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Hydrodynamical limit for a drift-diffusion system modeling large-population dynamics

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Abstract

In this paper we study the stability of the following nonlinear drift-diffusion system modeling large population dynamics $\partial_t \rho + \operatorname{div}(\rho U - \varepsilon \nabla \rho) = 0$, $\operatorname{div} U = \pm \rho$, with respect to the viscosity parameter ε. The sign in the second equation depends on the attractive or repulsive character of the field U. A proof of the compactness and convergence properties in the vanishing viscosity regime is given. The lack of compactness in the attractive case is caused by the blow-up of the solution which depends on the mass and on the space dimension. Our stability result is connected, depending of the character of the potentials, with models in semiconductor theory or in biological population.

Keywords: Hydrodynamic limit; Semiconductors; Electrodiffusion; Chemotaxis; Drift-diffusion; Approximation-diffusion

1. Introduction

The stochastic model governing the dynamics of many-particle systems in a surrounding bath is the well-known Vlasov–Poisson–Fokker–Planck system (VPFP). In terms of the thermal velocity and the thermal mean free path, the low-field limit of this system was analyzed by Poupaud and Soler in [13], who performed a parabolic limit which preserves the second-order diffusive term

$$\frac{\partial}{\partial t}\rho_\varepsilon + \operatorname{div}_x(\rho_\varepsilon U_\varepsilon - \varepsilon \nabla_x \rho_\varepsilon) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N, \tag{1.1}$$

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$$U_\varepsilon = -\nabla_x \Phi_\varepsilon, \quad -\Delta_x \Phi_\varepsilon = \theta\rho_\varepsilon, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N, \tag{1.2}$$

$$\rho_\varepsilon(0,\cdot) = \rho_{0,\varepsilon}, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where ε is a positive (viscosity) constant and $\theta = 1$ when we consider a self-consistent field U of electrostatic type produced by a charge density ρ (repulsive forces) or $\theta = -1$ for the gravitational case, in which the self-consistent field is due to the mass distribution (attractive forces).

The high-field limit corresponds to a different regime of the physical constants (thermal velocity and thermal mean free path) standing in the VPFP system. This limiting behavior was analyzed in [10] by Nieto, Poupaud and Soler obtaining the equations of pressureless gas dynamics:

$$\frac{\partial}{\partial t}\rho + \mathrm{div}_x(\rho U) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N, \tag{1.4}$$

$$U = -\nabla_x \Phi, \quad -\Delta_x \Phi = \theta\rho, \quad (t,x) \in [0,\infty) \times \mathbb{R}^N,$$

$$\rho(0,\cdot) = \rho_0, \quad x \in \mathbb{R}^N. \tag{1.5}$$

Then, we can consider the system (1.1)–(1.3) as a perturbation of (1.4)–(1.5) in which a viscosity term $\varepsilon\Delta$ has been introduced producing a smoothing effect on the density ρ. Therefore, in order to complete the framework of these stability results with respect to the physical constants, we shall try to connect these two different regimes by showing that the former converges to the latter when ε goes to 0 or, in other words, that the second model is stable under the perturbative viscosity method.

In fact, as will be shown in Section 3, in one dimension, a field U solution of (1.4)–(1.5) solves the Hopf–Burgers equation (to see that, at least formally, it is enough to take into account that in 1D, $U' = -\theta\rho$) and U_ε is the approximated solution given by the vanishing viscosity method. To justify the interest of the limit $\varepsilon \to 0$, let us comment some interesting phenomena modeled by these systems.

Both systems can be seen as hydrodynamic limits of the VPFP system and, as consequence, they model macroscopic regimes of many particle systems. We can also connect the drift-diffusion system (1.1)–(1.3) with an electrochemistry model for the electrodiffusion of charged ions in electrolytes filling the whole $\mathbb{R}^N$. In this direction we refer the reader to Choi and Lui [3], where the long-time behavior of a more general model for two species of charged particles is analyzed. Also, Biler and Dolbeault study in [1] the global stability of steady-state solutions of a multi-valued electrochemistry model on bounded domains. On the other hand, in the attractive case, the system (1.1)–(1.3) is a particular case of the Keller–Segel model (see [6]), which describes the aggregation of the slime mold amoebae due to an attractive chemical substance that they secrete when they lack nourishment. The blow-up behavior of (1.1)–(1.3) in the chemotactic case has been extensively analyzed by Herrero, Medina and Velázquez in [4,5] in dimension $N = 2$ and $N = 3$, respectively. Also, Nagai studies in [9] the blow-up of radially symmetric solutions in all dimensions for bounded domains. We remark that our system is considered in the whole space $\mathbb{R}^N$ in contrast with previous results concerning chemotaxis models. In the attractive case, the system (1.4)–(1.5) is also a particular case of the Keller–Segel model under some assumptions on ρ and Φ. We refer to Rascle and Ziti [15] for the analysis of these models, specially the study of blow-up in finite time.

When repulsive forces occur, the system (1.1)–(1.3) has been used in modeling semiconductor devices when the typical length is large enough with respect to the typical tunneling time which is for instance the case of silicium. However, due to the progressive miniaturization of semiconductors, the system (1.1)–(1.3) stopped being useful due to the hyperbolic character of the equations governing the electron transport. Then, (1.4)–(1.5) appears as the natural macroscopic model to describe the electron density transport.

The aim of this work is to study the behavior of (1.1)–(1.3) when the viscosity parameter ε goes to zero and recover the system (1.4)–(1.5). To do that, we consider a sequence of initial conditions $\rho_{0,\varepsilon}$ which converges to ρ_0 in a suitable space to be precised. We study the associated sequence ρ_ε of solutions to (1.1)–(1.3) with initial data $\rho_{0,\varepsilon}$. We shall apply the compactness techniques developed in [10,13] in dimension one and in higher dimensions when possible. Then, we shall prove that in the 1D case the anti-symmetry of the Poisson kernel and the global bound of its first derivative allow us to pass to the weak-$\star$ limit in the space of finite Radon measures uniformly on bounded time intervals. This result is obtained under the assumption that the initial condition converges only in $(\mathcal{M}(\mathbb{R})$, weak-$\star)$ and has a bounded first-order moment. In 1D, the attractive and repulsive case are treated simultaneously. In higher dimensions, we shall pass to the limit by using a uniform bound for $\|\rho_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R}^N)}$, with $1 \leqslant p \leqslant \infty$ and $t \in [0,T]$. Here the time interval $[0,T]$ is arbitrary in the repulsive case ($\theta = 1$) and, in the attractive one ($\theta = -1$), it depends on the initial data in the form $T < T^* = (\sup \|\rho_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^N)})^{-1} < \infty$.

The paper is structured as follows. In Section 2 we study the compactness properties of ρ_ε depending on the space dimension and on the attractive or repulsive character of the forces. Section 3 is devoted to the rigorous analysis of the limit.

2. Compactness properties and existence of solutions

In the previous literature some existence results for the system (1.1)–(1.3) can be found. We present here some additional properties of the system which do not depend on ε and some answers concerning the possible blow-up of solutions in the attractive case. The main result of this section is the following.

Theorem 2.1. *Let $\rho_{0,\varepsilon}$ be a sequence of nonnegative initial conditions satisfying*

$$L := \sup_{\varepsilon>0} \|\rho_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^N)} < \infty, \qquad M := \sup_{\varepsilon>0} \|\rho_{0,\varepsilon}\|_{L^1(\mathbb{R}^N)} < \infty, \quad \text{and}$$

$$\int_{\mathbb{R}^N} |x| \rho_{0,\varepsilon}(x)\,dx \leqslant C < \infty$$

where C is a positive constant independent on ε. Then, there exists a solution $(\rho_\varepsilon, U_\varepsilon)$ of (1.1)–(1.3) *verifying*:

(i) *Mass conservation*: $M_\varepsilon := \|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)} = \|\rho_{0,\varepsilon}\|_{L^1(\mathbb{R}^N)}$, $\forall t \geqslant 0$.

(ii) *If $N = 1$, the inequality*

$$\int_{\mathbb{R}} |x|\rho_\varepsilon(t,x)\,dx + \|U_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant C\,T,$$

holds for $t \in [0,T]$ with $T < \infty$.

(iii) *If $N > 1$, the inequality*

$$\|\rho_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} + \|U_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |x|\rho_\varepsilon(t,x)\,dx \leqslant C\,T$$

holds for $t \in [0,T]$ with $T < \infty$ if $\theta = 1$ and with $T < T^* = (1/L)$ if $\theta = -1$.

Moreover, for a fixed $\varepsilon > 0$, in the two-dimensional attractive case (*usually known as the critical case*) *this solution satisfies*: *if*

$$M_\varepsilon = \|\rho_{0,\varepsilon}\|_{L^1(\mathbb{R}^2)} < 4\varepsilon\frac{p}{(p+1)^2}, \tag{2.6}$$

for $1 < p < \infty$, then $\|\rho_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R}^2)} \leqslant \|\rho_{0,\varepsilon}\|_{L^p(\mathbb{R}^2)}$ for all positive t.

We devote the rest of this section to prove this result. We first study a priori estimates in L^∞ uniformly in ε by using a maximum principle in $\mathbb{R}^N$. Next, we derive (uniform in ε) bounds for the field and the first-order moment.

Also, for a fixed viscosity parameter ε, we study the behavior of solutions of (1.1)–(1.2) in the two-dimensional attractive case. To this aim, given $\varepsilon > 0$, (2.6) constitutes a sufficient condition to control the blow-up of the solutions in $L^p(\mathbb{R}^2)$. However, this condition cannot hold for the passage to the limit $\varepsilon \to 0$.

2.1. Uniform bounds and existence of solutions

We first observe that the field defined by (1.2) can be equivalently rewritten in convolution form $U_\varepsilon(t,x) = \theta K_N *_x \rho_\varepsilon(t,x)$, where

$$K_N = C_N\frac{x}{|x|^N}, \tag{2.7}$$

where C_N is a *positive* constant depending on the space dimension. We define T^* as in Theorem 2.1, that is, the maximum time until which we expect that the uniform bounds hold in the attractive case. Now, we fix $\varepsilon > 0$ and $T > 0$ (by choosing $T < T^*$ when $\theta = -1$) and take, for any natural number $n \in \mathbb{N}$ and for any $j = 0,\dots,n$, the partition $t_j = jT/n$. We consider the retarded-in-time sequence of Cauchy problems

$$\begin{aligned} &\frac{\partial}{\partial t}\rho_\varepsilon^{n,j} = \varepsilon\Delta_x\rho_\varepsilon^{n,j} - \operatorname{div}_x\left(U_\varepsilon^{n,j-1}\rho_\varepsilon^{n,j}\right), \quad t \in [t_{j-1}, t_j], \\ &\rho_\varepsilon^{n,j}(t_{j-1}) = \rho_\varepsilon^{n,j-1}(t_{j-1}), \quad j = 1,\dots,n, \end{aligned} \tag{2.8}$$

where

$$U_\varepsilon^{n,j}(t,x) := \theta\left[(K_N *_x \zeta^n) *_x \rho_\varepsilon^{n,j}\left(t - \frac{T}{n}, \cdot\right)\right](x),$$
$$\rho_\varepsilon^{n,0}(t,x) = (\zeta^n *_x \rho_{0,\varepsilon})(x).$$

Note that, in order to regularize the field and the initial condition, we have introduced a standard nonnegative mollifier $\zeta^n(x) = n^N \zeta(nx)$ with $\zeta \in C_0^\infty(\mathbb{R}^N)$ and $\int \zeta = 1$. Expanding the second term of (2.8) and using that $\operatorname{div}_x(U_\varepsilon^{n,j-1}(t,\cdot)) = \theta\, \zeta^n *_x \rho_\varepsilon^{n,j-1}(t - T/n, \cdot)$ we find

$$\frac{\partial}{\partial t}\rho_\varepsilon^{n,j} + \left(U_\varepsilon^{n,j-1} \cdot \nabla_x\right)\rho_\varepsilon^{n,j} - \varepsilon \Delta_x \rho_\varepsilon^{n,j} = -\theta\left(\zeta^n * \rho_\varepsilon^{n,j-1}\right)\rho_\varepsilon^{n,j}, \tag{2.9}$$

in $[t_{j-1}, t_j] \times \mathbb{R}^N$ where $\rho_\varepsilon^{n,j-1}$ is valued in $(t - T/n, \cdot)$. In order to study this retarded-in-time *linear* equation, let us focus our attention on a generic parabolic linear equation of the type

$$\frac{\partial p}{\partial t} + (a \cdot \nabla_x)p - \varepsilon \Delta_x p = -\theta f, \quad p(s_0, x) = p_0(x), \tag{2.10}$$

with given regular data a, f and p_0, where f is nonnegative and s_0 is fixed. A classical result based on the construction of the fundamental solution associated with (2.10) (see, for example, [8]), gives the existence of a smooth solution which is uniformly bounded in $[s_0, s_0 + T]$. But this bound is a priori strongly dependent on the coefficient ε and on the function a. To skip this dependence, we use a maximum principle as follows: define $\bar{p}$ as

$$\bar{p}(t,x) = \begin{cases} p(t,x) - \|p_0\|_{L^\infty(\mathbb{R}^N)}, & \text{if } \theta = 1, \\ p(t,x) - \left(\|p_0\|_{L^\infty(\mathbb{R}^N)} + \displaystyle\int_{s_0}^{t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^N)}\, ds\right), & \text{if } \theta = -1. \end{cases}$$

In both cases $\bar{p}$ verifies

$$\frac{\partial \bar{p}}{\partial t} + a \cdot \nabla \bar{p} - \varepsilon \Delta_x \bar{p} \leqslant 0, \quad \text{with } \bar{p}(s_0, x) = p_0(x) - \|p_0\|_{L^\infty(\mathbb{R}^N)} \leqslant 0.$$

Now, using the uniform bound for p, we deduce the following weak decreasing condition at infinity:

$$\liminf_{R\to\infty}\left(e^{-R^2} \sup\left\{\bar{p}(t,x)\colon t \in [s_0, T], |x| \leqslant R\right\}\right) = 0,$$

which allows to conclude that $\bar{p}(t,x) \leqslant 0$ in $[s_0, T] \times \mathbb{R}^N$ (see [14, Theorem 10, §3]) or, equivalently, that a solution of (2.10) verifies:

$$\text{if } \theta = 1, \qquad p(t,x) \leqslant \|p_0\|_{L^\infty(\mathbb{R}^N)}, \tag{2.11}$$

$$\text{if } \theta = -1, \quad p(t,x) \leqslant \|p_0\|_{L^\infty(\mathbb{R}^N)} + \int_{t_0}^{t} \|f(s,\cdot)\|_{L^\infty(\mathbb{R}^N)}\, ds. \tag{2.12}$$

We can apply this bound to the solutions of (2.9) with $p = \rho_\varepsilon^{n,j}$, $a = U_\varepsilon^{n,j-1}$, and $f = -\theta(\zeta^n * \rho_\varepsilon^{n,j-1})\rho_\varepsilon^{n,j}$. Then, we conclude that there exists $\rho_\varepsilon^{n,j}$ in $C(t_{j-1}, t_j; L^\infty(\mathbb{R}^N))$ solution of (2.9) verifying

$$\left\|\rho_\varepsilon^{n,j}(t,\cdot)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant \left\|\rho_\varepsilon^{n,j-1}\right\|_{L^\infty(\mathbb{R}^N)} \leqslant \cdots \leqslant \|\rho_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^N)}, \tag{2.13}$$

in the repulsive case and

$$\begin{aligned}\left\|\rho_\varepsilon^{n,j}(t,\cdot)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant{}& \left\|\rho_\varepsilon^{n,j-1}(t_{j-1},\cdot)\right\|_{L^\infty(\mathbb{R}^N)} \\ &+ \int_{t_{j-1}}^{t} \left\|\rho_\varepsilon^{n,j-1}\left(s-\frac{T}{n},\cdot\right)\right\|_{L^\infty(\mathbb{R}^N)} \left\|\rho_\varepsilon^{n,j}(s,\cdot)\right\|_{L^\infty(\mathbb{R}^N)}\,ds,\end{aligned}$$

in the attractive one. Here, Gronwall's Lemma and an inductive argument allow us to prove that

$$\left\|\rho_\varepsilon^{n,j}(t,\cdot)\right\|_{L^\infty(\mathbb{R}^N)} \leqslant \frac{\|\rho_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^N)}}{1-t\,\|\rho_{0,\varepsilon}\|_{L^\infty(\mathbb{R}^N)}}. \tag{2.14}$$

Now, we define

$$\rho_\varepsilon^n(t,x) = \rho_\varepsilon^{n,j}(t,x) \quad \text{in } [t_{j-1},t_j],$$

which is a continuous function in $[0,T]$. Analogously we define also U_ε^n as $U_\varepsilon^{n,j}$ on the corresponding time interval. Then, using the weak formulation of (2.8) we can write

$$\int_0^T\!\!\int_{\mathbb{R}^N} \left(\frac{\partial\psi}{\partial t} + \Delta_x\psi + U_\varepsilon^n\cdot\nabla_x\psi\right)\rho_\varepsilon^n\,dx\,dt = \int_{\mathbb{R}^N} \rho_{0,\varepsilon}(x)\psi(0,x)\,dx, \tag{2.15}$$

for every $\psi \in \mathcal{D}([0,T)\times\mathbb{R}^N)$.

Now, in order to take the limit $n\to\infty$ in (2.15), we need some additional bounds for ρ_ε^n. Firstly, using the same maximum principle as before, we deduce that $\rho_\varepsilon^n \geqslant 0$ so that we can estimate its L^1 norm by integrating in (2.8). In fact, the mass conservation property

$$\left\|\rho_\varepsilon^n(t,\cdot)\right\|_{L^1(\mathbb{R}^N)} = \left\|\rho_{0,\varepsilon}^n\right\|_{L^1(\mathbb{R}^N)} \leqslant M_\varepsilon \leqslant M,$$

holds. These bounds imply that the ρ_ε^n is bounded in $L^\infty(0,T;L^p(\mathbb{R}^N))$ for $1\leqslant p\leqslant\infty$ and then, that U_ε^n is bounded in $L^\infty(0,T;(W^{1,p}(\mathbb{R}^N))^N)$ for $2\leqslant p<\infty$. Using now Eq. (2.8), we deduce that $\partial_t\rho_\varepsilon^n(t,\cdot)$ is bounded in $W^{-2,\infty}(\mathbb{R}^N)\subseteq\mathcal{D}'(\mathbb{R}^N)$ uniformly with respect to t and n. Since $\mathcal{D}(\mathbb{R}^N)$ is separable and dense in $L^{p'}(\mathbb{R}^N)$ $(p\neq 1)$ we can then assure, by using standard arguments from the general theory of conservation laws, that ρ_ε^n lives in a compact set of $C([0,T];(L^p(\mathbb{R}^N),\text{weak-}\star))$ for all $1<p\leqslant\infty$ (this argument shall be carried out in detail in the next section).

Then, up to a subsequence, ρ_ε^n converges in $C([0,T];(L^2(\mathbb{R}^N),\text{weak-}\star))$ and U_ε^n converges in $L^2_{\text{loc}}([0,T]\times\mathbb{R}^N)$ when $n\to\infty$. Combining the convergence of ρ_ε^n and U_ε^n, we can pass to the limit in (2.15) to find a solution of (1.1). Finally, we can use estimates (2.13) and (2.14) to find the uniform bounds:

$$\begin{aligned}&\text{for } \theta=-1, \quad \|\rho_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \leqslant \frac{L}{1-tL}, \quad \forall\, 0\leqslant t<T^*=\frac{1}{L},\\ &\text{for } \theta=1, \qquad \|\rho_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \leqslant L, \quad \forall\, 0\leqslant t.\end{aligned}$$

Clearly (see [10]) this uniform bound is optimal because in the limit $\varepsilon\to 0$ it is satisfied by the solutions of the limiting system (1.4)–(1.5).

Remark 1. Even in the 2D attractive case ($\theta = -1$), but for fixed $\varepsilon > 0$, we can say something more about the absence of blow-up in $L^p(\mathbb{R}^2)$. $N = 2$ is known as the critical case, because there is a mass threshold for chemotactic collapse that seems to be absent for $N \geqslant 3$ (see [4] and references therein and [9] for the study on bounded domains). Using a Sobolev–Gagliardo–Nirenberg inequality (see [2, Chapter IX.3]),

$$\int_{\mathbb{R}^2} \rho^{p+1}(x)\,dx \leqslant \frac{(p+1)^2}{p^2}\left(\int_{\mathbb{R}^2} \rho(x)\,dx\right)\left(\int_{\mathbb{R}^2} \left|\nabla\left(\rho^{p/2}\right)(x)\right|^2 dx\right),$$

multiplying (1.1) by ρ_ε^{p-1} and integrating, we find

$$\frac{1}{p-1}\frac{\partial}{\partial t}\int_{\mathbb{R}^2} \rho_\varepsilon^p\,dx \leqslant \left(\frac{(p+1)^2}{p^2}M_\varepsilon - \frac{4\varepsilon}{p}\right)\int_{\mathbb{R}^2} \left|\nabla\left(\rho_\varepsilon^{p/2}\right)\right|^2 dx \tag{2.16}$$

for $1 < p < \infty$. Here we observe that the control of the mass implies the estimates on variation of $\|\rho_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R}^2)}$. In fact, if (2.6) holds, then the second term of (2.16) becomes negative and we conclude that $\|\rho_\varepsilon(t,\cdot)\|_{L^p(\mathbb{R}^2)} \leqslant \|\rho_{0,\varepsilon}\|_{L^p(\mathbb{R}^2)}$ for all $t \geqslant 0$.

2.2. *Field and moment estimates*

To estimate the field U_ε, we use the well-known bound

$$\|U_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} \leqslant C\left(\|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)} + \|\rho_\varepsilon(t,\cdot)\|_{L^r(\mathbb{R}^N)}\right), \quad r > N. \tag{2.17}$$

In particular, in the one-dimensional case, by using that $\nabla K_1 = \frac{1}{2}\operatorname{sign}(x)$ is bounded, we have

$$\|U_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant \frac{1}{2}\|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R})}, \tag{2.18}$$

which gives, for $N = 1$, a global uniform bound independent of the attractive or repulsive character of the Coulomb forces.

Finally, we shall find a bound for the first-order moment of ρ_ε to control the loss of mass at infinity and consequently to get compactness in L^1. To do that, we take an auxiliary function $g \in C^2(\mathbb{R}^N)$ such that $g(x) \geqslant |x|$ for all x and $g(x) = |x|$ for $|x| \geqslant 1$ (note that this implies that ∇g and Δg are bounded). Then, using Eq. (1.1) we find

$$\begin{aligned}\int_{\mathbb{R}^N} |x|\rho_\varepsilon(t,x)\,dx &\leqslant \int_{\mathbb{R}^N} g(x)\rho_\varepsilon(t,x)\,dx \\ &= \int_{\mathbb{R}^N} g(x)\rho_{\varepsilon,0}(x)\,dx + \int_0^t\int_{\mathbb{R}^N} \left(U_\varepsilon\cdot\nabla g + \varepsilon\Delta g\right)\rho_\varepsilon(s,x)\,dx\,ds.\end{aligned}$$

As consequence, for some constant C depending only on g, we have

$$\int_{\mathbb{R}^N} |x|\rho_\varepsilon(t,x)\,dx \leqslant C\left(\int_{\mathbb{R}^N} |x|\rho_{0,\varepsilon}\,dx + M_\varepsilon\right) + C\,M_\varepsilon\int_0^t \left(\|U_\varepsilon(s,\cdot)\|_{L^\infty(\mathbb{R}^N)} + 1\right)ds,$$

which gives an estimate in $[0,T]$ provided that U_ε is bounded. Then, following (2.17) and (2.18) and the bounds for ρ_ε, we have finally done with (ii) and (iii) of Theorem 2.1.

The following step is to pass to the limit rigorously in the sequence ρ_ε and in (1.1)–(1.3) in a weak sense as $\varepsilon \to 0$ and show that the limit ρ satisfies the system (1.4)–(1.5).

3. Passage to the limit

We first state the main result of this paper.

Theorem 3.1. *Under the same hypotheses of Theorem* 2.1, *the associated solution* $(\rho_\varepsilon, U_\varepsilon)$ *to the system* (1.1)–(1.3) *verify*

$$\begin{aligned}
&\rho_\varepsilon \rightharpoonup \rho, \quad \textit{in } C\big(0,T;\big(L^p\big(\mathbb{R}^N\big),\textit{weak-}\star\big)\big) \cap \big(L^1\big(\mathbb{R}^N\big),\textit{weak}\big),\ p \leqslant \infty,\\
&U_\varepsilon \to U, \quad \textit{in } L^q\big(0,T;L^p(\Omega)^N \cap C(\Omega)^N\big),\ 1 \leqslant q < \infty,\ 2 \leqslant p < \infty,
\end{aligned}$$

for every compact subset $\Omega \subset \mathbb{R}^N$ *and* $T>0$ *in the case of repulsive forces* $(\theta = 1)$ *and for* $T < T^*$ *in the case of attractive forces* $(\theta = -1)$.

The limit (ρ, U) *is the unique solution of* (1.4)–(1.5). *Moreover, in the one-dimensional case we have*

$$\rho_\varepsilon \rightharpoonup \rho, \quad \textit{in } C\big(0,T;\mathcal{M}(\mathbb{R})\textit{-weak-}\star\big)$$

for every $T>0$, *independently of* θ.

Proof. In the following $T>0$ will be a fixed time as before, that is, $T < T^* = 1/L$ in the attractive case and $T<\infty$ in the repulsive one or when we work in dimension one (independently on θ).

We first consider the system (1.1)–(1.3) in a weak form, i.e., for every test function $\psi \in \mathcal{D}([0,T)\times\mathbb{R}^N)$ we write

$$\int_{\mathbb{R}^N} \rho_{0,\varepsilon}(x)\psi(0,x)\,dx = \int_0^\infty \int_{\mathbb{R}^N} \rho_\varepsilon\left(\frac{\partial\psi}{\partial t} + \varepsilon\Delta_x\psi + U_\varepsilon\cdot\nabla\psi\right)dx\,dt. \tag{3.19}$$

We observe that the L^1 bound is enough to pass to the limit, weakly as measures, in the three first terms but not in the last one: the nonlinear term. Let us specify how to pass to the limit in the nonlinear term by studying the convergence of ρ_ε. For that we first take a test function $\phi \in \mathcal{D}(\mathbb{R}^N)$ and observe that

$$\begin{aligned}
&\alpha_\varepsilon[\phi](t) := \int_{\mathbb{R}}^{N} \rho_\varepsilon(t,x)\phi(x)\,dx \leqslant \|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)}\|\phi\|_{L^\infty(\mathbb{R}^N)},\\
&\frac{\partial}{\partial t}\alpha_\varepsilon[\phi](t) = \int_{\mathbb{R}^N} \rho_\varepsilon(\varepsilon\phi'' + U_\varepsilon\phi')\,dx \leqslant C\|\rho_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^N)}\big(1 + \|U_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^N)}\big).
\end{aligned}$$

Using Theorem 2.1, we conclude that $\alpha_\varepsilon[\phi](\cdot)$ is an ε-equi-continuous and bounded family of $C(0,T)$ and therefore, there exists a sub-sequence (depending on ϕ) such that $\alpha_\varepsilon[\phi](\cdot)$ converges to some $\alpha[\phi](\cdot)$ in $C(0,T)$. But using the separability of $\mathcal{D}(\mathbb{R}^N)$, we know that this sub-sequence can be chosen independently of the test ϕ.

Now, in a general dimension N, we use that $\mathcal{D}(\mathbb{R}^N)$ is dense in $L^{p'}(\mathbb{R}^N)$ for $1 < p \leqslant \infty$ and then, that $\alpha_\varepsilon[\,\cdot\,](t)$ defines a bounded family of linear continuous operators on the space $L^{p'}(\mathbb{R}^N)$. Moreover, using the density and the previous convergence, we can easily conclude in terms of ρ_ε (and identifying the limit α with a function ρ) that

$$\rho_\varepsilon \rightharpoonup \rho, \quad \text{in } C\big(0,T;\big(L^p\big(\mathbb{R}^N\big), \text{weak-}\star\big)\big), \quad 1 < p \leqslant \infty. \tag{3.20}$$

Analogously, in dimension one we observe that $\mathcal{D}(\mathbb{R})$ is dense in $C_{00}(\mathbb{R})$ and therefore, that $\alpha_\varepsilon[\,\cdot\,](t)$ become linear continuous operators on the space $C_{00}(\mathbb{R})$ for any t. In this case the convergence holds in the measure sense and the limit can be identified with a measure that we call $\rho(t)$. We have just proved that, up to a sub-sequence,

$$\rho_\varepsilon \rightharpoonup \rho, \quad \text{in } C\big(0,T;\big(\mathcal{M}(\mathbb{R})\text{-weak-}\star\big)\big). \tag{3.21}$$

Remark 2. We observe that (3.20) is also valid for $p = 1$ (with the weak topology) although $\mathcal{D}(\mathbb{R}^N)$ is not dense in $L^\infty(\mathbb{R}^N)$. This can be proved (see [13]) by using some results based on the Egorov Theorem and the bound for the first-order moment.

3.1. Passage to the limit in the general case

To pass to the limit in an arbitrary dimension N we will use the uniform estimates of ρ_ε to have the strong convergence of the field U_ε. From (3.20), Theorem 2.1(iii), and the estimates of harmonic analysis we know that

$$U_\varepsilon \text{ is uniformly bounded} \quad \text{in } L^\infty\big(0,T;\big(W^{1,p}\big(\mathbb{R}^N\big)\big)^N\big),\ 2 \leqslant p < \infty.$$

Thus $U_\varepsilon(t\,\cdot)$ is relatively compact in $L^p(\Omega) \cap C(\bar{\Omega})$, for every compact set $\Omega \subset \mathbb{R}^N$ and for every $t \in [0,T]$. Then, for every $t \in [0,T]$ there exists a sub-sequence converging to some $U(t,\cdot)$ in $L^p(\Omega) \cap C(\bar{\Omega})$. Now, using Eq. (1.3) and the convergence of ρ_ε (3.20), we find the form of the limit $U(t,x) = K_N *_x \rho(t,x)$ and conclude that all the sequence converges independently on t. Therefore, the dominated convergence theorem assures that

$$U_\varepsilon \to U, \quad \text{in } L^q\big(0,T;L^p(\Omega)^N \cap C(\Omega)^N\big), \tag{3.22}$$

for $1 \leqslant q < \infty$ and $2 \leqslant p < \infty$. The convergence (3.20) and (3.22) suffices to take the limit in (3.19) and find the announced system. Finally, the uniqueness of weak solutions of (1.4)–(1.5) in $L^1 \cap L^\infty$ (see [10]) concludes that the whole sequence converges and the result is proved.

3.2. Passage to the limit in dimension one

In this case we can use the techniques introduced in [10,13] based on the anti-symmetry properties of the kernel. Using the convolution form of U_ε in terms of the kernel K_1, we rewrite the nonlinear term of (3.19) as follows:

$$\int_0^\infty \int_{\mathbb{R}} \rho_\varepsilon(t,x) U_\varepsilon(t,x) \frac{\partial}{\partial x} \psi(t,x)\, dx\, dt \tag{3.23}$$
$$= \frac{\theta}{2} \int_0^\infty \int_{\mathbb{R}_x} \int_{\mathbb{R}_y} \left\{ \frac{x-y}{2|x-y|} \left(\frac{\partial \psi}{\partial x}(t,x) - \frac{\partial \psi}{\partial x}(t,y) \right) \right\} \rho_\varepsilon(t,x) \rho_\varepsilon(t,y)\, dy\, dx\, dt,$$

which can be taken as the *weak* definition of the product $\rho_\varepsilon U_\varepsilon$. Note that the function under brackets is continuous and bounded, and then, the duality with the measure $\rho_\varepsilon(t,x)\rho_\varepsilon(t,y)$ gives sense to this expression. Using here (3.21), we can easily conclude that

$$\rho_\varepsilon(t,x)\rho_\varepsilon(t,y) \rightharpoonup \rho(t,x)\rho(t,y), \quad \text{in } C\big(0,T; \big(\mathcal{M}(\mathbb{R}_x \times \mathbb{R}_y), \text{weak-}\star\big)\big).$$

Finally, a truncature argument as the one used in [10] together with the bound for the first-order moment given by Theorem 2.1 allow us to pass to the limit in (3.23). Then, defining the product ρU in this weak sense (as in [10]), we recover the system (1.4)–(1.5). □

Remark 3. The techniques of anti-symmetry have allowed to pass to the limit in the density ρ_ε and give sense to the product ρU. But we can skip the role of the density and study only the behavior of the field U_ε. Convoluting Eq. (1.1) with θK_1, we formally obtain

$$\frac{\partial}{\partial t} U_\varepsilon = -\partial_x\big(\theta K_N *_x (\rho_\varepsilon U_\varepsilon)\big) + \varepsilon \partial_{xx} U_\varepsilon = -\theta \rho_\varepsilon U_\varepsilon + \varepsilon \partial_{xx} U_\varepsilon = -\partial_x \frac{U_\varepsilon}{2} + \varepsilon \partial_{xx} U_\varepsilon$$

which is the Hopf–Burger equation. The technique of vanishing viscosity has been used to prove the existence of admissible solutions in the class of bounded solutions and the uniqueness in the sense of *entropy solutions* given by Stanislav Kruzskov [7] or the equivalent concept of *admissible solution* given by Olga Oleinik [12] (see also [11] for explanation). Then, in the one-dimensional case, the anti-symmetry method presented here is equivalent to the classical vanishing viscosity method.

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