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A stochastic model of AIDS and condom use

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Abstract

In this paper we introduce stochasticity into a model of AIDS and condom use via the technique of parameter perturbation which is standard in stochastic population modelling. We show that the model established in this paper possesses non-negative solutions as desired in any population dynamics. We also carry out a detailed analysis on asymptotic stability both in probability one and in pth moment. Our results reveal that a certain type of stochastic perturbation may help to stabilise the underlying system.
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1. Introduction

In 2003, almost five million people became newly infected with HIV, the greatest number in any one year since the beginning of the epidemic. At the global level, the number of people living with HIV continues to grow—from 35 million in 2001 to 38 million in 2003. In 2003, almost 3 million were killed by AIDS; over 20 million have died since the first cases of AIDS were identified in 1981. Country data indicate that the number of people living with HIV continues to rise in all parts of the world despite the fact that effective prevention strategies exist. Sub-Saharan Africa remains the hardest-hit region with extremely high HIV prevalence among pregnant women aged 15 to 24.

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Despite increased funding, political commitment and progress in expanding access to HIV treatment, the AIDS epidemic continues to outpace the global response. No region of the world has been spared. The epidemic remains extremely dynamic, growing and changing character as the virus exploits new opportunities for transmission. The epidemic varies in scale or impact within regions; some countries are more affected than others, and within countries there are usually wide variations in infection levels between different provinces, states or districts (see [17]).

The large amounts of work done on modelling the spread of HIV has been largely restricted to ordinary differential equations. Various models were studied by Blythe and Anderson [3–5] including heterogeneity in the population amongst whom the disease is spreading. They were able to derive distributions describing variation in the incubation period from a series of hazard functions. Also in Jacquez et al. [10,11] a compartmental model was presented for the spread of HIV in a homosexual population divided into n subgroups by levels of sexual activity, which they could examine using mixing patterns. These models do not take into account the inherent randomness that is associated with HIV. In this paper we propose to examine the effect of the introduction of environmental noise into such a system. Hence we propose a system of stochastic differential equations for modelling HIV.

Condom use is one of the important factors associated with the spread of the virus. In this paper we discuss modelling the spread of HIV among a homogeneous homosexual population which is split into distinct risk groups according to the tendency of individuals to use condoms. We discuss the modelling of such a population by stochastic differential equations.

2. Model derivation

Consider a two group model for the susceptible and infected populations. The individuals are assumed to belong to one of two distinct sexually active groups, defined by the risk levels α_1 and α_2. α_1 and α_2 are supposed to represent the probability of individuals not insisting on condom use during a partnership. For example, an individual from the first group with $\alpha_1 = 0$ will always insist on a condom use, whereas an individual from the first group with $\alpha_1 = 1$ will never insist on a condom use. Also, $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$. This means that the first group is safer than the second one. If an individual from group 1 has sex with an individual from group 2 then the probability of condom use is $1 - \alpha_1\alpha_2$.

Let us define the variables first.

$x_1(t)$ number of susceptible individuals at time t for the first group.
$x_2(t)$ number of infected individuals at time t for the first group.
$x_3(t)$ number of susceptible individuals at time t for the second group.
$x_4(t)$ number of infected individuals at time t for the second group.

Greenhalgh et al. [9] discussed such a model:

$$\frac{dx_1(t)}{dt} = \lambda_1 - \mu x_1(t) - B_1(t)x_1(t) + C_{21}x_3(t) - C_{12}x_1(t),$$
$$\frac{dx_2(t)}{dt} = B_1(t)x_1(t) - (\mu + \sigma + D_{12})x_2(t) + D_{21}x_4(t),$$
$$\frac{dx_3(t)}{dt} = \lambda_2 - \mu x_3(t) - B_2(t)x_3(t) + C_{12}x_1(t) - C_{21}x_3(t), \quad \text{and}$$
$$\frac{dx_4(t)}{dt} = B_2(t)x_3(t) - (\mu + \sigma + D_{21})x_4(t) + D_{12}x_2(t)$$

with suitable initial conditions. Here $B_i(t) = \beta c\alpha_i F(t)$, $i = 1, 2$, and $F(t) = (\alpha_1 x_2(t) + \alpha_2 x_4(t))/(x_1(t) + x_2(t) + x_3(t) + x_4(t))$.

The parameters involved in the system are described below:

λ_1 rate of recruitment of new susceptible individuals for group one;
λ_2 rate of recruitment of new susceptible individuals for group two;
μ per capita rate at which individuals leave the sexually mixing population for non-disease related reasons;
σ per capita rate at which infected individuals develop AIDS;
β average sexual risk per infected partner when bodily fluids are shared;
c mean number of partners for a typical individual per unit time;
α_1 probability of individual not insisting on condom use for group one;
α_2 probability of individual not insisting on condom use for group two;
C_{12} per capita rate of migration of susceptibles from safe group to unsafe group;
C_{21} per capita rate of migration of susceptibles from unsafe group to safe group;
D_{12} per capita rate of migration of infecteds from safe group to unsafe group;
D_{21} per capita rate of migration of infecteds from unsafe group to safe group.

We assume proportional mixing. An individual in group 1 has $c\Delta t + o(\Delta t)$ partners in $[t, t + \Delta t)$. The probability that a contact at time t is with an infectious partner who does not insist on condom use is $F(t)$. The number of infections of group 1 individuals in $[t, t + \Delta t)$ is therefore $\beta c\alpha_1 F(t)\Delta t + o(\Delta t)$ as the group 1 individual must also not insist on condom use and be infected. Similarly the number of infections of group 2 individuals in $[t, t + \Delta t)$ is $\beta c\alpha_2 F(t)\Delta t + o(\Delta t)$.

Greenhalgh et al. [9] derived an expression for the basic reproduction number for this model, defined as the expected number of secondary cases produced by a typical newly infected individual entering the disease free population, denoted by R_0, and show that this two group model has a unique disease free equilibrium solution which is locally asymptotically stable for $R_0 < 1$ and unstable for $R_0 > 1$. Moreover if $R_0 > 1$ then there is unique endemic equilibrium, but for $R_0 < 1$ two subcritical endemic equilibria can occur.

More complete results are obtained for a special case of this model, namely

$$\begin{aligned}
\frac{dx_1(t)}{dt} &= \lambda_1 - \mu x_1(t),\\
\frac{dx_2(t)}{dt} &= D_{21}x_4(t) - (\mu + \sigma)x_2(t),\\
\frac{dx_3(t)}{dt} &= \lambda_2 - \beta c\alpha_2^2 x_4(t)\frac{x_3(t)}{T(t)} - \mu x_3(t), \quad \text{and}\\
\frac{dx_4(t)}{dt} &= \beta c\alpha_2^2 x_4(t)\frac{x_3(t)}{T(t)} - (\mu + \sigma + D_{21})x_4(t),
\end{aligned}$$

where $T(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t)$. In this case individuals in the protected group (with $\alpha_1 = 0$) do not contribute to the spread of the virus in the population. The model is effectively a single group model with respect to viral spread.

We choose to examine the special case of the model where individuals in one of the population groups are completely safe in their sexual practices: that is $\alpha_1 = 0$. Also infecteds migrate from the unsafe group in the safe group, but not in the reverse direction, and there is no migration of susceptibles (hence $C_{12} = C_{21} = D_{12} = 0$). This particular form of migration is especially important as HIV infected individuals discover that they are infected through receiving a positive

HIV test and consequently adopt safer sexual practices. This has the advantage of providing a practically motivated yet tractable model where we can investigate the effects of environmental stochasticity. Ideally we would like to investigate the full model described earlier but here even the deterministic model was very complicated. Investigating this currently practically motivated yet tractable model may provide insight into incorporating the effects of environmental stochasticity into the model.

This model has $R_0 = \frac{\beta c \alpha_2^2 (1-\Lambda)}{\mu+\sigma+D_{21}}$ where $\Lambda = \frac{\lambda_1}{\lambda_1+\lambda_2}$. Greenhalgh et al. [9] found that the disease free solution is globally asymptotically stable for $R_0 \leqslant 1$ and unstable for $R_0 > 1$. This model also has a unique endemic equilibrium if $R_0 > 1$ which is locally asymptotically stable.

After getting the infection, persistent or severe symptoms may not appear for 10 years or more after HIV first enters the body in adults, or within two years in children born with HIV infection. This period of asymptomatic infection is highly individual. Some people may begin to have symptoms within a few months, while others may be symptom-free for more than 10 years. Because early HIV infection often causes no symptoms, it is very likely that a person may not seek medical advice. Hence the HIV infection may not be detected.

HIV antibodies generally do not reach detectable levels in the blood for one to three months following infection. It may take the antibodies as long as six months to be produced in quantities large enough to show up in standard blood tests. HIV disease, however, is not uniformly expressed in all individuals. A small proportion of persons infected with the virus develop AIDS and die within months following primary infection, while approximately 5 percent of HIV-infected individuals exhibit no signs of disease progression even after 12 or more years. Host factors such as age or genetic differences among individuals, the level of virulence of the individual strain of virus, as well as influences such as co-infection with other diseases may determine the rate and severity of HIV disease expression in different people (see [16]).

Recall that the parameter σ represents the per capita rate at which infected individuals develop AIDS. In practice we usually estimate it by an average value plus an error term and we may assume that the error term follows a normal distribution so that for a single infected individual the probability of an event in $[t, t+\Delta t)$ is approximately $N(\sigma\Delta t, \sigma_1^2\Delta t) + o(\Delta t)$. We may therefore replace σ by $\sigma + \sigma_1 \dot{B}(t)$, where $\dot{B}(t)$ is white noise (i.e. $B(t)$ is a Brownian motion). Here $\sigma_1 > 0$ (small compared with σ) is the intensity of the noise. This is a standard technique in stochastic population modelling and it also introduces stochasticity into the model and enables us to get analytical results (see [2,6,8,14,15]).

The new system is described by the following set of stochastic differential equations:

$$\frac{dx_1(t)}{dt} = \lambda_1 - \mu x_1(t), \tag{1}$$

$$dx_2(t) = \big[D_{21}x_4(t) - (\mu+\sigma)x_2(t)\big]\,dt - \sigma_1 x_2(t)\,dB(t), \tag{2}$$

$$\frac{dx_3(t)}{dt} = \lambda_2 - \beta c \alpha_2^2 x_4(t)\frac{x_3(t)}{T(t)} - \mu x_3(t), \quad \text{and} \tag{3}$$

$$dx_4(t) = \left[\beta c \alpha_2^2 x_4(t)\frac{x_3(t)}{T(t)} - (\mu+\sigma+D_{21})x_4(t)\right]dt - \sigma_1 x_4(t)\,dB(t), \tag{4}$$

with suitable initial conditions. Note that as the environmental stochasticity is through the parameter σ this means that the Brownian noise term is the same in both Eqs. (2) and (4). We find some analytical results in the next sections.

3. Non-negative solutions

In this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $B(t)$ be the one-dimensional Brownian motion defined on this probability space. Also $R^4_{++} = \{x \in R^4\colon\ x_i > 0 \text{ for all } 1 \leqslant i \leqslant 4\}$, $R^3_{++} = \{x \in R^3\colon\ x_i > 0 \text{ for all } 2 \leqslant i \leqslant 4\}$, $R^4_{+} = \{x \in R^4\colon\ x_i \geqslant 0 \text{ for all } 1 \leqslant i \leqslant 4\}$, and let $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$ and $y(t) = (x_2(t), x_3(t), x_4(t))^T$.

Before proving the main theorem we put forward a lemma.

Lemma 1. *The following inequality holds*:

$$u \leqslant 2\big(u + 1 - \log(u)\big) - (4 - 2\log 2) \quad \forall u > 0.$$

Proof. Define, for $u > 0$,

$$f(u) = u + 2 - 2\log(u).$$

$f(u)$ has a minimum at $u = 2$. The result follows. □

We now prove the main theorem.

Theorem 1. *Assume that* $\sigma_1, \mu, \sigma, D_{21}, \beta$ *are positive real numbers and* $0 < \lambda_1, \lambda_1, \alpha_2 < 1$. *Then for any initial value* $x_0 \in R^4_{++}$, *there is a unique solution* $x(t)$ *to Eqs.* (1)–(4) *on* $t \geqslant 0$ *and the solution will remain in* R^4_{++} *with probability* 1, *namely* $x(t) \in R^4_{++}$ *for all* $t \geqslant 0$ *almost surely.*

Proof. We separate out the proof for $x_1(t)$ and $y(t)$.

From (1) we can easily get,

$$x_1(t) = x_1(0)e^{-\mu t} + \frac{\lambda_1}{\mu}\big(1 - e^{-\mu t}\big). \tag{5}$$

We can clearly see that the solution will always be positive.

We now move on to $y(t)$. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial value $y_0 \in R^3_{++}$ there is a unique local solution $y(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time (see [1,7]). To show this solution is global, we need to show that $\tau_e = \infty$ a.s. Let $k_0 \geqslant 0$ be sufficiently large so that every component of y_0 lies within the interval $[1/k_0, k_0]$. For each integer $k \geqslant k_0$, define the stopping time

$$\tau_k = \inf\big\{t \in [0, \tau_e)\colon\ x_i(t) \notin (1/k, k) \text{ for some } i,\ 2 \leqslant i \leqslant 4\big\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leqslant \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s. then $\tau_e = \infty$ and $y(t) \in R^3_{++}$ a.s. for all $t \geqslant 0$. In other words, to complete the proof all we need to show is that $\tau_\infty = \infty$ a.s. For if this statement is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leqslant T\} > \epsilon.$$

Hence there is an integer $k_1 \geqslant k_0$ such that,

$$P\{\tau_k \leqslant T\} \geqslant \epsilon \quad \text{for all } k \geqslant k_1. \tag{6}$$

Define a C^2-function $V : R^3_{++} \to R_{++}$ by

$$V(x) = \sum_{i=2}^{4} \bigl[x_i + 1 - \log(x_i)\bigr].$$

The non-negativity of this function can be seen from $u + 1 - \log(u) \geqslant 0\ \forall u > 0$. Using Itô's formula we get,

$$dV\bigl(y(t)\bigr) \leqslant \left(c_1 + \left(1 - \frac{1}{x_2}\right) D_{21} x_4(t) + \left(1 - \frac{1}{x_3}\right)\lambda_2\right) dt + (2 - x_2 - x_4)\sigma_1\, dB(t),$$

where $c_1 = 3\mu + 2\sigma + D_{21} + \beta c\alpha_2^2 + \sigma_1^2$. Therefore,

$$dV\bigl(y(t)\bigr) \leqslant \bigl(c_1 + \lambda_2 + D_{21} x_4(t)\bigr)\, dt + (2 - x_2 - x_4)\sigma_1\, dB(t).$$

Now from Lemma 1 and definition of V,

$$x_4 \leqslant 2V(x) - (4 - 2\log 2).$$

Hence we get,

$$dV\bigl(y(t)\bigr) \leqslant \bigl(c_2\bigl(1 + V(x)\bigr)\bigr)\, dt + (2 - x_2 - x_4)\sigma_1\, dB(t),$$

where $c_2 = \max(c_1 + \lambda_2, 2D_{21})$. Therefore if $t_1 \leqslant T$,

$$\int_0^{\tau_k \wedge t_1} dV\bigl(y(t)\bigr) \leqslant \int_0^{\tau_k \wedge t_1} c_2\bigl(1 + V\bigl(x(t)\bigr)\bigr)\, dt + \int_0^{\tau_k \wedge t_1} (2 - x_2 - x_4)\sigma_1\, dB(t).$$

This implies that,

$$\begin{aligned}
E V\bigl(y(\tau_k \wedge t_1)\bigr) &\leqslant V(y_0) + E \int_0^{\tau_k \wedge t_1} c_2\bigl(1 + V\bigl(x(t)\bigr)\bigr)\, dt \\
&\leqslant V(y_0) + c_2 t_1 + c_2 E \int_0^{\tau_k \wedge t_1} V\bigl(x(t)\bigr)\, dt \\
&\leqslant V(y_0) + c_2 T + c_2 E \int_0^{t_1} V\bigl(x(\tau_k \wedge t)\bigr)\, dt \\
&= V(y_0) + c_2 T + c_2 \int_0^{t_1} E V\bigl(x(\tau_k \wedge t)\bigr)\, dt.
\end{aligned}$$

By the Gronwall inequality,

$$E V\bigl(y(\tau_k \wedge T)\bigr) \leqslant c_3, \tag{7}$$

where $c_3 = (V(y_0) + c_2 T)e^{c_2 T}$.

Set $\Omega_k = \{\tau_k \leqslant T\}$ for $k \geqslant k_1$ and by (6), $P(\Omega_k) \geqslant \epsilon$. Note that for every $\omega \in \Omega_k$, there is some i $(2 \leqslant i \leqslant 4)$ such that $x_i(\tau_k, \omega)$ equals either k or $1/k$, and hence $V(y(\tau_k, \omega))$ is no less than

$$k + 1 - \log(k) \quad \text{or} \quad (1/k) + 1 - \log(1/k) = (1/k) + 1 + \log(k).$$

Consequently,

$$V\big(y(\tau_k,\omega)\big) \geqslant \big[k+1-\log(k)\big] \wedge \big[(1/k)+1+\log(k)\big].$$

It then follows from (6) and (7) that,

$$c_3 \geqslant E\big[1_{\Omega_k}(\omega)V\big(y(\tau_k,\omega)\big)\big] \geqslant \epsilon\big(\big[k+1-\log(k)\big] \wedge \big[(1/k)+1+\log(k)\big]\big),$$

where 1_{Ω_k} is the indicator function of Ω_k. Letting $k \to \infty$ leads to the contradiction $\infty > c_3 = \infty$. So we must therefore have $\tau_\infty = \infty$ a.s. □

As real initial conditions can have some components zero it is both interesting and practically important to consider what happens when this occurs, i.e. $x_0 \in R_+^4$.

Theorem 2. *Assume that* $\sigma_1, \mu, \sigma, D_{21}, \beta$ *are positive real numbers and* $0 < \lambda_1, \lambda_1, \alpha_2 < 1$. *Then for any initial value* $x_0 \in R_+^4$, *the solution of Eqs.* (1)–(4) *will remain in* R_+^4 *with probability* 1, *namely* $x(t) \in R_+^4$ *for all* $t \geqslant 0$ *almost surely.*

Proof. The solution of $x_1(t)$ will always remain positive if the initial solution is in R_+^4, i.e. if $x_1(0) \in R_+^4$. Hence primarily $x_2(t)$ and $x_4(t)$ are of interest to us. Therefore we consider the following four cases:

(i) $x_2(0) = 0, x_4(0) = 0, x_1(0) \geqslant 0, x_3(0) \geqslant 0.$

If $x_2(0) = x_4(0) = 0$ then it is easy to see that $x_2(t) = x_4(t) = 0\ \forall t$ a.s. Using this result and from (1) and (3) we can clearly see that $x_1(t) \geqslant 0$ and $x_3(t) \geqslant 0$.

(ii) $x_2(0) > 0, x_4(0) = 0, x_1(0) \geqslant 0, x_3(0) \geqslant 0.$

If $x_4(0) = 0$ then again $x_4(t) = 0\ \forall t$ a.s. From (2) and using Itô's formula we get,

$$d\log x_2(t) = -\left(\mu+\sigma+\frac{\sigma_1^2}{2}\right)dt - \sigma_1\,dB(t)$$

$$\Longrightarrow \quad x_2(t) = x_2(0)\exp\left[-\left(\mu+\sigma+\frac{\sigma_1^2}{2}\right)t - \sigma_1 B(t)\right].$$

Therefore $x_2(t) > 0\ \forall t$ a.s. As in case (i) $x_1(t) \geqslant 0$ and $x_3(t) \geqslant 0$.

(iii) $x_2(0) = 0, x_4(0) > 0, x_1(0) \geqslant 0, x_3(0) \geqslant 0.$

We see that from (5) $x_1(\Delta t) > 0$ for Δt small and positive. Also given that $D_{21} > 0$,

$$x_2(\Delta t) = D_{21}x_4(0)\Delta t + o(\Delta t) > 0 \quad \text{a.s.}$$

From (3) we can write

$$\frac{dx_3(t)}{dt} \geqslant \lambda_2 - \big(\beta c\alpha_2^2 + \mu\big)x_3(t).$$

Solving further we get

$$x_3(t) \geqslant x_3(0)e^{-(\beta c\alpha_2^2+\mu)t} + \frac{\lambda_2}{\beta c\alpha_2^2+\mu}\big(1-e^{-(\beta c\alpha_2^2+\mu)t}\big).$$

Hence $x_3(\Delta t) > 0$ for Δt small and positive.

Now consider

$$\frac{dx_4(t)}{dt} = \beta c\alpha_2^2 x_4(t)\frac{x_3(t)}{T(t)} - (\mu + \sigma + D_{21})x_4(t) - \sigma_1 x_4(t)\dot{B}(t).$$

The solution of the above equations is

$$x_4(t) = x_4(0)\exp\left[\int_0^t \left(\beta c\alpha_2^2\frac{x_3(t)}{T(t)} - (\mu+\sigma+D_{21}) - \frac{\sigma_1^2}{2}\right)ds - \int_0^t \sigma_1\, dB(s)\right].$$

So $x_4(\Delta t) > 0$ a.s. for Δt small and positive. So by changing the time origin to Δt small and positive and then using Theorem 1 we get $x_i(t) > 0$ a.s. for $1 \leqslant i \leqslant 4$.

(iv) $x_2(0) > 0$, $x_4(0) > 0$, $x_1(0) \geqslant 0$, $x_3(0) \geqslant 0$.

By changing the time origin to Δt if necessary where Δt is small and positive without loss of generality $x_1(0) > 0$, $x_3(0) > 0$ and then using Theorem 1 we get $x_i(t) > 0\ \forall t$ a.s. $1 \leqslant i \leqslant 4$. □

4. Asymptotic behaviour and stability

We have a system of differential equations which is partially stochastic and partially deterministic. In the following theorem we try and analyse the equations individually. We try and look at the asymptotic behaviour of the variables.

To prove this theorem we state a lemma (see [12]).

Lemma 2. *Let $A(t)$ and $U(t)$ be two continuous adapted increasing processes on $t \geqslant 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real valued continuous local martingale with $M(0) = 0$ a.s. Let ξ be a non-negative $\mathcal{F}_0$ measurable random variable such that $E\xi < \infty$. Define*

$$X(t) = \xi + A(t) - U(t) + M(t) \quad \text{for } t \geqslant 0.$$

If $X(t)$ is non-negative, then

$$\left\{\lim_{t\to\infty} A(t) < \infty\right\} \subset \left\{\lim_{t\to\infty} X(t) < \infty\right\} \cap \left\{\lim_{t\to\infty} U(t) < \infty\right\} \quad \text{a.s.},$$

where $G \subset D$ a.s. means $P(G \cap D^c) = 0$. In particular, if $\lim_{t\to\infty} A(t) < \infty$ a.s. then for almost all $\omega \in \Omega$

$$\lim_{t\to\infty} X(t,\omega) < \infty, \qquad \lim_{t\to\infty} U(t,\omega) < \infty$$

and

$$\lim_{t\to\infty} M(t,\omega) \quad \text{exists and is finite.}$$

We now move on to proving the main theorem. Define $R_1 = \frac{\beta c\alpha_2^2(1-\Lambda)}{\mu+\sigma+D_{21}+\sigma_1^2/2}$.

Theorem 3. *If $R_1 < 1$, then the system of stochastic differential equations* (1)–(4) *is globally asymptotically stable in the sense that for any given any initial value $x_0 \in R_+^4$ it will tend to the equilibrium position of $(\frac{\lambda_1}{\mu}, 0, \frac{\lambda_2}{\mu}, 0)$ asymptotically with probability* 1.

Proof. We first look at the infected individuals in both the groups, i.e. $x_2(t)$ and $x_4(t)$. Before proceeding we need an upper bound on $\frac{x_3(t)}{T(t)}$. It can be easily shown from (1) that,

$$x_1(t) \to \frac{\lambda_1}{\mu} \quad \text{as } t \to \infty. \tag{8}$$

From (3) we can obtain,

$$\frac{dx_3(t)}{dt} \leqslant \lambda_2 - \mu x_3(t) \quad \text{a.s.}$$

Hence it is easy to prove that

$$\limsup_{t\to\infty} x_3(t) \leqslant \frac{\lambda_2}{\mu} \quad \text{a.s.}$$

It follows from (8) and (3) that,

$$\limsup_{t\to\infty} \frac{x_3(t)}{T(t)} \leqslant (1-\Lambda), \quad \text{a.s.},$$

where $\Lambda = \frac{\lambda_1}{\lambda_1+\lambda_2}$.

$\forall \epsilon > 0$ there exists t_1 such that,

$$\frac{x_3(t)}{T(t)} \leqslant (1-\Lambda+\epsilon) \quad \forall t \geqslant t_1 \text{ a.s.} \tag{9}$$

For $t \geqslant t_1$,

$$\frac{1}{t}\int_0^t \frac{x_3(s)}{T(s)}\,ds = \frac{1}{t}\int_0^{t_1} \frac{x_3(s)}{T(s)}\,ds + \frac{1}{t}\int_{t_1}^t \frac{x_3(s)}{T(s)}\,ds.$$

Hence using (9) we get,

$$\begin{aligned}\frac{1}{t}\int_0^t \frac{x_3(t)}{T(t)}\,ds &\leqslant \frac{1}{t}\int_0^{t_1} ds + \frac{1}{t}\int_{t_1}^t (1-\Lambda+\epsilon)\,ds = \frac{t_1}{t} + \frac{(1-\Lambda+\epsilon)(t-t_1)}{t} \\ &= \frac{t_1}{t}\big[1-(1-\Lambda+\epsilon)\big] + (1-\Lambda+\epsilon) \quad \text{a.s.}\end{aligned}$$

Therefore we have,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \frac{x_3(s)}{T(s)}\,ds \leqslant (1-\Lambda+\epsilon) \quad \text{a.s.}$$

Now since ϵ is arbitrary,

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \frac{x_3(s)}{T(s)}\,ds \leqslant (1-\Lambda) \quad \text{a.s.} \tag{10}$$

We now turn our attention to the infected populations, i.e. variables $x_2(t)$ and $x_4(t)$. For $x_4(t)$, if $x_4(0) > 0$, then let us define a function $\bar{V}(x_4) = \log(x_4)$, $x_4 \in (0,\infty)$. From Itô's formula,

$$d\bar{V}\big(x_4(t)\big) = \left(\beta c\alpha_2^2 \frac{x_3(t)}{T(t)} - (\mu+\sigma+D_{21}) - \frac{\sigma_1^2}{2}\right)dt - \sigma_1\,dB(t).$$

This means,

$$\int_0^t d\bar{V}\big(x_4(t)\big) = \int_0^t \left(\beta c\alpha_2^2 \frac{x_3(t)}{T(t)} - (\mu+\sigma+D_{21}) - \frac{\sigma_1^2}{2}\right) ds - \int_0^t \sigma_1\, dB(s).$$

Simplifying we get,

$$\frac{1}{t}\bar{V}\big(x_4(t)\big) = \frac{1}{t}\bar{V}\big(x_4(0)\big) + \frac{1}{t}\int_0^t \left(\beta c\alpha_2^2 \frac{x_3(s)}{T(s)} - (\mu+\sigma+D_{21}) - \frac{\sigma_1^2}{2}\right) ds - \frac{1}{t}\int_0^t \sigma_1\, dB(s).$$

We know that,

$$\lim_{t\to\infty} \frac{B(t)}{t} = 0 \quad \text{a.s.} \tag{11}$$

Using (10), (11) and substituting the value of $\bar{V}(x_4)$ we get,

$$\limsup_{t\to\infty} \frac{1}{t}\log x_4(t) \leqslant \kappa < 0 \quad \text{a.s.}, \tag{12}$$

where $\kappa = -\beta c\alpha_2^2(1-\Lambda) + \mu + \sigma + D_{21} + \frac{\sigma_1^2}{2}$. From (12),

$$x_4(t) \leqslant Me^{-\nu t} \quad \forall t \geqslant 0 \text{ a.s.}, \tag{13}$$

where $\nu = \frac{\kappa}{2}$ and M is a finite random variable a.s. However, if $x_4(0) = 0$ then $x_4(t) = 0\ \forall t$ a.s. Hence (13) holds, for all $x_4(0) \geqslant 0$.

For $x_2(t)$ we can write (2) as,

$$x_2(t) = x_2(0) + D_{21}\int_0^t x_4(s)\, ds - (\mu+\sigma)\int_0^t x_2(s)\, ds - \sigma_1 \int_0^t x_2\, dB(s).$$

From (13) we get,

$$\lim_{t\to\infty} \int_0^t x_4(s)\, ds \leqslant \int_0^\infty Me^{-\nu s}\, ds < \infty \quad \text{a.s.}$$

From Lemma 2,

$$\lim_{t\to\infty} x_2(t) < \infty \quad \text{a.s.}$$

and

$$\lim_{t\to\infty} \int_0^t x_2(s)\, ds = \int_0^\infty x_2(s)\, ds < \infty \quad \text{a.s.}$$

We need to show,

$$\lim_{t\to\infty} x_2(t) = 0 \quad \text{a.s.}$$

If not there is an $\Omega_1 \subset \Omega$ with $P(\Omega_1) > 0$ such that for all ω belonging to Ω_1,

$$\liminf_{t\to\infty} x_2(t,\omega) > 0.$$

Hence, for any fixed $\bar{\omega} \in \Omega_1$, we have,

$$\liminf_{t\to\infty} x_2(t,\bar{\omega}) = \rho(\bar{\omega}) > 0.$$

So there exists a $T_1 > 0$ such that, $x_2(t,\omega) > \frac{1}{2}\rho(\bar{\omega})$ for all $t \geqslant T_1$. Therefore,

$$\int_0^\infty x_2(s,\bar{\omega})\,ds = \int_0^{T_1} x_2(s,\bar{\omega})\,ds + \int_{T_1}^\infty x_2(s,\bar{\omega})\,ds \geqslant \int_{T_1}^\infty x_2(s,\bar{\omega})\,ds = \infty.$$

This implies that $\Omega_1 \subseteq \Omega_2$, where $\Omega_2 = \{\omega\colon \int_0^\infty x_2(s,\omega)\,ds = \infty\}$. Hence $P(\Omega_2) > 0$. Therefore, we must have

$$\lim_{t\to\infty} x_2(t) = 0 \quad \text{a.s.} \tag{14}$$

Coming back to the susceptible population in the second group, i.e. $x_3(t)$, using (13) we can write,

$$\lim_{t\to\infty} \frac{x_4(t)}{T(t)} = 0 \quad \text{a.s.} \tag{15}$$

Given $\epsilon > 0$ and using (15), there exists a random variable T_2 such that, with probability 1,

$$\beta c\alpha_2^2 \frac{x_4(t)}{T(t)} < \epsilon \quad \forall t \geqslant T_2.$$

Hence for $t \geqslant T_2$,

$$\frac{dx_3(t)}{dt} \geqslant \lambda_2 - (\mu+\epsilon)x_3(t) \quad \text{a.s.}$$

Now it is straightforward to prove that,

$$\liminf_{t\to\infty} x_3(t) \geqslant \frac{\lambda_2}{\mu} \quad \text{a.s.} \tag{16}$$

It follows from (3) and (16) that,

$$\lim_{t\to\infty} x_3(t) = \frac{\lambda_2}{\mu} \quad \text{a.s.}$$

Hence we have shown that as $t \to \infty$, $x_1(t) \to \frac{\lambda_1}{\mu}$, $x_2(t) \to 0$, $x_3(t) \to \frac{\lambda_2}{\mu}$, $x_4(t) \to 0$. The proof is thus complete. □

We see that the infective populations die out with time and the susceptible population stabilises at $\frac{\lambda_1}{\mu}$, $\frac{\lambda_2}{\mu}$ for the first and the second group, respectively.

5. Stability at the disease free equilibrium

We now examine different forms of stability at the disease free equilibrium. We divide this section into almost sure exponential stability and moment exponential stability. We first give the definitions of stability (see [13]).

Consider the general d-dimensional stochastic differential equation,

$$dx(t) = f\big(x(t),t\big)\,dt + g\big(x(t),t\big)\,dB(t) \tag{17}$$

on $t \geqslant 0$ with initial value $x(0) = x_0$. The solution is denoted by $x(t;x_0)$. Assume that

$$f(0,t) = 0 \quad \text{and} \quad g(0,t) = 0 \quad \text{for all } t \geqslant t_0.$$

So Eq. (17) has the solution $x(t) \equiv 0$ corresponding to the initial value $x(t_0) = 0$. This solution is called the *trivial solution* or *equilibrium position*.

Definition 1. The trivial solution of Eq. (17) is said to be almost surely exponentially stable if

$$\limsup_{t\to\infty} \frac{1}{t} \log\big|x(t;x_0)\big| < 0 \quad \text{a.s.}$$

for all $x_0 \in R^d$.

Definition 2. The trivial solution of Eq. (17) is said to be pth moment exponentially stable if there is a pair of positive constants λ and C such that

$$E\big|x(t;x_0)\big|^p \leqslant C|x_0|^p e^{-\lambda t} \quad \text{on } t \geqslant 0$$

for all $x_0 \in R^d$. When $p = 2$, it is usually said to be exponentially stable in mean square.

Consider the population near the disease free equilibrium, i.e. $x_1(t) = \frac{\lambda_1}{\mu} + \eta_1$, $x_3(t) = \frac{\lambda_2}{\mu} + \eta_3$, and $x_2(t)$, $x_4(t)$ small.

5.1. Almost sure exponential stability

We can write,

$$d\tilde{x}(t) \approx \begin{pmatrix} -(\mu+\sigma) & D_{21} \\ 0 & \beta c\alpha_2^2(1-\Lambda) - (\mu+\sigma+D_{21}) \end{pmatrix} \tilde{x}(t)\,dt - \sigma_1 \tilde{x}(t)\,dB(t),$$

where $\tilde{x}(t) = (x_2(t), x_4(t))^T$. The solution is given by,

$$\tilde{x}(t) = \exp\left[\left(F - \frac{1}{2}\sigma_1^2 I\right)t - \sigma_1 I B(t)\right]\tilde{x}_0,$$

where I is the identity matrix and

$$F = \begin{pmatrix} -(\mu+\sigma) & D_{21} \\ 0 & \beta c\alpha_2^2(1-\Lambda) - (\mu+\sigma+D_{21}) \end{pmatrix}.$$

Suppose first that $R_1 < 1$. As the eigenvalues of $F - \frac{1}{2}\sigma_1^2 I$ are real and negative

$$\left\|\exp\left[\left(F - \frac{1}{2}\sigma_1^2 I\right)t\right]\right\| \leqslant Ce^{-2\lambda t}$$

for some positive constants C and λ (see [13, p. 127]). Arguing as in Mao, [13, p. 127]

$$\limsup_{t\to\infty} \frac{1}{t}\log\bigl|\tilde{x}(t)\bigr| \leqslant -2\lambda \quad \text{a.s.}$$

Hence there exists t_0 such that for $t \geqslant t_0$

$$\bigl|\tilde{x}(t)\bigr| \leqslant \bigl|\tilde{x}(t_0)\bigr|e^{-\lambda(t-t_0)} \quad \text{a.s.}$$

Without loss of generality we may assume that $\lambda < \mu$.

Now from (5),

$$\left|x_1(t) - \frac{\lambda_1}{\mu}\right| \leqslant \left|x_1(0) - \frac{\lambda_1}{\mu}\right|e^{-\mu t}.$$

Next writing $\xi = x_3 - \frac{\lambda_2}{\mu}$ from (3),

$$\frac{d\xi}{dt} = -\mu\xi - \beta c\alpha_2^2 \frac{x_2 x_4}{T}.$$

Integrating for $t \geqslant t_0$,

$$\xi(t) = \xi(t_0)e^{-\mu(t-t_0)} - e^{-\mu t}\int_{t_0}^{t} \beta c\alpha_2^2 \frac{x_3(s)x_4(s)}{T(s)} e^{\mu s}\,ds,$$

$$\begin{aligned}\bigl|\xi(t)\bigr| &\leqslant \bigl|\xi(t_0)\bigr|e^{-\mu(t-t_0)} + \beta c\alpha_2^2 e^{-\mu t}\int_{t_0}^{t} \bigl|\tilde{x}(0)\bigr|e^{-\lambda s+\mu s}\,ds \\ &\leqslant \left(\bigl|\xi(t_0)\bigr| + \frac{\beta c\alpha_2^2|\tilde{x}(t_0)|}{\mu-\lambda}\right)e^{-\lambda(t-t_0)} \quad \text{a.s.}\end{aligned}$$

Hence for $t \geqslant t_0$

$$\left|x(t) - \left(\frac{\lambda_1}{\mu}, 0, \frac{\lambda_2}{\mu}, 0\right)\right| \leqslant C_1 e^{-\lambda(t-t_0)} \quad \text{a.s.}$$

for some constant C_1. Therefore the disease free equilibrium is almost surely locally exponentially stable.

If $R_1 > 1$ then choose $\tilde{x}(0) = (0, 1)^T$.

$$x_4(t) = \exp\bigl(-\kappa t - \sigma_1 I B(t)\bigr).$$

A similar argument shows that $\liminf_{t\to\infty} \frac{1}{t}\log|x_4(t)| > -\kappa > 0$ a.s. so the disease free equilibrium is a.s. exponentially unstable.

5.2. *Moment exponential stability*

Write

$$a_{11}(t) = -\left(\mu + \sigma + \frac{\sigma_1^2}{2}\right)t - \sigma_1 B(t),$$

$$a_{12}(t) = D_{12}t \quad \text{and}$$

$$a_{22}(t) = \left(\beta c\alpha_2^2(1-\Lambda) - (\mu + \sigma + D_{21}) - \frac{\sigma_1^2}{2}\right)t - \sigma_1 B(t).$$

Near the disease free equilibrium we have that

$$x_4(t) = x_4(0)e^{a_{22}(t)}.$$

Writing $\xi = x_3 - \frac{\lambda_2}{\mu}$,

$$\frac{d\xi}{dt} = -\mu\xi - \beta c\alpha_2^2 \frac{x_3 x_4}{T},$$

$$\xi(t) = \xi(0)e^{-\mu t} - e^{-\mu t}\int_0^t \beta c\alpha_2^2 \frac{x_3(s)x_4(s)}{T(s)} e^{\mu s}\,ds.$$

Note

$$\int_0^t \beta c\alpha_2^2 \frac{x_3(s)x_4(s)}{T(s)} e^{\mu s}\,ds \leqslant \beta c\alpha_2^2 \int_0^t x_4(0)e^{a_{22}(s)+\mu s}\,ds.$$

So

$$\begin{aligned}\big|\xi(t)\big| &\leqslant \big|\xi(0)\big|e^{-\mu t} + \beta c\alpha_2^2 x_4(0)e^{-\mu t}\int_0^t e^{a_{22}(s)+\mu s}\,ds \\ &\leqslant 2\max\left(\big|\xi(0)\big|e^{-\mu t}, \beta c\alpha_2^2 x_4(0)e^{-\mu t}\int_0^t e^{a_{22}(s)+\mu s}\,ds\right).\end{aligned}$$

It is straightforward to show that if $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{smallmatrix}\right)$ then

$$e^A = \begin{cases} \left(\begin{smallmatrix} e^{a_{11}} & \frac{a_{12}}{a_{11}-a_{22}}(e^{a_{11}}-e^{a_{22}}) \\ 0 & e^{a_{22}}\end{smallmatrix}\right) & \text{if } a_{11} \neq a_{22}, \\ \left(\begin{smallmatrix} e^{a_{11}} & a_{12}e^{a_{11}} \\ 0 & e^{a_{22}}\end{smallmatrix}\right) & \text{if } a_{11} = a_{22}.\end{cases}$$

The solution at the disease free equilibrium is given by

$$\tilde{x}(t) = \exp\left[\left(F - \frac{1}{2}\sigma_1^2 I\right)t - \sigma_1 I B t\right]\tilde{x}(0).$$

We have that

$$\left(F - \frac{1}{2}\sigma_1^2 I\right)t - \sigma_1 I B t = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ 0 & a_{22}(t)\end{pmatrix}. \tag{18}$$

First of all suppose that $\beta c\alpha_2^2(1-\Lambda) \neq D_{21}$ then

$$\tilde{x}(t) = \begin{pmatrix} e^{a_{11}(t)} & \frac{a_{12}(t)}{a_{11}(t)-a_{22}(t)}(e^{a_{11}(t)} - e^{a_{22}(t)}) \\ 0 & e^{a_{22}(t)}\end{pmatrix}\tilde{x}(0) = re^{a_{11}(t)} + se^{a_{22}(t)},$$

where r and s are constant vectors. Therefore

$$\big|\tilde{x}(t)\big| \leqslant K_0 e^{\max(a_{11}(t),a_{22}(t))}, \quad \text{where } K_0 = 2\max\big(|r|, |s|\big).$$

Moreover from (5),

$$\left|x_1(t) - \frac{\lambda_1}{\mu}\right| \leqslant \left|x_1(0) - \frac{\lambda_1}{\mu}\right|e^{-\mu t}.$$

Hence writing $\tilde{\tilde{x}}(t) = (x_1(t) - \frac{\lambda_1}{\mu}, x_2(t), x_3(t) - \frac{\lambda_2}{\mu}, x_4(t))$,

$$\begin{aligned}
\left|\tilde{\tilde{x}}(t)\right|^2 &= \left(x_1(t) - \frac{\lambda_1}{\mu}\right)^2 + x_2^2(t) + \left(x_3(t) - \frac{\lambda_2}{\mu}\right)^2 + x_4^2(t) \\
&\leqslant 3\max\left(\left|\tilde{x}(t)\right|^2, \left|x_1(t) - \frac{\lambda_1}{\mu}\right|^2, \left|x_3(t) - \frac{\lambda_2}{\mu}\right|^2\right) \\
&\leqslant 3\max\left(K_1 e^{\max(2a_{11}(t), 2a_{22}(t), -2\mu t)}, K_2 e^{-2\mu t}\left|\int_0^t e^{a_{22}(s)+\mu s}\,ds\right|^2\right),
\end{aligned}$$

where $K_1 = \max(K_0^2, 4|\xi(0)|^2, |x_1(0) - \frac{\lambda_1}{\mu}|^2)$ and $K_2 = 4|\beta c\alpha_2^2 x_4(0)|^2$. Hence for any $p > 0$

$$\left|\tilde{\tilde{x}}(t)\right|^p \leqslant 3^{\frac{p}{2}} \max\left(K_1^{\frac{p}{2}} e^{\max(pa_{11}(t), pa_{22}(t), -p\mu t)}, K_2^{\frac{p}{2}} e^{-p\mu t}\left|\int_0^t e^{a_{22}(s)+\mu s}\,ds\right|^p\right),$$

$$\begin{aligned}
E\left|\tilde{\tilde{x}}(t)\right|^p &\leqslant 3^{\frac{p}{2}} K_1^{\frac{p}{2}}\left(Ee^{pa_{11}(t)} + Ee^{pa_{22}(t)} + e^{-p\mu t}\right) \\
&\quad + K_2^{\frac{p}{2}} t^{p-1} e^{-p\mu t} E\left(\int_0^t e^{p(a_{22}(s)+\mu s)}\,ds\right) \\
&\qquad \text{for } p \geqslant 1 \text{ using Hölder's inequality,} \\
&\leqslant 3^{\frac{p}{2}} K_1^{\frac{p}{2}}\left(e^{-(\mu+\sigma+\frac{1}{2}\sigma_1^2)pt + \frac{p^2\sigma_1^2}{2}t} + e^{(\beta c\alpha_2^2(1-\Lambda) - (\mu+\sigma+D_{21}) - \frac{\sigma_1^2}{2})pt + \frac{p^2\sigma_1^2}{2}t} + e^{-p\mu t}\right) \\
&\quad + K_2^{\frac{p}{2}} t^{p-1} e^{-p\mu t}\int_0^t Ee^{p(a_{22}(s)+\mu s)}\,ds.
\end{aligned}$$

The last integral is

$$\begin{cases}
\dfrac{e^{p(\mu-\kappa)t + \frac{p^2\sigma_1^2}{2}t} - 1}{p(\mu-\kappa) + \frac{p^2\sigma_1^2}{2}}, & \text{if } \kappa \neq \mu + \dfrac{p\sigma_1^2}{2}, \\
t, & \text{if } \kappa = \mu + \dfrac{p\sigma_1^2}{2}.
\end{cases}$$

Hence

$$\begin{aligned}
E\left|\tilde{\tilde{x}}(t)\right|^p &\leqslant 3^{\frac{p}{2}} K_1^{\frac{p}{2}}\left(e^{-(\mu+\sigma+\frac{1}{2}\sigma_1^2)pt + \frac{p^2\sigma_1^2}{2}t}\right. \\
&\quad \left. + e^{(\beta c\alpha_2^2(1-\Lambda) - (\mu+\sigma+D_{21}) - \frac{\sigma_1^2}{2})pt + \frac{p^2\sigma_1^2}{2}t} + e^{-p\mu t}\right) \\
&\quad + K_3 t^{p-1} e^{-p\mu t} + K_4 t^{p-1} e^{(\beta c\alpha_2^2(1-\Lambda) - (\mu+\sigma+D_{21}) - \frac{\sigma_1^2}{2})pt + \frac{p^2\sigma_1^2}{2}t} \\
&\quad + K_5 t^p e^{-p\mu t}, \quad \text{for some constants } K_3, K_4 \text{ and } K_5.
\end{aligned}$$

For $(\mu+\sigma) - \max(0, \beta c\alpha_2^2(1-\Lambda) - D_{21}) > \frac{(p-1)\sigma_1^2}{2}$ the disease free equilibrium is pth moment exponentially stable. On the other hand, if $\frac{(p-1)\sigma_1^2}{2} > \mu + \sigma$, choose $\tilde{x}(0) = (1, 0)^T$, then

$$\tilde{x}(t) = \left(e^{a_{11}(t)}, 0\right)^T,$$
$$\left|\tilde{x}(t)\right|^p = e^{pa_{11}(t)},$$
$$E\left|\tilde{\tilde{x}}(t)\right|^p \geqslant Ee^{pa_{11}(t)} = e^{p[\frac{(p-1)\sigma_1^2}{2} - (\mu+\sigma)]t},$$

so in this case the disease free equilibrium is pth moment exponentially unstable (for any $p > 0$). Similarly if

$$\frac{(p-1)\sigma_1^2}{2} > (\mu + \sigma + D_{21}) - \beta c\alpha_2^2(1-\Lambda)$$

the disease free equilibrium is again pth moment exponentially unstable for any $p > 0$.

In the case where $\beta c\alpha_2^2(1-\Lambda) = D_{21}$ then

$$\tilde{x}(t) = \begin{pmatrix} e^{a_{11}(t)} & a_{12}(t)e^{a_{11}(t)} \\ 0 & e^{a_{22}(t)} \end{pmatrix} \tilde{x}(0)$$

and a similar argument shows that if

$$(\mu+\sigma) - \max\left(0, \beta c\alpha_2^2(1-\Lambda) - D_{21}\right) > \frac{(p-1)\sigma_1^2}{2}$$

the disease free equilibrium is pth moment exponentially stable for $p \geqslant 1$ and pth moment exponentially unstable for any $p > 0$ if

$$(\mu+\sigma) - \max\left(0, \beta c\alpha_2^2(1-\Lambda) - D_{21}\right) < \frac{(p-1)\sigma_1^2}{2}.$$

It is possible to show that the disease free equilibrium is locally pth moment exponentially stable if $0 < p < 1$ and $R_0 < 1$, but as the most important L^p-spaces are where p is a positive integer we do not pursue this here. One can also show that the disease free equilibrium is locally stochastically asymptotically stable in the large if $R_1 < 1$ and locally stochastically unstable if $R_1 > 1$ but the proofs are omitted because of lack of space (see [13, p. 110], for the definitions of stochastic asymptotic stability in the large and stochastic instability).

6. Results and discussion

In this paper we have discussed a model which represents the spread of HIV among homosexuals which also incorporates environmental noise in it. To do this, we used a four-dimensional model of stochastic differential equations. Our main aim was to check the effect of the noise in the system. We see that the noise term tends to stabilise the system for almost sure exponential stability and stability in probability, but not for pth moment exponential stability if $p \geqslant 1$.

Consider a single newly infected individual entering the disease free equilibrium. Let T denote the infectious period of this individual. Given $\mathcal{F}_{t+\Delta t}$ this individual leaves the infectious class during time $[t, t+\Delta t)$ with approximate probability $(\mu+\sigma+D_{21})\Delta t + \sigma_1(B(t+\Delta t) - B(t))$. Hence given $\mathcal{F}_t$ the total probability that the individual is still infectious at time t is

$$P(T \geqslant t \mid \mathcal{F}_t) \approx \min\left\{e^{-(\mu+\sigma+D_{21})t - \sigma_1 B(t)}, 1\right\}.$$

Therefore,

$$\begin{aligned} P(T \geqslant t) &\approx E\big(P(T \geqslant t \mid \mathcal{F}_t)\big) \\ &\approx E\big(\exp\big[-(\mu+\sigma+D_{21})t - \sigma_1 B(t)\big]\big) \\ &\approx \exp\left[-\left((\mu+\sigma+D_{21}) - \frac{\sigma_1^2}{2}\right)t\right]. \end{aligned}$$

So provided that $\mu+\sigma+D_{21} > \frac{\sigma_1^2}{2}$,

$$ET \approx \frac{1}{\mu+\sigma+D_{21}-\frac{\sigma_1^2}{2}},$$

and $R_2 \approx \beta c \alpha_2^2 (1-\Lambda) ET$ is the expected number of cases caused by a single infected individual entering the disease free equilibrium.

Hence for $p=2$ if $R_2 < 1$ and $(\mu+\sigma) > \frac{\sigma_1^2}{2}$ then the disease free equilibrium is mean square exponentially stable, but if $R_2 > 1$ or $\frac{\sigma_1^2}{2} > (\mu+\sigma)$ the disease free equilibrium is mean square exponentially unstable. In this sense R_2 has the traditional meaning of a threshold parameter between stability and instability.

We also see that the noise does not change the system too much. The significance of the biological parameters is not affected. One example is the importance of condom use in reducing the spread of the disease. It is still a very important parameter in the system of stochastic differential equations in the sense that it helps determine potential elimination of the disease from the population. However improvements can be made to the deterministic model with the addition of the noise term. As the real world is not deterministic it is important to examine the inclusion of stochastic effects into deterministic models and this paper is a step towards doing this. Our research indicates that introducing environmental noise into the deterministic model can have a stabilising effect.

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