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J. Math. Anal. Appl. 307 (2005) 153–166
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

The continuous symmetric Hahn polynomials found in Ramanujan's lost notebook

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Received 5 May 2004

Available online 9 February 2005

Submitted by B.C. Berndt

To Bruce C. Berndt on his 65th birthday

Abstract

Askey and Wilson found ${}_3F_2$ Hahn polynomials which are orthogonal with respect to a positive absolutely continuous weight function. More than a half century earlier, Ramanujan recorded the Stieltjes transform of this weight function in terms of a continued fraction in his lost notebook. We provide two different proofs for this integration. One applies theories of the Hamburger moment problem. The other uses elementary integration techniques and a couple of transformation formulas for hypergeometric functions.
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Keywords: Hamburger moment problems; Orthogonal polynomials; Continued fractions; Stieltjes transform; Hypergeometric functions; Integration

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[1] Supported by Korea Research Foundation Grant (KRF-2004-015-C00005).

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doi:10.1016/j.jmaa.2004.12.054

1. Introduction

Orthogonal polynomials are classes of polynomials $\{p_n\}$ which satisfy the orthogonality relation

$$\int_a^b p_n(x)\,p_m(x)w(x)\,dx = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases}$$

whereby the degree of p_n is equal to n, the weight function $w(x) \geqslant 0$ on the interval $[a, b]$, and $h_n > 0$. All of the classical polynomials such as those of Hermite, Laguerre and Jacobi have explicit representations in terms of hypergeometric series, which are defined by

$${}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where $(a)_k$ is the Pochhammer symbol or rising factorial

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = (a)(a+1)(a+2)\cdots(a+k-1).$$

In his PhD thesis [18], James Wilson found the most general orthogonal hypergeometric polynomials in the sense that all of the classical and many other polynomials can be expressed as special or limiting cases of these polynomials. The complete set of limit cases of Wilson polynomials is presented as a directed graph in the *Askey tableau*; see an appendix to [5]. In particular, the ${}_3F_2$ polynomials with 2 free parameters, the continuous symmetric Hahn polynomials in the tableau, were found by Richard Askey and Wilson [4]. They are defined by

$$P_n(x;\alpha,\beta) = i^n\,{}_3F_2\left(\begin{matrix} -n, n+2\alpha+2\beta-1, \beta-ix \\ \alpha+\beta, 2\beta \end{matrix}; 1\right) \tag{1.1}$$

and are orthogonal with respect to the positive absolutely continuous weight function

$$W(x) = \left|\Gamma(\alpha+ix)\Gamma(\beta+ix)\right|^2, \tag{1.2}$$

where $-\infty < x < \infty$ and $\alpha, \beta > 0$ or $\alpha = \bar{\beta}$ and $\operatorname{Re}\alpha > 0$.

Quoting Askey [3], "one reason certain definite integrals are interesting is that the integrand is the weight function for an important set of orthogonal polynomials." Without recognizing this importance, Ramanujan came close to the corresponding orthogonal polynomials of the weight function $W(x) = |\Gamma(\alpha+ix)\Gamma(\beta+ix)|^2$ at least three times. In the paper, 'Some definite integrals' [11,12], Ramanujan evaluated $\int_{-\infty}^{\infty} W(x)\,dx$ by using Fourier transform. He recorded this value of the integral in his second notebook, as well [10], [7, p. 225]. On page 327 in his lost notebook [13], he recorded without a proof the Stieltjes transform of this weight function $W(x)$ in terms of a continued fraction. He also wrote a special case of the integral where the continued fraction is written as an alternating series. We present these statements as Entry 1.1 and Entry 1.2 below with s instead of n.

Entry 1.1. *Let*

$$\varphi(a,x) := \frac{1}{\left\{1+\left(\frac{x}{a+1}\right)^2\right\}\left\{1+\left(\frac{x}{a+3}\right)^2\right\}\left\{1+\left(\frac{x}{a+5}\right)^2\right\}\cdots}. \tag{1.3}$$

For $a+1>0$ and $b+1>0$,

$$\int_0^\infty \varphi(a,x)\varphi(b,x)\frac{dx}{1+s^2x^2} = 2\sqrt{\pi}\,\frac{\Gamma(1+\frac{a}{2})\Gamma(1+\frac{b}{2})\Gamma(1+\frac{a+b}{2})}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)\Gamma\left(\frac{1+a+b}{2}\right)} \times \frac{1}{(a+b+1)} + \frac{1(a+1)(b+1)(a+b+1)s^2}{(a+b+3)} + \frac{2(a+2)(b+2)(a+b+2)s^2}{(a+b+5)} + \cdots. \tag{1.4}$$

Entry 1.2. *If $s=1$, the continued fraction in Entry* 1.1 *can be shown to be equal to the series*

$$\frac{1}{a+b+1}(1-A_1+A_1A_2-A_1A_2A_3+\cdots), \tag{1.5}$$

where

$$A_t = \frac{(a+t)(b+t)-ab\cos^2\frac{\pi t}{2}}{(a+1+t)(b+1+t)-ab\cos^2\frac{\pi t}{2}}. \tag{1.6}$$

If we set $\alpha=\frac{a+1}{2}$ and $\beta=\frac{b+1}{2}$ and replace the variable x with $2x$, then Entry 1.1 can be restated as:

Theorem 1.3. *Let*

$$\phi(\alpha,x) := \frac{1}{\left\{1+\left(\frac{x}{\alpha}\right)^2\right\}\left\{1+\left(\frac{x}{\alpha+1}\right)^2\right\}\left\{1+\left(\frac{x}{\alpha+2}\right)^2\right\}\cdots}. \tag{1.7}$$

For $\alpha>0$ and $\beta>0$,

$$\int_0^\infty \phi(\alpha,x)\phi(\beta,x)\frac{dx}{1+4s^2x^2} = \sqrt{\pi}\,\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\frac{1}{2})}\chi_1(s),$$

where

$$\chi_1(s) := \frac{1}{2} + \frac{2\cdot 1(2\alpha)(2\beta)s^2}{(2\alpha+2\beta+1)} + \frac{2(2\alpha+1)(2\beta+1)(2\alpha+2\beta)s^2}{(2\alpha+2\beta+3)} + \frac{3(2\alpha+2)(2\beta+2)(2\alpha+2\beta+1)s^2}{(2\alpha+2\beta+5)} + \cdots. \tag{1.8}$$

We provide two different proofs for Theorem 1.3 in this paper. In Section 2, first, we review some basic properties of continued fractions and orthogonal polynomials, in particular, of the continuous symmetric Hahn polynomials. Next, we present classical theories of the Hamburger moment problem, and finally, we establish a short proof of Theorem 1.3 for every complex number s off from the imaginary axis. In Section 3, we provide a more elementary proof than that of Section 2 for Theorem 1.3 for every real number s that uses only the Laplace and Fourier transforms and a couple of transformation formulas for hypergeometric functions. In Section 4, we prove Entry 1.2.

2. Moment problem and proof of Theorem 1.3

2.1. Continued fractions and orthogonal polynomials

Any set of orthogonal polynomials $\{p_n(x)\}$, orthogonal with respect to a positive measure, satisfies a three term recurrence relation

$$x p_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \tag{2.1}$$

with α_n, β_n, γ_n real and $\alpha_{n-1}\gamma_n > 0$, $n = 1, 2, \ldots$. Conversely, if a set of polynomials $\{p_n(x)\}$ satisfies (2.1) with α_n, β_n, γ_n real and $\alpha_{n-1}\gamma_n > 0$, $n = 1, 2, \ldots$, then there is a positive measure $d\psi(x)$ such that

$$\int_{-\infty}^{\infty} p_n(x) p_m(x)\, d\psi(x) = \begin{cases} 0, & m \neq n, \\ \frac{\gamma_n \cdots \gamma_1}{\alpha_{n-1} \cdots \alpha_0} \int_{-\infty}^{\infty} d\psi(x), & m = n. \end{cases} \tag{2.2}$$

See references in [1,5]. Called Farvard's Theorem, it shows the close connection between orthogonal polynomials and continued fractions.

Before proceeding, we review some basic properties of a continued fraction. Say f_n is the nth approximant of the continued fraction

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots,$$

i.e.,

$$f_n = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \quad \text{and} \quad f_n = \frac{U_n}{V_n}.$$

We call U_n and V_n nth numerator and denominator of the continued fraction, respectively. If we define $U_{-1} = 1$ and $V_{-1} = 0$, then the recurrence relations

$$b_n U_{n-1} + a_n U_{n-2} = U_n, \qquad b_n V_{n-1} + a_n V_{n-2} = V_n \tag{2.3}$$

for $n = 1, 2, 3, \ldots$ hold [17, p. 15]. Using the recurrence relation in (2.3) and mathematical induction, one can easily deduce the following equivalence transformation.

Proposition 2.1 [17, p. 19]. *Let $c_0 = 1$ and $c_i \neq 0$ for $i > 0$. Then the two continued fractions*

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \quad \textit{and} \quad c_0 b_0 + \frac{c_0 c_1 a_1}{c_1 b_1} + \frac{c_1 c_2 a_2}{c_2 b_2} + \frac{c_2 c_3 a_3}{c_3 b_3} + \cdots$$

have the same sequence of approximants.

The continuous symmetric Hahn polynomials $P_n(x)$ satisfy a three term recurrence relation [4]

$$x P_n(x) = \alpha_n P_{n+1}(x) + \gamma_n P_{n-1}(x), \tag{2.4}$$

where

$$\alpha_n = \frac{(n+2\beta)(n+2\alpha+2\beta-1)}{2(2n+2\alpha+2\beta-1)} \quad \text{and} \quad \gamma_n = \frac{n(n+2\alpha-1)}{2(2n+2\alpha+2\beta-1)}. \tag{2.5}$$

Hence by (2.3) and Proposition 2.1, the continued fraction corresponding to the orthogonal polynomials $P_n(x)$ with $\gamma_0 = -1$ is

$$\begin{aligned}\chi(x) &:= \frac{1}{x} - \frac{\alpha_0\gamma_1}{x} - \frac{\alpha_1\gamma_2}{x} - \frac{\alpha_2\gamma_3}{x} - \cdots \\ &= \frac{1}{x} - \frac{1\cdot(2\alpha)(2\beta)}{4x(2\alpha+2\beta+1)} - \frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)}{x(2\alpha+2\beta+3)} \\ &\quad - \frac{3\cdot(2\alpha+2)(2\beta+2)(2\alpha+2\beta+1)}{4x(2\alpha+2\beta+5)} - \cdots. \end{aligned} \tag{2.6}$$

That is, $P_n(x)$ is the nth denominator of $\chi(x)$.

On the other hand, Eq. (2.2) along with (2.4) and (2.5) gives the L^2-norm of the continuous symmetric Hahn polynomials [4, p. 653] as

$$\int_{-\infty}^{\infty} \big[P_n(x;\alpha,\beta)\big]^2 W(x)\,dx = \frac{(1)_n(2\alpha)_n(\alpha+\beta-\frac{1}{2})_n W_I}{(2\beta)_n(2\alpha+2\beta-1)_n(\alpha+\beta+\frac{1}{2})_n},$$

where

$$W_I = \int_{-\infty}^{\infty} W(x)\,dx = \sqrt{\pi}\,\frac{\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})\Gamma(\beta)\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}. \tag{2.7}$$

The integral in (2.7) is a special case of Barnes's beta integral [6]. This particular evaluation was also established by Ramanujan [11,12] and Ranjan Roy [14] by using the Fourier transform and the Mellin transform, respectively.

It follows from (2.7) that the normalized weight function of the continuous symmetric Hahn polynomials $P_n(x)$ is

$$W_{\mathbb{N}}(x) := \frac{\Gamma(\alpha+\beta+\frac{1}{2})|\Gamma(\alpha+ix)\Gamma(\beta+ix)|^2}{\sqrt{\pi}\,\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})\Gamma(\beta)\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}. \tag{2.8}$$

Since

$$\phi(\alpha,x) = \frac{|\Gamma(\alpha+ix)|^2}{\Gamma^2(\alpha)} \tag{2.9}$$

[11], [12, p. 54], Theorem 1.3 is equivalent to:

Entry 2.2. *For $\alpha > 0$ and $\beta > 0$,*

$$\int_0^{\infty} \frac{W_{\mathbb{N}}(x)\,dx}{1+4s^2x^2} = \frac{1}{2} + \frac{2(2\alpha)(2\beta)s^2}{(2\alpha+2\beta+1)} + \frac{2(2\alpha+1)(2\beta+1)(2\alpha+2\beta)s^2}{(2\alpha+2\beta+3)} + \cdots.$$

Entry 2.2 gives the representation of the Stieltjes transform of the weight function of continuous symmetric Hahn polynomials in terms of a continued fraction. A more general continued fraction with five free parameters was found by Ismail, Letessier, Valent, and Wimp [9]. Using contiguous relations for generalized hypergeometric functions of the type ${}_7F_6$, they derived explicit representations for associated Wilson polynomials (ILVW polynomials) and computed the corresponding continued fraction.

2.2. Hamburger moment problem

The Hamburger moment problem is to find a bounded non-decreasing function $\psi(x)$ in the interval $[-\infty, \infty]$ satisfying the equations

$$\mu_n = \int_{-\infty}^{\infty} x^n \, d\psi(x) \quad (n = 0, 1, 2, \ldots) \tag{2.10}$$

for a given sequence of real numbers μ_n. Throughout this section, it is assumed that the solution of a Hamburger moment problem (2.10) has an infinite number of points of increase. If its solution is unique, the moment problem is called determinate and otherwise, it is called indeterminate.

For any solution $\psi(x)$ of the moment problem (2.10), let

$$I(z, \psi) = \int_{-\infty}^{\infty} \frac{d\psi(x)}{z - x} \tag{2.11}$$

and $\mathcal{H}$ denote the upper half plane. The following two lemmas will show that there is a one-to-one correspondence between the elements of a certain class of functions to which $I(z, \psi)$ belongs and those of the class of solutions $\psi(x)$ of the moment problem (2.10).

Lemma 2.3 [15, Theorem 2.1]. *$I(z, \psi)$ is analytic and $\operatorname{Im} I(z, \psi) \leqslant 0$ on $\mathcal{H}$, and $I(z, \psi)$ is asymptotically represented by the series $\sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}$ in any sector*

$$0 < \varepsilon \leqslant \arg z \leqslant \pi - \varepsilon, \quad 0 < \varepsilon < \pi/2. \tag{2.12}$$

Lemma 2.4 [15, Theorem 2.1]. *If $F(z)$ is analytic and $\operatorname{Im} F(z) \leqslant 0$ on $\mathcal{H}$, and if $F(z)$ is asymptotically represented by the series $\sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}$ in any sector* (2.12)*, then there exists a unique solution $\psi(x)$ of the moment problem* (2.10) *such that $F(z) = I(z, \psi)$.*

The integral $I(z, \psi)$ is also closely related to a continued fraction.

Lemma 2.5 [15, Theorem 2.4]. *There exists a function $F(z)$ such that $F(z)$ is analytic, $\operatorname{Im} F(z) \leqslant 0$, and $F(z)$ is asymptotically represented by the series $\sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}$ on $\mathcal{H}$ if and only if there is an associated continued fraction*

$$b_0 + \frac{a_0}{b_1 + z} - \frac{a_1}{b_2 + z} - \frac{a_2}{b_3 + z} - \cdots \tag{2.13}$$

such that all $a_n > 0$ and $b_n \in \mathbb{R}$ and

$$F(z) = b_0 + \frac{a_0}{b_1 + z} - \frac{a_1}{b_2 + z} - \cdots - \frac{a_n}{F_{n+1}(z) + z},$$

where $F_{n+1}(z)$ is an arbitrary analytic function, $\operatorname{Im} F_{n+1}(z) \leqslant 0$, and $F_{n+1}(z) = o(z)$ as $z \to \infty$ on $\mathcal{H}$.

In fact, the nth approximant of the continued fraction (2.13), say $q_n(z)/p_n(z)$, is expanded to [15, p. 46]

$$\frac{q_n(z)}{p_n(z)} = \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \cdots + \frac{\mu_{2n-1}}{z^{2n}} + \frac{\mu'_{2n}}{z^{2n+1}} + \frac{\mu'_{2n+1}}{z^{2n+2}} + \cdots. \tag{2.14}$$

As we have seen in Section 2.1, nth denominators $p_n(z)$ is a set of orthogonal polynomials of degree n by (2.3) and Eqs. (2.1) and (2.2) in Farvard's Theorem. Moreover, the orthogonality relation

$$\int_{-\infty}^{\infty} p_n(x)p_m(x)\,d\psi(x) = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases} \tag{2.15}$$

is satisfied by the solution $\psi(x)$ of the moment problem (2.10), [15, p. 35].

Now, we will state a couple of theorems which present a necessary or sufficient condition for the uniqueness of the solution of the moment problem (2.10).

Lemma 2.6 [15, Theorem 2.9]. *The moment problem* (2.10) *is determinate if* $\sum_{n=0}^{\infty} |p_n(z)|^2$ *diverges at a non-real number* z.

Lemma 2.7 [15, Theorem 2.10]. *If the moment problem* (2.10) *is determinate, then the associated continued fraction* (2.13) *converges for all complex number* z.

2.3. Proof of Theorem 1.3

Using the lemmas in Section 2.2, we are going to prove the following theorem.

Theorem 2.8. *For* $\alpha > 0$ *and* $\beta > 0$, *and non-real complex number* z,

$$\int_{-\infty}^{\infty} \frac{W_{\mathbb{N}}(x)\,dx}{z-x} = \frac{1}{z} - \frac{1(2\alpha)(2\beta)}{4z(2\alpha+2\beta+1)} - \frac{2(2\alpha+1)(2\beta+1)(2\alpha+2\beta)}{z(2\alpha+2\beta+3)} - \cdots,$$

where $W_{\mathbb{N}}(x)$ *is defined in* (2.8).

Since

$$\int_{-\infty}^{\infty} \frac{W_{\mathbb{N}}(x)}{z-x}\,dx = 2z\int_{0}^{\infty} \frac{W_{\mathbb{N}}(x)}{z^2-x^2}\,dx,$$

Theorem 2.8 is equivalent to

$$\int_{0}^{\infty} \frac{W_{\mathbb{N}}(x)\,dx}{z^2-x^2} = \frac{1}{2z^2} - \frac{2(2\alpha)(2\beta)}{4(2\alpha+2\beta+1)} - \frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)}{z^2(2\alpha+2\beta+3)} - \frac{3\cdot(2\alpha+2)(2\beta+2)(2\alpha+2\beta+1)}{4(2\alpha+2\beta+5)} - \cdots \tag{2.16}$$

from which Theorem 1.3 or Entry 2.2 immediately follows after replacing z by $i/2s$. Therefore, Theorem 2.8 implies that Theorem 1.3 holds for every complex number s except for a purely imaginary number.

In order to complete the proof of Theorem 2.8, we need one more lemma by Stieltjes which gives the power series representation of a continued fraction of the type in (2.13).

Lemma 2.9 [17, Theorem 53.1]. *The coefficients in the J-fractions*

$$\frac{1}{b_1+z}-\frac{a_1}{b_2+z}-\frac{a_2}{b_3+z}-\cdots$$

and its power series expansion $\sum_{n=0}^{\infty}\frac{(-1)^n c_n}{z^{n+1}}$ are connected by the relations $c_{p+q}=k_{0,p}k_{0,q}+a_1k_{1,p}k_{1,q}+a_1a_2k_{2,p}k_{2,q}+\cdots$, where $k_{0,0}=1$, $k_{r,s}=0$ if $r>s$, and where the $k_{r,s}$ for $s\geqslant r$ are given recurrently by the matrix equations

$$\begin{pmatrix} k_{00} & 0 & 0 & 0 & \cdots \\ k_{01} & k_{11} & 0 & 0 & \cdots \\ k_{02} & k_{12} & k_{22} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}\cdot\begin{pmatrix} b_1 & 1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & 1 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}$$

$$=\begin{pmatrix} k_{01} & k_{11} & 0 & 0 & 0 & \cdots \\ k_{02} & k_{12} & k_{22} & 0 & 0 & \cdots \\ k_{03} & k_{13} & k_{23} & k_{33} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

Proof of Theorem 2.8. Note that the continued fraction in Theorem 2.8 is $\chi(z)$ in (2.6), the continued fraction corresponding to the continuous symmetric Hahn polynomials $P_n(z)$. In the case of $\chi(z)$, $a_n=\alpha_{n-1}\gamma_n>0$ and $b_n=0$ for all n. It is hence easy to find that $k_{ij}=0$ where $i+j$ is odd and thus $c_{2n+1}=0$ and $c_{2n}>0$. Let $Q_n(z)$ be the nth numerator of $\chi(z)$. Then for positive real numbers c_{2n} obtained from Lemma 2.9,

$$\frac{Q_n(z)}{P_n(z)}=\frac{1}{z}+\frac{c_2}{z^3}+\cdots+\frac{c_{2n-2}}{z^{2n-1}}+\frac{c'_{2n}}{z^{2n+1}}\cdots.$$

Consider the moment problem

$$c_n=\int_{-\infty}^{\infty}x^n\,d\psi(x)\quad(n=0,1,2,\ldots) \tag{2.17}$$

for the sequence of real numbers c_n found above.

Observe that $P_0(\beta i)=1$, $P_1(\beta i)=-i$, $P_2(\beta i)=-1$, and $P_3(\beta i)=i$. Using mathematical induction and the recurrence relation (2.4) after replacing α_n and γ_n by $-\alpha_n$ and $-\gamma_n$, respectively, we can prove that $P_{4n}(\beta i)=1$ for all n. Hence $\sum_{n=0}^{\infty}|P_n(\beta i)|^2$ diverges, and thus the moment problem (2.17) has a unique solution $W_{\mathbb{N}}(x)$ by (2.15) and Lemma 2.6. Now, it follows from Lemmas 2.3–2.5, 2.7, and (2.14) that the continued fraction $\chi(z)$ converges for every non-real number z to $I(z,\psi)$ where $d\psi(x)=W_{\mathbb{N}}(x)\,dx$. □

3. Second proof of Theorem 1.3

For $t > 0$, we set

$$\Phi(\alpha, \beta, t) := \int_0^\infty \phi(\alpha, x)\phi(\beta, x) \cos tx \, dx, \tag{3.1}$$

where $\phi(\alpha, x)$ is as defined in (1.7). Then by the Laplace transform of the cosine function, for $x > 0$ and $s > 0$,

$$\int_0^\infty \cos(xt) e^{-st} \, dt = \frac{s}{s^2 + x^2},$$

the integral in Theorem 1.3 becomes

$$\mathbb{I} := \int_0^\infty \phi(\alpha, x)\phi(\beta, x) \left(\frac{1}{2s} \int_0^\infty e^{-\frac{t}{2s}} \cos xt \, dt \right) dx.$$

Since $\int_0^\infty \phi(\alpha, x)\phi(\beta, x) \, dx$ is bounded by (2.7) and (2.9),

$$\begin{aligned} \mathbb{I} &= \frac{1}{2s} \int_0^\infty e^{-\frac{t}{2s}} \left(\int_0^\infty \phi(\alpha, x)\phi(\beta, x) \cos tx \, dx \right) dt \\ &= \frac{1}{2s} \int_0^\infty e^{-\frac{t}{2s}} \Phi(\alpha, \beta, t) \, dt. \end{aligned} \tag{3.2}$$

By term-by-term integration of partial fractions of the integrand, [11], [12, p. 53], we have

$$\int_0^\infty \phi(\alpha, x) \cos yx \, dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha)} \operatorname{sech}^{2\alpha}\left(\frac{y}{2} \right) \tag{3.3}$$

for $y > 0$. Hence by the Fourier transform,

$$\int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2} \right) \cos xy \, dy = \sqrt{\pi} \, \frac{\Gamma(\alpha)}{\Gamma(\alpha + \frac{1}{2})} \phi(\alpha, x) \tag{3.4}$$

for $x > 0$. Consequently,

$$\phi(\alpha, x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha)} \int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2} \right) \cos xy \, dy. \tag{3.5}$$

Applying (3.5) to (3.1) yields

$$\Phi(\alpha, \beta, t) = \frac{1}{\pi} \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha)} \frac{\Gamma(\beta + \frac{1}{2})}{\Gamma(\beta)} \mathcal{T}, \tag{3.6}$$

where $\mathcal{T}$ is the triple integral

$$\mathcal{T} = \int_0^\infty \int_0^\infty \int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right) \operatorname{sech}^{2\beta}\left(\frac{z}{2}\right) \cos xy \cos xz \cos tx \, dz \, dy \, dx. \tag{3.7}$$

It follows from $2\cos xy \cos xz = (\cos(y+z)x + \cos(y-z)x)$ that

$$\begin{aligned}
\mathcal{T} &= \frac{1}{2} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right) \operatorname{sech}^{2\beta}\left(\frac{z}{2}\right) \cos(y+z)x \cos tx \, dz \, dy \, dx \\
&= \frac{1}{2} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right) \operatorname{sech}^{2\beta}\left(\frac{y-u}{2}\right) \cos ux \cos tx \, du \, dy \, dx \\
&= \frac{1}{2} \int_0^\infty \int_0^\infty \int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right) \\
&\quad \times \left(\operatorname{sech}^{2\beta}\left(\frac{y+u}{2}\right) + \operatorname{sech}^{2\beta}\left(\frac{y-u}{2}\right)\right) \cos ux \cos tx \, du \, dx \, dy,
\end{aligned} \tag{3.8}$$

after changing z by $-z$ in one of the integral in the first equality, replacing $y+z$ with a variable u in the second equality, exchanging the order of the integrations with respect to x and y and then changing u by $-u$ in one of the integral in the last equality above. Utilize the Fourier integral formula

$$\int_0^\infty \cos nx \, dx \int_0^\infty f(u) \cos ux \, du = \frac{\pi}{2} f(n)$$

in (3.8) to derive that

$$\mathcal{T} = \frac{\pi}{4} \int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right) \left(\operatorname{sech}^{2\beta}\left(\frac{y+t}{2}\right) + \operatorname{sech}^{2\beta}\left(\frac{y-t}{2}\right)\right) dy. \tag{3.9}$$

So far, we have shown from (3.1), (3.6), and (3.9) that

$$\begin{aligned}
\Phi(\alpha,\beta,t) &= \frac{\sqrt{\pi}}{4} \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)} \frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta)} \\
&\quad \times \int_0^\infty \operatorname{sech}^{2\alpha}\left(\frac{y}{2}\right) \left(\operatorname{sech}^{2\beta}\left(\frac{y+t}{2}\right) + \operatorname{sech}^{2\beta}\left(\frac{y-t}{2}\right)\right) dy.
\end{aligned} \tag{3.10}$$

The equality (3.10) which is a generalization of the integral of $W(x)$ in (2.7) was found in [7, p. 226] as a consequence of Parseval's theorem, (3.4) in this paper, and Legendre's duplication formula. In [7, p. 226], it is also indicated that M.L. Glasser [8] had shown how to evaluate integrals like that on the right side in (3.10). Glasser used the contour integration for the evaluation, but we will use the binomial theorem and Euler's beta integral as below.

From

$$\begin{aligned}&\operatorname{sech}^{2\beta}\left(\frac{y+t}{2}\right)+\operatorname{sech}^{2\beta}\left(\frac{y-t}{2}\right)\\&\quad=\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\operatorname{sech}^{2\beta}\left(\frac{y}{2}\right)\\&\qquad\times\left\{\left(\frac{1}{1+\tanh\left(\frac{y}{2}\right)\tanh\left(\frac{t}{2}\right)}\right)^{2\beta}+\left(\frac{1}{1-\tanh\left(\frac{y}{2}\right)\tanh\left(\frac{t}{2}\right)}\right)^{2\beta}\right\}\end{aligned}\tag{3.11}$$

and the binomial theorem,

$$\mathcal{T}=\frac{\pi}{2}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\int_0^\infty\operatorname{sech}^{2\alpha+2\beta}\left(\frac{y}{2}\right)\sum_{n=0}^{\infty}\frac{(2\beta)_{2n}}{(2n)!}\tanh^{2n}\left(\frac{t}{2}\right)\tanh^{2n}\left(\frac{y}{2}\right)dy.\tag{3.12}$$

After substituting v for $\tanh^2\left(\frac{y}{2}\right)$ in (3.12), we obtain

$$\mathcal{T}=\frac{\pi}{2}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\sum_{n=0}^{\infty}\frac{(2\beta)_{2n}}{(2n)!}\tanh^{2n}\left(\frac{t}{2}\right)\int_0^1(1-v)^{\alpha+\beta-1}v^{n-\frac{1}{2}}\,dv.\tag{3.13}$$

Using Euler's beta integral $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$, we deduce that

$$\begin{aligned}\mathcal{T}&=\frac{\pi}{2}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\sum_{n=0}^{\infty}\frac{(2\beta)_{2n}}{(2n)!}\tanh^{2n}\left(\frac{t}{2}\right)\frac{\Gamma(n+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+n+\frac{1}{2})}\\&=\frac{\pi}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right)\sum_{n=0}^{\infty}\frac{(2\beta)_{2n}\left(\frac{1}{2}\right)_n}{(2n)!(\alpha+\beta+\frac{1}{2})_n}\tanh^{2n}\left(\frac{t}{2}\right)\\&=\frac{\pi}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right){}_2F_1\left(\beta,\beta+\frac{1}{2};\alpha+\beta+\frac{1}{2};\tanh^2\left(\frac{t}{2}\right)\right).\end{aligned}\tag{3.14}$$

Set $w=\tanh\left(\frac{t}{4}\right)$ so that

$$\begin{aligned}F(t)&:=\operatorname{sech}^{2\beta}\left(\frac{t}{2}\right){}_2F_1\left(\beta,\beta+\frac{1}{2};\alpha+\beta+\frac{1}{2};\tanh^2\left(\frac{t}{2}\right)\right)\\&=\left(\frac{1-w^2}{1+w^2}\right)^{2\beta}{}_2F_1\left(\beta,\beta+\frac{1}{2};\alpha+\beta+\frac{1}{2};\frac{4w^2}{(1+w^2)^2}\right).\end{aligned}\tag{3.15}$$

By the quadratic transformation [2, p. 128, Eq. (3.1.9)],

$${}_2F_1(a,b;a-b+1;z)=(1+z)^{-a}\,{}_2F_1\left(\frac{a}{2},\frac{a+1}{2};a-b+1;\frac{4z}{(1+z)^2}\right)$$

with $z=w^2$, $a=2\beta$, and $b=\beta-\alpha+\frac{1}{2}$, we find that

$$F(t)=\left(1-w^2\right)^{2\beta}{}_2F_1\left(-\alpha+\beta+\frac{1}{2},2\beta;\alpha+\beta+\frac{1}{2};w^2\right),$$

and by Pfaff's transformation formula [2, Theorem 2.2.5],

$$(1-z)^a\,{}_2F_1(a,b;c;z) = {}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right)$$

with $a=2\beta$, $b=-\alpha+\beta+\frac{1}{2}$, and $c=\alpha+\beta+\frac{1}{2}$,

$$F(t) = {}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};\frac{w^2}{1-w^2}\right). \tag{3.16}$$

Therefore, by (3.14)–(3.16),

$$\mathcal{T} = \frac{\pi}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})}\,{}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};-\sinh^2\frac{t}{4}\right). \tag{3.17}$$

From (3.6) and (3.17), we now have

$$\begin{aligned}\Phi(\alpha,\beta,t) = {}& \frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\frac{1}{2})}\\ &\times {}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};-\sinh^2\frac{t}{4}\right).\end{aligned} \tag{3.18}$$

Then, it follows from (3.2) and (3.18) that

$$\begin{aligned}\mathbb{I} = {}& \frac{\sqrt{\pi}}{4s}\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+\frac{1}{2})}\\ &\times\int_0^\infty {}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};-\sinh^2\frac{t}{4}\right)e^{-\frac{t}{2s}}\,dt.\end{aligned} \tag{3.19}$$

Recall the continued fraction expansion formula by Stieltjes [16], [17, p. 370],

$$\begin{aligned}&\int_0^\infty {}_2F_1\left(a,b;\frac{a+b+1}{2};-\sinh^2 t\right)e^{-tz}\,dt\\ &\quad=\frac{1}{z}+\frac{1\cdot ab\cdot 4}{(a+b+1)z}+\frac{2\cdot(a+1)(b+1)(a+b)\cdot 4}{(a+b+3)z}\\ &\qquad+\frac{3\cdot(a+2)(b+2)(a+b+1)\cdot 4}{(a+b+5)z}+\cdots,\end{aligned} \tag{3.20}$$

for $\operatorname{Re} z>0$, to deduce that

$$\begin{aligned}&\int_0^\infty {}_2F_1\left(2\alpha,2\beta;\alpha+\beta+\frac{1}{2};-\sinh^2\frac{t}{4}\right)e^{-\frac{t}{2s}}\,dt\\ &\quad=\frac{4}{\frac{2}{s}}+\frac{1\cdot(2\alpha)(2\beta)\cdot 4}{(2\alpha+2\beta+1)\frac{2}{s}}+\frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)\cdot 4}{(2\alpha+2\beta+3)\frac{2}{s}}+\cdots\end{aligned} \tag{3.21}$$

for $s > 0$. By (3.19) and (3.21), we finally obtain

$$\mathbb{I} = \sqrt{\pi}\,\frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)}\,\frac{\Gamma(\beta+\frac{1}{2})}{\Gamma(\beta)}\,\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+\frac{1}{2})} \times\left(\frac{1}{2}+\frac{2\cdot 1\cdot(2\alpha)(2\beta)s^2}{(2\alpha+2\beta+1)}+\frac{2\cdot(2\alpha+1)(2\beta+1)(2\alpha+2\beta)s^2}{(2\alpha+2\beta+3)}+\cdots\right),$$

which completes the proof of Theorem 1.3 for $s > 0$. By (2.7), Theorem 1.3 holds for $s = 0$. Since both sides of Theorem 1.3 are even functions of s, Theorem 1.3 is valid for all real s.

4. Proof of Entry 1.2

It is easy to show by mathematical induction on n with the recurrence relation (2.3) that for all integers $n \geqslant 1$,

$$\frac{1}{a+b+1}+\frac{1\cdot(a+1)(b+1)(a+b+1)}{a+b+3}+\cdots+\frac{n\cdot(a+n)(b+n)(a+b+n)}{a+b+(2n+1)} =\frac{1}{a+b+1}\big(1-A_1+A_1A_2-A_1A_2A_3+\cdots+(-1)^nA_1A_2\cdots A_n\big),$$

where A_n is as defined in (1.6), which proves Entry 1.2.

Acknowledgment

The authors are very grateful to Professor Mourad Ismail for his suggestions for our proof in Section 2.

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