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Note

Weyl type theorems for operators satisfying the single-valued extension property

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Abstract

Let T be a bounded linear operator acting on a Banach space X such that T or its adjoint T^* has the single-valued extension property. We prove that the spectral mapping theorem holds for the B-Weyl spectrum, and we show that generalized Browder's theorem holds for $f(T)$ for every analytic function f defined on an open neighborhood U of $\sigma(T)$. Moreover, we give necessary and sufficient conditions for such T to satisfy generalized Weyl's theorem. Some applications are also given.

Keywords: B-Fredholm operator; B-Weyl spectrum; Generalized Weyl's theorem; Single-valued extension property

1. Introduction

Throughout this note, X denotes an infinite dimensional complex Banach space and $\mathcal{L}(X)$ denotes the algebra of all bounded linear operators on X. For an operator $T \in \mathcal{L}(X)$, write T^*, $\sigma(T)$, $\rho(T)$, $\sigma_p(T)$, $\operatorname{iso}\sigma(T)$ and $\operatorname{acc}\sigma(T)$ for the adjoint, spectrum, resolvent set, point spectrum of T, isolated points and accumulation points of $\sigma(T)$, respectively. By $\alpha(T)$ and $\beta(T)$ we denote the dimension of the Kernel $N(T)$ and the codimension of the range $R(T)$, respectively. If both $\alpha(T)$ and $\beta(T)$ are finite, then T is called a *Fredholm* operator and the *index* of T is defined by

$$\operatorname{ind}(T) = \alpha(T) - \beta(T).$$

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A bounded linear operator $T \in \mathcal{L}(X)$ is said to be a *Weyl* operator if it is Fredholm of index 0. Recall that the *ascent* $p := p(T)$ of an operator T is the smallest nonnegative integer p such that $N(T^p) = N(T^{p+1})$. If such integer does not exist we put $p(T) = \infty$. Analogously, the *descent* $q := q(T)$ of an operator T is the smallest nonnegative integer q such that $R(T^q) = R(T^{q+1})$, and if such integer does not exist we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [15, Proposition 38.3]. $T \in \mathcal{L}(X)$ is said to be a *Browder* operator if T is Fredholm with $p(T) = q(T) < \infty$. Note that if T is Browder then T is Weyl, see [15, Proposition 38.5]. We shall henceforth abbreviate $T - \lambda I$ to $T - \lambda$. The Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ are defined by (see [14])

$$\sigma_w(T) := \{\lambda \in \mathbb{C}\colon\ T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) := \{\lambda \in \mathbb{C}\colon\ T - \lambda \text{ is not Browder}\}.$$

For a bounded linear operator T and a nonnegative integer n define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_0 = T$). If for some integer n the range space $R(T^n)$ is closed and T_n is a Fredholm operator then T is called a *B-Fredholm* operator. The class of B-Fredholm operators contains the class of Fredholm operators as a proper subclass [4]. Let T be a B-Fredholm operator and let n be any integer such that T_n is a Fredholm operator. Then T_m is a Fredholm operator and $\operatorname{ind}(T_n) = \operatorname{ind}(T_m)$ for each $m \geqslant n$. The index of T, $\operatorname{ind}(T)$, is defined to be the index of the Fredholm operator T_n, see [4, Definition 2.3]. If T is a B-Fredholm operator of index 0, then T is called a *B-Weyl* operator. The B-Weyl spectrum $\sigma_{\mathrm{BW}}(T)$ of T is defined by (see [5])

$$\sigma_{\mathrm{BW}}(T) = \{\lambda \in \mathbb{C}\colon\ T - \lambda \text{ is not B-Weyl operator}\}.$$

We write $\rho_{\mathrm{BW}}(T) := \mathbb{C} \setminus \sigma_{\mathrm{BW}}(T)$ for the resolvent B-Weyl set.

In [4, Theorem 2.7] it is proved that T is B-Fredholm if and only if there exists two closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T|_M$ is a Fredholm operator and $T|_N$ is a nilpotent operator. The proof is based on the decomposition of quasi-Fredholm operators of Labrousse [18] which was proved only for Hilbert-spaces operators. This gap was subsequently filled by Müller in [26, Theorem 7].

The classical Weyl's theorem initiated by Hermann Weyl in [29], asserts that if T is a self-adjoint operator acting on Hilbert space, then we have $\sigma_w(T) = \sigma(T) \setminus E_0(T)$, where $E_0(T)$ is the set of isolated eigenvalues of finite multiplicity of T. Note that $T \in \mathcal{L}(X)$ satisfies *Weyl's theorem* if

$$\sigma_w(T) = \sigma(T) \setminus E_0(T).$$

Analogously T satisfies *Browder's* theorem if

$$\sigma_w(T) = \sigma(T) \setminus \pi_0(T),$$

where $\pi_0(T)$ is the set of poles of the resolvent of T of finite rank. A generalization of these two notions to the class of B-Fredholm operators are given in [8]; precisely, $T \in \mathcal{L}(X)$ satisfies *generalized Weyl's theorem* if

$$\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus E(T),$$

where $E(T)$ is the set of all isolated eigenvalues of T, and T satisfies *generalized Browder's theorem* if

$$\sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus \pi(T),$$

where $\pi(T)$ is the set of all poles of the resolvent of T. Note that if T satisfies generalized Weyl's theorem then T satisfies generalized Browder's theorem, see [5, Corollary 2.6]. Moreover, in [8] it is shown that if T satisfies generalized Weyl's theorem, then it satisfies Weyl's theorem, and if T satisfies generalized Browder's theorem, then it satisfies Browder's theorem.

Generalized Weyl's theorem has been studied in [5,8]. It has been established for operators T acting on a Hilbert space such that T is hyponormal [7]. In this paper, we study generalized Weyl's theorem and generalized Browder's theorem for operators T acting on a Banach space such that T or T^* has the SVEP. In Section 2, we prove that the spectral mapping theorem holds for the B-Weyl spectrum $\sigma_{\mathrm{BW}}(T)$, and we show that generalized Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$, where $H(\sigma(T))$ denotes the set of all analytic functions defined on an open neighborhood U of $\sigma(T)$. Section 3 is devoted to an application of the results obtained in the previous section.

2. Main results

Let T be a bounded linear operator on X. We say that T has the *single-valued extension property* at λ_0, SVEP (for short), if for every open neighborhood U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation

$$(T-\lambda)f(\lambda) = 0 \quad \text{for all } \lambda \in U$$

is the function $f \equiv 0$. We say that T has the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$ (see [20]).

In [7] it is shown that the spectral mapping theorem holds for the B-Weyl spectrum $\sigma_{\mathrm{BW}}(T)$ whenever T is hyponormal. In the following we will give more for Banach space operators. For this we start with the next result.

Proposition 2.1. *Let $T \in \mathcal{L}(X)$. Then*

(i) *If T has the SVEP, then* $\operatorname{ind}(T-\lambda) \leqslant 0$ *for every* $\lambda \in \rho_{\mathrm{BF}}(T)$.
(ii) *If T^* has the SVEP, then* $\operatorname{ind}(T-\lambda) \geqslant 0$ *for every* $\lambda \in \rho_{\mathrm{BF}}(T)$.

Proof. (i) If $\lambda \in \rho_{\mathrm{BF}}(T)$, then $T-\lambda$ is B-Fredholm. For some n large enough, $T-(\lambda+\frac{1}{n})$ is a Fredholm operator and $\operatorname{ind}(T-(\lambda+\frac{1}{n})) = \operatorname{ind}(T-\lambda)$, see [6, Remark A]. If T has the SVEP, then $T-(\lambda+\frac{1}{n})$ also has the SVEP. By virtue of [3, Theorem 2.6], we conclude that $\operatorname{ind}(T-(\lambda+\frac{1}{n})) \leqslant 0$. Thus $\operatorname{ind}(T-\lambda) \leqslant 0$, which prove (i).

(ii) Follows from [3, Theorem 2.6] and the fact that $\operatorname{ind}(T^*) \geqslant 0$ whenever T^* has the SVEP. □

Theorem 2.1. *If T or T^* has the SVEP, then $f(\sigma_{\mathrm{BW}}(T)) = \sigma_{\mathrm{BW}}(f(T))$, for every $f \in H(\sigma(T))$.*

Proof. This follows directly from Proposition 2.1 and [7, Theorem 2.4]. □

The analytic core of an operator $T \in \mathcal{L}(X)$ is the subspace

$$K(T) := \big\{x \in X\colon\ Tx_{n+1} = x_n,\ Tx_1 = x,\ \|x_n\| \leqslant c^n\|x\| \\ (n = 1, 2, \ldots) \text{ for some } c > 0,\ x_n \in X\big\}.$$

The quasi-nilpotent part of T is the subspace

$$H_0(T) := \left\{x \in X\colon \lim_{n\to\infty} \left\|T^n x\right\|^{\frac{1}{n}} = 0\right\}.$$

The spaces $K(T)$ and $H_0(T)$ are hyperinvariant under T and satisfy $T^{-n}(0) \subset H_0(T)$, $K(T) \subset T^n(X)$ for all $n \in \mathbb{N}$ and $TK(T) = K(T)$, see the recent book of Aiena [1] and [23,25] for more information about these subspaces.

Next, we shall consider the class of operators $T \in \mathcal{L}(X)$ for which the condition $K(T) = \{0\}$ holds. This class was introduced by Mbekhta in [24] in the case of Hilbert space and studied in more general setting of Banach spaces, see [1]. Such condition is verified by every weighted unilateral right shift T on $l^p(\mathbb{N})$ $(1 \leqslant p < \infty)$ defined by

$$Te_n = \omega_n e_{n+1},$$

where the weight $(w_n)_{n\in\mathbb{N}}$ is a bounded sequence of positive numbers, and $(e_n)_{n\in\mathbb{N}}$ stands for the canonical basis of $l^p(\mathbb{N})$. In fact, for these operators it is easily seen that $K(T) = \{0\}$.

Lemma 2.1. *Let $T \in \mathcal{L}(X)$. If T is a quasi-nilpotent and B-Fredholm, then T is a nilpotent.*

Proof. Suppose that T is a B-Fredholm operator. Then there exists two closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $T|_M$ is a Fredholm operator and $T|_N$ is a nilpotent operator [26, Theorem 7]. If T is quasi-nilpotent, then $T|_M$ is quasi-nilpotent, and by [23, Corollary 2.15] we conclude that $T|_M$ is nilpotent. So T is a nilpotent operator. □

Proposition 2.2. *Let $T \in \mathcal{L}(X)$. If $K(T) = \{0\}$, then*

$$\sigma(T) = \sigma_{\mathrm{BW}}(T).$$

Proof. Suppose that $K(T) = \{0\}$. Since we have $\sigma_{\mathrm{BW}}(T) \subseteq \sigma(T)$, then it suffices to show that $\sigma(T) \subseteq \sigma_{\mathrm{BW}}(T)$. If $\lambda \notin \sigma_{\mathrm{BW}}(T)$, then $T - \lambda$ is a B-Fredholm operator of index 0, and hence by [26, Theorem 7] there exists two closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $(T - \lambda)|_M$ is a Fredholm operator and $(T - \lambda)|_N$ is a nilpotent operator. By the argument used in the proof of [6, Lemma 4.1], we conclude that $(T - \lambda)|_M$ is a Fredholm operator of index 0. If $\lambda \neq 0$, then

$$N(T - \lambda) \subseteq K(T) = \{0\}.$$

Hence $N = 0$ and $T - \lambda = (T - \lambda)|_M$ is a Fredholm operator of index 0. From [1, Theorem 3.116], we get that $\lambda \notin \sigma(T)$.

If $\lambda = 0$, then $T|_M$, is Fredholm of index 0, and hence $0 \notin \sigma_w(T|_M)$. Since $K(T|_M) = \{0\}$, then $\sigma_w(T|_M) = \sigma(T|_M)$ and $0 \in \sigma(T|_M)$, see [1, Theorem 3.116] and [1, Theorem 2.82]. Hence $0 \in \sigma_w(T|_M)$, which is a contradiction. This implies that $0 \in \sigma_{\mathrm{BW}}(T)$. Since $0 \in \sigma(T)$ (see [1, Theorem 2.82]), then we conclude that $\sigma(T) \subseteq \sigma_{\mathrm{BW}}(T)$. □

Theorem 2.2. *Let $T \in \mathcal{L}(X)$. If there exists a complex number $\lambda_0 \in \operatorname{acc}\sigma(T)$ such that $K(T - \lambda_0) = \{0\}$ or $K(T^* - \lambda_0) = \{0\}$, then the generalized Weyl's theorem holds for both $f(T)$ and $f(T^*)$ for every $f \in H(\sigma(T))$.*

Proof. If $K(T) = \{0\}$ or $K(T^*) = \{0\}$, then T or T^* has the SVEP and $\sigma(T)$ is connected and contains 0, see [1, Theorem 2.82] and [1, Theorem 3.116]. Also, if $0 \in \operatorname{acc}\sigma(T)$, then $\sigma(T)$ does not have any isolated point, otherwise $\sigma(T) = \{0\}$. Let $f \in H(\sigma(T))$. Since the identity operator satisfies generalized Weyl's theorem, we may assume that the function f is nonconstant. Hence

$$f\big(\sigma(T)\big) = \sigma\big(f(T)\big) = \sigma\big(f(T^*)\big)$$

is a connected subset of $\mathbb{C}$ without isolated points. So

$$E\big(f(T)\big) = E\big(f(T^*)\big) = \emptyset.$$

Moreover, by Theorem 2.1 and Proposition 2.2 above we have that

$$\sigma\big(f(T)\big) = f\big(\sigma(T)\big) = f\big(\sigma_{\mathrm{BW}}(T)\big) = \sigma_{\mathrm{BW}}\big(f(T)\big) = \sigma_{\mathrm{BW}}\big(f(T^*)\big).$$

Finally, $f(T)$ and $f(T^*)$ satisfy generalized Weyl's theorem. This completes the proof, since generalized Weyl's theorem is translation-invariant. $\square$

As a consequence of this theorem we have the following result.

Corollary 2.1. *If $T \in \mathcal{L}(X)$ is a non-quasi-nilpotent weighted unilateral right shift T on $l^p(\mathbb{N})$ $(1 \leqslant p < \infty)$, then T satisfies generalized Weyl's theorem.*

In general, we cannot expect that generalized Weyl's theorem holds for an operator satisfying the SVEP.

Example 1. If $T \in l^2(\mathbb{N})$ is defined by

$$T(x_0, x_1, \ldots) = \left(\frac{1}{2}x_1, \frac{1}{3}x_2, \ldots\right) \quad \text{for all } (x_n) \in l^2(\mathbb{N}),$$

then T is quasi-nilpotent and hence has the SVEP. But T does not satisfy generalized Weyl's theorem, since $\sigma(T) = \sigma_{\mathrm{BW}}(T) = \{0\}$ and $E(T) = \{0\}$.

However, for generalized Browder's theorem we have the following theorem.

Remark 2.1. In [5, Theorem 2.4] it is shown that if T is a bounded linear operator acting on a Hilbert space then T satisfies generalized Browder's theorem if and only if T^* does. This result is also valid in more general setting of Banach spaces. Since the proof of this assertion is based on the fact that T is B-Fredholm if and only if T^* is B-Fredholm (see [6, Remark B]), which is also valid in Banach spaces by [20, Theorem A.1.10].

Theorem 2.3. *If T or T^* has the SVEP, then generalized Browder's theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.*

Proof. Suppose that T has the SVEP. $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{BW}}(T)$ implies that $T - \lambda$ is a B-Fredholm operator of index 0. By [26, Theorem 7] there exists two closed T-invariant subspaces M and N of X such that $X = M \oplus N$ and $(T - \lambda)|_M$ is a Fredholm operator and $(T - \lambda)|_N$ is a nilpotent operator. By the argument used in the proof of [6, Lemma 4.1], we conclude that $(T - \lambda)|_M$ is a Fredholm operator of index 0. Since $T|_M$ has the SVEP, then λ is a pole of $T|_M$ of finite rank (see [27, Theorem 2.9]), and hence λ is isolated in $\sigma(T|_M)$. This implies that if $\mu \in \mathbb{C}$ is such that $|\lambda - \mu| < \epsilon$, then $T - \mu|_M$ is invertible. Since $T - \mu|_N$ is invertible then $T - \mu$ is invertible. Hence λ is isolated in $\sigma(T)$. From [5, Theorem 2.3], we conclude that $\lambda \in \pi(T)$. Hence $\sigma(T) \setminus$

$\sigma_{BW}(T) \subseteq \pi(T)$. For the reverse inclusion, if $\lambda \in \pi(T)$, then $\lambda \in E(T)$, and hence λ is isolated in $\sigma(T)$. From [5, Theorem 2.3], we conclude that $T - \lambda$ is B-Fredholm of index 0. So

$$\lambda \in \sigma(T) \setminus \sigma_{BW}(T).$$

Consequently, $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$. That is T satisfies generalized Browder's theorem. If T^* has the SVEP, then by the above argument generalized Browder's theorem holds for T^*, hence it holds for T, see Remark 2.1. Finally, if $f \in H(\sigma(T))$, then by [20, Theorem 3.3.6], $f(T)$ or $f(T^*)$ satisfies the SVEP, and hence the above argument implies that generalized Browder's theorem holds for $f(T)$. □

Let $T \in \mathcal{L}(X)$. By Remark 2.1, generalized Browder's theorem holds for T if and only if it holds for T^*. However, generalized Weyl's theorem does not pass from an operator to its adjoint as shown by the following example:

Example 2. Let us consider the weighted unilateral right shift T on $l^2(\mathbb{N})$, defined by

$$Te_n = \omega_n e_{n+1},$$

where the bounded weight sequence $(\omega_n)_{n\in\mathbb{N}}$ of positive real numbers satisfying $\lim_{n\to\infty}(\sup_k | \omega_k\omega_{k+1}\cdots\omega_{k+n-1}|)^{1/n} = 0$. Since T is quasi-nilpotent, then $K(T) = 0$, and hence by Proposition 2.2,

$$\sigma(T) = \sigma_{BW}(T) = \{0\}.$$

Since $E(T) = \emptyset$, then T satisfies generalized Weyl's theorem. However the adjoint T^* of T defined by

$$T^*e_0 = 0 \quad \text{and} \quad T^*e_n = \omega_{n-1}e_{n-1}$$

does not satisfy generalized Weyl's theorem, since $E(T^*) = \{0\}$.

Let $T \in \mathcal{L}(X)$ such that T or T^* has the SVEP. In what follows, we will give a necessary and sufficient conditions for T to satisfy generalized Weyl's theorem.

Proposition 2.3. *If T or T^* has the SVEP, then*

(i) *generalized Weyl's theorem holds for T if and only if*

$$E(T) = \pi(T),$$

(ii) *generalized Weyl's theorem holds for T^* if and only if*

$$E(T^*) = \pi(T^*).$$

Proof. If T or T^* has the SVEP, then by Theorem 2.3,

$$\sigma_{BW}(T) = \sigma(T) \setminus \pi(T).$$

(i) If T satisfies generalized Weyl's theorem, then $E(T) = \pi(T)$, see [5, Corollary 2.6]. Conversely, if $E(T) = \pi(T)$, then we have

$$\sigma_{BW}(T) = \sigma(T) \setminus \pi(T) = \sigma(T) \setminus E(T),$$

which completes the proof of (i).

Using the same argument we prove (ii). □

Recall that $T \in \mathcal{L}(X)$ is said to have growth condition G_d ($d \geqslant 1$) [28], if there is a constant $c > 0$ such that

$$\left\|(T-\lambda)^{-1}\right\| \leqslant \frac{c}{\operatorname{dis}(\lambda, \sigma(T))^d} \quad \text{for } \lambda \notin \sigma(T).$$

The condition G_d does not generally, ensure that T has the SVEP, see [21]. But from the proof of [27, Corollary 2.14], we conclude that if T satisfies G_d, then for all $\lambda \in \operatorname{iso}\sigma(T)$ there exists an integer $d_\lambda \geqslant 1$ such that $H_0(T-\lambda) = N(T-\lambda)^{d_\lambda}$ and T is isoloid. Here, we recall that T is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T. This leads us to give the following result.

Proposition 2.4. *If T or T^* has the SVEP and for all $\lambda \in \operatorname{iso}\sigma(T)$ there exists an integer $d_\lambda \geqslant 1$ such that $H_0(T-\lambda) = N(T-\lambda)^{d_\lambda}$, then T satisfies generalized Weyl's theorem.*

Proof. Suppose that T or T^* has the SVEP. From Proposition 2.3, it suffices to show that

$$E(T) = \pi(T).$$

If $\lambda \in E(T)$, then $\lambda \in \operatorname{iso}\sigma(T)$. Since $H_0(T-\lambda) = N(T-\lambda)^{d_\lambda}$ for $d_\lambda \geqslant 1$, then by [23, Theorem 1.6] and [17, Theorem 3.4] we conclude that λ is a pole of T of order d_λ. Thus, $\lambda \in \pi(T)$. Hence $E(T) \subseteq \pi(T)$. As we always have $\pi(T) \subseteq E(T)$, then we get the desired equality. □

Theorem 2.4. *Let $T \in \mathcal{L}(X)$ such that T or T^* has the SVEP and for all $\lambda \in \operatorname{iso}\sigma(T)$ there exists an integer $d_\lambda \geqslant 1$ such that $H_0(T-\lambda) = N(T-\lambda)^{d_\lambda}$, then generalized Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T))$.*

Proof. Under the hypothesis and from Theorem 2.1 and Proposition 2.4, we conclude that

$$\sigma_{\mathrm{BW}}\big(f(T)\big) = f\big(\sigma_{\mathrm{BW}}(T)\big) \quad \text{and} \quad \sigma_{\mathrm{BW}}(T) = \sigma(T) \setminus E(T).$$

Hence

$$\sigma_{\mathrm{BW}}\big(f(T)\big) = f\big(\sigma_{\mathrm{BW}}(T)\big) = f\big[\sigma(T) \setminus E(T)\big].$$

Since T is isoloid, then by [7, Lemma 2.9] we have

$$f\big[\sigma(T) \setminus E(T)\big] = \sigma\big(f(T)\big) \setminus E\big(f(T)\big).$$

So

$$\sigma_{\mathrm{BW}}\big(f(T)\big) = f\big(\sigma_{\mathrm{BW}}(T)\big) = f\big[\sigma(T) \setminus E(T)\big] = \sigma\big(f(T)\big) \setminus E\big(f(T)\big).$$

Thus $f(T)$ satisfies generalized Weyl's theorem. □

As a consequence of this theorem we have the following result:

Corollary 2.2. *Let $T \in \mathcal{L}(X)$ such that T or T^* has the SVEP. If T satisfies the growth condition G_d ($d \geqslant 1$), then generalized Weyl's theorem holds for $f(T)$, for every $f \in H(\sigma(T))$.*

3. Applications

One such class which has attracted the attention of a number of authors is the set $\mathcal{P}(X)$ of all operators $T \in \mathcal{L}(X)$ such that for every complex number λ there exists an integer $d_\lambda \geqslant 1$ for which the following condition holds

$$H_0(T-\lambda) = N(T-\lambda)^{d_\lambda}.$$

The class $\mathcal{P}(X)$ contains the classes of subscalar, algebraically totally paranormal and transaloid operators on a Banach space, $*$-totally paranormal, M-hyponormal, p-hyponormal $(0 < p < 1)$ and log-hyponormal operators on a Hilbert space (see [9,10,12,13,16,22,27]).

It is known that if $H_0(T-\lambda)$ is closed for every complex number λ, then T has the SVEP, see [2,19]. So that, the SVEP is shared by all the operators of $\mathcal{P}(X)$.

For an hyponormal operator T, it is shown in [7] that generalized Weyl's theorem hold for $f(T)$ for every $f \in H(\sigma(T))$. In the following we give more.

Corollary 3.1. *If $T \in \mathcal{P}(X)$. Then $f(T)$ satisfies generalized Weyl's theorem for every $f \in H(\sigma(T))$.*

Proof. Apply Theorem 2.4. □

Now, let us consider an elementary operator $d_{A,B} \in \mathcal{L}(\mathcal{L}(\mathcal{H}))$, where $\mathcal{H}$ is a Hilbert space, $A, B \in \mathcal{L}(\mathcal{H})$, and $d_{A,B}$ is either the generalized derivation $\delta_{A,B}(X) = AX - XB$ or the elementary operator $\Delta_{A,B}(X) = AXB - X$ $(X \in \mathcal{H})$.

The following corollary extends [11, Theorem 3.1].

Corollary 3.2. *Let $A, B \in \mathcal{L}(\mathcal{H})$. If A and B^* are hyponormal operators, then $f(d_{A,B})$ satisfies generalized Weyl's theorem for every $f \in H(\sigma(T))$.*

Proof. From [11, Corollary 2.4], we have $p(d_{A,B}) \leqslant 1$ for all complex numbers λ, hence $d_{A,B}$ has the SVEP. On the other hand, from the proof of [11, Theorem 2.7] we can deduce that $H_0(d_{A,B} - \lambda) = N(d_{A,B} - \lambda)$ for all $\lambda \in \operatorname{iso}\sigma(d_{A,B})$. Applying Theorem 2.4, we get the desired result. □

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References

[1] P. Aiena, Fredholm Theory and Local Spectral Theory, with Applications to Multipliers, Kluwer Academic, 2004.
[2] P. Aiena, M.L. Colasante, M. González, Operators which have a closed quasi-nilpotent part, Proc. Amer. Math. Soc. 130 (2002) 2701–2710.
[3] P. Aiena, O. Monsalve, The single valued extension property and the generalized Kato decomposition property, Acta Sci. Math. (Szeged) 67 (2001) 461–477.
[4] M. Berkani, On a class of quasi-Fredholm operators, Integral Equations Operator Theory 34 (1999) 244–249.
[5] M. Berkani, B-Weyl spectrum and poles of the resolvent, J. Math. Anal. Appl. 272 (2002) 596–603.

[6] M. Berkani, Index of B-Fredholm operators and generalization of a Weyl theorem, Proc. Amer. Math. Soc. 130 (2002) 1717–1723.
[7] M. Berkani, A. Arroud, Generalized Weyl's theorem and hyponormal operators, J. Aust. Math. Soc. 76 (2004) 291–302.
[8] M. Berkani, J.J. Koliha, Weyl type theorems for bounded linear operators, Acta Sci. Math. (Szeged) 69 (2003) 359–376.
[9] L.A. Coburn, Weyl's theorem for nonnormal operators, Michigan Math. J. 13 (1966) 285–288.
[10] R.E. Curto, Y.M. Han, Weyl's theorem, a-Weyl's theorem, and local spectral theory 2002, J. London Math. Soc. (2) 67 (2003) 499–509.
[11] B.P. Duggal, Weyl's theorem for a generalized derivation and an elementary operator derivation, Math. Vestnik 54 (2002) 71–81.
[12] B.P. Duggal, S.V. Djordjević, Dunford's property and Weyl's theorem, Integral Equations Operator Theory 43 (2002) 290–297.
[13] Y.M. Han, A.H. Kim, A note on $*$-paranormal operators, Integral Equations Operator Theory 49 (2004) 435–444.
[14] R.E. Harte, Fredholm, Weyl and Browder theory, Proc. R. Irish Acad. A 85 (2) (1985) 151–176.
[15] H. Heuser, Functional Analysis, Marcel Dekker, New York, 1982.
[16] J.C. Kim, On Weyl spectra of algebraically totally-paranormal operators, Bull. Korean Math. Soc. 39 (2002) 571–575.
[17] J.J. Koliha, Isolated spectral points, Proc. Amer. Math. Soc. 124 (1996) 3417–3424.
[18] J.P. Labrousse, Les opérateurs quasi-Fredholm: une généralisation des opérateurs semi-Fredholm, Rend. Circ. Math. Palermo 29 (1980) 161–258.
[19] K.B. Laursen, Operators with finite ascent, Pacific J. Math. 152 (1992) 323–336.
[20] L. Laursen, M.M. Neumann, An Introduction to Local Spectral Theory, London Math. Soc. Monographs, vol. 20, Clarendon Press, Oxford, 2000.
[21] S.H. Lee, W.Y. Lee, A spectral mapping theorem for the Weyl spectrum, Glasgow Math. J. 38 (1996) 61–64.
[22] L. Chen, R. Yingbin, Y. Zikun, p-hyponormal operators are subscalar, Proc. Amer. Math. Soc. 131 (2003) 2753–2759.
[23] M. Mbekhta, Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux, Glasgow Math. J. 29 (1987) 159–175.
[24] M. Mbekhta, Sur la théorie spectrale locale et limite de nilpotents, Proc. Amer. Math. Soc. 110 (1990) 621–631.
[25] M. Mbekhta, Ouahab, Opérateurs s-regulier dans un espace de Banach et théorie spectrale, Acta Sci. Math. (Szeged) 59 (1994) 525–543.
[26] V. Müller, On the Kato-decomposition of quasi-Fredholm and B-Fredholm operators, Preprint ESI 1013, Vienna, 2001.
[27] M. Oudghiri, Weyl's and Browder's theorem for operators satisfying the SVEP, Studia Math. 163 (1) (2004).
[28] B.L. Wadhwa, Spectral, M-hyponormal and decomposable operators, Ph.D. thesis, Indiana University, 1971.
[29] H. Weyl, Über beschränkte quadratische Formen, deren Differenz vollstetig ist, Rend. Circ. Mat. Palermo 27 (1909) 373–392.