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Existence of positive solutions for higher-order functional differential equations

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Abstract

Under suitable conditions on $f(t, y(t+\theta))$, the boundary value problem of higher-order functional differential equation (FDE) of the form

$$\text{(BVP)}\quad \begin{cases} \text{(FDE)} & y^{(n)}(t) + f(t, y(t+\theta)) = 0, \quad t \in [0,1],\ \theta \in [-\tau, a], \\ \text{(BC)} & \begin{cases} y^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-3, \\ \begin{cases} \alpha y^{(n-2)}(t) - \beta y^{(n-1)}(t) = \eta(t), & t \in [-\tau, 0], \\ \gamma y^{(n-2)}(t) + \delta y^{(n-1)}(t) = \xi(t), & t \in [1, 1+a], \end{cases} \end{cases} \end{cases}$$

has at least one positive solution, where $\theta \in [-\tau, a]$ is a fixed constant. Moreover, we also apply this main result to establish several existence theorems which guarantee (BVP) has multiple positive solutions.

Keywords: Higher-order; Functional differential equation; Green's function; Positive solution; Cone; Fixed point

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1. Introduction

In this article, we consider the existence of positive solutions for boundary value problems of an nth-order functional differential equation (FDE) of the form

$$
\text{(BVP)} \quad \begin{cases} \text{(FDE)} & y^{(n)}(t) + f(t, y(t+\theta)) = 0, \quad t \in [0,1],\ \theta \in [-\tau, a], \\ \text{(BC)} & \begin{cases} y^{(i)}(0) = 0, \quad 0 \leqslant i \leqslant n-3, \\ \begin{cases} \alpha y^{(n-2)}(t) - \beta y^{(n-1)}(t) = \eta(t), & t \in [-\tau, 0], \\ \gamma y^{(n-2)}(t) + \delta y^{(n-1)}(t) = \xi(t), & t \in [1, 1+a], \end{cases} \end{cases} \end{cases}
$$

where

(i) $\tau, a, \alpha, \beta, \gamma$ and δ are nonnegative constants satisfying $0 \leqslant \tau + a < 1$ and $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$. Moreover, $\theta \in [-\tau, a]$ is a fixed constant.
(ii) $\eta \in C([-\tau, 0], \Re)$, $\xi \in C([1, b], \Re)$, $\eta(0) = \xi(1) = 0$, where $b = 1 + a$.

Let $C = C^{n-2}([-\tau, a], \Re)$ be a space with norm $\|\psi\|_{[-\tau,a]} = \sup_{-\tau \leqslant x \leqslant a} |\psi^{(n-2)}(x)|$ for $\psi \in C$. Let

$$C^+ = \{\psi \in C\colon\ \psi(x) \geqslant 0,\ x \in [-\tau, a]\},$$

$C^* := \{\psi \in C^+\colon\ 0 < c\|\psi\|_{[-\tau,a]} \leqslant \psi^{(n-2)}(x) \text{ for } x \in [-\tau, a]\}$ for some constant $c \in (0,1)$ depending on ψ, and let $E = \{t \in [0,1]\colon\ 0 \leqslant t + \theta \leqslant 1\}$ possesses nonzero measure.

In the last twenty five years, many authors considered the boundary value problem (BVP) with $n = 2$ under the situation that $\tau = a = 0$, in this case, (BVP) becomes the two-points boundary value problem for second-order ordinary differential equation which has strong background in the fields of mechanics, physics, and applied mathematics; see [1,4,11].

Recently, Erbe and Kong [3] investigated the boundary value problem of functional differential equations of the form $y''(t) + Q(t, y(w(t))) = 0$. For example, $Q(t, y) = (1/y^n)$ $(n > 0)$ has singularity at $y = 0$. As pointed out by the authors of [3], the study of (BVP) is of significance since it arises and has applications in variational problems in control theory and other areas of applied mathematics. In fact, the existence of positive solutions of (BVP) with $n = 2$ had been studied by many authors; see, for example, Henderson and Hudson [5], Hong et al. [7], Lee and O'Regan [9,10], Ntouyas et al. [12] and Weng and Jiang [14].

Recently, the study of higher-order functional differential equation has received more attention from some authors; see, for example, Davis et al. [2], Henderson and Yin [6] and Taunton and Yin [13]. Some of them, see [2,6], examined the existence of solutions of (FDE) under the boundary conditions

$$
\text{(BC}^*) \quad \begin{cases} u(s) = \phi(s), & -\tau \leqslant s \leqslant 0, \\ u^{(i)}(0) = 0, & 0 \leqslant i \leqslant k-1, \\ u^{(j)}(1) = 0, & 0 \leqslant j \leqslant n-k-1. \end{cases}
$$

Here $\phi \in C[-\tau, 0]$ which satisfies $\phi(0) = 0$ and $0 \leqslant k \leqslant n$ is a positive integer.

The purpose of this paper is to establish the existence of positive solutions of the functional differential equation (FDE) with boundary conditions (BC) under suitable conditions on f.

2. Definitions and lemmas

In order to abbreviate our discussion, throughout this paper, we suppose that the following assumptions hold:

(C_1) $G(t,s)$ is the Green's function of the differential equation

$$-u^{(n)}(t)=0 \quad \text{in } (0,1)$$

subject to the boundary conditions (BC) with $\tau=a=0$.

(C_2) $g(t,s)$ is the Green's function of the differential equation

$$-u''(t)=0 \quad \text{in } (0,1)$$

subject to the boundary conditions

$$\begin{cases} \alpha u(0)-\beta u'(0)=0, \\ \gamma u(1)+\delta u'(1)=0, \end{cases}$$

where α, β, γ and δ are defined as in (i).

(C_3) $f\in C([0,1]\times C^+;[0,\infty))$.

(C_4) $0<\int_{E_M} g(1/2,s)\,ds<\infty$, where $E_M=\{s\in E\colon\ M\leqslant s+\theta\leqslant 1-M\}$ for some small enough constant $M\in(0,1/2)$ and $\theta\in[-\tau,a]$ is given as in (ii).

Note. It is easy to see that

$$\frac{\partial^{n-2}}{\partial t^{n-2}}G(t,s)=g(t,s), \qquad t,s\in[0,1].$$

In order to establish our main result (Theorem 2.1 below), we need the following two useful lemmas. The first lemma is due to Hong et al. [7]. The second lemma is due to Krasnoselski [8].

Lemma A [7]. *Suppose that $g(t,s)$ is defined as in* (C_2) *and M defined as in* (C_4). *Then we have the following results*:

$$\begin{cases} (\mathrm{R}_1) \quad \frac{g(t,s)}{g(s,s)}\leqslant 1, & t\in[0,1],\ s\in[0,1], \\ (\mathrm{R}_2) \quad \frac{g(t,s)}{g(s,s)}\geqslant M, & t\in[M,1-M],\ s\in[0,1]. \end{cases}$$

Lemma B (Krasnoselskii [8]). *Let K be a cone in a Banach space E. Assume that Ω_1, Ω_2 are open subsets of E with $0\in\Omega_1$, $\bar{\Omega}_1\subset\Omega_2$. If*

$$A: K\cap(\bar{\Omega}_2\backslash\Omega_1)\to K$$

is a completely continuous operator such that either

(i) $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$, *or*
(ii) $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_2$,

then A has a fixed point in $K \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

Suppose that $y(t)$ is a solution of (BVP); then it can be written as

$$y(t) = \begin{cases} y(-\tau; t), & -\tau \leqslant t \leqslant 0, \\ \int_0^1 G(t,s) f(s, y(s+\theta))\, ds, & 0 \leqslant t \leqslant 1, \\ y(b; t), & 1 \leqslant t \leqslant b, \end{cases}$$

where $y(-\tau; t)$ and $y(b; t)$ satisfy

$$y^{(n-2)}(-\tau; t) = e^{\frac{\alpha}{\beta} t} \left(\frac{1}{\beta} \int_t^0 e^{-\frac{\alpha}{\beta} s} \eta(s)\, ds + y^{(n-2)}(0) \right), \quad t \in [-\tau, 0],$$

and

$$y^{(n-2)}(b; t) = e^{-\frac{\gamma}{\delta} t} \left(\frac{1}{\delta} \int_t^1 e^{\frac{\gamma}{\delta} s} \xi(s)\, ds + e^{\frac{\gamma}{\delta}} y^{(n-2)}(1) \right), \quad t \in [1, b],$$

respectively.

Throughout this paper, we assume that $u_0(t)$ is the solution of (BVP) with $f \equiv 0$. Clearly, it satisfies

$$u_0^{(n-2)}(t) = \begin{cases} \frac{1}{\beta} e^{\frac{\alpha}{\beta} t} \int_t^0 e^{-\frac{\alpha}{\beta} s} \eta(s)\, ds, & -\tau \leqslant t \leqslant 0, \\ 0, & 0 \leqslant t \leqslant 1, \\ \frac{1}{\delta} e^{-\frac{\gamma}{\delta} t} \int_t^1 e^{\frac{\gamma}{\delta} s} \xi(s)\, ds, & 1 \leqslant t \leqslant b. \end{cases}$$

Let $y(t)$ be a solution of (BVP) and $u(t) = y(t) - u_0(t)$. Note that $u(t) \equiv y(t)$ for $0 \leqslant t \leqslant 1$, and $u(t)$ satisfies

$$u^{(n-2)}(t) = \begin{cases} e^{\frac{\alpha}{\beta} t} u^{(n-2)}(0), & -\tau \leqslant t \leqslant 0, \\ \int_0^1 g(t,s) f(s, u(s+\theta) + u_0(s+\theta))\, ds, & 0 \leqslant t \leqslant 1, \\ e^{-\frac{\gamma}{\delta}(t-1)} u^{(n-2)}(1), & 1 \leqslant t \leqslant b, \end{cases}$$

which implies

$$u(t) = \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-2} e^{\frac{\alpha}{\beta} t} u^{(n-2)}(0), & -\tau \leqslant t \leqslant 0, \\ \int_0^1 G(t,s) f(s, u(s+\theta) + u_0(s+\theta))\, ds, & 0 \leqslant t \leqslant 1, \\ \left(-\frac{\delta}{\gamma}\right)^{n-2} e^{-\frac{\gamma}{\delta}(t-1)} u^{(n-2)}(1), & 1 \leqslant t \leqslant b. \end{cases}$$

Now, we can state and prove our main result.

Theorem 2.1 (Main result). *Assume that there exist two distinct positive constants* λ, μ *such that*

$$(\mathrm{H}_1) \quad f(t, \psi) \leqslant \lambda \left(\int_0^1 g(s,s) ds \right)^{-1} \quad \text{on } [0,1] \times C^+_{[0, \lambda + M_0]}$$

and

$$(\mathrm{H}_2)\quad f(t,\psi) \geqslant \mu\left(\int_{E_M} g\left(\frac{1}{2}, s\right) ds\right)^{-1} \quad \text{on } E_M \times C^+_{[M\mu,\mu+M_0]},$$

where $C^+_{[0,\lambda+M_0]} := \{\psi \in C^+\colon 0 \leqslant \|\psi\|_{[-\tau,a]} \leqslant \lambda + M_0\}$, $C^+_{[M\mu,\mu+M_0]} := \{\psi \in C^+\colon M\mu \leqslant \|\psi\|_{[-\tau,a]} \leqslant \mu + M_0\}$ *and* $M_0 := \|u_0\|_{[-\tau,b]}$. *Then* (BVP) *has at least one positive solution* ψ *such that* $\|\psi\|_{[-\tau,a]}$ *lies between* $\lambda + M_0$ *and* $\mu + M_0$.

Proof. Without loss of generality, we may assume that $\lambda < \mu$. Let $E = C^{(n-2)}([-\tau, b]; R)$ with norm $\|u\|_{[-\tau,b]} = \sup_{-\tau \leqslant t \leqslant b} |u^{(n-2)}(t)|$. Clearly, E is a Banach space. Define a cone K in the Banach space E by

$$K := \Big\{u \in E\colon \min_{t\in[M,1-M]} u^{(n-2)}(t) \geqslant M\|u\|_{[-\tau,b]}\Big\}, \quad \text{where } M \text{ is defined as in } (\mathrm{C}_4).$$

Define a mapping $\Phi : K \to K$ as follows:

$$(\Phi u)(t) := \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-2} e^{\frac{\alpha}{\beta}t} u^{(n-2)}(0), & -\tau \leqslant t \leqslant 0,\\ \int_0^1 G(t,s) f(s, u(s+\theta) + u_0(s+\theta))\,ds, & 0 \leqslant t \leqslant 1,\\ \left(-\frac{\delta}{\gamma}\right)^{n-2} e^{-\frac{\gamma}{\delta}(t-1)} u^{(n-2)}(1), & 1 \leqslant t \leqslant b. \end{cases}$$

Thus,

$$(\Phi u)^{(n-2)}(t) = \begin{cases} e^{\frac{\alpha}{\beta}t} \int_0^1 g(0,s) f(s, u(s+\theta) + u_0(s+\theta))\,ds, & -\tau \leqslant t \leqslant 0,\\ \int_0^1 g(t,s) f(s, u(s+\theta) + u_0(s+\theta))\,ds, & 0 \leqslant t \leqslant 1,\\ e^{-\frac{\gamma}{\delta}(t-1)} \int_0^1 g(1,s) f(s, u(s+\theta) + u_0(s+\theta))\,ds, & 1 \leqslant t \leqslant b. \end{cases}$$

It follows from

$$0 \leqslant (\Phi u)^{(n-2)}(t) \leqslant (\Phi u)^{(n-2)}(0) \quad \text{for } -\tau \leqslant t \leqslant 0$$

and

$$0 \leqslant (\Phi u)^{(n-2)}(t) \leqslant (\Phi u)^{(n-2)}(1) \quad \text{for } 1 \leqslant t \leqslant b$$

that $\|\Phi u\|_{[-\tau,b]} = \|\Phi u\|_{[0,1]}$. It follows from the definition of K and Lemma A that

$$\|\Phi u\|_{[-\tau,b]} = \|\Phi u\|_{[0,1]} \leqslant \int_0^1 g(s,s) f\big(u(s+\theta) + u_0(s+\theta)\big)\,ds \quad (\text{using } (\mathrm{R}_1)).$$

It follows from

$$\begin{aligned} \min_{M\leqslant t\leqslant 1-M} (\Phi u)^{(n-2)}(t) &= \min_{M\leqslant t\leqslant 1-M} \int_0^1 g(t,s) f\big(s, u(s+\theta) + u_0(s+\theta)\big)\,ds \\ &\geqslant M \int_0^1 g(s,s) f\big(s, u(s+\theta) + u_0(s+\theta)\big)\,ds \quad (\text{using}(R_2)) \end{aligned}$$

that

$$\min_{M\leqslant t\leqslant 1-M}(\Phi u)^{(n-2)}(t)\geqslant M\|\Phi u\|_{[0,1]}=M\|\Phi u\|_{[-\tau,b]},$$

which implies $\Phi(K)\subseteq K$. It is easy to see that $\Phi: K\to K$ is completely continuous.

In order to complete the proof, we separate the rest of the proof into the following two steps.

Step (I). Let $\Omega_1 := \{u\in K\colon \|u\|_{[-\tau,b]}<\lambda\}$. If $u\in\partial\Omega_1$, then $|u(s+\theta)+u_0(s+\theta)|\leqslant \lambda+M_0$, i.e., $u(s+\theta)+u_0(s+\theta)\in C^+_{[0,\lambda+M_0]}$ for all $s\in[0,1]$. It follows from (H_1) that, for $u\in\partial\Omega_1$ and $t\in[0,1]$,

$$\begin{aligned}(\Phi u)^{(n-2)}(t)&=\int_0^1 g(t,s)f\big(s,u(s+\theta)+u_0(s+\theta)\big)\,ds\\&\leqslant\int_0^1 g(s,s)f\big(s,u(s+\theta)+u_0(s+\theta)\big)\,ds\\&\leqslant\lambda\left(\int_0^1 g(s,s)\,ds\right)^{-1}\left(\int_0^1 g(s,s)\,ds\right)\frac{\|u\|_{[-\tau,b]}}{\lambda}=\|u\|_{[-\tau,b]}.\end{aligned}$$

Hence

$$\|\Phi u\|_{[-\tau,b]}\leqslant\|u\|_{[-\tau,b]}\quad\text{for } u\in\partial\Omega_1.$$

Step (II). Let $\Omega_2 := \{u\in K\colon \|u\|_{[-\tau,b]}<\mu\}$. If $u\in\partial\Omega_2$, then

$$u^{(n-2)}(t+\theta)\geqslant\min_{s\in[M,1-M]}u^{(n-2)}(s)\geqslant M\|u\|_{[-\tau,b]}=M\mu\quad\text{for } t\in E_M.$$

Thus

$$M\mu\leqslant u^{(n-2)}(t+\theta)\leqslant\mu\quad\text{for } t\in E_M,$$

and hence

$$M\mu\leqslant u^{(n-2)}(t+\theta)+u_0^{(n-2)}(t+\theta)\leqslant\mu+M_0\quad\text{for } t\in E_M.$$

Then

$$\begin{aligned}(\Phi u)^{(n-2)}\left(\frac12\right)&=\int_0^1 g\left(\frac12,s\right)f\big(s,u(s+\theta)+u_0(s+\theta)\big)\,ds\\&\geqslant\int_{E_M} g\left(\frac12,s\right)f\big(s,u(s+\theta)+u_0(s+\theta)\big)\,ds\\&\geqslant\mu\left(\int_{E_M} g\left(\frac12,s\right)ds\right)^{-1}\left(\int_{E_M} g\left(\frac12,s\right)ds\right)\frac{\|u\|_{[-\tau,b]}}{\mu}=\|u\|_{[-\tau,b]}.\end{aligned}$$

It follows from (H_2) that

$$\|\Phi u\|_{[-\tau,b]} \geqslant \|u\|_{[-\tau,b]} \quad \text{for } u \in \partial\Omega_2.$$

Therefore, by the first part of Lemma B, the proof is complete. □

Remark 2.2. Let

$$\max f_0 := \lim_{\psi\in C^*,\ \|\psi\|_{[-\tau,a]}\to 0^+} \max_{t\in[0,1]} \frac{f(t,\psi)}{\|\psi\|_{[-\tau,a]}},$$
$$\min f_0 := \lim_{\psi\in C^*,\ \|\psi\|_{[-\tau,a]}\to 0^+} \min_{t\in[0,1]} \frac{f(t,\psi)}{\|\psi\|_{[-\tau,a]}},$$
$$\max f_\infty := \lim_{\psi\in C^*,\ \|\psi\|_{[-\tau,a]}\to \infty} \max_{t\in[0,1]} \frac{f(t,\psi)}{\|\psi\|_{[-\tau,a]}},$$
$$\min f_\infty := \lim_{\psi\in C^*,\ \|\psi\|_{[-\tau,a]}\to \infty} \min_{t\in[0,1]} \frac{f(t,\psi)}{\|\psi\|_{[-\tau,a]}},$$
$$\left(\int_0^1 G(s,s)\,ds\right)^{-1} := D_1 \quad\text{and}\quad \left(\int_{E_M} G\left(\frac{1}{2},s\right)ds\right)^{-1} := D_2.$$

Then, we have the following results.

(a) Suppose that $\eta(t)\equiv 0$, $\xi(t)\equiv 0$ and $\max f_0 := C_1 \in [0, D_1)$. It is clear that $u_0(t)\equiv 0$ for $t\in[-\tau,b]$, thus $M_0 = 0$. Taking $\epsilon := D_1 - C_1$ (>0), there exists $\lambda_1 > 0$ (λ_1 can be chosen arbitrarily small) such that

$$\max_{t\in[0,1]} \frac{f(t,\psi)}{\|\psi\|_{[-\tau,a]}} \leqslant \epsilon + C_1 = D_1 \quad \text{on } C^+_{(0,\lambda_1]}.$$

Hence

$$f(t,\psi) \leqslant D_1\|\psi\|_{[-\tau,a]} \leqslant D_1\lambda_1 \quad \text{on } [0,1]\times C^+_{(0,\lambda_1]},$$

which satisfies the hypothesis (H_1) of Theorem 2.1.

(b) Suppose that $\min f_\infty := C_2 \in (D_2/M, \infty)$. Taking $\epsilon := C_2 - D_2/M$ (>0), there exists $\mu_1 > 0$ (μ_1 can be chosen arbitrarily large) such that

$$\min_{t\in E_M} \frac{f(t,\psi)}{\|\psi\|_{[-\tau,a]}} \geqslant -\epsilon + C_2 = \frac{D_2}{M} \quad \text{on } C^+_{[M\mu_1,\infty)}.$$

Hence

$$f(t,\psi) \geqslant \frac{D_2}{M}\|\psi\|_{[-\tau,a]} \geqslant \frac{D_2}{M} M\mu_1 = D_2\mu_1$$

on $E_M \times C^+_{[M\mu_1,\mu_1+M_0]} \subseteq E_M \times C^+_{[M\mu_1,\infty)}$, which satisfies the hypothesis (H_2) of Theorem 2.1.

(c) Suppose that $\min f_0 := C_3 \in (D_2/M, \infty)$. Taking $\epsilon := C_3 - D_2/M$ (>0), there exists $\mu_2 > 0$ (μ_2 can be chosen arbitrarily small) such that

$$\min_{t\in E_M} \frac{f(t,\psi)}{\|\psi\|_{[-\tau,a]}} \geqslant -\epsilon + C_3 = \frac{D_2}{M} \quad \text{on } C^+_{(0,\mu_2]}.$$

Hence

$$f(t,\psi) \geqslant \frac{D_2}{M}\|\psi\|_{[-\tau,a]} \geqslant \frac{D_2}{M} M\mu_2 = D_2\mu_2$$

on $E_M \times C^+_{[M\mu_2,\mu_2]} \subseteq E_M \times C^+_{[0,\mu_2]}$, which satisfies the hypothesis (H_2) of Theorem 2.1.

(d) Suppose that $\eta(t) \equiv 0$, $\xi(t) \equiv 0$ and $\max f_\infty := C_4 \in [0, D_1)$. It is clear that $u_0(t) \equiv 0$ for $t \in [-\tau, b]$, thus $M_0 = 0$. Taking $\epsilon := D_1 - C_4$ (> 0), there exists $\lambda > 0$ (λ can be chosen large arbitrarily) such that

$$\max_{t\in[0,1]} \frac{f(t,\psi)}{\|\psi\|_{[-\tau,a]}} \leqslant \epsilon + C_4 = D_1 \quad \text{on } C^+_{[\lambda,\infty)}.$$

Hence, we have the following two cases.

Case (I). Assume that $\max_{t\in[0,1]} f(t,\psi)$ is bounded, say,

$$f(t,\psi) \leqslant L \quad \text{on } [0,1] \times C^+_{(0,\infty)}.$$

Taking $\lambda_1 = L/D_1$ (since L can be chosen arbitrarily large, λ_1 can be chosen arbitrarily large, too),

$$f(t,\psi) \leqslant L = D_1\lambda_1 \quad \text{on } [0,1] \times C^+_{(0,\lambda_1]} \subseteq [0,1] \times C^+_{(0,\infty)}.$$

Case (II). Assume that $\max_{t\in[0,1]} f(t,\psi)$ is unbounded, hence, there exists $\lambda_2 \geqslant \rho$ (λ_2 can be chosen arbitrarily large) and $t_0 \in [0,1]$ such that

$$f(t,\psi) \leqslant f(t_0,\lambda_2) \quad \text{on } [0,1] \times C^+_{(0,\lambda_2]}.$$

It follows from $\lambda_2 \geqslant \rho$ that

$$f(t,\psi) \leqslant f(t_0,\lambda_2) \leqslant D_1\lambda_2 \quad \text{on } [0,1] \times C^+_{(0,\lambda_2]}.$$

By Cases (I) and (II), the hypothesis (H_1) of Theorem 2.1 is satisfied.

By Remark 2.2, we have the following three corollaries.

Corollary 2.3. *Let D_1 and D_2 be defined as in Remark* 2.2 *and $\xi(t) \equiv \eta(t) \equiv 0$. Then,* (BVP) *has at least one positive solution if one of the following conditions hold*:

(1) $\max f_0 = C_1 \in [0, D_1)$ *and* $\min f_\infty = C_2 \in (D_2/M, \infty]$, *or*
(2) $\min f_0 = C_3 \in (D_2/M, \infty]$ *and* $\max f_\infty = C_4 \in [0, D_1)$.

Proof. It follows from Remark 2.2 and Theorem 2.1 that the desired result holds, immediately. □

Corollary 2.4. *Let D_1 and D_2 be defined as in Remark* 2.2. *If the following hypotheses hold*:

(H_1) $\min f_\infty = C_2$, $\min f_0 = C_3 \in (D_2/M, \infty]$,
(H_2) *there exists $\lambda^* > 0$ such that*

$$f(t,\psi) \leqslant D_1\lambda^* \quad \textit{on } [0,1] \times C^+_{[0,\lambda^*+M_0]},$$

then (BVP) *has at least two positive solutions* ψ_1 *and* ψ_2 *such that*

$$0 < \|\psi_1\| < \lambda^* + M_0 < \|\psi_2\|.$$

Proof. It follows from Remark 2.2 that there exist two real numbers μ_1 and μ_2 satisfying

$$0 < \mu_2 < \lambda^* < \mu_1,$$
$$f(t,\psi) \geqslant D_2\mu_1 \quad \text{on } [0,1] \times C^+_{[M\mu_1,\mu_1+M_0]},$$

and

$$f(t,\psi) \geqslant D_2\mu_2 \quad \text{on } [0,1] \times [M\mu_2, \mu_2 + M_0].$$

Hence, by Theorem 2.1, we see that (BVP) has two positive solutions ψ_1 and ψ_2 such that

$$\mu_2 + M_0 < \|\psi_1\| < \lambda^* + M_0 < \|\psi_2\| < \mu_1 + M_0.$$

Thus, we complete the proof. □

Corollary 2.5. *Let* D_1 *and* D_2 *be defined as in Remark* 2.2. *If the following hypotheses hold*:

(H$_3$) $\max f_0 = C_1$, $\max f_\infty = C_4 \in [0, D_1)$ *and* $\xi(t) \equiv \eta(t) \equiv 0$,
(H$_4$) *there exists* $\mu^* > 0$ *such that*

$$f(t,\psi) \geqslant D_2\mu^* \quad \text{on } [0,1] \times C^+_{[M\mu^*,\mu^*]},$$

then (BVP) *has at least two positive solutions* ψ_1 *and* ψ_2 *such that*

$$0 < \|\psi_1\| < \mu^* < \|\psi_2\|.$$

Proof. It follows from Remark 2.2 that there exist two real numbers λ_1 and λ_2 satisfying

$$0 < \lambda_1 < \mu^* < \lambda_2,$$
$$f(t,\psi) \leqslant D_1\lambda_1 \quad \text{on } [0,1] \times C^+_{[0,\lambda_1]},$$

and

$$f(t,\psi) \leqslant D_1\lambda_2 \quad \text{on } [0,1] \times C^+_{[0,\lambda_2]}.$$

Hence, by Theorem 2.1, (BVP) has two positive solutions ψ_1 and ψ_2 such that

$$\lambda_1 < \|\psi_1\| < \mu^* < \|\psi_2\| < \lambda_2.$$

Thus, we complete the proof. □

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