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On characterizations of unique extremality ☆

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Abstract

In this paper, we make some further considerations of the characteristic conditions of (infinitesimally) unique extremality for Beltrami coefficients obtained by Bozin et al., and find some sufficient conditions simpler in form for a Beltrami coefficient μ with nonconstant absolute value to be (infinitesimally) uniquely extremal.

Keywords: Beltrami coefficient; Uniquely extremal; Infinitesimally uniquely extremal; Strebel point

1. Introduction

Let $\Delta = \{z\colon |z| < 1\}$ be the unit disk in the complex plane, and $f(z)$ be the quasiconformal mapping of Δ onto itself. We denote its complex dilatation (or Beltrami coefficient) by

$$\mu_f = \frac{f_{\bar{z}}}{f_z}, \qquad \|\mu_f\|_\infty < 1,$$

and its maximal dilatation by

$$K[f] = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}.$$

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The boundary homeomorphism $f|\partial\Delta$ then determines the extremal maximal dilatation $K_0[f] \geqslant 1$, defined as

$$K_0[f] = \inf\{K[g]\colon\ g|\partial\Delta = f|\partial\Delta\}.$$

To avoid triviality, we always assume that $K_0[f] > 1$, i.e., $f|\partial\Delta$ is not the boundary values of a conformal mapping. Then f is called extremal if $K[f] = K_0[f]$, and uniquely extremal if $K[g] > K_0[f]$ for any other g with $g|\partial\Delta = f|\partial\Delta$.

Let $\mu(z) \in M(\Delta) = \{\mu\colon\ \mu \in L^\infty(\Delta),\ \|\mu(z)\|_\infty < 1\}$, and denote by f^μ the normalized quasiconformal mapping of Δ onto itself with complex dilatation (or Beltrami coefficient) μ, which fix three points (for example, 1, i and -1) on the unit circle $\partial\Delta$. If $f^\nu|\partial\Delta = f^\mu|\partial\Delta$ for another $\nu(z) \in L^\infty(\Delta)$ with $\|\nu(z)\|_\infty < 1$, we say that μ and ν are in the same Teichmüller class, or simply say that they are equivalent, and denote $\mu \sim \nu$. We also denote by $[\mu]$ the class of Beltrami coefficients equivalent to μ and

$$k([\mu]) = \inf\{\|\nu\|_\infty,\ \nu \sim \mu\}, \qquad K([\mu]) = \frac{1 + k([\mu])}{1 - k([\mu])}.$$

Corresponding to the extremality and uniquely extremality of the quasiconformal mappings, we say that μ is extremal in its Teichmüller class (or extremal in T, or simply extremal) if $\|\mu\|_\infty \leqslant \|\nu\|_\infty$ for all $\nu \sim \mu$ (i.e., $\|\mu\|_\infty = k([\mu])$), and μ is uniquely extremal in its Teichmüller class (or uniquely extremal in T, or simply uniquely extremal) if $\|\mu\|_\infty < \|\nu\|_\infty$ for any other $\nu \sim \mu$. It is well known that there always exists at least one extremal Beltrami coefficient in a Teichmüller class.

An equivalent class $[\mu]$ is called a Strebel point [2,7] if $K_0[f^\mu] > H(f^\mu|\partial\Delta)$, where $H(f^\mu|\partial\Delta)$ is the dilatation of the boundary correspondence $f^\mu|\partial\Delta$, which is the infimum of the maximal dilatations of all quasiconformal extensions of $f^\mu|\partial\Delta$ in any neighborhood of $\partial\Delta$ in Δ. Evidently, if we denote $H(f^\mu|\partial\Delta)$ by $H([\mu])$, then the condition for $[\mu]$ to be a Strebel point can be written as $H([\mu]) < K([\mu])$.

Let $L_a^1(\Delta)$ be the set of analytic functions belonging to $L^1(\Delta)$. When $\varphi \in L_a^1(\Delta)$, we denote its L^1-norm by $\|\varphi\| = \iint_\Delta |\varphi|\,dx\,dy$. It follows from Strebel's frame mapping theorem [7] that if $[\mu]$ is a Strebel point, then there exists a unit vector φ in $L_a^1(\Delta)$ such that μ and $k\bar{\varphi}/|\varphi|$ are equivalent, where $k = k([\mu])$. Let $\mu(z), \nu(z) \in L^\infty(\Delta)$. If $\iint_\Delta \mu\varphi\,dx\,dy = \iint_\Delta \nu\varphi\,dx\,dy$ for all $\varphi \in L_a^1(\Delta)$, we say that μ and ν are in the same infinitesimal Teichmüller class, or simply say that they are infinitesimally equivalent, and denote $\mu \approx \nu$. We also denote by $[\mu]$ the class of Beltrami coefficients infinitesimally equivalent to μ whenever there is no ambiguity and $\|\mu\| = \inf\{\|\nu\|_\infty,\ \nu \approx \mu\}$. We say that μ is extremal in its infinitesimal Teichmüller class (or extremal in B, or simply infinitesimally extremal) if $\|\mu\|_\infty \leqslant \|\nu\|_\infty$ for all $\nu \approx \mu$ (i.e., $\|\mu\|_\infty = \|\mu\|$), and μ is uniquely extremal in its infinitesimal Teichmüller class (or uniquely extremal in B, or simply infinitesimally uniquely extremal) if $\|\mu\|_\infty < \|\nu\|_\infty$ for any other $\nu \approx \mu$. It is also known that there always exists at least one infinitesimal extremal Beltrami coefficient in an infinitesimal Teichmüller class.

Corresponding to the boundary dilatation for the Teichmüller class, we can also define the boundary seminorm $b([\mu])$ for the infinitesimal Teichmüller class: $b([\mu]) = \inf\{\|\nu|_{\Delta - F}\|_\infty,\ \nu \approx \mu,\ F$ is compact in $\Delta\}$. An infinitesimally equivalent class $[\mu]$ is called an infinitesimal Strebel point if $\|\mu\| > b([\mu])$. It follows from the infinitesimal frame mapping theorem (see Theorem 2.4 in [3]) that if $[\mu]$ is an infinitesimal Strebel point, then there exists a unit vector φ in $L^1_a(\Delta)$ such that μ and $\|\mu\|\bar{\varphi}/|\varphi|$ are infinitesimally equivalent.

We also need the following definitions: The extremal set $X(\mu)$ of a Beltrami coefficient μ is the set where $|\mu(z)| = \|\mu\|_\infty$. A Beltrami coefficient η is called an admissible variation of μ if it does not increase the L^∞-norm of μ and it is allowed to differ from μ only on the set where $|\mu(z)| \leqslant \text{constant} < \|\mu\|_\infty$. Let $r > 0$, and let E be a compact subset of Δ; the Beltrami coefficient

$$\mu\chi_E + \frac{1}{1+r}\mu\chi_{\Delta - E} = \begin{cases} \mu, & E, \\ \frac{\mu}{1+r}, & \Delta - E, \end{cases}$$

is called the truncation of μ to E.

In [1], Bozin et al. gave a series of characteristic conditions for a Beltrami coefficient μ to be uniquely extremal or infinitesimally uniquely extremal. For simplicity, we state the characteristic conditions in the specialized situation when the domain of the mappings is the unit disk Δ. The following are parts of them:

Theorem A [1]. *Let μ be a Beltrami coefficient in $M(\Delta)$ with constant absolute value. Then the following conditions are equivalent*:

(1) *μ is uniquely extremal in its class T*;
(2) *μ is uniquely extremal in its class B*;
(3) *For every measurable subset E of Δ with nonzero measure, there exists a sequence of unit vectors φ_n in $L^1_a(\Delta)$ such that*

$$\frac{1}{\iint_E |\varphi_n|\,dx\,dy}\left(\|\mu\|_\infty - \operatorname{Re}\iint\limits_{\Delta} \mu\varphi_n\,dx\,dy\right) \to 0 \quad (n \to \infty);$$

(4) *μ is extremal in its class T, and for every compact subset E of Δ with nonzero measure and every $r > 0$, $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ is a Strebel point in T*;
(5) *μ is extremal in its class B, and for every compact subset E of Δ with nonzero measure and every $r > 0$, $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ is an infinitesimal Strebel point in B.*

Theorem B [1]. *Let μ be a Beltrami coefficient in $M(\Delta)$. Then the following conditions are equivalent*:

(1) *μ is uniquely extremal in its class T*;
(2) *μ is uniquely extremal in its class B*;
(3) *For every admissible variation η of μ, and every compact subset E of $X(\eta)$ with nonzero measure, there exists a sequence of unit vectors φ_n in $L_a^1(\Delta)$ such that*
$$\frac{1}{\iint_E |\varphi_n|\,dx\,dy}\left(\|\eta\|_\infty - \operatorname{Re}\iint\limits_\Delta \eta\varphi_n\,dx\,dy\right) \to 0 \quad (n\to\infty);$$
(4) *μ is extremal in its class T, and for every $r>0$, every admissible variation η of μ, and every compact subset E of $X(\eta)$ with nonzero measure, $[\eta\chi_E + (1/(1+r))\eta\chi_{\Delta-E}]$ is a Strebel point in T*;
(5) *μ is extremal in its class B, and for every $r>0$, every admissible variation η of μ, and every compact subset E of $X(\eta)$ with nonzero measure, $[\eta\chi_E + (1/(1+r))\eta\chi_{\Delta-E}]$ is an infinitesimal Strebel point in B.*

In this paper, we make some further considerations of the conditions in the above two theorems and find some sufficient conditions for a Beltrami coefficient μ with nonconstant absolute value to be (infinitesimally) uniquely extremal, which are similar in certain sense to that in Theorems A and B, but simpler in form. In fact, we prove that the conditions in Theorems A and B are still sufficient when the condition of “constant absolute value of μ” in Theorem A and the condition “for every admissible variation η of μ” in Theorem B are removed.

The remainder of this paper is organized as follows: Section 2 provides the preliminaries and some lemmas; Section 3 provides some theorems regarding the (infinitesimally) unique extremality of Beltrami coefficients.

2. Preliminaries and some lemmas

Our main tools in this paper are some basic inequalities, which we now state as follows:

Theorem C (Main Inequality) [6]. *Let f and g be two quasiconformal mappings of Δ onto itself, and denote by $\mu, \nu, \tilde{\mu}, \tilde{\nu}$ the complex dilatations of f, g, f^{-1}, g^{-1}, respectively. If $\mu\sim\nu$, then*
$$\iint\limits_\Delta |\varphi|\,dx\,dy \leqslant \iint\limits_\Delta |\varphi|\frac{|1-\mu\varphi/|\varphi||^2}{1-|\mu|^2}\,\frac{\left|1+\mu\frac{\tilde{\nu}\circ f}{\tilde{\mu}\circ f}\frac{\varphi}{|\varphi|}\frac{1-\bar{\mu}\bar{\varphi}/|\varphi|}{1-\mu\varphi/|\varphi|}\right|^2}{1-|\tilde{\nu}\circ f|^2}\,dx\,dy$$
holds for any $\varphi\in L_a^1(\Delta)$.

The following is a consequence of the Main Inequality, known as the Reich–Strebel's fundamental inequality:

Theorem D [6]. *If there exists $\varphi_0 \in L_a^1(\Delta)$ with $\|\varphi_0\| = 1$ and $[\mu] = [k\bar{\varphi}_0/|\varphi_0|]$ for some $k \in (0,1)$ in T, then*

$$\frac{1+k}{1-k} \leqslant \iint_{\Delta} |\varphi_0| \frac{|1+\mu\varphi_0/|\varphi_0||^2}{1-|\mu|^2}\, dx\, dy. \tag{2.1}$$

The following two inequalities, which are Theorems 3 and 4 in paper [1], can also be obtained from the Main Inequality.

Theorem E [1]. *If μ and ν are two equivalent Beltrami coefficients with $\|\nu\|_\infty \leqslant \|\mu\|_\infty = k < 1$, then there is a constant C depending only on k such that*

$$\iint_{\Delta} |\tilde{\mu} \circ f - \tilde{\nu} \circ f|^2 |\varphi|\, dx\, dy \leqslant C\left(k\|\varphi\| - \operatorname{Re} \iint_{\Delta} \mu\varphi\, dx\, dy\right) \tag{2.2}$$

holds for all $\varphi \in L_a^1(\Delta)$, where $f = f^\mu$ and $\tilde{\mu}, \tilde{\nu}$ are the complex dilatations of $(f^\mu)^{-1}$, $(f^\nu)^{-1}$, respectively.

Theorem F [1]. *If μ and ν are two infinitesimally equivalent Beltrami coefficients with $\|\nu\|_\infty \leqslant \|\mu\|_\infty = k < \infty$, then there is a constant C depending only on k such that*

$$\iint_{\Delta} |\mu - \nu|^2 |\varphi|\, dx\, dy \leqslant C\left(k\|\varphi\| - \operatorname{Re} \iint_{\Delta} \mu\varphi\, dx\, dy\right) \tag{2.3}$$

holds for all $\varphi \in L_a^1(\Delta)$.

Before we state our main results, we need to prove the following two lemmas, the first of that was inspired by the lemma in [5]:

Lemma 1 [5]. *If $\mu \in M(\Delta)$ is extremal with $\|\mu\|_\infty = k$, then for every compact subset E of Δ with nonzero measure and every $r > 0$, the truncation $\mu_r = \mu\chi_E + (1/(1+r))\mu\chi_{\Delta - E}$ of μ to E has the property $k([\mu_r]) \geqslant k/(1+r)$.*

Proof. Let η be equivalent to μ_r and η is extremal, and denote by f^μ, f^{μ_r}, f^η the normalized quasiconformal mappings of Δ onto itself with complex dilatation μ, μ_r, η. Since f^η has the same boundary values as f^{μ_r}, it follows that $f^\mu \circ (f^{\mu_r})^{-1} \circ f^\eta$ has the same boundary values as f^μ. Since f^μ is extremal by hypothesis, it therefore follows that

$$\frac{1+k}{1-k} = K[f^\mu] \leqslant K\left[f^\mu \circ (f^{\mu_r})^{-1} \circ f^\eta\right] \leqslant K[h]K[f^\eta],$$

where $h = f^{\mu} \circ (f^{\mu_r})^{-1}$. Since

$$\left|\mu_h\left(f^{\mu_r}(z)\right)\right| = \left|\frac{\mu(z) - \mu_r(z)}{1 - \bar{\mu}_r(z)\mu(z)}\right| = \begin{cases} \frac{r|\mu(z)|}{1+r-|\mu(z)|^2}, & z \in \Delta - E, \\ 0, & z \in E, \end{cases}$$

we have

$$\left|\mu_h\left(f^{\mu_r}(z)\right)\right| \leqslant \frac{rk}{1+r-k^2}, \quad z \in \Delta.$$

Thus

$$K[h] \leqslant \frac{1+k}{1-k}\frac{1+r-k}{1+r+k}.$$

Hence we conclude that

$$K[f^{\eta}] \geqslant \frac{1+r+k}{1+r-k} = \frac{1+\frac{k}{1+r}}{1-\frac{k}{1+r}},$$

which proves the lemma. □

Lemma 2. *If $\mu \in L^{\infty}(\Delta)$ is infinitesimally extremal with $\|\mu\|_{\infty} = k$, then for every compact subset E of Δ with nonzero measure and every $r > 0$, the truncation $\alpha_r = \mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}$ of μ to E has the property $\|\alpha_r\| \geqslant k/(1+r)$.*

Proof. Suppose that α_r is infinitesimally equivalent to some infinitesimally extremal Beltrami coefficient η. Then μ is infinitesimally equivalent to $\eta + \mu - \alpha_r$, and

$$\mu - \alpha_r = \begin{cases} \frac{r\mu(z)}{1+r}, & z \in \Delta - E, \\ 0, & z \in E, \end{cases} \qquad \|\mu - \alpha_r\|_{\infty} \leqslant \frac{rk}{1+r}.$$

Then we have

$$\|\mu\|_{\infty} \leqslant \|\eta\|_{\infty} + \|\mu - \alpha_r\|_{\infty} \leqslant \|\eta\|_{\infty} + \frac{rk}{1+r}.$$

Hence

$$\|\eta\|_{\infty} \geqslant k - \frac{rk}{1+r} = \frac{k}{1+r},$$

so the lemma follows. □

3. Unique extremality theorems

We have the following theorems:

Theorem 1. (1) *If $\mu \in M(\Delta)$ is extremal, and for every compact subset E of Δ and every $r > 0$, $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ is a Strebel point in T, then μ is uniquely extremal.*

(2) *If $\mu \in M(\Delta)$ is uniquely extremal, then for every compact subset E of Δ and every $r > 0$, either $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ is a Strebel point in T, or $\mu_r = \mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}$ is uniquely extremal.*

Proof. (1) Denote $\mu_r = \mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}$ and $\|\mu\|_\infty = k$. Then by Lemma 1 we have $k([\mu_r]) \geqslant s = (k/(1+r))$. Because $[\mu_r]$ is a Strebel point in T, then, by Strebel's frame mapping theorem, there exists $s_r = k([\mu_r]) \geqslant s$ and a unit vector φ in $L_a^1(\Delta)$ such that μ_r and $s_r\bar{\varphi}/|\varphi|$ are equivalent. Therefore, by Reich–Strebel's fundamental inequality (2.1),

$$\frac{1+s}{1-s} \leqslant \frac{1+s_r}{1-s_r} = K([\mu_r]) \leqslant \iint_\Delta |\varphi| \frac{|1+\mu_r\varphi/|\varphi||^2}{1-|\mu_r|^2}\,dx\,dy.$$

By letting $\mu_1 = \mu/(1+r)$, we have

$$\begin{aligned}\frac{1+s}{1-s} &\leqslant \iint_{\Delta-E} |\varphi| \frac{|1+\mu_1\varphi/|\varphi||^2}{1-|\mu_1|^2}\,dx\,dy + \iint_E |\varphi| \frac{|1+\mu\varphi/|\varphi||^2}{1-|\mu|^2}\,dx\,dy \\ &\leqslant \iint_\Delta |\varphi| \frac{|1+\mu_1\varphi/|\varphi||^2}{1-|\mu_1|^2}\,dx\,dy + C_1(k)r\iint_E |\varphi|\,dx\,dy.\end{aligned}$$

Thus

$$\frac{1+s}{1-s} \leqslant \frac{1+s^2+2\,\mathrm{Re}\iint_\Delta \mu_1\varphi\,dx\,dy}{1-s^2} + C_1(k)r\iint_E |\varphi|\,dx\,dy,$$

$$2\left(s - \mathrm{Re}\iint_\Delta \mu_1\varphi\,dx\,dy\right) \leqslant C_1(k)r\iint_E |\varphi|\,dx\,dy.$$

Hence we obtain

$$k - \mathrm{Re}\iint_\Delta \mu\varphi\,dx\,dy \leqslant C_2(k)r\iint_E |\varphi|\,dx\,dy.$$

On the other hand, suppose μ is not uniquely extremal. Then there exists a Beltrami coefficient ν with $\|\nu\|_\infty \leqslant \|\mu\|_\infty$ and distinct from μ such that μ and ν are equivalent. Then there exists $\varepsilon > 0$ and a compact subset E in Δ of positive measure such that $|\tilde{\mu} \circ f - \tilde{\nu} \circ f| \geqslant \varepsilon$ on E where $f = f^\mu$ and $\tilde{\mu}, \tilde{\nu}$ are the complex dilatations of $(f^\mu)^{-1}, (f^\nu)^{-1}$, respectively. By inequality (2.2), we have

$$\frac{\varepsilon^2}{4}\iint_E |\varphi|\,dx\,dy \leqslant C\left(k - \mathrm{Re}\iint_\Delta \mu\varphi\,dx\,dy\right) \leqslant C_3(k)r\iint_E |\varphi|\,dx\,dy,$$

whenever $\|\varphi\| = 1$, which leads to a contradiction provided that r is sufficiently small.

(2) If $\mu \in M(\Delta)$ is uniquely extremal, and $[\mu_r]$ is not a Strebel point, then it follows from Lemma 1 that

$$K([\mu_r]) = \frac{1 + \frac{k}{1+r}}{1 - \frac{k}{1+r}} = H([\mu_r]).$$

Let $\eta \sim \mu_r$, and let η be extremal. Then $f^\mu \sim f^\mu \circ (f^{\mu_r})^{-1} \circ f^\eta$.

If $\eta \neq \mu_r$, then

$$\frac{1+k}{1-k} = K[f^\mu] < K\big[f^\mu \circ (f^{\mu_r})^{-1}\big]K[f^\eta] \leqslant \frac{1+k}{1-k}\frac{1+r+k}{1+r-k}K[f^\eta].$$

Hence

$$K([\mu_r]) = K([f^\eta]) > \frac{1 + \frac{k}{1+r}}{1 - \frac{k}{1+r}}.$$

This contradiction implies that μ_r is uniquely extremal. □

Theorem 2. *If $\mu \in M(\Delta)$ is extremal, and for every compact subset E of Δ there exists a sequence of unit vectors φ_n in $L_a^1(\Delta)$ such that*

$$\frac{1}{\iint_E |\varphi_n|\,dx\,dy}\left(\|\mu\|_\infty - \operatorname{Re}\iint\limits_{\Delta} \mu\varphi_n\,dx\,dy\right) \to 0 \quad (n \to \infty),$$

then μ is uniquely extremal.

Proof. If μ is not uniquely extremal, then there exists a Beltrami coefficient ν with $\|\nu\|_\infty \leqslant \|\mu\|_\infty$ and distinct from μ such that μ and ν are equivalent. Then there exists $\varepsilon > 0$ and a compact subset E in Δ of positive measure such that $|\tilde{\mu} \circ f - \tilde{\nu} \circ f| \geqslant \varepsilon$ on E where $f = f^\mu$ and $\tilde{\mu}, \tilde{\nu}$ are the complex dilatations of $(f^\mu)^{-1}$, $(f^\nu)^{-1}$, respectively. By inequality (2.2), we have

$$\begin{aligned}\varepsilon^2 \iint\limits_E |\varphi_n|\,dx\,dy &\leqslant \iint\limits_\Delta |\tilde{\mu} \circ f - \tilde{\nu} \circ f|^2 |\varphi_n|\,dx\,dy \\ &\leqslant C\left(\|\mu\|_\infty - \operatorname{Re}\iint\limits_\Delta \mu\varphi_n\,dx\,dy\right).\end{aligned}$$

Obviously it leads to a contradiction provided n is sufficiently large. □

Theorem 3. (1) *If $\mu \in L^\infty(\Delta)$ is infinitesimally extremal, and for every compact subset E of Δ and every $r > 0$, $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ is an infinitesimal Strebel point, then μ is infinitesimally uniquely extremal.*

(2) *If $\mu \in L^\infty(\Delta)$ is infinitesimally uniquely extremal, then for every compact subset E of Δ and every $r > 0$, either $[\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}]$ is an infinitesimally Strebel point, or $\alpha_r = \mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}$ is infinitesimally uniquely extremal.*

Proof. (1) Denote $\alpha_r = \mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}$ and $\|\mu\|_\infty = k$. Then by Lemma 2 we have $\|\alpha_r\| \geqslant s = k/(1+r)$. Because $[\alpha_r]$ is an infinitesimal Strebel point, then by the infinitesimal frame mapping theorem there exists $s_r = \|\alpha_r\| \geqslant s$ and a unit vector φ in $L_a^1(\Delta)$ such that α_r and $s_r\bar{\varphi}/|\varphi|$ are infinitesimally equivalent and $\iint_\Delta \alpha_r\varphi\,dx\,dy \geqslant k/(1+r)$. Therefore

$$\frac{k}{1+r} \leqslant \iint_E \mu\varphi\,dx\,dy + \iint_{\Delta-E} \varphi\frac{\mu}{1+r}\,dx\,dy.$$

Hence

$$k - \operatorname{Re}\iint_\Delta \mu\varphi\,dx\,dy \leqslant kr\iint_E |\varphi|\,dx\,dy.$$

Suppose μ is not infinitesimally uniquely extremal. Then there exists a Beltrami coefficient ν with $\|\nu\|_\infty \leqslant \|\mu\|_\infty$ and distinct from μ such that μ and ν are infinitesimally equivalent. Then there exists $\varepsilon > 0$ and a compact subset E in Δ of positive measure such that $|\mu - \nu| \geqslant \varepsilon$ on E. By inequality (2.3), we have

$$\begin{aligned}
\varepsilon^2\iint_E |\varphi|\,dx\,dy &\leqslant \iint_\Delta |\mu-\nu|^2|\varphi|\,dx\,dy\\
&\leqslant C\left(k\|\varphi\| - \operatorname{Re}\iint_\Delta \mu\varphi\,dx\,dy\right) \leqslant Ckr\iint_E |\varphi|\,dx\,dy,
\end{aligned}$$

whenever $\|\varphi\| = 1$, which leads to a contradiction provided that r is sufficiently small.

(2) If $\mu \in L^\infty(\Delta)$ is infinitesimally uniquely extremal, and $[\alpha_r]$ is not an infinitesimally Strebel point, then it follows from Lemma 2 that

$$\|\alpha_r\| = \frac{k}{1+r} = b([\alpha_r]).$$

Let $\eta \approx \alpha_r$, and let η be infinitesimally extremal. Then $\mu \approx \mu - \alpha_r + \eta$.

If $\eta \neq \alpha_r$, then

$$k = \|\mu\|_\infty < \|\mu - \alpha_r + \eta\|_\infty \leqslant \|\mu - \alpha_r\|_\infty + \|\eta\|_\infty \leqslant \frac{r}{1+r}k + \|\eta\|_\infty.$$

Hence

$$\|\alpha_r\| = \|\eta\|_\infty > \frac{k}{1+r}.$$

This contradiction implies that α_r is infinitesimally uniquely extremal. □

Theorem 4. *If $\mu \in L^\infty(\Delta)$ is infinitesimally extremal, and for every compact subset E in Δ there exists a sequence of unit vectors φ_n in $L_a^1(\Delta)$ such that*

$$\frac{1}{\iint_E |\varphi_n|\,dx\,dy}\left(\|\mu\|_\infty - \operatorname{Re}\iint_\Delta \mu\varphi_n\,dx\,dy\right) \to 0 \quad (n \to \infty),$$

then μ is infinitesimally uniquely extremal.

Proof. If μ is not infinitesimally uniquely extremal, then there exists a Beltrami coefficient ν with $\|\nu\|_\infty \leqslant \|\mu\|_\infty$ and distinct from μ such that μ and ν are infinitesimally equivalent. Then there exists $\varepsilon > 0$ and a compact subset E in Δ of positive measure such that $|\mu - \nu| \geqslant \varepsilon$ on E. By inequality (2.3), we have

$$\begin{aligned}\varepsilon^2 \iint_E |\varphi_n|\,dx\,dy &\leqslant \iint_\Delta |\mu-\nu|^2|\varphi_n|\,dx\,dy \\ &\leqslant C\left(\|\mu\|_\infty - \operatorname{Re}\iint_\Delta \mu\varphi_n\,dx\,dy\right).\end{aligned}$$

Obviously it leads to a contradiction provided n is sufficiently large. □

Remarks. (a) It should be noticed that conditions (1) in Theorems 1, 3 and conditions (4), (5) in Theorems A, B have the following difference: The set of truncations $\{\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}\}$ in Theorems 1, 3 and the set of truncations $\{\eta\chi_E + (1/(1+r))\eta\chi_{\Delta-E}\}$ in Theorems A, B are not the same. They do not contain each other.

(b) We have the following example for the result (2) in Theorems 1 and 3: In [1], Bozin et al. gave a counterexample of a Beltrami coefficient which is (infinitesimally) uniquely extremal, but the absolute value of $\mu(z)$ is not a constant (also see the counterexample in [4]). Because $\mu(z)$ equals zero in a Merglyan subset E, which is a compact connected subset of Δ with empty interior that does not disconnect Δ, the truncation $\mu\chi_E + (1/(1+r))\mu\chi_{\Delta-E}$ of $\mu(z)$ to E is equal to $(1/(1+r))\mu$, which obviously does not represent a (infinitesimal) Strebel point, but is (infinitesimally) uniquely extremal.

(c) From the theorems proved above, we know that if all the truncations of an extremal Beltrami coefficient μ are Strebel points, then in the infinitesimal setting all the truncations of μ are infinitesimal Strebel points.

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