Available online at www.sciencedirect.com
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J. Math. Anal. Appl. 314 (2006) 631–643
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Strong convergence of an iterative method for nonexpansive and accretive operators

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Received 15 February 2005

Available online 2 June 2005

Submitted by T.D. Benavides

Abstract

Let X be a Banach space and A an m-accretive operator with a zero. Consider the iterative method that generates the sequence $\{x_n\}$ by the algorithm $x_{n+1} = \alpha_n u + (1-\alpha_n) J_{r_n} x_n$, where $\{\alpha_n\}$ and $\{r_n\}$ are two sequences satisfying certain conditions, and J_r denotes the resolvent $(I + rA)^{-1}$ for $r > 0$. Strong convergence of the algorithm $\{x_n\}$ is proved assuming X either has a weakly continuous duality map or is uniformly smooth.

Keywords: Iterative method; Nonexpansive mapping; m-Accretive operator; Weakly continuous duality map; Uniformly smooth Banach space

1. Introduction

Let X be a Banach space, let C be a nonempty closed convex subset of X, and let $T: C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$). Denote by $\mathrm{Fix}(T)$ the set of fixed points of T (i.e., $\mathrm{Fix}(T) = \{x \in C\colon\ Tx = x\}$). One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive

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[1] Supported in part by the National Research Foundation of South Africa.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.04.082

mapping (Browder [2] and Reich [8]). More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by

$$T_t x = tu + (1-t)Tx, \quad x \in C,$$

where $u \in C$ is a fixed point. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in C. It is unclear, in general, what is the behavior of x_t as $t \to 0$, even if T has a fixed point. However, in the case of T having a fixed point, Browder [2] proved that if X is a Hilbert space, then x_t does converges strongly to the fixed point of T that is nearest to u. Reich [8] extended Browder's result to the setting of Banach spaces and proved that if X is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the (unique) sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$. The first result of this paper says that Reich's result holds in a Banach space which has a weakly continuous duality map.

Recall that an operator A with domain $D(A)$ and range $R(A)$ in X is said to be *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there is a $j \in J(x_2 - x_1)$ such that

$$\langle y_2 - y_1, j\rangle \geqslant 0,$$

where J is the duality map from X to the dual space X^* given by

$$J(x) = \left\{x^* \in X^*\colon \left\langle x, x^*\right\rangle = \|x\|^2 = \left\|x^*\right\|^2\right\}, \quad x \in X.$$

An accretive operator A is m-accretive if $R(I + \lambda A) = X$ for all $\lambda > 0$.

Denote by F the zero set of A; i.e.,

$$F := A^{-1}(0) = \left\{x \in D(A)\colon\ 0 \in Ax\right\}.$$

Throughout the rest of this paper it is always assumed that A is m-accretive and F is nonempty.

Denote by J_r the resolvent of A for $r > 0$:

$$J_r = (I + rA)^{-1}.$$

It is known that J_r is a nonexpansive mapping from X to $C := \overline{D(A)}$ which will be assumed convex (this is so provided X is uniformly smooth and uniformly convex).

An interesting topic is to find a point in F via iterative methods. In [4], the authors study iterative solutions of m-accretive operators in a Banach space that is uniformly smooth and has a weakly continuous duality map. The iterative method studied in [4] generates a sequence by the algorithm

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geqslant 0, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{r_n\}$ is a sequence of positive numbers, and the initial guess $x_0 \in C$ is arbitrarily chosen. Theorem 2.5 of [4] asserts that if X is uniformly smooth and has a weakly continuous duality map, then the sequence $\{x_n\}$ given in (1.1) converges strongly to a point in F provided the sequences $\{\alpha_n\}$ and $\{r_n\}$ satisfy certain conditions.

The main purpose of this paper is to remove either the uniform smoothness assumption or the assumption of a weak continuous duality map in the above mentioned result.

2. Preliminaries

We require the following lemmas. The proof of Lemma 2.1 can be found in [9,10]. Lemma 2.2 is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$. Lemma 2.3 is the resolvent identity which can be found in [1]. Lemma 2.4 can be found in [7].

Lemma 2.1. *Let (a_n) be a sequence of nonnegative real numbers that satisfies the condition*:

$$a_{n+1} \leqslant (1-\lambda_n)a_n + \lambda_n\mu_n, \quad n \geqslant 0,$$

where the sequences $\{\lambda_n\} \subset (0,1)$ and $\{\mu_n\}$ satisfy the conditions:

(i) $\lim_{n\to\infty} \lambda_n = 0$;
(ii) $\sum_{n=1}^{\infty} \lambda_n = \infty$;
(iii) $\limsup_{n\to\infty} \mu_n \leqslant 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. *In a smooth Banach space X there holds the inequality*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, J(x+y)\rangle, \quad x, y \in X.$$

Lemma 2.3 (The Resolvent Identity). *For $\lambda, \mu > 0$, there holds the identity*:

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right), \quad x \in X.$$

Lemma 2.4. *Assume that $c_2 \geqslant c_1 > 0$. Then $\|J_{c_1}x - x\| \leqslant 2\|J_{c_2}x - x\|$ for all $x \in X$.*

Recall that if C and D are nonempty subsets of a Banach space X such that C is nonempty closed convex and $D \subset C$, then a map $Q : C \to D$ is called a retraction from C onto D provided $Q(x) = x$ for all $x \in D$. A retraction $Q : C \to D$ is sunny provided $Q(x + t(x - Q(x))) = Q(x)$ for all $x \in C$ and $t \geqslant 0$ whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction which is also nonexpansive. Sunny nonexpansive retractions are characterized as follows (cf. [5]): If X is a smooth Banach space, then $Q : C \to D$ is a sunny nonexpansive retraction if and only if there holds the inequality

$$\langle x - Qx, J(y - Qx)\rangle \leqslant 0 \quad \text{for all } x \in C \text{ and } y \in D. \tag{2.1}$$

Reich [8] showed that if X is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there is a unique sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Lemma 2.5 [8]. *Let X be a uniformly smooth Banach space and let $T : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0,1)$, the* *unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1-t)Tx$ converges strongly as $t \to 0$ to a fixed point of T. Define $Q : C \to \mathrm{Fix}(T)$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto $\mathrm{Fix}(T)$; that is, Q satisfies the property*:

$$\big\langle u - Qu, J(z - Qu)\big\rangle \leqslant 0, \quad u \in C,\ z \in \mathrm{Fix}(T). \tag{2.2}$$

Recall that a gauge is a continuous strictly increasing function $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) \to \infty$ as $t \to \infty$. Associated to a gauge φ is the duality map $J_\varphi : X \to X^*$ defined by

$$J_\varphi(x) = \big\{x^* \in X^*\colon \langle x, x^*\rangle = \|x\|\varphi\big(\|x\|\big),\ \big\|x^*\big\| = \varphi\big(\|x\|\big)\big\}, \quad x \in X.$$

Following Browder [3], we say that a Banach space X has a weakly continuous duality map if there exists a gauge φ for which the duality map J_φ is single-valued and weak-to-weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in X weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weak*ly to $J_\varphi(x)$). It is known that l^p has a weakly continuous duality map for all $1 < p < \infty$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)\, d\tau, \quad t \geqslant 0.$$

Then

$$J_\varphi(x) = \partial\Phi\big(\|x\|\big), \quad x \in X,$$

where ∂ denotes the subdifferential in the sense of convex analysis. The first part of the next lemma is an immediate consequence of the subdifferential inequality and the proof of the second part can be found in [6].

Lemma 2.6. *Assume that X has a weakly continuous duality map J_φ with gauge φ.*

(i) *For all $x, y \in X$, there holds the inequality*

$$\Phi\big(\|x + y\|\big) \leqslant \Phi\big(\|x\|\big) + \big\langle y, J_\varphi(x + y)\big\rangle.$$

(ii) *Assume a sequence $\{x_n\}$ in X is weakly convergent to a point x. Then there holds the identity*

$$\limsup_{n\to\infty} \Phi\big(\|x_n - y\|\big) = \limsup_{n\to\infty} \Phi\big(\|x_n - x\|\big) + \Phi\big(\|y - x\|\big), \quad x, y \in X.$$

Notation: '$\rightharpoonup$' stands for weak convergence and '$\to$' for strong convergence.

3. Weakly continuous duality map and sunny nonexpansive retraction

Recall that C is a nonempty closed convex subset of a Banach space X and $T : C \to C$ is a nonexpansive mapping with a nonempty fixed point set. Recall also that for $t \in (0,1)$ and $u \in C$, x_t is the unique solution to the fixed point equation

$$x_t = tu + (1-t)Tx_t. \tag{3.1}$$

It is known that (Reich [8]) if X is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of T and the limit defines the sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$. Our first result shows that Reich's result holds in a Banach space which has a weakly continuous duality map.

Theorem 3.1. *Let X be a reflexive Banach space and have a weakly continuous duality map J_φ with gauge φ. Let C be a closed convex subset of X and let $T : C \to C$ be a nonexpansive mapping. Fix $u \in C$ and $t \in (0,1)$. Let $x_t \in C$ be the unique solution in C to Eq.* (3.1). *Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 0^+$, and in this case, $\{x_t\}$ converges as $t \to 0^+$ strongly to a fixed point of T.*

Proof. Assume first that $F(T) \neq \emptyset$. Take $p \in F(T)$ to deduce that, for $t \in (0,1)$,

$$\begin{aligned} \|x_t - p\| &= \big\|t(u-p) + (1-t)(Tx_t - p)\big\| \\ &\leqslant t\|u-p\| + (1-t)\|Tx_t - p\| \\ &\leqslant t\|u-p\| + (1-t)\|x_t - p\|. \end{aligned}$$

Hence

$$\|x_t - p\| \leqslant \|u-p\|$$

and $\{x_t\}$ is bounded.

Next assume that $\{x_t\}$ is bounded as $t \to 0^+$. Assume $t_n \to 0^+$ and $\{x_{t_n}\}$ is bounded. Since X is reflexive, we may assume that $x_{t_n} \rightharpoonup z$ for some $z \in C$. Since J_φ is weakly continuous, we have by Lemma 2.6,

$$\limsup_{n\to\infty} \Phi\big(\|x_{t_n} - x\|\big) = \limsup_{n\to\infty} \Phi\big(\|x_{t_n} - z\|\big) + \Phi\big(\|x - z\|\big), \quad \forall x \in X.$$

Put

$$f(x) = \limsup_{n\to\infty} \Phi\big(\|x_{t_n} - x\|\big), \quad x \in X.$$

It follows that

$$f(x) = f(z) + \Phi\big(\|x - z\|\big), \quad x \in X.$$

Since

$$\|x_{t_n} - Tx_{t_n}\| = \frac{t_n}{1-t_n}\|u - x_{t_n}\| \to 0,$$

we obtain

$$\begin{aligned} f(Tz) &= \limsup_{n\to\infty} \Phi\big(\|x_{t_n} - Tz\|\big) = \limsup_{n\to\infty} \Phi\big(\|Tx_{t_n} - Tz\|\big) \\ &\leqslant \limsup_{n\to\infty} \Phi\big(\|x_{t_n} - z\|\big) = f(z). \end{aligned} \tag{3.2}$$

On the other hand, however,

$$f(Tz) = f(z) + \Phi\big(\|Tz - z\|\big). \tag{3.3}$$

Combining Eqs. (3.2) and (3.3) yields

$$\Phi\big(\|Tz - z\|\big) \leqslant 0.$$

Hence $Tz = z$ and $z \in F(T)$.

Finally we prove that $\{x_t\}$ converges strongly to a fixed point of T provided it remains bounded when $t \to 0$.

Let $\{t_n\}$ be a sequence in $(0,1)$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup z$ as $n \to \infty$. Then the argument above shows that $z \in F(T)$. We next show that $x_{t_n} \to z$. As a matter of fact, we have by Lemma 2.6,

$$\begin{aligned}\Phi\big(\|x_{t_n} - z\|\big) &= \Phi\big(\big\|t_n(Tx_{t_n} - z) + (1-t_n)(u-z)\big\|\big)\\ &\leqslant \Phi\big(t_n\|Tx_{t_n} - z\|\big) + (1-t_n)\big\langle u - z, J_\varphi(x_{t_n} - z)\big\rangle\\ &\leqslant t_n\Phi\big(\|x_{t_n} - z\|\big) + (1-t_n)\big\langle u - z, J_\varphi(x_{t_n} - z)\big\rangle.\end{aligned}$$

This implies that

$$\Phi\big(\|x_{t_n} - z\|\big) \leqslant \big\langle u - z, J_\varphi(x_{t_n} - z)\big\rangle.$$

Now observing that $x_{t_n} \rightharpoonup z$ implies $J_\varphi(x_{t_n} - z) \rightharpoonup 0$, we conclude from the last inequality that

$$\Phi\big(\|x_{t_n} - z\|\big) \to 0.$$

Hence $x_{t_n} \to z$.

We finally prove that the entire net $\{x_t\}$ converges strongly. Towards this end, we assume that two null sequences $\{t_n\}$ and $\{s_n\}$ in $(0,1)$ are such that

$$x_{t_n} \to z \quad \text{and} \quad x_{s_n} \to z'.$$

We have to show $z = z'$. Indeed, for $p \in F(T)$, it is easy to see that

$$\begin{aligned}\big\langle x_t - Tx_t, J_\varphi(x_t - p)\big\rangle &= \Phi\big(\|x_t - p\|\big) + \big\langle p - Tx_t, J_\varphi(x_t - p)\big\rangle\\ &\geqslant \Phi\big(\|x_t - p\|\big) - \|p - Tx_t\| \cdot \big\|J_\varphi(x_t - p)\big\|\\ &\geqslant \Phi\big(\|x_t - p\|\big) - \Phi\big(\|x_t - p\|\big)\\ &= 0.\end{aligned}$$

On the other hand, since

$$x_t - Tx_t = \frac{t}{1-t}(u - x_t),$$

we get for $t \in (0,1)$ and $p \in F(T)$,

$$\big\langle x_t - u, J_\varphi(x_t - p)\big\rangle \leqslant 0.$$

In particular,

$$\big\langle x_{t_n} - u, J_\varphi(x_{t_n} - p)\big\rangle \leqslant 0 \quad \text{and} \quad \big\langle x_{s_n} - u, J_\varphi(x_{s_n} - p)\big\rangle \leqslant 0.$$

Passing on to the limits as $n \to \infty$, we obtain

$$\big\langle z - u, J_\varphi(z - v)\big\rangle \leqslant 0 \quad \text{and} \quad \big\langle z' - u, J_\varphi(z' - v)\big\rangle \leqslant 0.$$

Adding up gets

$$\langle z - z', J_\varphi(z - z')\rangle \leqslant 0.$$

Hence $z = z'$ and $\{x_t\}$ converges strongly. $\square$

Under the condition of Theorem 3.1, we can define a map $Q : C \to F(T)$ by

$$Q(u) := \lim_{t \to 0} x_t, \quad u \in C.$$

The next result shows that Q is the sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$.

Theorem 3.2. *Under the conditions of Theorem* 3.1*, Q defines the sunny nonexpansive retraction from C onto $F(T)$.*

Proof. Since we have proved that, for all $t \in (0,1)$ and $p \in F(T)$,

$$\langle x_t - u, J_\varphi(x_t - p)\rangle \leqslant 0,$$

letting $t \to 0$, we obtain that

$$\big\langle Q(u) - u, J_\varphi\big(Q(u) - p\big)\big\rangle \leqslant 0.$$

This implies that

$$\big\langle Q(u) - u, J\big(Q(u) - p\big)\big\rangle \leqslant 0$$

since $J_\varphi(x) = (\varphi(\|x\|)/\|x\|)J(x)$ for $x \neq 0$. Now by the characterization inequality (2.2) of Lemma 2.5 we see that Q is sunny nonexpansive. $\square$

4. Zeros of *m*-accretive operators

Next consider the problem of finding a zero of an m-accretive operator A in a Banach space X,

$$0 \in Ax. \tag{4.1}$$

Recall that the resolvent and Yosida approximation of A are respectively defined by

$$J_r = (I + rA)^{-1} \quad \text{and} \quad A_r = \frac{1}{r}(I - J_r).$$

Assume

$$F := \{x \in X \colon 0 \in Ax\} = A^{-1}(0) \neq \emptyset.$$

Write $C = \overline{D(A)}$. Consider the following algorithm

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \quad n \geqslant 0, \tag{4.2}$$

where $u \in C$ is arbitrarily fixed, $\{\alpha_n\}$ is a sequence in $(0,1)$, and $\{r_n\}$ is a sequence of positive numbers.

Algorithm (4.2) has been investigated in [4] in which strong convergence is proved provided the space X is uniformly smooth and has a weakly continuous duality map may J_φ

for some gauge φ. The purpose of this section is to remove either the uniform smoothness assumption or the weak continuous duality map assumption.

Theorem 4.1. *Suppose that X is reflexive and has a weakly continuous duality map J_φ with gauge φ. Suppose that A is an m-accretive operator in X such that $C = \overline{D(A)}$ is convex. Assume*

(i) $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $r_n \to \infty$.

Then $\{x_n\}$ converges strongly to a point in F.

Proof. First notice that $\{x_n\}$ is bounded. Indeed, take $p \in F$ to get

$$\begin{aligned}\|x_{n+1} - p\| &\leqslant \alpha_n \|u - p\| + (1 - \alpha_n)\|J_{r_n} x_n - p\| \\ &\leqslant \alpha_n \|u - p\| + (1 - \alpha_n)\|x_n - p\|.\end{aligned}$$

An induction gives that

$$\|x_n - p\| \leqslant \max\bigl\{\|u - p\|,\ \|x_0 - p\|\bigr\} \quad \text{for all } n \geqslant 0.$$

This implies that $\{x_n\}$ is bounded and hence

$$\|x_{n+1} - J_{r_n} x_n\| = \alpha_n \|u - J_{r_n} x_n\| \to 0.$$

We next prove that

$$\limsup_{n\to\infty} \bigl\langle u - p, J_\varphi(x_n - p)\bigr\rangle \leqslant 0, \quad p \in F. \tag{4.3}$$

By Theorem 3.1, we have the sunny nonexpansive retraction $Q : C \to \mathrm{Fix}(T)$. Put $p = Q(u)$ and take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \bigl\langle u - p, J_\varphi(x_n - p)\bigr\rangle = \lim_{k\to\infty} \bigl\langle u - p, J_\varphi(x_{n_k} - p)\bigr\rangle. \tag{4.4}$$

Since X is reflexive, we may further assume that $x_{n_k} \rightharpoonup \tilde{x}$. Moreover, since

$$\|x_{n+1} - J_{r_n} x_n\| \to 0,$$

we obtain

$$J_{r_{n_k - 1}} x_{n_k - 1} \rightharpoonup \tilde{x}.$$

Taking the limit as $k \to \infty$ in the relation

$$[J_{r_{n_k - 1}} x_{n_k - 1}, A_{r_{n_k - 1}} x_{n_k - 1}] \in A,$$

we get $[\tilde{x}, 0] \in A$. That is, $\tilde{x} \in F$. Hence by (4.4) and (2.1) we have

$$\limsup_{n\to\infty} \bigl\langle u - p, J_\varphi(x_n - p)\bigr\rangle = \bigl\langle u - p, J_\varphi(\tilde{x} - p)\bigr\rangle \leqslant 0.$$

That is, (4.3) holds. Finally to prove that $x_n \to p$, we apply Lemma 2.6 to get

$$\begin{aligned}\Phi\big(\|x_{n+1}-p\|\big) &= \Phi\big(\big\|(1-\alpha_n)(J_{r_n}x_n-p)+\alpha_n(u-p)\big\|\big)\\ &\leqslant \Phi\big((1-\alpha_n)\|J_{r_n}x_n-p\|\big)+\alpha_n\big\langle u-p, J_\varphi(x_{n+1}-p)\big\rangle\\ &\leqslant (1-\alpha_n)\Phi\big(\|x_n-p\|\big)+\alpha_n\big\langle u-p, J_\varphi(x_{n+1}-p)\big\rangle.\end{aligned}$$

An application of Lemma 2.1 yields that $\Phi(\|x_n-p\|)\to 0$; that is, $\|x_n-p\|\to 0$. □

Another result which differs from Theorem 2.5 of [4] in terms of the conditions on $\{\alpha_n\}$ and $\{r_n\}$.

Theorem 4.2. *Suppose that X is reflexive and has a weakly continuous duality map J_φ with gauge φ. Suppose that A is an m-accretive operator in X such that $C=\overline{D(A)}$ is convex. Assume*

(i) $\alpha_n\to 0$, $\sum_{n=0}^\infty \alpha_n=\infty$, *and* $\sum_{n=1}^\infty |\alpha_{n+1}-\alpha_n|<\infty$ (*e.g.*, $\alpha_n=\frac{1}{n}$);
(ii) $r_n\geqslant\varepsilon$ *for all n and* $\sum_{n=1}^\infty |r_{n+1}-r_n|<\infty$ (*e.g.*, $r_n=1+\frac{1}{n}$).

Then $\{x_n\}$ converges strongly to a point in F.

Proof. We only include the differences. We have

$$x_{n+1}=\alpha_n u+(1-\alpha_n)J_{r_n}x_n,\qquad x_n=\alpha_{n-1}u+(1-\alpha_{n-1})J_{r_{n-1}}x_{n-1}.$$

Thus,

$$x_{n+1}-x_n=(\alpha_n-\alpha_{n-1})(u-J_{r_{n-1}}x_{n-1})+(1-\alpha_n)(J_{r_n}x_n-J_{r_{n-1}}x_{n-1}). \tag{4.5}$$

If $r_{n-1}\leqslant r_n$, using the resolvent identity

$$J_{r_n}x_n=J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_n+\left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}x_n\right),$$

we obtain

$$\begin{aligned}\|J_{r_n}x_n-J_{r_{n-1}}x_{n-1}\| &\leqslant \frac{r_{n-1}}{r_n}\|x_n-x_{n-1}\|+\left(1-\frac{r_{n-1}}{r_n}\right)\|J_{r_n}x_n-x_{n-1}\|\\ &\leqslant \|x_n-x_{n-1}\|+\left(\frac{r_n-r_{n-1}}{r_n}\right)\|J_{r_n}x_n-x_{n-1}\|\\ &\leqslant \|x_n-x_{n-1}\|+(1/\varepsilon)|r_{n-1}-r_n|\|J_{r_n}x_n-x_{n-1}\|.\end{aligned}$$

It follows from (4.5) that

$$\|x_{n+1}-x_n\|\leqslant M\big(|\alpha_n-\alpha_{n-1}|+|r_{n+1}-r_n|\big)+(1-\alpha_n)\|x_n-x_{n-1}\|, \tag{4.6}$$

where $M>0$ is some appropriate constant. Similarly we can prove (4.6) if $r_{n-1}\geqslant r_n$. By assumptions (i) and (ii) and Lemma 2.1, we conclude that

$$\|x_{n+1}-x_n\|\to 0.$$

This implies that

$$\|x_n-J_{r_n}x_n\|\leqslant \|x_{n+1}-x_n\|+\|x_{n+1}-J_{r_n}x_n\|\to 0 \tag{4.7}$$

since $\|x_{n+1} - J_{r_n} x_n\| = \alpha_n \|u - J_{r_n} x_n\| \to 0$. It follows that

$$\|A_{r_n} x_n\| = \frac{1}{r_n}\|x_n - J_{r_n} x_n\| \leqslant \frac{1}{\varepsilon}\|x_n - J_{r_n} x_n\| \to 0.$$

Now if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ converging weakly to a point $\tilde{x}$, then taking the limit as $k \to \infty$ in the relation

$$[J_{r_{n_k}} x_{n_k}, A_{r_{n_k}} x_{n_k}] \in A,$$

we get $[\tilde{x}, 0] \in A$; i.e., $\tilde{x} \in F$. We therefore conclude that all weak limit points of $\{x_n\}$ are zeros of A.

The rest of the proof follows that of Theorem 4.1. □

Finally, we consider the framework of uniformly smooth Banach spaces. Assume $r_n \geqslant \varepsilon$ for some $\varepsilon > 0$ (not necessarily $r_n \to \infty$). Since F is the fixed point set of the nonexpansive mapping J_r for all $r > 0$, there exists a unique sunny nonexpansive retraction Q from C onto F and this retraction Q can be constructed as in Reich [8]. In particular, for each integer $n \geqslant 1$, we have

$$Q(u) = \lim_{t \to 0} z_{t,n}, \quad u \in C, \tag{4.8}$$

where $z_{t,n} \in C$ is the unique point in C such that

$$z_{t,n} = tu + (1-t) J_{r_n} z_{t,n}. \tag{4.9}$$

Note that $\{z_{t,n}\}$ is uniformly bounded; indeed, $\|z_{t,n} - p\| \leqslant \|u - p\|$ for all $t \in (0, 1)$, $n \geqslant 1$ and $p \in F$. A key component of the proof of the next theorem is the following lemma.

Lemma 4.3. *The limit in* (4.8) *is uniform for* $n \geqslant 1$.

Proof. It suffices to show that for any positive integer n_t (which may depend on $t \in (0, 1)$), if $z_{t,n_t} \in C$ is the unique point in C that satisfies the property

$$z_{t,n_t} = tu + (1-t) J_{r_{n_t}} z_{t,n_t}, \tag{4.10}$$

then $\{z_{t,n_t}\}$ converges as $t \to 0$ to a point in F. For simplicity put

$$w_t = z_{t,n_t} \quad \text{and} \quad V_t = J_{r_{n_t}}.$$

It follows that

$$w_t = tu + (1-t) V_t w_t. \tag{4.11}$$

Note that $\mathrm{Fix}(V_t) = F$ for all t. Note also that $\{w_t\}$ is bounded; indeed, we have $\|w_t - p\| \leqslant \|u - p\|$ for all $t \in (0, 1)$ and $p \in F$. Since $\{V_t w_t\}$ is bounded, it is easy to see that

$$\|w_t - V_t w_t\| = t\|u - V_t w_t\| \to 0, \quad \text{as } t \to 0.$$

Since $r_n \geqslant \varepsilon$ for all n, by Lemma 2.4, we have

$$\|w_t - J_\varepsilon w_t\| \leqslant 2\|w_t - J_{r_{n_t}} w_t\| = 2\|w_t - V_t w_t\| \to 0. \tag{4.12}$$

Let $\{t_k\}$ be a sequence in (0,1) such that $t_k \to 0$ as $k \to \infty$. Define a function f on C by

$$f(w) = \mathrm{LIM}_k \frac{1}{2}\|w_{t_k} - w\|^2, \quad w \in C,$$

where LIM denotes a Banach limit on l^∞. Let

$$K := \bigl\{w \in C\colon\ f(w) = \min\bigl\{f(y)\colon\ y \in C\bigr\}\bigr\}.$$

Then K is a nonempty closed convex bounded subset of C. We claim that K is also invariant under the nonexpansive mapping J_ε. Indeed, noting (4.12), we have for $w \in K$,

$$\begin{aligned} f(J_\varepsilon w) &= \mathrm{LIM}_k \frac{1}{2}\|w_{t_k} - J_\varepsilon w\|^2 = \mathrm{LIM}_k \frac{1}{2}\|J_\varepsilon w_{t_k} - J_\varepsilon w\|^2 \\ &\leqslant \mathrm{LIM}_k \frac{1}{2}\|w_{t_k} - w\|^2 = f(w). \end{aligned}$$

Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings and since J_ε is a nonexpansive self-mapping of C, J_ε has a fixed point in K, say w'. Now since w' is also a minimizer of f over C, it follows that, for $w \in C$,

$$\begin{aligned} 0 &\leqslant \frac{f(w' + \lambda(w - w')) - f(w')}{\lambda} \\ &= \mathrm{LIM}_k \frac{\frac{1}{2}\|(w_{t_k} - w') + \lambda(w' - w)\|^2 - \frac{1}{2}\|w_{t_k} - w'\|^2}{\lambda}. \end{aligned}$$

Since X is uniformly smooth, the duality map J is uniformly continuous on bounded sets, letting $\lambda \to 0^+$ in the last equation yields

$$0 \leqslant \mathrm{LIM}_k \bigl\langle w' - w, J(w_{t_k} - w')\bigr\rangle, \quad w \in C. \tag{4.13}$$

Since

$$w_{t_k} - w' = t_k(u - w') + (1 - t_k)(V_{t_k} w_{t_k} - w'),$$

we obtain

$$\begin{aligned} \|w_{t_k} - w'\|^2 &= t_k\bigl\langle u - w', J(w_{t_k} - w')\bigr\rangle + (1 - t_k)\bigl\langle V_{t_k} w_{t_k} - w', J(w_{t_k} - w')\bigr\rangle \\ &\leqslant t_k\bigl\langle u - w', J(w_{t_k} - w')\bigr\rangle + (1 - t_k)\|V_{t_k} w_{t_k} - w'\| \cdot \bigl\|J(w_{t_k} - w')\bigr\| \\ &\leqslant t_k\bigl\langle u - w', J(w_{t_k} - w')\bigr\rangle + (1 - t_k)\|w_{t_k} - w'\|^2. \end{aligned}$$

It follows that

$$\|w_{t_k} - w'\|^2 \leqslant \bigl\langle u - w', J(w_{t_k} - w')\bigr\rangle. \tag{4.14}$$

Upon letting $w = u$ in (4.13), we see that the last equation implies

$$\mathrm{LIM}_k \|w_{t_k} - w'\|^2 \leqslant 0. \tag{4.15}$$

Therefore $\{w_{t_k}\}$ contains a subsequence, still denoted $\{w_{t_k}\}$, converging strongly to w_1 (say). By virtue of (4.12), w_1 is a fixed point of J_ε; i.e., a point in F.

To prove that the entire net $\{w_t\}$ converges strongly, assume $\{s_k\}$ is another null subsequence in $(0, 1)$ such that $w_{s_k} \to w_2$ strongly. Then $w_2 \in F$.

Repeating the argument of (4.14) we obtain

$$\|w_t - w'\|^2 \leqslant \langle u - w', J(w_t - w')\rangle, \quad \forall w' \in F. \tag{4.16}$$

In particular,

$$\|w_2 - w_1\|^2 \leqslant \langle u - w_1, J(w_2 - w_1)\rangle \tag{4.17}$$

and

$$\|w_1 - w_2\|^2 \leqslant \langle u - w_2, J(w_1 - w_2)\rangle. \tag{4.18}$$

Adding up the last two equations gives

$$\|w_1 - w_2\|^2 \leqslant 0.$$

That is, $w_1 = w_2$. This concludes the proof. □

We conclude this paper with the main result of this section which removes the assumption of a weak continuous duality map of Theorem 2.5 of [4].

Theorem 4.4. *Suppose that X is a uniformly smooth Banach space. Suppose that A is an m-accretive operator in X such that $C = \overline{D(A)}$ is convex. Assume*

(i) $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ (*e.g.*, $\alpha_n = \frac{1}{n}$);
(ii) $r_n \geqslant \varepsilon$ *for all* n *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ (*e.g.*, $r_n = 1 + \frac{1}{n}$).

Then $\{x_n\}$ converges strongly to a point in F.

Proof. Since $z_{t,n} = tu + (1-t)J_{r_n} z_{t,n}$, we have

$$z_{t,n} - x_n = t(u - x_n) + (1-t)(J_{r_n} z_{t,n} - x_n).$$

Thus by Lemma 2.2,

$$\begin{aligned}
\|z_{t,n} - x_n\|^2 &\leqslant (1-t)^2 \|J_{r_n} z_{t,n} - x_n\|^2 + 2t\langle u - x_n, J(z_{t,n} - x_n)\rangle \\
&\leqslant (1-t)^2 \big(\|J_{r_n} z_{t,n} - J_{r_n} x_n\| + \|J_{r_n} x_n - x_n\|\big)^2 \\
&\quad + 2t\langle u - z_{t,n}, J(z_{t,n} - x_n)\rangle + 2t\|z_{t,n} - x_n\|^2 \\
&\leqslant \big(1 + t^2\big)\|z_{t,n} - x_n\|^2 + \|J_{r_n} x_n - x_n\|\big(2\|z_{t,n} - x_n\| + \|J_{r_n} x_n - x_n\|\big) \\
&\quad + 2t\langle u - z_{t,n}, J(z_{t,n} - x_n)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\langle u - z_{t,n}, J(x_n - z_{t,n})\rangle \\
&\quad \leqslant \frac{t}{2}\|z_{t,n} - x_n\|^2 + \frac{1}{2t}\|J_{r_n} x_n - x_n\|\big(2\|z_{t,n} - x_n\| + \|J_{r_n} x_n - x_n\|\big).
\end{aligned}$$

Remember that Eq. (4.7) still holds. Letting $n \to \infty$ in the last inequality, we obtain

$$\limsup_{n\to\infty} \langle u - z_{t,n}, J(x_n - z_{t,n})\rangle \leqslant \limsup_{n\to\infty} \frac{t}{2}\|z_{t,n} - x_n\|^2 \leqslant \mu t,$$

where $\mu > 0$ is a constant such that $\mu \geqslant 2\|z_{t,n} - x_n\|^2$ for all $n \geqslant 1$ and $t \in (0,1)$. Hence

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle u - z_{t,n}, J(x_n - z_{t,n})\rangle \leqslant 0. \tag{4.19}$$

Furthermore, noticing the fact that the duality map J is uniformly continuous on bounded sets and the uniform convergence of $\{z_{t,n}\}$ to $Q(u)$ (Lemma 4.3), we can interchange the two limits above and deduce that

$$\limsup_{n\to\infty}\langle u - z, J(x_n - z)\rangle \leqslant 0, \tag{4.20}$$

where $z = \lim_{t\to 0} z_{t,n} = Q(u)$.

Finally to prove that $x_n \to z$ strongly, we write

$$x_{n+1} - z = \alpha_n(u - z) + (1 - \alpha_n)(J_{r_n}x_n - z).$$

Apply Lemma 2.2 to get

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leqslant (1-\alpha_n)^2\|J_{r_n}x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle \\ &\leqslant (1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, J(x_{n+1} - z)\rangle.\end{aligned}$$

By Lemma 2.1 and (4.20), we see that $x_n \to z$. □

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