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Hardy–Littlewood maximal function of τ-measurable operators

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Abstract

We define the Hardy–Littlewood maximal function of τ-measurable operators and obtain weak $(1,1)$-type and (p,p)-type inequalities for the Hardy–Littlewood maximal function.

Keywords: τ-Measurable operator; Hardy–Littlewood maximal function; von Neumann algebra

0. Introduction

Nelson [2] defined the measure topology of τ-measurable operators affiliated with a semi-finite von Neumann algebra. Fack and Kosaki [1] studied generalized s-numbers of τ-measurable operators, proved dominated convergence theorems for a gage and convexity (or concavity) inequalities.

We will study the Hardy–Littlewood maximal function of a τ-measurable operator T. More precisely, let $\mathcal{M}$ be a semi-finite von Neumann algebra with a normal faithful semi-finite trace τ. For an operator T in $\mathcal{M}$, the Hardy–Littlewood maximal function of T is defined by

$$MT(x) = \sup_{r>0} \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau\big(|T| E_{[x-r,x+r]}\big(|T|\big)\big).$$

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Classically $Mf(x)$ is defined as

$$Mf(x) = \sup_{r>0} \frac{1}{m([x-r, x+r])} \int\limits_{[x-r,x+r]} \left|f(t)\right| dt,$$

for the case $f:\mathbb{R} \to \mathbb{R}$ and m a Lebesgue measure on $(-\infty, +\infty)$ (cf. [3]). A natural generalization of this is the case $f:\mathbb{R} \to \mathbb{R}$ and μ a Borel measure on $(-\infty, +\infty)$ where

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu([x-r, x+r])} \int\limits_{[x-r,x+r]} \left|f(t)\right| d\mu(t).$$

Let $\mu(A) = \tau(E_A(|T|))$, where A is a Borel subset of $(-\infty, +\infty)$. Then μ is a Borel measure and

$$MT(x) = \sup_{r>0} \frac{1}{\mu([x-r, x+r])} \int\limits_{[x-r,x+r]} t\, d\mu(t),$$

i.e., $MT(x)$ is the Hardy–Littlewood maximal function $M_\mu f(x)$ of $f:\mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} t\chi_{\sigma(|T|)}(t), & t \in \sigma(|T|), \\ 0, & t \notin \sigma(|T|), \end{cases}$$

with respect to μ.

Via spectral theory $|T|$ is represented as

$$f(t) = \begin{cases} t\chi_{\sigma(|T|)}(t), & t \in \sigma(|T|), \\ 0, & t \notin \sigma(|T|), \end{cases}$$

and $MT(|T|)$ is represented as $MT(x)$. Then for T, we consider $MT(|T|)$ as the operator analogue of the Hardy–Littlewood maximal function in the classical case. We will give some results similar to the classical case. Hence roughly speaking, $MT(|T|)$ stands in relation to T as $Mf(x)$ stands in relation to f in classical analysis.

Section 1 consists of some preliminaries. In Section 2, we give some properties of the Hardy–Littlewood maximal function of τ-measurable operators. In Section 3, we prove weak $(1,1)$-type and (p,p)-type inequalities for the Hardy–Littlewood maximal function.

1. Preliminaries

Throughout this paper, we denote by $\mathcal{M}$ a semi-finite von Neumann algebra on the Hilbert space $\mathcal{H}$ with a normal faithful semi-finite trace τ. The closed densely defined linear operator T in $\mathcal{H}$ with domain $D(T)$ is said to be affiliated with $\mathcal{M}$ if and only if $U^*TU = T$ for all unitary operators U which belong to the commutant $\mathcal{M}'$ of $\mathcal{M}$. If T is affiliated with $\mathcal{M}$, then T is said to be τ-measurable if for every $\varepsilon > 0$ there exists a projection $P \in \mathcal{M}$ such that $P(H) \sqsubseteq D(T)$ and $\tau(P^\perp) < \varepsilon$ (where for any projection P we let $P^\perp = 1 - P$). The set of all τ-measurable operators will be denoted by $\overline{\mathcal{M}}$. The set $\overline{\mathcal{M}}$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product. For a positive self-adjoint operator $T = \int_0^\infty \lambda\, dE_\lambda$ affiliated with $\mathcal{M}$, we set

$$\tau(T) = \sup_n \tau\left(\int\limits_0^n \lambda\, dE_\lambda\right) = \int\limits_0^\infty \lambda \tau(E_\lambda).$$

For $0 < p < \infty$, $L^p(\mathcal{M};\tau)$ is defined as the set of all τ-measurable operators T affiliated with $\mathcal{M}$ such that

$$\|T\|_p = \tau\big(|T|^p\big)^{1/p} < \infty.$$

In addition, we put $L^\infty(\mathcal{M};\tau) = \mathcal{M}$ and denote by $\|\cdot\|_\infty$ ($= \|\cdot\|$), the usual operator norm. It is well known that $L^p(\mathcal{M};\tau)$ is a Banach space under $\|\cdot\|_p$ ($1 \leqslant p \leqslant \infty$) satisfying all the expected properties such as duality.

For a positive operator T, let $E_{(t,\infty)}(T)$ be the spectral projection of T corresponding to the interval (t,∞). We state for easy reference the following fact that will be applied below.

Theorem A *(Besicovitch). Let F be a bounded subset of $[0,\infty)$ and suppose to each $x \in F$ we associate a number $r(x) > 0$. Then we can take a sequence of intervals $\{[x_k - r(x_k), x_k + r(x_k)]\}$ such that*

$$F \subset \bigcup_k \big[x_k - r(x_k), x_k + r(x_k)\big] \tag{1}$$

and

$$\sum_k \chi_{[x_k - r(x_k), x_k + r(x_k)]} \leqslant 4, \quad \forall x \in [0,\infty). \tag{2}$$

2. Maximal function

Let $L_{\mathrm{loc}}(\mathcal{M};\tau)$ be the set of all τ-measurable operators such that

$$\tau\big(|T|E_I\big(|T|\big)\big) < +\infty,$$

for all bounded intervals $I \in [0,+\infty)$.

Definition 1. For $T \in L_{\mathrm{loc}}(\mathcal{M};\tau)$, we define the maximal function of T by

$$MT(x) = \sup_{r>0} \frac{1}{\tau(E_{[x-r,x+r]}(|T|))} \tau\big(|T|E_{[x-r,x+r]}\big(|T|\big)\big)$$

(let $\frac{0}{0} = 0$). M is called the Hardy–Littlewood maximal operator.

The maximal function of a τ-measurable operator has the following property.

Lemma 1. *Let $T \in L_{\mathrm{loc}}(\mathcal{M};\tau)$.*

(i) *If the map: $t \in [0,\infty) \to E_{(t,\infty)}(|T|)$ is strongly continuous, then $MT(x)$ is a lower semi-continuous function on $[0,\infty)$.*
(ii) *For all $T \in L^\infty(\mathcal{M};\tau)$, we have*

$$\big\|MT\big(|T|\big)\big\|_\infty \leqslant \|T\|_\infty. \tag{3}$$

Proof. (i) It needs to be proved that

$$F_{MT}(t) = \big\{x \in [0,\infty):\ MT(x) > t\big\}, \quad \forall t > 0,$$

is an open set. In other words, if $\{x_k\}$ is a sequence in $[0,\infty)\backslash F_{MT}(t)$, converging to x, then $x\in[0,\infty)\backslash F_{MT}(t)$, i.e., for all $r>0$ with $E_{[x-r,x+r]}(|T|)\neq 0$, we have

$$\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[x-r,x+r]}\big(|T|\big)\big)\leqslant t. \tag{4}$$

Let

$$T_k=|T|E_{[x_k-r,x_k+r]\Delta[x-r,x+r]}\big(|T|\big),$$

where

$$[x_k-r,x_k+r]\Delta[x-r,x+r]=\big([x_k-r,x_k+r]\backslash[x-r,x+r]\big)\\ \cup\big([x-r,x+r]\backslash[x_k-r,x_k+r]\big)$$

and $k=1,2,3,\ldots.$ It is clear that

$$T_k\leqslant\lambda E_{[x_k-r,x_k+r]\Delta[x-r,x+r]}, \tag{5}$$

for some $\lambda>0$ and all k.

(a) If τ is finite, we use σ-strong continuity of the trace and the fact that the strong and σ-strong topologies agree on the unit ball of $\mathcal{M}$, to obtain the continuity of

$$[0,\infty)\to[0,\infty):s\to\tau\big(E_{(s,\infty)}\big(|T|\big)\big).$$

By (5) and the previous continuity, we get

$$\lim_{k\to\infty}\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T_k|E_{[x-r,x+r]}\big(|T|\big)\big)=0,$$
$$\lim_{k\to\infty}\tau\big(E_{[x_k-r,x_k+r]}\big(|T|\big)\big)=\tau\big(E_{[x-r,x+r]}\big(|T|\big)\big).$$

Hence, for $\delta>0$, there exists k_0 such that

$$\begin{aligned}&\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[x_k-r,x_k+r]}\big(|T|\big)\big)\\ &\quad=\frac{\tau(E_{[x_k-r,x_k+r]}(|T|))}{\tau(E_{[x-r,x+r]}(|T|))}\frac{1}{\tau(E_{[x_k-r,x_k+r]}(|T|))}\tau\big(|T|E_{[x_k-r,x_k+r]}\big(|T|\big)\big)\\ &\quad\leqslant\frac{\tau(E_{[x_k-r,x_k+r]}(|T|))}{\tau(E_{[x-r,x+r]}(|T|))}t<t+\delta,\quad\forall k\geqslant k_0.\end{aligned}$$

Thus for $k\geqslant k_0$, we have

$$\begin{aligned}&\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[x-r,x+r]}\big(|T|\big)\big)\\ &\quad\leqslant\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[x_k-r,x_k+r]\Delta[x-r,x+r]}\big(|T|\big)\big)\\ &\qquad+\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[x_k-r,x_k+r]}\big(|T|\big)\big)\\ &\quad\leqslant\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T_k|E_{[x-r,x+r]}\big(|T|\big)\big)+t+\delta.\end{aligned}$$

Letting $k\to\infty$, $\delta\to 0$, one obtains that

$$\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[x-r,x+r]}\big(|T|\big)\big)\leqslant t.$$

(b) In the general case, for $T \in L_{\mathrm{loc}}(\mathcal{M};\tau)$, for any $\varepsilon > 0$ and any bounded interval $I \subset [\varepsilon,\infty)$, we have that

$$\varepsilon\tau\big(E_I\big(|T|\big)\big) \leqslant \tau\big(|T|E_I\big(|T|\big)\big) < \infty.$$

On the other hand, we may assume $\tau(E_{[x-r,x+r]}(|T|)) < \infty$ (since otherwise the inequality (4) automatically holds). Hence $\tau(E_{[x-r-\varepsilon_0,x+r+\varepsilon_0]}(|T|)) < \infty$ for $\varepsilon_0 > 0$ small enough. Since $x_k \to x$ $(k \to \infty)$, for the above $\varepsilon_0 > 0$ there exists an integer $K > 0$ such that

$$[x_k - r, x_k + r] \cup [x - r, x + r] \subset [x - r - \varepsilon_0, x + r + \varepsilon_0], \quad k \geqslant K.$$

Without loss of generality we can replace $\mathcal{M}$ by

$$E_{[x-r-\varepsilon_0,x+r+\varepsilon_0]}\big(|T|\big)\mathcal{M}E_{[x-r-\varepsilon_0,x+r+\varepsilon_0]}\big(|T|\big).$$

Hence, by the case (a), we obtain (4).

(ii) From

$$\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[x-r,x+r]}\big(|T|\big)\big) \leqslant \|T\|_\infty,$$

we get $MT(x) \leqslant \|T\|_\infty$. Hence (3) follows from

$$\begin{aligned}\big(MT\big(|T|\big)y,y\big) &= \int_{\sigma(|T|)} MT(t)\,d\big(E_t\big(|T|\big)y,y\big) \leqslant \int_{\sigma(|T|)} \|T\|_\infty\,d\big(E_t\big(|T|\big)y,y\big)\\ &= \|T\|_\infty \int_{\sigma(|T|)} d\big(E_t\big(|T|\big)y,y\big) = \|T\|_\infty(y,y), \quad \forall y \in D(T). \qquad \square\end{aligned}$$

3. Inequalities of the Hardy–Littlewood maximal function

For a τ-measurable operator T and a positive function f, we define

$$F_f(t) = \big\{x \in [0,\infty):\ f(x) > t\big\}$$

and

$$f_*\big(|T|\big)(t) = \tau\big(E_{F_f(t)}\big(|T|\big)\big).$$

Lemma 2. *Let T be a τ-measurable operator and f be a positive function on* $[0,\infty)$.

(i) *$f_*(|T|)$ is non-increasing on* $[0,\infty)$.

(ii) *If $f(|T|) \in L^p(\mathcal{M};\tau)$, $1 < p < \infty$, then*

$$\lim_{t\to\infty} t^p f_*\big(|T|\big)(t) = \lim_{t\to 0} t^p f_*\big(|T|\big)(t) = 0. \tag{6}$$

(iii) *If $\int_0^\infty t^{p-1} f_*(|T|)(t)\,dt < \infty$, then*

$$\lim_{t\to\infty} t^p f_*\big(|T|\big)(t) = \lim_{t\to 0} t^p f_*\big(|T|\big)(t) = 0. \tag{7}$$

Proof. (i) Follows immediately from the definition of $f_*(|T|)$.

(ii) From

$$t^p f_*(t)\big(|T|\big) \leqslant \tau\big(E_{F_f(t)}\big(|T|\big)f\big(|T|\big)^p\big) \leqslant \tau\big(f\big(|T|\big)^p\big),$$

we get $f_*(|T|)(t) = \tau(E_{F_f(t)}(|T|)) \to 0$ $(t \to \infty)$. Hence, we obtain that

$$\lim_{t\to\infty} \tau\big(E_{F_f(t)}\big(|T|\big) f\big(|T|\big)^p\big) = 0,$$

so that

$$t^p f_*\big(|T|\big)(t) \to 0 \quad (t \to \infty).$$

Fix $\delta > 0$. Then for $t < \delta$, we have

$$\begin{aligned}\lim_{t\to 0} t^p f_*\big(|T|\big)(t) &= \lim_{t\to 0} t^p \big(f_*\big(|T|\big)(t) - f_*\big(|T|\big)(\delta)\big)\\ &= \lim_{t\to 0} t^p \tau\big(E_{\{x\in[0,\infty):\ \delta\geqslant f(x)>t\}}\big(|T|\big)\big)\\ &\leqslant \tau\big(E_{\{x\in[0,\infty):\ \delta\geqslant f(x)>0\}}\big(|T|\big) f\big(|T|\big)^p\big).\end{aligned}$$

Letting $\delta \to 0$, we obtain

$$t^p f_*\big(|T|\big)(t) \to 0 \quad (t \to 0).$$

(iii) Follows immediately from the following fact:

$$\begin{aligned}p\int_{t/2}^{t} s^{p-1} f_*\big(|T|\big)(s)\,ds \geqslant p\int_{t/2}^{t} s^{p-1} f_*\big(|T|\big)(t)\,ds &= f_*\big(|T|\big)(t)\bigg(t^p - \Big(\frac{t}{2}\Big)^p\bigg)\\ &= f_*\big(|T|\big)(t) t^p\big(1 - 2^{-p}\big). \qquad \square\end{aligned}$$

Lemma 3. *Let* $1 < p < \infty$.

(i) *If for* $t > 0$, *we have* $f_*(|T|)(t) < \infty$, *then*

$$\tau\big(f\big(|T|\big)^p\big) = -\int_0^{+\infty} t^p\, df_*\big(|T|\big)(t). \tag{8}$$

(ii) *If* T *is a measurable operator, then*

$$\tau\big(f\big(|T|\big)^p\big) = p\int_0^{+\infty} t^{p-1} f_*\big(|T|\big)(t)\,dt. \tag{9}$$

Proof. (i) Let

$$0 < \varepsilon < 2\varepsilon < \cdots < n\varepsilon < \cdots$$

and

$$E_j = E_{\{x\in[0,\infty):\ j\varepsilon\geqslant f(x)>(j-1)\varepsilon\}}\big(|T|\big), \quad j = 1, 2, \ldots.$$

Then we get

$$\tau\big(f\big(|T|\big)^p\big) = \int_0^{+\infty} f(t)^p\, d\tau\big(E_t\big(|T|\big)\big)$$

$$= \lim_{\varepsilon \to 0} \sum_{j=1}^{+\infty} \big((j-1)\varepsilon\big)^p \tau(E_j)$$

$$= -\lim_{\varepsilon \to 0} \sum_{j=1}^{+\infty} \big((j-1)\varepsilon\big)^p \big[f_*\big(|T|\big)(j\varepsilon) - f_*\big(|T|\big)\big((j-1)\varepsilon\big)\big]$$

$$= -\int_0^{+\infty} t^p \, df_*\big(|T|\big)(t).$$

(ii) If two sides of (9) are infinity, then the result follows. Let one side of (9) be finite. Then by Lemma 2 we have $f_*(|T|)(t) < \infty$, $\forall t > 0$. Therefore (8) holds. On the other hand, by (6), (7), we get

$$-\int_0^{+\infty} t^p \, df_*\big(|T|\big)(t) = p\int_0^{+\infty} t^{p-1} f_*\big(|T|\big)(t)\, dt - t^p f_*\big(|T|\big)(t)\big|_0^\infty$$

$$= p\int_0^{+\infty} t^{p-1} f_*\big(|T|\big)(t)\, dt.$$

Thus we obtain (9). $\square$

Theorem 1. *For all $t > 0$ and $T \in L^1(\mathcal{M}; \tau)$, we have*

$$\tau\big(E_{\{x\in[0,\infty):\ MT(x)>t\}}\big(|T|\big)\big) \leqslant \frac{4}{t}\|T\|_1.$$

Proof. Let

$$F_{MT}(t) = \big\{x \in [0,\infty):\ MT(x) > t\big\}.$$

Then from the definition of $MT(x)$, for every $x \in F_{MT}(t)$, there is a $r(x) > 0$, such that

$$\frac{1}{\tau(E_{[x-r(x),x+r(x)]}(|T|))}\tau\big(|T|E_{[x-r(x),x+r(x)]}\big(|T|\big)\big) > t. \tag{10}$$

Take

$$F_n = F_{MT}(t) \cap [0,n], \quad n = 1, 2, 3, \ldots.$$

We apply Theorem A to F_n, to obtain a sequence of intervals $\{[x_k - r(x_k), x_k + r(x_k)]\}$ such that

$$F_n \subset \bigcup_k \big[x_k - r(x_k), x_k + r(x_k)\big] \quad \text{and} \quad \sum_k \chi_{[x_k-r(x_k),x_k+r(x_k)]} \leqslant 4.$$

Notice that every $[x_k - r(x_k), x_k + r(x_k)]$ satisfies (10), so

$$\tau\big(E_{F_n}\big(|T|\big)\big) \leqslant \tau\big(E_{\bigcup_k [x_k-r(x_k),x_k+r(x_k)]}\big(|T|\big)\big)$$

$$\leqslant \sum_k \tau\big(E_{[x_k-r(x_k),x_k+r(x_k)]}\big(|T|\big)\big)$$

$$\leqslant \sum_k \frac{1}{t}\tau\big(|T|E_{[x_k-r(x_k),x_k+r(x_k)]}\big(|T|\big)\big)$$

$$
\begin{aligned}
&= \frac{1}{t}\sum_k \tau\big(|T|E_{[x_k-r(x_k),x_k+r(x_k)]}\big(|T|\big)\big)\\
&= \frac{1}{t}\sum_k \tau\Bigg(\int_0^\infty s\chi_{[x_k-r(x_k),x_k+r(x_k)]}\,dE_s\big(|T|\big)\Bigg)\\
&= \frac{1}{t}\tau\Bigg(\int_0^\infty s\sum_k \chi_{[x_k-r(x_k),x_k+r(x_k)]}\,dE_s\big(|T|\big)\Bigg)\\
&\leqslant \frac{4}{t}\tau\Bigg(\int_0^\infty s\,dE_s\big(|T|\big)\Bigg) = \frac{4}{t}\tau\big(|T|\big) = \frac{4}{t}\|T\|_1,
\end{aligned}
$$

i.e.,

$$\tau\big(E_{F_n}\big(|T|\big)\big) \leqslant \frac{4}{t}\|T\|_1.$$

On the other hand, we have

$$F_1 \subset F_2 \subset \cdots \subset F_n \cdots \quad \text{and} \quad F_{MT}(t) \subset \bigcup_{n=1}^{\infty} F_n.$$

Hence, we get

$$\tau\big(E_{F_{MT(t)}}\big(|T|\big)\big) = \lim_{n\to\infty}\tau\big(E_{F_n}\big(|T|\big)\big) \leqslant \frac{4}{t}\|T\|_1. \qquad \square$$

Lemma 4. *Let $T \in L_{\mathrm{loc}}(\mathcal{M};\tau)$. Then*

$$\tau\big(E_{F_{MT(t)}}\big(|T|\big)\big) \leqslant \frac{8}{t}\tau\big(|T|E_{(\frac{t}{2},+\infty)}\big(|T|\big)\big), \quad \forall t > 0. \tag{11}$$

Proof. We set

$$T_1 = TE_{[0,\frac{t}{2}]}\big(|T|\big), \qquad T_2 = T - T_1.$$

Then since

$$
\begin{aligned}
&\frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[x-r,x+r]}\big(|T|\big)\big)\\
&\quad\leqslant \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[0,\frac{t}{2}]}E_{[x-r,x+r]}\big(|T|\big)\big)\\
&\qquad+ \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{(\frac{t}{2},+\infty)}E_{[x-r,x+r]}\big(|T|\big)\big)\\
&\quad\leqslant \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T_1|E_{[x-r,x+r]}\big(|T|\big)\big)\\
&\qquad+ \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T_2|E_{[x-r,x+r]}\big(|T|\big)\big),
\end{aligned}
$$

it follows that

$$MT(x) \leqslant MT_1(x) + MT_2(x) \leqslant MT_2(x) + \frac{t}{2}.$$

Hence,

$$\begin{aligned}\tau\big(E_{F_{MT(t)}}\big(|T|\big)\big) &\leqslant \tau\big(E_{\{x\in[0,\infty):\ MT_2(x)>\frac{t}{2}\}}\big(|T|\big)\big)\\ &\leqslant \frac{8}{t}\|T_2\|_1 = \frac{8}{t}\tau\big(|T_2|\big) = \frac{8}{t}\tau\big(|T|E_{(\frac{t}{2},+\infty)}\big(|T|\big)\big). \qquad \Box\end{aligned}$$

Theorem 2. *Let* $1 < p < \infty$. *Then there is a constant* $C = C(p) > 0$ *such that*

$$\big\|MT\big(|T|\big)\big\|_p \leqslant C\|T\|_p, \quad \forall T \in L^p(\mathcal{M};\tau). \tag{12}$$

Proof. For $MT(x)$ we use Lemmas 3 and 4 to obtain that

$$\begin{aligned}\big\|MT\big(|T|\big)\big\|_p^p &= \tau\big(MT\big(|T|\big)^p\big) = \tau\left(\int_0^\infty MT(s)^p d\big(E_s\big(|T|\big)\big)\right)\\ &= \int_0^\infty MT(s)^p\, d\tau\big(E_s\big(|T|\big)\big)\\ &= p\int_0^\infty t^{p-1}\tau\big(E_{F_{MT(t)}}\big(|T|\big)\big)\,dt\\ &\leqslant 8p\int_0^\infty t^{p-2}\tau\big(|T|E_{(\frac{t}{2},+\infty)}\big(|T|\big)\big)\,dt\\ &= 8p\int_0^\infty t^{p-2}\left[\int_0^\infty s\chi_{(\frac{t}{2},+\infty)}\,d\tau\big(E_s\big(|T|\big)\big)\right]dt\\ &= 8p\int_0^\infty s\left[\int_0^{2s} t^{p-2}\,dt\right]d\tau\big(E_s\big(|T|\big)\big)\\ &= 8\left(2^{p-1}\frac{p}{p-1}\right)\int_0^\infty s^p\,d\tau\big(E_s\big(|T|\big)\big)\\ &= 8\left(2^{p-1}\frac{p}{p-1}\right)\|T\|_p^p. \qquad \Box\end{aligned}$$

Theorem 3. *For* $T \in L_{\mathrm{loc}}(\mathcal{M};\tau)$, $r > 0$ *define*

$$L_rT(x) = \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[x-r,x+r]}\big(|T|\big)\big)$$

(*let* $\frac{0}{0} = 0$). *Then we have*

(i) $\lim_{r\to 0} L_rT(x) = x\chi_{\sigma(|T|)}(x)$,

(ii) $|T| \leqslant MT\big(|T|\big)$.

Proof. If $x \in \sigma(|T|)$, then for all $r > 0$, we have $E_{[x-r,x+r]}(|T|) \neq 0$. So from

$$\tau\big(|T|E_{[x-r,x+r]}\big(|T|\big)\big) = \int\limits_{x-r}^{x+r} s\, d\tau\big(E_s\big(|T|\big)\big),$$

we obtain

$$x - r \leqslant \frac{1}{\tau(E_{[x-r,x+r]}(|T|))}\tau\big(|T|E_{[x-r,x+r]}\big(|T|\big)\big) \leqslant x + r.$$

Hence,

$$\lim_{r\to 0} L_r T(x) = x.$$

If $x \notin \sigma(|T|)$, then for enough small $r > 0$, we have $E_{[x-r,x+r]}(|T|) = 0$, so that

$$\lim_{r\to 0} L_r T(x) = 0.$$

(ii) It follows from

$$\begin{aligned}
\big(|T|x, x\big) &= \int\limits_{\sigma(|T|)} t\, d\big(E_t\big(|T|\big)x, x\big) \\
&\leqslant \liminf_{r\to 0} \int\limits_{\sigma(|T|)} L_r T(t)\, d\big(E_t\big(|T|\big)x, x\big) \\
&\leqslant \int\limits_{\sigma(|T|)} MT(t)\, d\big(E_t\big(|T|\big)x, x\big) \\
&= \big(MT\big(|T|\big)x, x\big), \quad \forall x \in D(T). \qquad \square
\end{aligned}$$

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