

# Existence and asymptotic behavior of ground states for quasilinear singular equations involving Hardy–Sobolev exponents<sup>☆</sup>

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## Abstract

We study the existence and decaying rate of solutions for the quasilinear problem

$$\begin{cases} -\Delta_p u = \rho(x)f(u) + \frac{\lambda}{|x|^\theta}g(u) & \text{in } \mathbf{R}^N, \\ u > 0 & \text{in } \mathbf{R}^N, \quad u(x) \xrightarrow{|x| \rightarrow \infty} 0, \end{cases}$$

where  $\Delta_p$  stands for the  $p$ -Laplacian operator,  $1 < p < N$ ,  $\rho: \mathbf{R}^N \rightarrow [0, \infty)$  is continuous and not identically zero,  $\lambda \geq 0$  is a parameter,  $|x|$  is the Euclidean norm of  $x$ ,  $0 \leq \theta \leq p$ ,  $f, g: [0, \infty) \rightarrow [0, \infty)$  are continuous and nondecreasing,  $f$  has sublinear growth and the Hardy–Sobolev exponent  $p_\theta^* := p(N - \theta)/(N - p)$  bounds the growth of  $g$ . We deal with variational methods and the lower and upper solutions technique.

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**Keywords:** Quasilinear singular equations; Ground states; Variational methods; Lower–upper solutions

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## 1. Introduction

We study the existence and the decaying rate of solutions for the problem

$$\begin{cases} -\Delta_p u = \rho(x)f(u) + \frac{\lambda}{|x|^\theta}g(u) & \text{in } \mathbf{R}^N, \\ u > 0 & \text{in } \mathbf{R}^N, \quad u(x) \xrightarrow{|x| \rightarrow \infty} 0, \end{cases} \quad (1.1)$$

where  $\Delta_p$  is the  $p$ -Laplacian,  $1 < p < N$ ,  $\rho: \mathbf{R}^N \rightarrow [0, \infty)$  is continuous and not identically zero,  $\lambda \geq 0$  is a parameter,  $|x|$  is the Euclidean norm of  $x$ ,  $0 \leq \theta \leq p$ ,  $f, g: [0, \infty) \rightarrow [0, \infty)$  are continuous and nondecreasing,  $f$  has sublinear growth and the Hardy–Sobolev exponent  $p_\theta^* := p(N - \theta)/(N - p)$  bounds the growth of  $g$ .

Let  $\Omega \subset \mathbf{R}^N$  be a smooth domain. Given  $s \in [1, \infty)$  set  $L_\theta^s(\Omega) := L^s(\Omega, 1/|x|^\theta dx)$  and  $L_\theta^s := L_\theta^s(\mathbf{R}^N)$ . The usual notations  $L^s(\Omega)$  and  $L^s$  are used in the case  $\theta = 0$ , while the corresponding norms are denoted by  $|\cdot|_{s,\Omega}$ ,  $|\cdot|_s$ . Now,  $D_0^{1,p}(\Omega)$  denotes the closure of  $C_0^\infty(\Omega)$  under the norm

$$\|\phi\|^p = \int_\Omega |\nabla \phi|^p dx, \quad \phi \in C_0^\infty(\Omega),$$

$D^{1,p} := D_0^{1,p}(\mathbf{R}^N)$ .  $W_{\text{rad}}^{1,p}(\Omega)$  is the closed subspace of radially symmetric functions of the usual Sobolev space  $W^{1,p}(\Omega)$  when  $\Omega$  is a ball and  $W_{\text{rad}}^{1,p} := W_{\text{rad}}^{1,p}(\mathbf{R}^N)$ . To close this set of notations,  $B_r(0)$  is the ball of radius  $r$  centered at the origin of  $\mathbf{R}^N$ ,  $\omega_N$  is the volume of the unit ball,  $\int := \int_{\mathbf{R}^N}$ ,  $u_+$ ,  $u_-$  are, respectively, the positive and negative parts of a measurable function  $u$ , and  $C_1, C_2, \dots$  will denote positive constants.

By Caffarelli, Kohn and Nirenberg [6], the embedding  $D_0^{1,p}(\Omega) \xrightarrow{\text{cont}} L_\theta^{p_\theta^*}(\Omega)$  has as best constant:

$$S_\theta := \inf \left\{ \frac{\int_\Omega |\nabla u|^p dx}{\left( \int_\Omega \frac{|u|^{p_\theta^*}}{|x|^\theta} dx \right)^{p/p_\theta^*}} \mid u \in D_0^{1,p}(\Omega), u \neq 0 \right\}.$$

Notice that  $p_\theta^* = Np/(N - p) := p^*$  and  $S_0$  is the best constant in the Sobolev inequality. According to Ghoussoub and Yuan [14], if  $0 \leq \theta < p$  and  $\Omega = \mathbf{R}^N$ ,  $S_\theta$  is attained at

$$w_\epsilon(x) = \left( \epsilon(N - \theta) \left( \frac{N - p}{p - 1} \right)^{p-1} \right)^{\frac{N-p}{p(p-\theta)}} \left( \epsilon + |x|^{\frac{p-\theta}{p-1}} \right)^{\frac{p-N}{p-\theta}},$$

where  $\epsilon > 0$  and  $\{w_\epsilon\}$  are the only positive radial solutions of

$$-\Delta_p u = \frac{1}{|x|^\theta} u^{p_\theta^*-1} \quad \text{in } \mathbf{R}^N.$$

As a consequence,

$$\|w_\epsilon\|^p = |w_\epsilon|_{\theta, p_\theta^*}^{p_\theta^*} = S_\theta^{\frac{N-\theta}{p-\theta}}, \quad (1.2)$$

where

$$|w_\epsilon|_{\theta, p_\theta^*} := \left( \int \frac{|w_\epsilon|^{p_\theta^*}}{|x|^\theta} dx \right)^{\frac{1}{p_\theta^*}}.$$

To date, there is a broad literature on the present class of singular problems. Regarding smooth bounded domains  $\Omega \subset \mathbf{R}^N$ , Brézis and Cabré [3] showed that if  $\rho \in L^1(\Omega, \delta_0(x) dx)$ , where  $\delta_0(x) := \text{dist}(x, \partial\Omega)$ , then the problem

$$-\Delta u = \rho(x) + \frac{u^2}{|x|^2} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

has no distribution solution  $u \in L^1_{\text{loc}}(\Omega)$ . On the other hand, Montefusco [20] showed that if  $\rho$  satisfies both  $\rho(x) \xrightarrow{|x| \rightarrow \infty} 0$ ,  $\rho \geq \epsilon_0$  in some open subset of  $\mathbf{R}^N$  for some  $\epsilon_0 > 0$  and  $0 < \lambda < S_p$ , then the equation

$$-\Delta_p u = \rho(x) + \lambda \frac{|u|^{p-2} u}{|x|^p} \quad \text{in } \mathbf{R}^N$$

admits a solution  $u \in D^{1,p}$ . Back to the case of bounded domains, Ghoussoub and Yuan [14] proved that the problem

$$-\Delta_p u = |u|^{q-2} u + \frac{\lambda}{|x|^\theta} |u|^{p_\theta^*-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

admits at least a positive weak solution when  $0 \leq \theta < p$ ,  $p^2 \leq N$ ,  $p < q < p^*$ ,  $\lambda > 0$  and further has infinitely many weak solutions, one of which is positive, when  $\theta = p$ ,  $1 \leq p < N$ ,  $p < q < p^*$  and  $\lambda \in (0, S_p)$ . On the other hand, Chen and Li [9] showed that if the set  $\{x \in \mathbf{R}^N \mid \rho(x) > 0\}$  has positive Lebesgue measure, then the equation

$$-\Delta_p u = \rho(x) |u|^{q-2} u + \frac{\lambda}{|x|^\theta} |u|^{p_\theta^*-2} u \quad \text{in } \mathbf{R}^N,$$

admits an infinite sequence  $\{u_m\}$  of weak solutions with energy  $I$  satisfying  $I(u_m) \nearrow 0$ , provided  $0 \leq \theta < p$ ,  $1 < q < p$ ,  $\lambda$  is small and has an infinite sequence  $\{u_m\}$  of weak solutions with unbounded energy  $I(u_m)$  if  $\theta = p$ ,  $p < q < p^*$  and  $\lambda \in (0, S_p)$ .

The reader is further referred to Smets and Tesei [23], Dupaigne and Nedev [12], Dávila and Dupaigne [10] and their references.

We point out that [9,14,20] employ direct variational methods. In this paper this does not seem to be possible due the presence of the more general nonlinearities  $f$  and  $g$ . We instead use lower and upper solutions whose construction employ variational methods.

The following conditions will be required in some of our results:

$$\begin{aligned} \text{(i)} \quad & f(t) \leq t^q, \quad \text{where } 0 \leq q < p-1, \\ \text{(ii)} \quad & \frac{f(t)}{t^{p-1}} \text{ is decreasing,} \\ \text{(iii)} \quad & \frac{f(t)}{t^{p-1}} \xrightarrow{t \rightarrow 0} \infty, \end{aligned} \tag{1.3}$$

$$g(t) \leq t^{p_\theta^*-1}, \tag{1.4}$$

setting  $\hat{\rho}(r) := \max_{|x|=r} \rho(x)$ , assume that

$$\hat{\rho} \in L^{\hat{\mu}}, \quad \text{where } \hat{\mu} := p^*/(p^* - q - 1). \tag{1.5}$$

Additionally,  $\hat{\rho}$  will be required to satisfy

$$\beta \left[ \frac{(p-q-1)\alpha}{(p_\theta^*-p)} \frac{p_\theta^*-p}{\beta} \right]^{\frac{p_\theta^*-p}{p_\theta^*-q-1}} + \alpha \left[ \frac{(p-q-1)\alpha}{(p_\theta^*-p)} \frac{\alpha}{\beta} \right]^{\frac{q+1-p}{p_\theta^*-q-1}} < \frac{1}{p}, \tag{1.6}$$

where

$$\alpha := \frac{|\hat{\rho}|_{\hat{\mu}}}{(q+1)S_0^{(q+1)/p}}, \quad \beta := \frac{1}{p_\theta^* S_\theta^{p_\theta^*/p}}.$$

Notice that  $\hat{\rho}$  is radially symmetric that is  $\hat{\rho}(x) = \hat{\rho}(|x|)$ ,  $x \in \mathbf{R}^N$ .  
Our main result is

**Theorem 1.1.** Assume (1.3)–(1.6). If in addition, one of the conditions

- (i)  $0 \leq \theta < p$  and  $0 \leq \lambda \leq 1$ ,
- (ii)  $\theta = p$  and  $0 \leq \lambda < S_p$

holds, then there is  $u \in D^{1,p}$  with  $u > 0$  satisfying

$$\int |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \int \left( \rho(x) f(u) + \frac{\lambda}{|x|^\theta} g(u) \right) \phi \, dx, \quad \phi \in D^{1,p}. \quad (1.7)$$

Moreover,

$$\left[ u(x)^{q+1} \int_{B_{|x|}(0)} \rho(y) \, dy + \frac{\lambda \omega_N}{N-\theta} |x|^{N-\theta} u(x)^{p_\theta^*} \right] \text{ is bounded in } \mathbf{R}^N \setminus \{0\}. \quad (1.8)$$

Two key auxiliary results are established below. The first one extends to singular problems a result by Cañada et al. [7].

**Theorem 1.2.** Let  $\Omega \subset \mathbf{R}^N$  be a smooth bounded domain. Assume (1.3)–(1.5) and  $\lambda \geq 0$ . If  $v, \omega \in L_\theta^{p_\theta^*}(\Omega) \cap W^{1,p}(\Omega)$  satisfy  $\omega = 0$  and  $v \geq 0$  on  $\partial\Omega$ ,  $0 \leq \omega \leq v$  in  $\Omega$ ,

$$\int_\Omega |\nabla \omega|^{p-2} \nabla \omega \nabla \phi \, dx \leq \int_\Omega \left( \rho(x) f(\omega) + \frac{\lambda}{|x|^\theta} g(\omega) \right) \phi \, dx \quad (1.9)$$

and

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \phi \, dx \geq \int_\Omega \left( \rho(x) f(v) + \frac{\lambda}{|x|^\theta} g(v) \right) \phi \, dx \quad (1.10)$$

for  $\phi \in D_0^{1,p}(\Omega)$  with  $\phi \geq 0$ , then there is  $u \in D_0^{1,p}(\Omega)$  such that both  $\omega \leq u \leq v$  and

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \int_\Omega \left( \rho(x) f(u) + \frac{\lambda}{|x|^\theta} g(u) \right) \phi \, dx, \quad \phi \in D_0^{1,p}(\Omega). \quad (1.11)$$

The second one aims constructing an upper solution of (1.1). It, in fact, gives existence of a solution in the closed subspace  $D_{\text{rad}}^{1,p}$  of radially symmetric functions of  $D^{1,p}$  for the problem

$$\begin{cases} -\Delta_p v = \hat{\rho}(x) v^q + \frac{\lambda}{|x|^\theta} v^{p_\theta^*-1} & \text{in } \mathbf{R}^N, \\ v > 0 & \text{in } \mathbf{R}^N, \quad v(x) \xrightarrow{|x| \rightarrow \infty} 0. \end{cases} \quad (1.12)$$

**Theorem 1.3.** Assume (1.5)–(1.6). If in addition, one of the conditions

- (i)  $0 \leq \theta < p$  and  $\lambda = 1$ ,
- (ii)  $\theta = p$  and  $0 \leq \lambda < S_p$

holds, then there is  $v \in D_{\text{rad}}^{1,p} \cap C^2(\mathbf{R}^N \setminus \{0\})$  satisfying

$$\int |\nabla v|^{p-2} \nabla v \nabla \phi \, dx = \int \left( \hat{\rho} v^q + \frac{\lambda}{|x|^\theta} v^{p_\theta^*-1} \right) \phi \, dx, \quad \phi \in D_{\text{rad}}^{1,p}, \quad (1.13)$$

$$v > 0 \quad \text{in } \mathbf{R}^N, \quad \nabla v(x) \cdot x < 0 \quad \text{in } \mathbf{R}^N \setminus \{0\}, \quad (1.14)$$

and

$$v(x)^{q+1} \int_{B_{|x|}(0)} \hat{\rho}(y) \, dy + \frac{\lambda \omega_N}{N-\theta} |x|^{N-\theta} v(x)^{p_\theta^*} \leq \|v\|^p, \quad x \in \mathbf{R}^N \setminus \{0\}. \quad (1.15)$$

The main result of this paper as well as its proof were greatly inspired by Brézis and Nirenberg [5] and Ambrosetti et al. [2]. The proof of Theorem 1.1 in fact consists in three steps. In a first one, we construct an upper solution  $v$  of (1.1) with the aid of Theorem 1.3. In a second step we construct a sequence of functions, say  $\{u_k\} \subset D_0^{1,p}(B_k)$ , with  $0 < u_k < v$  in  $B_k$ , where  $B_k := B_k(0)$ , satisfying (1.11). In the last step we pass to the limit in  $k$  getting to a solution of (1.1).

## 2. Proof of Theorem 1.2

Denoting by  $D^{-1,p'}(\Omega)$  the dual space of  $D_0^{1,p}(\Omega)$ , where  $1/p + 1/p' = 1$ , consider  $T : D^{-1,p'}(\Omega) \rightarrow D_0^{1,p}(\Omega)$  defined by  $T(\psi) := u$ , where  $u \in D_0^{1,p}(\Omega)$  is the unique solution of the equation  $-\Delta_p u = \psi$  in  $\Omega$ . As is well known,  $T$  is continuous and monotone nondecreasing. Next, consider the set

$$\mathcal{C} := \{u \in D_0^{1,p}(\Omega) \mid \omega \leq u \leq v\}$$

endowed with the pointwise convergence topology and the mapping  $S$  defined by

$$\langle Su, \phi \rangle := \int_{\Omega} \zeta(x, u) \phi \, dx, \quad u \in \mathcal{C}, \quad \phi \in D_0^{1,p}(\Omega),$$

where

$$\zeta(x, u) := \rho(x) f(u) + \frac{\lambda}{|x|^\theta} g(u). \quad (2.1)$$

We claim that  $S(\mathcal{C}) \subset D^{-1,p'}(\Omega)$  and  $S : \mathcal{C} \rightarrow D^{-1,p'}(\Omega)$  is continuous and nondecreasing. Indeed, take  $u \in \mathcal{C}$ ,  $\phi \in D_0^{1,p}(\Omega)$  and notice that by (1.3)(i) and (1.4),

$$|\langle Su, \phi \rangle| \leq \int_{\Omega} \rho(x) f(u) |\phi| \, dx + \lambda \int_{\Omega} \frac{u^{p_\theta^*-1}}{|x|^{\frac{\theta}{p_\theta^*}-1}} \frac{|\phi|}{|x|^{\frac{\theta}{p_\theta^*}}} \, dx.$$

Remarking that  $f(u) \leq 1$  when  $q = 0$ , we have by first applying Hölder's inequality in the integrals above and subsequently applying the Sobolev and Hardy–Sobolev inequalities that

$$\begin{aligned}
|\langle Su, \phi \rangle| &\leq \int_{\Omega} \rho(x) f(u) |\phi| dx + \lambda \left( \int_{\Omega} \frac{u^{p_{\theta}^*}}{|x|^{\theta}} \right)^{\frac{p_{\theta}^*-1}{p_{\theta}^*}} \left( \int_{\Omega} \frac{|\phi|^{p_{\theta}^*}}{|x|^{\theta}} \right)^{\frac{1}{p_{\theta}^*}} \\
&\leq C_1 \|u\|^q \|\phi\| + C_2 \|u\|^{p_{\theta}^*-1} \|\phi\|,
\end{aligned}$$

showing that  $S(\mathcal{C}) \subset D^{-1,p'}(\Omega)$  and in addition  $S$  maps  $D_0^{1,p}(\Omega)$ -bounded subsets of  $\mathcal{C}$  into bounded subsets of  $D^{-1,p'}(\Omega)$ .

To show that  $S$  is continuous let  $\{u_n\}$  be a sequence in  $\mathcal{C}$  such that  $u_n \rightarrow u$  a.e. in  $\Omega$  for some  $u \in \mathcal{C}$ . If  $\phi \in D_0^{1,p}(\Omega)$ , we have

$$|\langle Su_n - Su, \phi \rangle| \leq \int_{\Omega} \rho(x) |f(u_n) - f(u)| |\phi| dx + \lambda \int_{\Omega} \frac{1}{|x|^{\theta}} |g(u_n) - g(u)| |\phi| dx. \quad (2.2)$$

By (1.3)(i), (1.4), the definition  $\mathcal{C}$ , the Hölder and Hardy–Sobolev inequalities and the Lebesgue theorem,

$$|\langle Su_n - Su, \phi \rangle| \leq o_n(1) \|\phi\|, \quad \text{where } o_n(1) \xrightarrow{n} 0. \quad (2.3)$$

Hence  $\|Su_n - Su\|_{D^{-1,p'}(\Omega)} \rightarrow 0$ , showing that  $S$  is continuous.

To show that  $S$  is nondecreasing notice that if  $u_1 \leq u_2$  and  $\phi \in D_0^{1,p}(\Omega)$  is nonnegative,

$$\langle Su_1, \phi \rangle = \int_{\Omega} \zeta(x, u_1) \phi dx \leq \int_{\Omega} \zeta(x, u_2) \phi dx \leq \langle Su_2, \phi \rangle.$$

Next let  $F: \mathcal{C} \rightarrow D_0^{1,p}(\Omega)$  given by  $F(u) := T(Su)$ . Notice that  $u \in \mathcal{C}$  satisfies (1.11) if and only if  $u = F(u)$  that is  $u = T(Su)$ . In order to find a fixed point  $u$  of  $F$  consider the sequence  $\omega_{n+1} := F(\omega_n)$ , where  $\omega_1 := F(\omega)$ .

We claim that

$$\omega \leq \omega_1 \leq \dots \leq \omega_n \leq \dots \leq v.$$

Indeed, taking  $\phi \in D_0^{1,p}(\Omega)$  with  $\phi \geq 0$ , we notice that

$$\int_{\Omega} |\nabla \omega|^{p-2} \nabla \omega \nabla \phi dx \leq \int_{\Omega} \zeta(x, \omega) \phi dx = \int_{\Omega} |\nabla \omega_1|^{p-2} \nabla \omega_1 \nabla \phi dx.$$

So the comparison principle for  $\Delta_p$  gives  $\omega \leq \omega_1$ . Iterating this argument, we have  $\omega_1 \leq \omega_2 \leq \dots \leq \omega_n \leq \dots$ . By a similar argument  $w_n \leq v$  and the claim follows. Next noticing that  $\omega_n \rightarrow u$  pointwisely, one has of course that  $\omega \leq u \leq v$ . We claim that  $u \in \mathcal{C}$  and  $u = F(u)$ . Indeed, estimating as we did in (2.2) to get to (2.3), we find a constant  $C_{\omega,v} > 0$  such that

$$\|S\omega_n - S\omega_m\|_{D^{-1,p'}(\Omega)} \leq C_{\omega,v} |\omega_n - \omega_m|_{L_{\theta}^{p_{\theta}^*}(\Omega)}.$$

On the other hand, reminding that  $\{\omega_n\} \subset \mathcal{C}$  we find by applying Lebesgue's theorem that  $|\omega_n - \omega_m|_{L_{\theta}^{p_{\theta}^*}(\Omega)} \rightarrow 0$ . Therefore,  $S\omega_n$  is a Cauchy sequence in  $D^{-1,p'}(\Omega)$  and thus  $S\omega_n \rightarrow \bar{u}$

for some  $\bar{u} \in D^{-1,p'}(\Omega)$ . By the continuity of  $T$ ,  $T(S\omega_n) \rightarrow T(\bar{u})$  in  $D_0^{1,p}(\Omega)$ . Since  $\omega_{n+1} = F(\omega_n) = T(S\omega_n)$ , we get

$$\omega_{n+1} \xrightarrow{D_0^{1,p}(\Omega)} T(\bar{u}).$$

Therefore,  $\omega_{n+1} \rightarrow T(\bar{u})$  pointwisely showing that  $u = T(\bar{u})$  and so  $u \in \mathcal{C}$ . To end the verification of the claim notice that

$$u = T(\bar{u}) = \lim \omega_{n+1} = \lim F(\omega_n) = F(T(\bar{u})) = F(u).$$

This proves Theorem 1.2.

### 3. Proof of Theorem 1.3

The energy functional associated with (1.12), namely

$$I(v) = \frac{1}{p} \int |\nabla v|^p dx - \frac{1}{q+1} \int \hat{\rho}(x) v_+^{q+1} dx - \frac{\lambda}{p_\theta^*} \int \frac{v_+^{p_\theta^*}}{|x|^\theta} dx, \quad v \in D_{\text{rad}}^{1,p},$$

belongs to  $C^1(D_{\text{rad}}^{1,p}, \mathbf{R})$  and

$$\langle I'(v), \phi \rangle = \int |\nabla v|^{p-2} \nabla v \nabla \phi dx - \int \hat{\rho} v_+^q \phi dx - \lambda \int \frac{v_+^{p_\theta^*-1}}{|x|^\theta} \phi dx, \quad \phi \in D_{\text{rad}}^{1,p}.$$

In the case  $0 \leq \theta < p$ , we shall apply the technique by Brézis and Nirenberg [5] and, as a matter of fact, arguments in Alves and Goncalves [1]. This will be accomplished through the use of the Hardy–Sobolev inequality. In this regard, given  $e \in D_{\text{rad}}^{1,p}$  let

$$c := \inf_{p \in \mathcal{P}} \max_{0 \leq t \leq 1} I(p(t)),$$

where

$$\mathcal{P} := \{p \in C([0, 1], D_{\text{rad}}^{1,p}) \mid p(0) = 0, p(1) = e\}.$$

In the case  $\theta = p$  minimization arguments will be explored.

In both cases we will make use of the following result on the concentration–compactness principle (limit case) of Lions [18,19], see also Montefusco [20] and Smets [22].

**Lemma 3.1.** Assume  $0 \leq \theta \leq p$  and let  $v_n \in D_{\text{rad}}^{1,p}$  be a sequence with  $v_n \xrightarrow{D_{\text{rad}}^{1,p}} v$ . Then

$$v_{n+} \xrightarrow{D_{\text{rad}}^{1,p}} v_+, \quad |\nabla v_n|^p dx \rightharpoonup \mu, \quad \frac{v_{n+}^{p_\theta^*}}{|x|^\theta} dx \rightharpoonup \nu$$

for some Radon measures  $\mu$  and  $\nu$ . In addition, there are an at most denumerable set  $J_\theta$ , a subset  $\{x_j\}_{j \in J_\theta} \subset \mathbf{R}^N$  and positive numbers  $\mu_j, \nu_j$  such that

$$\begin{aligned} \nu &= \frac{v_+^{p_\theta^*}}{|x|^\theta} dx + \sum_{j \in J_\theta} \delta_{x_j} \nu_j, \\ \mu &\geq |\nabla v|^p dx + \sum_{j \in J_\theta} \delta_{x_j} \mu_j, \\ \mu_j &\geq S_\theta v_j^{p/p_\theta^*}, \end{aligned}$$

where  $\delta_{x_j}$  means the Dirac mass at  $x_j$ . If  $0 < \theta \leq p$  then  $x_j = 0$  and  $J_\theta$  is a singleton.

The lemma below establishes the Mountain Pass Geometry in the case  $0 \leq \theta < p$ , gives an estimate to the critical level  $c$  and establishes that  $I$  is coercive when  $\theta = p$ .

**Lemma 3.2.** Assume (1.5)–(1.6). If  $0 \leq \theta < p$  and  $\lambda = 1$ , then there are  $\eta, \hat{v} > 0$  and  $e \in D_{\text{rad}}^{1,p}$  such that

$$I(v) \geq \eta, \quad \|v\| = \hat{v}, \quad (3.1)$$

$$\|e\| > \hat{v}, \quad I(e) \leq 0, \quad (3.2)$$

and furthermore,

$$0 < c < \frac{p - \theta}{p(N - \theta)} S_{\theta}^{(N - \theta)/(p - \theta)}. \quad (3.3)$$

If, on the other hand,  $\theta = p$  and  $0 \leq \lambda < S_p$ , then

$$I \text{ is coercive and } I(\phi_p) < 0 \text{ for some } \phi_p \in D_{\text{rad}}^{1,p}.$$

**Proof.** If  $0 \leq \theta < p$  let  $v \in D_{\text{rad}}^{1,p}$ . We have, firstly, by (1.5) and Hölder's inequality,

$$\int \hat{\rho} v_+^{q+1} dx \leq \left( \int \hat{\rho}^{\hat{\mu}} \right)^{1/\hat{\mu}} \left( \int |v|^{p^*} \right)^{(q+1)/p^*} \quad (3.4)$$

and secondly, by the Hardy–Sobolev inequality,

$$\left( \int |v|^{p^*} \right)^{(q+1)/p^*} \leq S_0^{-(q+1)/p} \|v\|^{q+1} \quad \text{and} \quad \int \frac{|v_+|^{p_{\theta}^*}}{|x|^{\theta}} \leq S_{\theta}^{-p_{\theta}^*/p} \|v\|^{p_{\theta}^*}. \quad (3.5)$$

Now, applying (3.4) and (3.5) we find

$$\begin{aligned} I(v) &\geq \frac{1}{p} \|v\|^p - \frac{|\hat{\rho}|_{\hat{\mu}}}{(q+1)S_0^{(q+1)/p}} \|v\|^{q+1} - \frac{1}{p_{\theta}^* S_{\theta}^{p_{\theta}^*/p}} \|v\|^{p_{\theta}^*} \\ &= \|v\|^p \left( \frac{1}{p} - \alpha \|v\|^{(q+1-p)} - \beta \|v\|^{(p_{\theta}^*-p)} \right). \end{aligned}$$

Using (1.6), (3.1) follows. Next (3.2) and (3.3) will be shown. In order to show (3.2), take  $w_{\epsilon}$  as in (1.2). Picking  $t := t_0 > 0$  large enough in the expression below,

$$I(tw_{\epsilon}) = \frac{t^p}{p} \int |\nabla w_{\epsilon}|^p dx - \frac{t^{q+1}}{q+1} \int \hat{\rho} w_{\epsilon}^{q+1} dx - \frac{t^{p_{\theta}^*}}{p_{\theta}^*} \int \frac{w_{\epsilon}^{p_{\theta}^*}}{|x|^{\theta}} dx, \quad t \geq 0,$$

and setting  $e := t_0 w_{\epsilon}$ , we get  $I(e) \leq 0$ . Next we infer by (1.2) and adapting arguments in Alves and Gonçalves [1] that

$$\max_{t \geq 0} I(tw_{\epsilon}) < \left( \frac{1}{p} - \frac{1}{p_{\theta}^*} \right) S_{\theta}^{(N - \theta)/(p - \theta)}.$$

Reminding the definition of  $c$ , we get (3.3). Now, if  $\theta = p$ , observing that

$$|v_+|^p \leq |v|^p \quad \text{and} \quad \int_{\Omega} \frac{v_+^p}{|x|^p} dx \leq S_p^{-1} \int_{\Omega} |\nabla v|^p dx,$$

we get to the following estimate:



$$\begin{aligned} I(v) &\geq \frac{1}{p} \|v\|^p - \frac{C_3}{q+1} |\hat{\rho}|_{\hat{\mu}} \|v\|^{q+1} - \frac{\lambda}{p S_p} \|v\|^p \\ &= \frac{1}{p} \left( 1 - \frac{\lambda}{S_p} \right) \|v\|^p - \frac{C_3}{q+1} |\hat{\rho}|_{\hat{\mu}} \|v\|^{q+1}, \end{aligned}$$

which shows that  $I$  is coercive. Now, choosing  $\phi \in D_{\text{rad}}^{1,p}$  with  $\phi > 0$  and taking  $t > 0$ ,

$$I(t\phi) = \frac{t^p}{p} \left( \int |\nabla \phi|^p dx - \lambda \int \frac{\phi^p}{|x|^p} dx \right) - \frac{t^{q+1}}{q+1} \int \hat{\rho} \phi^{q+1} dx.$$

Since  $\lambda \in [0, S_p)$ , the second term in the inequality above is positive and so  $I(t_0\phi) < 0$  for some  $t_0 > 0$ . Setting  $\phi_p := t_0\phi$  ends the proof.  $\square$

**Lemma 3.3.** Assume (1.5). If  $\theta = p$  and  $\lambda \in [0, S_p)$ , then  $I : D_{\text{rad}}^{1,p} \rightarrow \mathbf{R}$  is weakly sequentially lower semicontinuous.

**Proof.** Let  $v_n \xrightarrow{D_{\text{rad}}^{1,p}} v$  so that  $v_{n+} \xrightarrow{D_{\text{rad}}^{1,p}} v_+$ . We point out that Lemma 3.1 applies to the sequence  $v_{n+}$ . Using the fact that  $\hat{\rho} \in L^{\hat{\mu}} = (L^{p/(q+1)})'$ , we find

$$\begin{aligned} \lim_n I(v_n) &= -\overline{\lim}_n (-I(v_n)) \\ &\geq -\frac{1}{p} \overline{\lim}_n \left( -\int |\nabla v_n|^p dx \right) - \frac{1}{q+1} \overline{\lim}_n \int \hat{\rho} v_{n+}^{q+1} dx - \frac{\lambda}{p} \overline{\lim}_n \int \frac{v_{n+}^p}{|x|^p} dx \\ &= \frac{1}{p} \lim_n \int |\nabla v_n|^p dx - \frac{1}{q+1} \int \hat{\rho} v_+^{q+1} dx - \frac{\lambda}{p} \lim_n \int \frac{v_{n+}^p}{|x|^p} dx. \end{aligned} \quad (3.6)$$

It follows by Evans [13, Theorem 3], Lemma 3.1, properties of Radon measures and the fact that  $J_p$  is a singleton, say  $J_p = \{o\}$  that

$$\lim_n \int |\nabla v_n|^p dx \geq \int |\nabla v|^p dx + \mu_o \quad \text{and} \quad \overline{\lim}_n \int \frac{v_{n+}^p}{|x|^p} dx \leq \int \frac{v_+^p}{|x|^p} dx + \nu_o.$$

Taking these into (3.6), using Lemma 3.1 and the assumption  $\lambda \in [0, S_p)$ , we get

$$\begin{aligned} \lim_n I(v_n) &\geq \frac{1}{p} \int |\nabla v|^p dx - \frac{1}{q+1} \int \hat{\rho} v_+^{q+1} dx - \frac{\lambda}{p} \int \frac{v_+^p}{|x|^p} dx + \frac{1}{p} (\mu_o - \lambda \nu_o) \\ &\geq I(v) + \frac{1}{p} \left( 1 - \frac{\lambda}{S_p} \right) \mu_o \\ &\geq I(v). \end{aligned}$$

Thus  $I$  is weakly sequentially lower semicontinuous, proving Lemma 3.3.  $\square$

**Remark 1.** (i)  $0 \leq \theta < p$ . Using Lemma 3.2 and the Mountain Pass Theorem there is a sequence  $v_n \in D_{\text{rad}}^{1,p}$  satisfying

$$I(v_n) \rightarrow c \geq \eta \quad \text{and} \quad I'(v_n) \rightarrow 0.$$

(ii)  $\theta = p$ . Using Lemmas 3.2 and 3.3 there is  $v \in D_{\text{rad}}^{1,p}$  such that  $I(v) = \min_{D_{\text{rad}}^{1,p}} I$  and  $I(v) < 0$ . Thus  $v \neq 0$  and verifies  $I'(v) = 0$  because  $I$  is a  $C^1$ -functional.

**Lemma 3.4.** Assume  $0 \leq \theta < p$ . Then there is  $v \in D_{\text{rad}}^{1,p}$  such that  $v \geq 0$ ,

$$v_n \xrightarrow{D_{\text{rad}}^{1,p}} v \quad \text{and} \quad v_n \xrightarrow{\text{a.e.}} v.$$

**Proof.** Remarking that  $p < p_\theta^*$ , we have

$$\begin{aligned} I(v_n) - \frac{1}{p_\theta^*} \langle I'(v_n), v_n \rangle &= \left( \frac{1}{p} - \frac{1}{p_\theta^*} \right) \|v_n\|^p + \left( \frac{1}{p_\theta^*} - \frac{1}{q+1} \right) \int \hat{\rho} v_n^{q+1} \\ &\geq \left( \frac{1}{p} - \frac{1}{p_\theta^*} \right) \|v_n\|^p + \left( \frac{1}{p_\theta^*} - \frac{1}{q+1} \right) S_\theta^{-(q+1)/p} |\hat{\rho}|_{\hat{\mu}} \|v_n\|^{q+1}. \end{aligned}$$

Taking  $n$  large enough gives

$$\left( \frac{1}{p} - \frac{1}{p_\theta^*} \right) \|v_n\|^p + \left( \frac{1}{p_\theta^*} - \frac{1}{q+1} \right) S_\theta^{-(q+1)/p} |\hat{\rho}|_{\hat{\mu}} \|v_n\|^{q+1} \leq \|v_n\| + C_4,$$

showing that  $v_n$  is bounded in  $D_{\text{rad}}^{1,p}$ . As a consequence there is some  $v \in D_{\text{rad}}^{1,p}$  such that

$$v_n \xrightarrow{D_{\text{rad}}^{1,p}} v. \text{ Now, since}$$

$$\langle I'(v_n), v_n \rangle = o_n(1),$$

we infer that  $\|v_n\| \rightarrow 0$ . Now since  $1 < p < N$ , by the Rellich–Kondrachov theorem,  $D_{\text{rad}}^{1,p}(B) \xrightarrow{\text{compactly}} L^q(B)$  for each ball  $B \subset \mathbf{R}^N$  and for some  $q > 1$ . By a diagonal argument one infers (eventually passing to a further subsequence) that  $v_n \xrightarrow{\text{a.e.}} v$ . As a consequence  $v \geq 0$ .  $\square$

**Lemma 3.5.**  $\nabla v_n \xrightarrow{\text{a.e.}} \nabla v$ .

The proof of Lemma 3.5 is quite technical and will be left to the end of this section.

**Remark 2.** Due to the fact that  $\|v_n\| \rightarrow 0$  and Lemma 3.5, we will assume on the proof of Theorem 1.3 below that  $v_n \geq 0$ .

We establish below Lemma 3.6. It slightly improves a remark by Brézis and Lieb [4] and will be used, for instance, in the proof of Theorem 1.3. Its proof is similar to that of [17, Lemma 4.8] and will be omitted.

**Lemma 3.6.** Let  $\Omega$  be an open subset of  $\mathbf{R}^N$  and let  $u_n$  be a bounded sequence in  $L_\theta^s(\Omega)$  where  $1 < s < \infty$  and  $0 \leq \theta \leq p$ . If in addition,  $u_n \xrightarrow{\text{a.e.}} u$  for some  $u \in L_\theta^s(\Omega)$ , then  $u_n \rightharpoonup u$  in  $L_\theta^s(\Omega)$ .

**Proof of Theorem 1.3 (Completed).** Regarding case (i) we have,  $v_n \xrightarrow{D_{\text{rad}}^{1,p}} v$  and hence  $|\nabla v_n|^{p-2} \nabla v_n$  is bounded in  $(L^{p/(p-1)})^N$ . Assuming first that  $0 < q < p-1$  and using the Sobolev and Hardy–Sobolev inequalities it follows that  $v_n^q$  and  $v_n^{p_\theta^*-1}$  are bounded in  $L^{p^*/q}$  and  $L_\theta^{p_\theta^*/(p_\theta^*-1)}$ , respectively. By Lemma 3.4,  $v_n \xrightarrow{\text{a.e.}} v$  and by Lemma 3.5,  $\nabla v_n \xrightarrow{\text{a.e.}} \nabla v$ . By Lemma 3.6,

$$\begin{aligned} |\nabla v_n|^{p-2} \nabla v_n &\rightharpoonup |\nabla v|^{p-2} \nabla v \quad \text{in } (L^{p/(p-1)})^N, \\ v_n^{p_\theta^*-1} &\rightharpoonup v^{p_\theta^*-1} \quad \text{in } L_\theta^{p_\theta^*/(p_\theta^*-1)} \quad \text{and} \quad v_n^q \rightharpoonup v^q \quad \text{in } L^{p^*/q}, \end{aligned}$$

which means that for  $\phi \in D_{\text{rad}}^{1,p}$ ,

$$\begin{aligned} \int |\nabla v_n|^{p-2} \nabla v_n \nabla \phi \, dx &\rightarrow \int |\nabla v|^{p-2} \nabla v \nabla \phi \, dx, \\ \int \frac{v_n^{p_\theta^*-1}}{|x|^\theta} \phi \, dx &\rightarrow \int \frac{v^{p_\theta^*-1}}{|x|^\theta} \phi \, dx \quad \text{and} \quad \int \hat{\rho} v_n^q \phi \, dx \rightarrow \int \hat{\rho} v^q \phi \, dx. \end{aligned}$$

If  $q = 0$ , the last convergence holds true, of course. Thus  $\langle I'(v_n), \phi \rangle \rightarrow \langle I'(v), \phi \rangle$ . As for the case (ii),  $v$  is also a solution because it is a minimizer of  $I$ . As a consequence, in both cases,

$$\int |\nabla v|^{p-2} \nabla v \nabla \phi \, dx = \int \left[ \hat{\rho} v^q + \lambda \frac{v^{p_\theta^*-1}}{|x|^\theta} \right] \phi \, dx, \quad \phi \in D_{\text{rad}}^{1,p}, \quad (3.7)$$

that is,  $v$  is a weak solution of the equation in (1.12).

We claim that  $v \neq 0$ . In case (i), assume by the way of contradiction, that  $v = 0$ . By the boundedness of  $v_n$ ,

$$\int |\nabla v_n|^p \, dx \rightarrow \ell \geq 0.$$

But from the fact that  $I'(v_n) \rightarrow 0$ ,

$$\int |\nabla v_n|^p = \int \left[ \hat{\rho} v_n^{q+1} + \frac{v_n^{p_\theta^*}}{|x|^\theta} \right] dx + o_n(1). \quad (3.8)$$

Arguing as we have done before and using (3.8),

$$\int \hat{\rho} v_n^{q+1} \rightarrow 0 \quad \text{and} \quad \int \frac{v_n^{p_\theta^*}}{|x|^\theta} dx \rightarrow \ell.$$

Now using the definition of  $\ell$  and the fact that  $I(v_n) \rightarrow c$ ,

$$\left( \frac{1}{p} - \frac{1}{p_\theta^*} \right) \ell = c, \quad (3.9)$$

which gives  $\ell > 0$  since  $\theta \in [0, p)$ . On the other hand, passing to the limit in

$$\int |\nabla v_n|^p \geq S_\theta \left( \int \frac{v_n^{p_\theta^*}}{|x|^\theta} dx \right)^{p/p_\theta^*}$$

leads to  $\ell \geq S_\theta \ell^{p/p_\theta^*}$  which in turn gives

$$\ell \geq S_\theta^{(N-\theta)/(p-\theta)}. \quad (3.10)$$

Thus, by (3.9)–(3.10),

$$c = \frac{p-\theta}{p(N-\theta)} \ell \geq \frac{p-\theta}{p(N-\theta)} S_\theta^{(N-\theta)/(p-\theta)},$$

a contradiction. So  $v \neq 0$ . Concerning case (ii), by Remark 1,  $I(v) < 0$  and so  $v \neq 0$  as well. Moreover, by construction,  $v \geq 0$  in both cases.

For each  $\epsilon$ ,  $r > 0$ , consider the function

$$v_{r,\epsilon}(t) := \begin{cases} 1 & \text{if } 0 \leq t \leq r, \\ \text{linear} & \text{if } r \leq t \leq r + \epsilon, \\ 0 & \text{if } t \geq r + \epsilon. \end{cases}$$

Set  $\phi(x) := v_{r,\epsilon}(|x|)$  so that  $\phi \in D_{\text{rad}}^{1,p}$ . Replacing  $\phi$  in (3.7), we get

$$\begin{aligned} \frac{-1}{\epsilon} \int_r^{r+\epsilon} t^{N-1} |v'|^{p-2} v' dt &= \int_0^r t^{N-1} (\hat{\rho} v^q + \lambda t^{-\theta} v^{p_\theta^*-1}) dt \\ &+ \int_r^{r+\epsilon} t^{N-1} (\hat{\rho} v^q + \lambda t^{-\theta} v^{p_\theta^*-1}) v_{r,\epsilon} dt. \end{aligned}$$

Making  $\epsilon \rightarrow 0$  gives

$$-r^{N-1} |v'(r)|^{p-2} v'(r) = \int_0^r (t^{N-1} \hat{\rho} v^q + \lambda t^{N-\theta-1} v^{p_\theta^*-1}) dt, \quad (3.11)$$

which leads to  $v' \leq 0$ . But this and the facts that  $v \neq 0$  and  $v \geq 0$  already shown give  $v' < 0$  in  $(0, \infty)$  and  $v > 0$  in  $(0, \infty)$ . From (3.11),

$$(-v'(r))^{p-1} = r^{1-N} \int_0^r (t^{N-1} \hat{\rho} v^q + \lambda t^{N-\theta-1} v^{p_\theta^*-1}) dt \quad (3.12)$$

so that  $v \in C^2((0, \infty))$ . At this point, we have

$$v \in D_{\text{rad}}^{1,p} \cap C^2(\mathbf{R}^N \setminus \{0\}) \quad \text{and both} \quad v > 0 \quad \text{and} \quad \nabla v(x) \cdot x < 0 \quad \text{in} \quad \mathbf{R}^N \setminus \{0\}, \quad (3.13)$$

which lead to (1.14). Setting  $\phi = v$  in (3.7) and using radial symmetry, we have

$$\begin{aligned} \|v\|^p &= \omega_N \int_0^\infty t^{N-1} (\hat{\rho}(t) v(t)^{q+1} + \lambda t^{-\theta} v(t)^{p_\theta^*}) dt \\ &\geq \omega_N v(r)^{q+1} \int_0^r t^{N-1} \hat{\rho}(t) dt + \frac{\lambda \omega_N}{(N-\theta)} r^{N-\theta} v(r)^{p_\theta^*} \\ &= v(x)^{q+1} \int_{B_{|x|}(0)} \hat{\rho}(y) dy + \frac{\lambda \omega_N}{(N-\theta)} |x|^{N-\theta} v(x)^{p_\theta^*}. \end{aligned} \quad (3.14)$$

Theorem 1.3 is proved.  $\square$

It remains to show Lemma 3.5.

**Proof of Lemma 3.5.** To begin with we claim that either  $J_\theta$  is a singleton or  $J_\theta = \emptyset$ . (We write  $J_\theta = \{o\}$  in the first case.) Indeed, if  $0 < \theta \leq p$  it follows by Lemma 3.1 that both  $J_\theta = \{o\}$  and  $x_0 = 0$ . If on the other hand  $\theta = 0$ , by Lemma 3.1,

$$\int v_{n+}^{p^*} \phi dx \rightarrow \int v^{p^*} \phi dx + \sum_{j \in J_0} v_j \langle \delta_{x_j}, \phi \rangle, \quad \phi \in C_{0,\text{rad}}^\infty. \quad (3.15)$$

If for some  $j \in J_0$ ,  $x_j \neq 0$ , choose a ball  $B := B(x_j)$  which does not contain 0,  $\phi \in C_{0,\text{rad}}^\infty$ , with  $\phi \geq 0$  in  $B$ ,  $\phi > 0$  in a smaller ball say  $B'$  centered at  $x_j$  and  $\phi = 0$  on  $B \setminus B'$ . Now observing that

$$|\nabla(v_{n+}\phi)|^p \leq 2^p(|\phi \nabla v_{n+}|^p + |v_{n+} \nabla \phi|^p) \quad \text{a.e. in } \mathbf{R}^N,$$

and using the embedding  $D_{\text{rad}}^{1,p} \xrightarrow{\text{cont}} L^p(B)$ , we have the following inequalities:

$$\int_B |v_{n+} \nabla \phi|^p dx \leq C_5 \int_B |v_n|^p dx, \quad \int_B |\phi \nabla v_{n+}|^p dx \leq C_6 \|v_n\|^p.$$

These inequalities show that the sequence  $\tilde{v}_n := v_{n+}\phi$  is bounded in  $W^{1,p}(B)$ . In fact,  $\tilde{v}_n \rightarrow \tilde{v}$  a.e. in  $\mathbf{R}^N$  with  $\tilde{v} = v\phi$ .

Using the fact that there is a positive constant  $C(N, p)$  such that for all  $u \in W_{\text{rad}}^{1,p}$ ,

$$|u(x)| \leq C(N, p)|x|^{-\frac{(N-1)}{p}}(|\nabla u|_p + |u|_p), \quad |x| \geq \epsilon,$$

where  $\epsilon > 0$ , we get

$$\begin{aligned} |(v_{n+}\phi)^{p^*}(x) - (v\phi)^{p^*}(x)| &\leq C_7(|\tilde{v}_n|_{W_{\text{rad}}^{1,p}}^{p^*} + |\tilde{v}|_{W_{\text{rad}}^{1,p}}^{p^*})|x|^{-\frac{(N-1)}{p}} \\ &\leq C_8|x|^{-\frac{(N-1)}{p}} := g(x). \end{aligned}$$

Notice that  $g \in L^1(B(x_j))$  because  $\{0\} \cap B(x_j) = \emptyset$ . By Lebesgue's theorem,

$$\int_B v_{n+}^{p^*} \phi^{p^*} dx \rightarrow \int_B v^{p^*} \phi^{p^*} dx.$$

It follows as a consequence of this and (3.15) that  $\sum_{j \in J_0} v_j \langle \delta_{x_j}, \phi^{p^*} \rangle = 0$ , impossible. Therefore,  $x_j = 0$  for  $j \in J_0$ . Next we consider two cases:

**Case 1.**  $J_\theta = \{o\}$ . Given  $\delta > 0$  consider the set  $A_\delta := B_{1/2\delta} \setminus B_\delta$  and take  $\rho \in (0, \delta)$ . We claim that

$$\int_{A_\rho} (|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v)(\nabla v_n - \nabla v) dx \rightarrow 0. \quad (3.16)$$

Indeed, let  $\phi \in C_0^\infty$  such that  $\phi = 1$  in  $B_{1/2}$ ,  $\phi = 0$  in  $B_1^c$  and  $0 \leq \phi \leq 1$ . Set

$$\psi_\epsilon(x) := \phi(\epsilon x) - \phi(x/\epsilon), \quad x \in \mathbf{R}^N,$$

where  $\epsilon \in (0, \rho)$  and notice that

$$\psi_\epsilon(x) = \begin{cases} 0 & \text{if } x \in B_{\epsilon/2}, \\ 1 & \text{if } x \in A_\epsilon. \end{cases}$$

By Simon [21], there is a constant  $C_p > 0$  such that

$$\langle |x|^{p-2}x - |y|^{p-2}y, x - y \rangle \geq \begin{cases} C_p|x - y|^p & \text{if } p \geq 2, \\ C_p \frac{|x-y|^2}{(1+|x|+|y|)^{2-p}} & \text{if } 0 < p < 1, \end{cases} \quad (3.17)$$

for all  $x, y \in \mathbf{R}^N$ . Applying (3.17), we get

$$(|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v)(\nabla v_n - \nabla v) \geq 0 \quad \text{a.e. in } \mathbf{R}^N.$$

Taking into account that  $A_\rho \subset A_\epsilon$ , we have

$$\begin{aligned} & \int_{A_\rho} (|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v)(\nabla v_n - \nabla v) dx \\ & \leq \int_{A_\epsilon} (|\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v)(\nabla v_n - \nabla v) \psi_\epsilon dx := \Upsilon_{n,\epsilon}. \end{aligned}$$

Employing arguments similar to those in Jianfu and Xiping [16], one shows that

$$\lim_{\epsilon \rightarrow 0} \lim_n \Upsilon_{n,\epsilon} = 0. \quad (3.18)$$

By (3.17) and (3.18),  $\nabla v_n \xrightarrow{\text{a.e.}} \nabla v$ .

**Case 2.**  $J_\theta = \emptyset$ . In this case  $\psi_\epsilon(x) := \phi(\epsilon x)$  and  $A_\rho = B_{1/2\rho}$ . Arguments similar to the ones above apply, showing (3.16) in this case as well. It follows as above that  $\nabla v_n \xrightarrow{\text{a.e.}} \nabla v$ .  $\square$

#### 4. Proof of Theorem 1.1

The lower and upper solutions technique will be applied.

*Construction of an upper solution of (1.1).* Let  $v$  as in Theorem 1.3. Since  $v(x) = v(r)$  and  $v \in C^2((0, \infty))$ , it follows by (3.11) that

$$-(r^{N-1} |v'(r)|^{p-2} v(r)')' = r^{N-1} \left( \hat{\rho} v(r)^q + \frac{\lambda v^{p_\theta^*-1}}{r^\theta} \right), \quad r > 0,$$

which shows in both cases (i) and (ii) that

$$-\Delta_p v = \hat{\rho} v^q + \frac{\lambda v^{p_\theta^*-1}}{|x|^\theta} \quad \text{in } \mathbf{R}^N \setminus \{0\} \quad (4.1)$$

in the classical sense. Multiplying (4.1) by  $\phi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$  and integrating gives

$$\int |\nabla v|^{p-2} \nabla v \nabla \phi dx = \int \left( \hat{\rho}(x) v^q + \frac{\lambda}{|x|^\theta} v^{p_\theta^*-1} \right) \phi dx. \quad (4.2)$$

Pick  $\epsilon > 0$  and a  $C^\infty$ -function  $\eta$  with  $0 \leq \eta \leq 1$ ,  $\eta(x) = 0$  if  $|x| \leq 1$  and  $\eta(x) = 1$  if  $|x| \geq 2$ .

Consider the function  $\psi_\epsilon(x) = \eta(x/\epsilon)$ . If  $\phi \in C_0^\infty$  then  $\psi_\epsilon \phi \in C_0^\infty(\mathbf{R}^N \setminus \{0\})$ . Replacing  $\phi$  in (4.2) with this function gives

$$\int |\nabla v|^{p-2} \nabla v \nabla \phi \psi_\epsilon + \int |\nabla v|^{p-2} \nabla v \nabla \psi_\epsilon \phi = \int \left( \hat{\rho} v^q \phi \psi_\epsilon + \frac{\lambda}{|x|^\theta} v^{p_\theta^*-1} \phi \psi_\epsilon \right). \quad (4.3)$$

We claim that

$$\int |\nabla v|^{p-2} \nabla v \nabla \phi \psi_\epsilon dx \rightarrow \int |\nabla v|^{p-2} \nabla v \nabla \phi dx, \quad (4.4)$$

$$\int |\nabla v|^{p-2} \nabla v \nabla \psi_\epsilon \phi dx \rightarrow 0, \quad (4.5)$$

$$\int \hat{\rho} v^q \phi \psi_\epsilon dx \rightarrow \int \hat{\rho} v^q \phi dx, \quad (4.6)$$

$$\int \frac{v^{p_\theta^*-1}}{|x|^\theta} \phi \psi_\epsilon dx \rightarrow \int \frac{v^{p_\theta^*-1}}{|x|^\theta} \phi dx, \quad (4.7)$$

as  $\epsilon \rightarrow 0$ . Indeed, (4.4), (4.6) and (4.7) follow as a straightforward application of Lebesgue's theorem. Regarding (4.5), we have by applying Hölder's inequality,

$$\begin{aligned} \left| \int |\nabla v|^{p-2} \nabla v \nabla \psi_\epsilon \phi dx \right| &\leq C_9 \int |\nabla v|^{p-1} |\nabla \psi_\epsilon| dx \\ &\leq C_{10} \left( \int |\nabla v|^p dx \right)^{\frac{p-1}{p}} \left( \int |\nabla \psi_\epsilon|^p dx \right)^{\frac{1}{p}} \\ &\leq C_{11} \left( \int_{|x| \leq 2\epsilon} |\nabla \psi_\epsilon|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

Setting  $z := x/\epsilon$  and  $\eta(z) := \psi_\epsilon(x(z))$ , we get  $\frac{\partial \eta}{\partial z_j} = \epsilon \frac{\partial \psi_\epsilon}{\partial x_j}$  and  $|\nabla \eta|^p = \epsilon^p |\nabla \psi_\epsilon|^p$ . Using Hölder's inequality,

$$\left| \int |\nabla v|^{p-2} \nabla v \nabla \psi_\epsilon \phi dx \right| \leq C_{12} \left( \int_{|z| \leq 2} |\nabla \eta|^p dz \right)^{\frac{1}{p}} \epsilon^{\frac{N-p}{p}} \rightarrow 0.$$

Making  $\epsilon \rightarrow 0$  in (4.3) and using (4.4)–(4.7), we get to

$$\int |\nabla v|^{p-2} \nabla v \nabla \phi dx = \int \left( \hat{\rho} v^q + \frac{\lambda}{|x|^\theta} v^{p_\theta^*-1} \right) \phi dx, \quad \phi \in D^{1,p},$$

where we remind that  $\lambda = 1$  if  $\theta \in [0, p)$  and  $\lambda \in [0, S_p)$  if  $\theta = p$ . So by (1.3)(i), (1.4), (1.5) and (i), (ii) in Theorem 1.1, we get

$$\int |\nabla v|^{p-2} \nabla v \nabla \phi dx \geq \int \left( \rho(x) f(v) + \frac{\lambda}{|x|^\theta} g(v) \right) \phi dx \quad \phi \in D^{1,p}, \quad \phi \geq 0. \quad (4.8)$$

**Remark 3.** As a consequence of (4.8)  $v$  satisfies (1.10) with  $\Omega = B_k$  and  $\lambda$  in accordance with either (i) or (ii) in Theorem 1.1.

*Construction of a family of lower solutions of (1.1).* In what follows we will refer several times to (1.8), (1.9), (1.11) and unless otherwise stated we mean  $\Omega = B_k$  and  $\lambda = 0$  in those expressions. At this point of the proof we adapt some arguments in Carrião, Gonçalves and Miyagaki [8].

Using (1.3)–(1.5), we get by Díaz and Saa [11, Theorems 1, 2] an only solution of (1.11) here labeled  $\omega_k$  with  $\omega_k \in D_0^{1,p}(B_k)$ ,  $\omega_k \geq 0$ ,  $\omega_k \neq 0$ . Moreover, by Guedda and Veron [15, Corollary 1.1],  $\omega_k \in C^1(\bar{B}_k)$ . Next, applying the maximum principle by Vázquez [24, Theorem 5], one infers that  $\omega_k > 0$  in  $B_k$ .

We contend that  $\omega_k \leq v$  in  $B_k$ . Indeed, by Theorem 1.3,  $v$  is positive and continuous. So there is some  $\tau_k \in (0, 1)$  such that  $\tau_k \max_{\bar{B}_k} \omega_k < \min_{\bar{B}_k} v$ . Thus  $\tau_k \omega_k < v$  in  $\bar{B}_k$ . Now, given  $\phi \in D_0^{1,p}(B_k)$  with  $\phi \geq 0$ ,

$$\begin{aligned}
& \int_{B_k} |\nabla(\tau_k \omega_k)|^{p-2} \nabla(\tau_k \omega_k) \nabla \phi \, dx - \int_{B_k} \rho(x) f(\tau_k \omega_k) \phi \, dx \\
&= \tau_k^{p-1} \int_{B_k} |\nabla \omega_k|^{p-2} \nabla \omega_k \nabla \phi \, dx - \int_{B_k} \rho(x) \frac{f(\tau_k \omega_k)}{(\tau_k \omega_k)^{p-1}} (\tau_k \omega_k)^{p-1} \phi \, dx \\
&\leq \tau_k^{p-1} \left( \int_{B_k} |\nabla \omega_k|^{p-2} \nabla \omega_k \nabla \phi \, dx - \int_{B_k} \rho(x) f(\omega_k) \phi \, dx \right) = 0,
\end{aligned}$$

showing that  $\tau_k \omega_k$  satisfies (1.9). On the other hand, by Remark 3,  $v$  satisfies (1.10). Applying Theorem 1.2 with  $\lambda = 0$ , there is  $\hat{\omega}_k \in D_0^{1,p}(B_k)$  satisfying (1.11). By uniqueness,  $\hat{\omega}_k = \omega_k$  and so  $\omega_k \leq v$  in  $B_k$ .

We claim that  $\omega_k \leq \omega_{k+1}$  in  $B_k$ . Indeed, since for each  $k$ ,  $\omega_k$  is continuous on  $\bar{B}_k$  and positive in  $B_k$  there is  $\delta_k \in (0, 1)$  such that  $\delta_k \omega_k < \omega_{k+1}$  in  $B_k$ .

But as above,  $\delta_k \omega_k$  satisfies (1.9) and  $\omega_{k+1}$  satisfies (1.10). So there is by Theorem 1.2 some  $\tilde{\omega}_k \in D_0^{1,p}(B_k)$  satisfying (1.11). By the Díaz and Saa theorem referred to above,  $\tilde{\omega}_k = \omega_k$  which further shows that  $\omega_k \leq \omega_{k+1}$  in  $B_k$ .

Making  $\omega_k = 0$  outside  $B_k$ , we have

$$0 \leq \omega_1 \leq \omega_2 \leq \dots \leq \omega_k \leq \dots \leq v. \quad (4.9)$$

Since  $\omega_k$  and  $v$  respectively satisfy (1.9) and (1.10), we get by Theorem 1.2 some  $u_k$  in  $D_0^{1,p}(B_k)$  with  $\omega_k \leq u_k \leq v$  satisfying (1.11). We make  $u_k = 0$  outside  $B_k$ .

We claim that  $\{u_k\}$  is  $D^{1,p}$ -bounded. Indeed, set  $\phi = u_k$  in (1.11) and remark that

$$\zeta(x, u_k) u_k \leq \zeta(x, v) v,$$

where  $\zeta(x, v)$  was defined in (2.1). Noticing that  $\zeta(x, v) v \in L^1$  it follows by (3.7) that  $\{u_k\}$  is bounded in  $D^{1,p}$  and hence  $u_k \xrightarrow{D^{1,p}} u$ . It will be shown that  $u$  is a solution of (1.1).

Consider the functional  $Z_k : D^{1,p} \rightarrow \mathbf{R}$  defined by

$$\langle Z_k, \phi \rangle := \int |\nabla u_k|^{p-2} \nabla u_k \nabla \phi \, dx, \quad \phi \in D^{1,p},$$

and notice that the sequence  $\{Z_k\}$  is bounded in  $(D^{1,p})'$ . By compactness,  $Z_k \xrightarrow{*} \chi$  for some  $\chi \in (D^{1,p})'$ , that is

$$\langle Z_k, \phi \rangle \rightarrow \langle \chi, \phi \rangle, \quad \phi \in D^{1,p}.$$

We claim that  $\chi = -\Delta_p u$ . Given  $\psi \in D^{1,p}$ , by the monotonicity of  $-\Delta_p$ ,

$$0 \leq \langle -\Delta_p u_k - (-\Delta_p \psi), u_k - \psi \rangle = \langle -\Delta_p u_k, u_k \rangle - \langle -\Delta_p u_k, \psi \rangle - \langle -\Delta_p \psi, u_k - \psi \rangle.$$

Hence,

$$0 \leq \int \zeta(x, u_k) u_k \, dx - \langle -\Delta_p u_k, \psi \rangle - \langle -\Delta_p \psi, u_k - \psi \rangle.$$

Passing to the limit in  $k$ ,

$$0 \leq \int \zeta(x, u) u \, dx - \langle \chi, \psi \rangle - \langle -\Delta_p \psi, u - \psi \rangle.$$



Using the fact that  $u_k$  satisfies (1.11) and passing to the limit in  $k$ , we get

$$\langle \chi, u \rangle = \int \zeta(x, u) u \, dx.$$

Setting  $\psi := u - tw$ , where  $t > 0$  and  $w \in D^{1,p}$ ,

$$0 \leq \int \zeta(x, u) u \, dx - \langle \chi, u - tw \rangle - \langle -\Delta_p(u - tw), tw \rangle$$

which gives

$$0 \leq \langle \chi, w \rangle - \langle -\Delta_p(u - tw), w \rangle, \quad t > 0.$$

Making  $t \rightarrow 0$ , we get

$$0 \leq \langle \chi + \Delta_p u, w \rangle,$$

which gives  $\chi = -\Delta_p u$ . As a consequence,  $u$  satisfies (1.7). The facts that  $u_k \xrightarrow{\text{a.e.}} u$ ,  $u_k \geq \omega_k$  and (4.9) easily lead to  $u > 0$ . From  $0 \leq u \leq v$  and (1.15), (1.8) follows. Theorem 1.1 is proved.

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