}, \qquad 0 \le k_j \le \mu_j \ (j = 2, \dots, m),$$

where

$$I(k_1, k_2, \dots, k_m) = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12}^T & I_{22} & I_{23} \\ I_{13}^T & I_{23}^T & I_{33} \end{pmatrix}_{2n \times 2n}$$

$I_{11} = -w_{11}k_1^2 - 2u_{11}k_1 l_1 + g_{11} l_1^2 \in R$, $I_{12} = -k_1(U_{21}^T + W_{21}^T \overline{K}) + G_{21}^T l_1 L_2 - U_{12}^T \overline{K} l_1 \in R^{(m-1)\times 1}$, $I_{13} = -k_1 U_{31}^T + l_1 G_{31}^T \in R^{(2n-m)\times 1}$, $I_{22} = L_2^T G_{22} L_2 - \overline{K} W_{22} \overline{K} - U_{22} L_2 \overline{K} - \overline{K} L_2 U_{22}^T \in R^{(m-1)\times(m-1)}$, $I_{23} = G_{32}^T L_2 - \overline{K} U_{32}^T \in R^{(m-1)\times(2n-m)}$, $I_{33} = G_{33} \in R^{(2n-m)\times(2n-m)}$, $L_2 = \operatorname{diag}(l_2, \dots, l_m)$.

In the following, we introduce several key lemmas

LEMMA 1. *The necessary and sufficient condition for the functional* $V(\psi)$ *of* (1.4) *satisfying condition* (1.6) *is that*

$$\det I(k_1, \dots, k_m) > 0 \qquad \text{for any } k_i \in [0, \mu_i] \ (i = 1, 2, \dots, m).$$

LEMMA 2. *Suppose that conditions* (i) *and* (ii) *in Theorem* 1 *are satisfied. Then* $I(\alpha_1, k_2, \dots, k_m) > 0$ $(i.e.,$ *the positive definite*$)$ $(\alpha_1 = 0, \mu_1)$, *for any* $k_i \in [0, \mu_i]$ $(i = 2, \dots, m)$ *and* $I(k_1, 0, \dots, 0) > 0$ *for any* $k_1 \in [0, \mu_1]$.

The proof of Lemmas 1, 2 is similar to those of [7].

LEMMA 3. *Suppose that conditions* (*i*) *and* (*ii*) *hold*, *and let*

$$R(\overline{K}) = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix}_{(2n+1)\times(2n+1)} . \tag{3.2}$$

Then $\det R(\overline{K}) \le 0$ *for* $0 \le k_i \le \mu_i$ $(i = 2, \dots, m)$, *where* $r_{11} = -w_{11} l_1$, $r_{12} = -u_{11} l_1$, $r_{13} = -U_{21}^T L_2 - l_1 W_{21}^T \overline{K} L_2$, $r_{14} = -l_1 U_{31}^T$, $r_{21} = -l_1 u_{11}$, $r_{22} = g_{11} l_1^2$, $r_{23} = l_1 G_{21}^T L_2 - U_{21} \overline{K} l_1$, $r_{24} = l_1 G_{31}^T$, $r_{31} = -U_{21} - l_1 W_{21} \overline{K} L_2$, $r_{32} = l_1 L_2 G_{21} - l_1 \overline{K} L_2 U_{21}$, $r_{33} = L_2^T G_{22} L_2 - \overline{K} W_{22} \overline{K} - \overline{K} L_2 U_{22} - U_{22}^T L_2 \overline{K}$, $r_{34} = G_{32}^T L_2 - \overline{K} U_{32}^T$, $r_{41} = -U_{31} l_1$, $r_{42} = l_1 G_{31}$, $r_{43} = L_2 G_{32} - U_{32} \overline{K}$, $r_{44} = G_{33}$.

Proof. (1) By means of Lemma 2 in [7], we get $R[\begin{smallmatrix}1\\1\end{smallmatrix}] = I(0, k_2, \dots, k_m)$ is positive definite; if $\theta_1 = 0$, we have $w_{11} = \theta_1 c_1^T b = 0$, further, we have $\det R(\overline{K}) \le 0$.

(2) If $\theta_1 > 0$, we are going to prove that $\det R(\overline{K}) \le 0$ as follows; if not, that is, $\det R(\overline{K}) > 0$, then we have $R(\overline{K}) > 0$.

We consider the linear delay system

$$\begin{cases} \dfrac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} = A\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + E\begin{bmatrix} x_1(t-\tau_1) \\ x_2(t-\tau_2) \\ \vdots \\ x_n(t-\tau_n) \end{bmatrix} + \sum_{i=2}^{m} b_i l_i k_i x_i(t) + b_1 f_1 \\ \dfrac{df_1}{dt} = l_1 c_1^T \left[Ax(t) + Ex(t-\tau) + \sum_{i=2}^{m} b_i l_i k_i x_i(t) + b_1 f_1 \right], \end{cases} \tag{3.3}$$

where $B = (b_1, b_2, \dots, b_m)$ and b_i is the ith column of B, $i = 1, 2, \dots, m$.

Take the Lyapunov functional of system (3.3) as

$$\overline{V}(\psi, f_1) = \psi^T(0) P \psi(0) + \sum_{i=1}^{n} \int_{-\tau_i}^{0} \psi_i(u) \sum_{j=1}^{n} n_{ij} \psi_j(u)\, du$$
$$+ \frac{1}{2}\theta_1 f_1^2 + \frac{1}{2} \sum_{i=2}^{m} \theta_i l_i k_i \psi_i^2(0).$$

We denote

$$\psi(0) = \begin{bmatrix} \psi_1(0) \\ \overline{\psi}(0) \end{bmatrix}, \qquad \overline{\psi}(0) = \operatorname{col}(\psi_2(0), \dots, \psi_n(0)).$$

Then we have

$$-\dot{\overline{V}}(\psi)\mid_{(3.3)} = \begin{bmatrix} \psi(0) \\ \psi(-\tau) \end{bmatrix}^T G \begin{bmatrix} \psi(0) \\ \psi(-\tau) \end{bmatrix} + \psi^T(0)(2PB) \begin{bmatrix} f_1 \\ l_2 k_2 \psi_2(0) \\ \vdots \\ l_m k_m \psi_m(0) \end{bmatrix}$$
$$+ \theta_1 f_1 c_1^T \left[A\psi(0) + E\psi(-\tau) + \sum_{i=2}^{m} b_i l_i k_i \psi_i(0) + b_1 f_1 \right]$$
$$+ \sum_{i=2}^{m} \theta_i l_i k_i \psi_i(0) \left[\sum_{j=1}^{n} a_{ij} \psi_j(0) + \sum_{j=1}^{n} e_{ij} \psi_j(-\tau_j) \right.$$
$$\left. + \sum_{j=2}^{n} b_{ij} k_j \psi_j(0) + b_{i1} f_1 \right]$$
$$= \begin{bmatrix} f_1 \\ \psi_1(0) \\ \overline{\psi}(0) \\ \psi(-\tau) \end{bmatrix}^T R(\overline{K}) \begin{bmatrix} f_1 \\ \psi_1(0) \\ \overline{\psi}(0) \\ \psi(-\tau) \end{bmatrix}.$$

Since $-\dot{\overline{V}}(\psi, f_1)\mid_{(3.3)}$ is positive definite and infinitely large with respect to variables ψ and f_1, the zero solution of (3.3) is asymptotically stable, which is a contradiction with the determinant of the coefficient of system (3.3) being zero. This proves Lemma 3.

Remark 1. The matrix $R = [{}^1_1]$ is a matrix R removing the first row, the first column.

With the above Lemmas 1–3 and Lemma 3 of [7], we can prove Theorem 1 easily by similar methods as in [7].

Case 2. Suppose that rank $C = k < m$. Then there must exist a proper transformation $x(t) = Ts(t)$ (T is an $n \times n$ matrix, and rank $T = n$) such that (1.1) becomes

$$\begin{cases} \dot{s} = \overline{A}s + \overline{E}s(t-\tau) + \overline{B}f(\sigma) \\ \sigma = \overline{C}^T s - Df(\sigma), \end{cases} \tag{$\star$}$$

where $\overline{A} = T^{-1}AT$, $\overline{E} = T^{-1}ET$, $\overline{B} = T^{-1}B$, $\overline{C} = T^T C = [\begin{smallmatrix} E_k & \star \\ 0 & 0 \end{smallmatrix}]$. Similar to Case 1, we can study the absolute stability of time lag control system ($\star$). Thus, we also obtain the conclusion of Theorem 1.

Therefore, the proof of Theorem 1 is complete.

To prove Theorem 2, we introduce the following lemma.

LEMMA 4. *If $m = 1$, $0 < \mu_1 < +\infty$, then necessary and sufficient conditions for $\dot{V}|_{(1.1)} < 0$ (for any $f_1(\sigma_1) \in A_{\mu_1}$; $|\psi(0)| \neq 0$ and $|\psi(-\tau)| \neq 0$) are*

(i) *There exist $n \times n$ positive definite matrices P and N such that*

$$G = \begin{bmatrix} -(A^T P + PA) - N & -PE \\ -E^T P & N \end{bmatrix}$$

is positive definite.

(ii) *There exists a number $\theta_1 > 0$ such that*

$$\frac{1}{\mu_1} > \left\{ \left[\theta_1 c_1^T b_1 + u_1^T G u_1 \right] \left[\bar{c}_1^T G^{-1} \bar{c}_1 \right] \right\}^{1/2} + \bar{c}_1 G^{-1} u_1,$$

where $\bar{c}_1^T = (c_1^T, 0, \dots, 0)$ and $2u_1^T = (2b_1^T P + b_1 l_1^* c_1^T A, \theta_1 c_1^T E)$.

The proof of Lemma 4 is similar to those of Lemma 5 of [7].

Proof of Theorem 2. From Lemma 4, we know that Theorem 2 holds for $m = 1$. Suppose that it holds for $m = r$. We are going to show that it holds for $m = r + 1$.

In fact, it is not difficult to show that the necessary and sufficient condition for (i) in Theorem 1 is that

$$\frac{1}{\mu_i} > \max_{\alpha \in D_i} Z_i[\alpha] \qquad (i = 2, \dots, r+1),$$

by the inductive method, and from the fact that Theorem 2 holds for $m = 1$, we can easily deduce that the necessary and sufficient condition for condition (ii) in Theorem 1 is

$$\frac{1}{\mu_1} > Z_1[\operatorname{diag}(0, \dots, 0)].$$

This completes the proof of Theorem 2.

Proof of Theorem 3. From Theorem 2 and the functional $V(\psi)$ of (1.4) being a positive definite infinite-large function, we can easily prove Theorem 3.

4. EXAMPLE

Consider the 2-dimension delay general Lurie control system with 2 non-linearities

$$\begin{cases} \dot{x}_1(t) = -2x_1(t) - x_1(t-\tau_1) - f_1(\sigma_1(t)) \\ \dot{x}_2(t) = -2x_2(t) - x_2(t-\tau_2) - f_2(\sigma_2(t)) \\ \sigma_1(t) = x_1(t) - 10f_1(\sigma_1(t)) \\ \sigma_2(t) = x_2(t) - 10f_2(\sigma_2(t)). \end{cases} \tag{4.1}$$

From system (4.1), we have $A = -2I$, $B = E = -I$, $C = I$, $D = 10I$, and select $P = I$, $N = 2I$, $\theta = \frac{1}{2}I$, where I is a 2×2 unit matrix. Then

$$G = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

Since $m = 2$, we have $D_1 = \{\text{diag}(0,0)\}$, $D_2 = \{\text{diag}(0,0), \text{diag}(\mu_1, 0)\}$. By calculating, we obtain

$$G[\text{diag}(0,0)] = G = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix},$$

$$u_1[\text{diag}(0,0)] = u_1 = \text{col}(-1.5, 0, -0.25, 0),$$

$u_2[\text{diag}(0,0)] = u_2 = \text{col}(0, -1.5, 0, -0.25)$,
Furthermore, we have

$$Z_1[\text{diag}(0,0)] = -2.0235 < 0,\ Z_2[\text{diag}(0,0)] = -2.0235 < 0,$$

$G[\text{diag}(\mu_1, 0)]$

$$= \begin{bmatrix} -2\left(-2 - \dfrac{\mu_1}{1+10\mu_1}\right)\left(1 + \dfrac{\mu_1}{4(1+10\mu_1)}\right) - 2 & 0 & 1 + \dfrac{\mu_1}{4(1+10\mu_1)} & 0 \\ 0 & 2 & 0 & 1 \\ 1 + \dfrac{\mu_1}{4(1+10\mu_1)} & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

When μ_1 is large enough, we can obtain

$$G[\mathrm{diag}(\mu_1, 0) \approx \lim_{\mu_1 \to +\infty} G[\mathrm{diag}(\mu_1, 0) = \begin{bmatrix} 2.305 & 0 & 1.025 & 0 \\ 0 & 2 & 0 & 1 \\ 1.025 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$$G^{-1}[\mathrm{diag}(\mu_1, 0)] \approx \begin{bmatrix} 0.5619 & 0 & -0.2880 & 0 \\ 0 & 0.6667 & 0 & -0.33331 \\ -0.2880 & 0 & 0.6476 & 0 \\ 0 & -0.3333 & 0 & 0.6667 \end{bmatrix}$$

and $u_1[\mathrm{diag}(\mu_1, 0)] \approx \mathrm{col}(-1.5215, 0, -0.25, 0)$, $u_2[\mathrm{diag}(\mu_1, 0)] \approx \mathrm{col}(0, -1.5, 0, -0.25)$, $Z_2[\mathrm{diag}(\mu_1, 0)] \approx -0.2466 < 0$,

$$\max_{\alpha \in D_1} Z_1[\alpha] = Z_1[\mathrm{diag}(0,0)] = -2.0235 < 0$$

$$\max_{\alpha \in D_2} Z_2[\alpha] = \{Z_2[\mathrm{diag}(0,0)], Z_2[\mathrm{diag}(\mu_1, 0)]\} = -2.0235 < 0.$$

Obviously, the conditions of Theorem 2 are satisfied. From Theorem 2, the functional

$$V(\psi) = \psi^T(0)\psi(0) + 2\sum_{i=1}^{2} \int_{-\tau_i}^{0} \psi_i(u)^2\, du + \frac{1}{2}\sum_{i=1}^{2} \int_0^{\sigma_i} f_i(\xi)\, d\xi + \frac{10}{4}\sum_{i=1}^{2} f_i^2(\sigma_i) \tag{4.2}$$

is a Lyapunov functional of (4.1) with negative definite derivative.

By Theorem 2, we can show that (1.4) is the Lyapunov functional, which ensures the absolute stability of system (1.1) in a finite sector $\mu = \mathrm{diag}(\mu_1, \mu_2)$ with μ_1, μ_2 being any large finite numbers.

5. CONCLUSIONS

In this paper, some necessary and sufficient conditions for existence of the Lyapunov functional of extended Lurie form for a general Lurie delay system with $\dot{V} < 0$ are obtained under the condition of a finite sector. Those conditions are described by $2^m - 1$ inequalities for the parameters contained in $\dot{V}$. Hence, the new criteria are easy. From these conditions, a very simple algebraic criterion for absolute stability of general Lurie multiple delay lag systems with multiple non-linearities is obtained, which extends the results of [6, 7]. A concise example illustrates the effectiveness of the present results. Similar to the previous results, it is necessary to indicate that we can study the stabilizability and robust absolute stability for general Lurie systems by means of the methods of this paper.

Moreover, in view of the results of [8, 9], we can obtain a frequency domain criterion of absolute stability for system (1.1), which is a sufficient condition. We will introduce it in another paper.

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