

Singular solutions for semi-linear parabolic equations on nonsmooth domains

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Abstract

We prove the existence of positive singular solutions for the semi-linear parabolic equation $\Delta_x u - \frac{\partial}{\partial t} u + \mu u^p = 0$ on $\Omega = D \times]0, \infty[$, where $p > 1$, D is a bounded NTA-domain in \mathbb{R}^n , $n \geq 2$, and μ is in a general class of signed Radon measures on D covering the elliptic Kato class of potentials adopted by Zhang and Zhao. A new proof of the result based on a simple fixed point theorem is also given.

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1. Introduction

In the recent years, the study of the semi-linear parabolic equation $\Delta_x u - \frac{\partial}{\partial t} u + V u^p = 0$ has been the object of many works (see for example [5,10–12] and [6]). In particular, the existence of global positive solutions for this equation has been studied in different cases of domains and classes of potentials V . In this paper, we will study the existence of positive continuous solutions for the semi-linear parabolic problem

$$(\mathcal{P}) \quad \begin{cases} \Delta_x u(x, t) - \frac{\partial}{\partial t} u(x, t) + \mu u^p(x, t) = 0, & (x, t) \in D \setminus \{0\} \times]0, \infty[, \\ \lim_{x \rightarrow 0} u(x, t) = \infty, & t > 0, \\ u(x, t) = 0, & (x, t) \in \partial D \times]0, \infty[, \\ u(x, 0) = u_0(x), & x \in D, \end{cases}$$

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where $p > 1$, D is a bounded non-tangentially accessible (NTA)-domain (see the definition in [4]) containing 0 in \mathbb{R}^n , $n \geq 2$, and μ is a signed Radon measure on D . By solutions we mean solutions in the distributional sense. We recall that in the dimension $n \geq 3$, Zhang and Zhao [12] studied the problem (\mathcal{P}) in the special cases $\mu = V(x)dx$ and D is a bounded Lipschitz domain; and they proved that when u_0 and V are nonnegative functions with $V(x)/|x|^{(n-2)(p-1)}$ in the elliptic Kato class K_n , there exists a constant $M > 0$ such that if $u_0(x) \leq MG(x, 0)$ the problem (\mathcal{P}) has a positive continuous solution u satisfying $u(x, t) \leq CG(x, 0)$ for all $t > 0$, where G is the Laplacian Green function with Dirichlet boundary condition on D . Our first aim in this work is to extend this result to NTA-domains and more general class of signed Radon measures. More precisely, we prove the existence of positive solutions for the parabolic problem (\mathcal{P}) when the measure $G^{p-1}(\cdot, 0)\mu$ is in the new class $\mathcal{K}_n^{\text{loc}}(D)$ introduced in [9] in the study of the Schrödinger equation on bounded Lipschitz domains. The second aim is to present a new and short proof based on a fixed point theorem simpler than the Schauder theorem used by Zhang and Zhao in [12]. We also obtain, by studying the large time behavior of the solutions of the problem (\mathcal{P}) , the existence of positive solutions for the counterpart elliptic problem. Before stating our main results, we first recall the definition of the class $\mathcal{K}_n^{\text{loc}}(D)$.

Definition 1.1. Let μ be a signed Radon measure on D . We say that μ is in the class $\mathcal{K}_n^{\text{loc}}(D)$ if it satisfies

$$\|\mu\| \equiv \sup_{x \in D} \int_D \frac{\varphi(y)}{\varphi(x)} G(x, y) |\mu|(dy) < \infty, \quad (1.1)$$

and for any compact subset $E \subset D$,

$$\limsup_{r \rightarrow 0} \sup_{x \in E} \int_{D \cap (|x-y| < r)} G(x, y) |\mu|(dy) = 0, \quad (1.2)$$

where G is the Laplacian Green function with Dirichlet boundary condition on D and $\varphi(x) = 1 \wedge G(x, 0)$.

Our main results are the followings.

Theorem 1.2. Let μ be a nonnegative Radon measure on D such that $G^{p-1}(\cdot, 0)\mu$ is in the class $\mathcal{K}_n^{\text{loc}}(D)$. Then, there exists a number $\lambda_0 > 0$ such that for any $\lambda \in]0, \lambda_0[$ and any non-negative function u_0 satisfying $u_0(x) \leq \lambda G(x, 0)$, for all $x \in D$, the problem (\mathcal{P}) has a positive continuous solution u satisfying for all $(x, t) \in \Omega = D \times]0, \infty[$,

$$u(x, t) \leq 3\lambda G(x, 0).$$

Remark 1.3. The solutions of the homogenous problem (\mathcal{P}) corresponding to $\mu \equiv 0$ are the functions $v_\lambda(x, t) = \lambda \int_0^t g(x, s; 0, 0) ds + \int_D g(x, t; y, 0) u_0(y) dy$, $\lambda > 0$, where $g(x, t; y, 0)$ is the Dirichlet heat kernel on D . The solution u obtained in Theorem 1.2 also satisfies the lower bound $v_\lambda(x, t) \leq u(x, t)$, for all $(x, t) \in \Omega$.

Corollary 1.4. *Under the assumptions in Theorem 1.2, there exists a sequence (t_k) , $t_k \rightarrow \infty$, such that $u(x, t_k)$ converges, as $k \rightarrow \infty$, locally uniformly on $D \setminus \{0\}$ to a continuous solution u_∞ of the elliptic problem*

$$(E) \quad \begin{cases} \Delta_x u(x) + \mu u^p(x) = 0, & x \in D \setminus \{0\}, \\ \lim_{x \rightarrow 0} u(x) = \infty, \\ u(x) = 0, & x \in \partial D, \end{cases}$$

satisfying

$$\lambda G(x, 0) \leq u_\infty(x) \leq 3\lambda G(x, 0).$$

Remark 1.5. Note that the measure μ may be singular with respect to the Lebesgue measure. Moreover, if we denote by *Lip* and *NTA*, the sets of bounded Lipschitz domains and bounded NTA-domains, respectively; we have $Lip \subsetneq NTA$. In particular our result extends the one proved in [12] to a more general class of potentials and a more general class of domains.

Remark 1.6. Let us denote by *Unif* and *Jor*, the sets of bounded uniform domains (see the definition in [1]) in \mathbb{R}^n , $n \geq 2$, and bounded Jordan domains (see the definition in [2]) in \mathbb{R}^2 , respectively. We have $NTA \subsetneq Unif$ for $n \geq 2$, $NTA \not\subseteq Jor$ and $Jor \not\subseteq NTA$, for $n = 2$. In contrast to NTA-domains, the uniform and Jordan domains may be irregular for the Dirichlet problem and their Green functions vanish outside a polar set of ∂D (see [1]). Replacing the condition “ $u(x, t) = 0$, $(x, t) \in \partial D \times]0, \infty[$ ” by “ $u(x, t) = 0$ outside a polar set on $\partial D \times]0, \infty[$ ” and using the same arguments, based on the 3G-Theorems established in [3] and [8], we may prove the existence of positive solutions for the problem (\mathcal{P}) on uniform domains in \mathbb{R}^n , $n \geq 3$ and Jordan domains in \mathbb{R}^2 .

2. Preliminaries and known results

Throughout the paper D means a bounded non-tangentially accessible (NTA)-domain (see [4]) in \mathbb{R}^n , $n \geq 2$, containing 0. For $x, y \in D$ and $t, s \in \mathbb{R}$, $t > s$, let $g = g(x, t; y, s)$ be the Green function of the heat operator $\Delta_x - \frac{\partial}{\partial t}$ for the initial Dirichlet problem on $\Omega = D \times]0, \infty[$. We know that g satisfies the reproducing property

$$g(x, t; y, s) = \int_D g(x, t; z, \tau) g(z, \tau; y, s) dz,$$

for all $x, y \in D$ and $s < \tau < t$, and

$$G(x, y) = \int_0^\infty g(x, t; y, 0) dt \tag{2.1}$$

is the Green function of the Laplacian Δ_x with Dirichlet boundary condition on D . The following 3G-Theorem is proved on Lipschitz domains (see [9]) and NTA-domains (see [3]) in \mathbb{R}^n , $n \geq 3$ and on Jordan domains in \mathbb{R}^2 (see [8]). It is also true for NTA-domains in \mathbb{R}^2 by similar arguments as in [3].

Theorem 2.1 (3G-Theorem). *There exists a constant $C_0 = C_0(n, D) > 0$ such that for all $x, y, z \in D$,*

$$\frac{G(x, z)G(z, y)}{G(x, y)} \leq C_0 \left[\frac{\varphi(z)}{\varphi(x)} G(x, z) + \frac{\varphi(z)}{\varphi(y)} G(z, y) \right].$$

Corollary 2.2. Let $\mu \in \mathcal{K}_n^{\text{loc}}(D)$. Then, for every nonnegative Δ -superharmonic function h on D , we have

$$\sup_{x \in D} \int_D \frac{h(y)}{h(x)} G(x, y) |\mu|(dy) \leq 2C_0 \|\mu\|.$$

In particular,

$$\sup_{x \in D} \int_D \frac{G(y, 0)}{G(x, 0)} G(x, y) |\mu|(dy) \leq 2C_0 \|\mu\| \quad (2.2)$$

and

$$\sup_{x \in D} \int_D G(x, y) |\mu|(dy) \leq 2C_0 \|\mu\|. \quad (2.3)$$

Proof. The inequality holds easily by writing $h(y) = \sup_m \int_D G(y, z) f_m(z) dz$ for some increasing sequence $(f_m)_m$ of nonnegative measurable functions on D , using the Fubini's theorem and the 3G-Theorem. (2.2) and (2.3) hold by taking $h = G(\cdot, 0)$ and $h = 1$, respectively. \square

Proposition 2.3. We have, for all $x \in D$ and $t > 0$,

$$\int_D g(x, t; y, 0) G(y, 0) dy \leq G(x, 0).$$

Proof. By (2.1) and the reproducing property of g , we have

$$\begin{aligned} \int_D g(x, t; y, 0) G(y, 0) dy &= \int_D g(x, t; y, 0) \int_0^\infty g(y, s; 0, 0) ds dy \\ &= \int_0^\infty \int_D g(x, t+s; y, s) g(y, s; 0, 0) dy ds \\ &= \int_0^\infty g(x, t+s; 0, 0) ds = \int_t^\infty g(x, s; 0, 0) ds \\ &\leq \int_0^\infty g(x, s; 0, 0) ds = G(x, 0). \quad \square \end{aligned}$$

Proposition 2.4. Let $f \in C_b(\Omega)$ and $\mu \in \mathcal{K}_n^{\text{loc}}(D)$, $\mu \geq 0$. Then, the function

$$F(x, t) = \int_0^t \int_D \frac{G(y, 0)}{G(x, 0)} g(x, t; y, s) |f(y, s)| \mu(dy) ds$$

is continuous on Ω . Moreover, the functions $\{F(\cdot, t), t > 0\}$ are equi-continuous on D .

Proof. By (2.1) and (2.2), we have

$$\begin{aligned} F(x, t) &= \int_D \frac{G(y, 0)}{G(x, 0)} \int_0^t g(x, s; y, 0) |f(y, t-s)| ds \mu(dy) \\ &\leq \|f\|_\infty \int_D \frac{G(y, 0)G(x, y)}{G(x, 0)} \mu(dy) \\ &\leq 2C_0 \|\mu\| \|f\|_\infty, \end{aligned}$$

and so F is a real finite valued function. Let $(x_0, t_0) \in \Omega$ be fixed and $\delta \in]0, t_0[$ such that $E \equiv \bar{B}(x_0, \delta) \subset D$. For $(x, t) \in \Omega$, we have

$$|F(x, t) - F(x_0, t_0)| \leq |F(x, t) - F(x_0, t)| + |F(x_0, t) - F(x_0, t_0)|. \quad (2.4)$$

Moreover

$$\begin{aligned} &|F(x_0, t) - F(x_0, t_0)| \\ &\leq \|f\|_\infty \int_D \frac{G(y, 0)}{G(x_0, 0)} \int_{t \wedge t_0}^{t \vee t_0} g(x_0, s; y, 0) ds \mu(dy) \\ &\quad + \int_D \frac{G(y, 0)}{G(x_0, 0)} \int_0^{t_0} g(x_0, s; y, 0) |f(y, t-s) - f(y, t_0-s)| ds \mu(dy), \end{aligned}$$

and so by (2.1), the continuity of f , (2.2) and the dominated convergence theorem, it follows that

$$|F(x_0, t) - F(x_0, t_0)| \rightarrow 0 \quad \text{as } |t - t_0| \rightarrow 0. \quad (2.5)$$

On the other hand, by (1.2) and (2.3), for $\varepsilon > 0$, there exists $r \in]0, \delta[$ small so that

$$\sup_{x \in E} \int_{D \cap B(x, r)} G(x, y) \mu(dy) < \varepsilon \quad (2.6)$$

and

$$\int_{D \cap B(x_0, r)} G(0, y) \mu(dy) < \varepsilon. \quad (2.7)$$

Let $M = \sup_{x \in E} \frac{1}{\varphi(x)} < \infty$. For $|x - x_0| \leq \frac{r}{2}$, $t > 0$, we have

$$\begin{aligned} |F(x, t) - F(x_0, t)| &\leq \|f\|_\infty \int_D G(y, 0) \int_0^\infty \left| \frac{g(x, s; y, 0)}{G(x, 0)} - \frac{g(x_0, s; y, 0)}{G(x_0, 0)} \right| ds \mu(dy) \\ &\equiv \|f\|_\infty \left(\int_{B(x_0, \frac{r}{2})} \cdots \mu(dy) + \int_{D \cap B^c(x_0, \frac{r}{2})} \cdots \mu(dy) \right) \\ &\equiv \|f\|_\infty (I_1(x, x_0) + I_2(x, x_0)). \end{aligned} \quad (2.8)$$

From (2.1) and the 3G-Theorem, we have

$$\begin{aligned}
I_1(x, x_0) &\leq C_0 \int_{B(x_0, \frac{r}{2})} \left[\frac{G(y, 0)G(x, y)}{G(x, 0)} + \frac{G(y, 0)G(x_0, y)}{G(x_0, 0)} \right] \mu(dy) \\
&\leq C_0 \int_{B(x_0, \frac{r}{2})} \left[2 \frac{\varphi(y)}{\varphi(0)} G(y, 0) + \frac{\varphi(y)}{\varphi(x)} G(x, y) + \frac{\varphi(y)}{\varphi(x_0)} G(x_0, y) \right] \mu(dy) \\
&\leq 2C_0 \int_{B(x_0, \frac{r}{2})} G(0, y) \mu(dy) + MC_0 \int_{D \cap B(x, r)} G(x, y) \mu(dy) \\
&\quad + MC_0 \int_{B(x_0, \frac{r}{2})} G(x_0, y) \mu(dy),
\end{aligned}$$

and so by (2.6) and (2.7), it follows that

$$I_1(x, x_0) \leq 2C_0(1 + M)\varepsilon. \quad (2.9)$$

Now, we estimate $I_2(x, x_0)$. By (2.1), we have

$$\begin{aligned}
I_2(x, x_0) &= \int_{D \cap B^c(x_0, \frac{r}{2})} G(y, 0) \int_0^\infty \left| \frac{g(x, s; y, 0)}{G(x, 0)} - \frac{g(x_0, s; y, 0)}{G(x_0, 0)} \right| ds \mu(dy) \\
&\leq \frac{1}{G(x_0, 0)} \int_{D \cap B^c(x_0, \frac{r}{2})} G(y, 0) \int_0^\infty |g(x, s; y, 0) - g(x_0, s; y, 0)| ds \mu(dy) \\
&\quad + \left| \frac{1}{G(x, 0)} - \frac{1}{G(x_0, 0)} \right| \int_{D \cap B^c(x_0, \frac{r}{2})} G(y, 0) G(x, y) \mu(dy) \\
&\equiv I_{21}(x, x_0) + I_{22}(x, x_0).
\end{aligned} \quad (2.10)$$

By using the inequalities,

$$G(x, y) \leq \begin{cases} \frac{\Gamma(n/2-1)}{4(\pi)^{n/2}} \frac{1}{|x-y|^{n-2}} & \text{if } n \geq 3, \\ \frac{1}{2\pi} (1 \vee \text{Log}(\frac{1}{|x-y|})) & \text{if } n = 2, \end{cases}$$

it follows that $G(x, y) \leq \frac{c_n}{r^{n-1}}$, for $x \in B(x_0, \frac{r}{4})$ and $y \in D \cap B^c(x_0, \frac{r}{2})$, where c_n is a constant depending only on n . Combining this inequality with (2.3), we obtain

$$I_{22}(x, x_0) \leq 2C_0 \|\mu\| \frac{c_n}{r^{n-1}} \left| \frac{1}{G(x, 0)} - \frac{1}{G(x_0, 0)} \right|.$$

Since $\frac{1}{G(\cdot, 0)}$ is continuous on D , we have

$$I_{22}(x, x_0) \rightarrow 0 \quad \text{as } |x - x_0| \rightarrow 0. \quad (2.11)$$

On the other hand,

$$I_{21}(x, x_0) = \frac{1}{G(x_0, 0)} \int_{D \cap B^c(x_0, \frac{r}{2})} G(y, 0) \int_0^\infty |g(x, s; y, 0) - g(x_0, s; y, 0)| ds \mu(dy)$$

$$\begin{aligned} &\leq M \left(\int_{D \cap B^c(x_0, \frac{r}{2})} G(y, 0) \int_0^1 \cdots ds \mu(dy) + \int_{D \cap B^c(x_0, \frac{r}{2})} G(y, 0) \int_1^\infty \cdots ds \mu(dy) \right) \\ &\equiv M(J_1(x, x_0) + J_2(x, x_0)). \end{aligned} \quad (2.12)$$

For $y \in D \cap B^c(x_0, \frac{r}{2})$, the function $g(\cdot, \cdot; y, 0)$ is a continuous solution of the heat equation on $B(x_0, \frac{r}{4}) \times]0, \infty[$ satisfying

$$g(x, s; y, 0) \leq (4\pi s)^{-n/2} \exp\left(-\frac{|x-y|^2}{4s}\right) \leq \frac{(\frac{n}{2\pi e})^{n/2}}{|x-y|^n} \leq \left(\frac{8n}{\pi e r^2}\right)^{n/2}, \quad (2.13)$$

where $(4\pi s)^{-n/2} \exp(-\frac{|x-y|^2}{4s})$ is the Dirichlet heat kernel on the hole space \mathbb{R}^n ; and by the well-known Harnack inequality [7], there is $C = C(\delta) > 0$, such that for all $s > 1$ and all $y \in D \cap B^c(x_0, \frac{r}{2})$,

$$\sup_{|x-x_0| \leq \frac{r}{4}} g(x, s; y, 0) \leq C g(x_0, s+1; y, 0). \quad (2.14)$$

By (2.13), (2.3) and the dominated convergence theorem, we have

$$J_1(x, x_0) \rightarrow 0, \quad \text{as } |x - x_0| \rightarrow 0. \quad (2.15)$$

By (2.14), (2.1) and the dominated convergence theorem, we have

$$J_2(x, x_0) \rightarrow 0, \quad \text{as } |x - x_0| \rightarrow 0. \quad (2.16)$$

Combining (2.12), (2.15) and (2.16), we obtain

$$I_{21}(x, x_0) \rightarrow 0, \quad \text{as } |x - x_0| \rightarrow 0. \quad (2.17)$$

From (2.10), (2.11) and (2.17), we have

$$I_2(x, x_0) \rightarrow 0, \quad \text{as } |x - x_0| \rightarrow 0. \quad (2.18)$$

Combining (2.8), (2.9) and (2.18), we obtain

$$\sup_{t>0} |F(x, t) - F(x_0, t)| \rightarrow 0 \quad \text{as } |x - x_0| \rightarrow 0. \quad (2.19)$$

Therefore the continuity of F on Ω holds by combining (2.4), (2.5) and (2.19). The equi-continuity of $\{F(\cdot, t), t > 0\}$ on D follows from (2.19). \square

3. Proof of Theorem 1.2 and Corollary 1.4

Proof of Theorem 1.2. We will show that there exists a number $\lambda_0 > 0$ such that for any $\lambda \in]0, \lambda_0[$ and any nonnegative function u_0 satisfying $u_0(x) \leq \lambda G(x, 0)$, there is a function $u \in C(D \setminus \{0\} \times]0, \infty[)$ satisfying

$$\begin{aligned} u(x, t) &= \lambda \int_0^t g(x, t; s) ds + \int_D g(x, t; y, 0) u_0(y) dy \\ &\quad + \int_0^t \int_D g(x, t; y, s) u^p(y, s) \mu(dy) ds \end{aligned} \quad (3.1)$$

and

$$0 \leq u(x, t) \leq 3\lambda G(x, 0). \quad (3.2)$$

Clearly if u satisfies (3.1), then u is a distributional solution of the problem (\mathcal{P}) . It is then enough to show that there exists a function $w \in C(\Omega)$ satisfying

$$\begin{aligned} w(x, t) = & \frac{\lambda}{G(x, 0)} \int_0^t g(x, t; 0, s) ds + \int_D \frac{G(y, 0)}{G(x, 0)} g(x, t; y, 0) v_0(y) dy \\ & + \int_0^t \int_D \frac{G(y, 0)}{G(x, 0)} g(x, t; y, s) w^p(y, s) \sigma(dy) ds \end{aligned} \quad (3.3)$$

and

$$0 \leq w(x, t) \leq 3\lambda, \quad (3.4)$$

where $v_0(y) = \frac{u_0(y)}{G(y, 0)}$ and $\sigma = G^{p-1}(\cdot, 0)\mu$.

This is clear because if w satisfies (3.3) and (3.4), then the function $u(x, t) = G(x, 0)w(x, t)$ satisfies (3.1) and (3.2). Let $C_b(\Omega)$ be the set of all bounded continuous functions on Ω . For $\lambda > 0$, consider the set

$$S = \{w \in C_b(\Omega): 0 \leq w(x, t) \leq 3\lambda\}.$$

Then S is a nonempty closed set in the Banach space $(C_b(\Omega), \|\cdot\|_\infty)$. For simplicity, let

$$h(x, t) = \frac{1}{G(x, 0)} \int_0^t g(x, t; 0, s) ds \quad \text{and} \quad H(x, t) = \int_D \frac{G(y, 0)}{G(x, 0)} g(x, t; y, 0) v_0(y) dy.$$

Define the operator T on S by

$$Tw(x, t) = \lambda h(x, t) + H(x, t) + \int_0^t \int_D \frac{G(y, 0)}{G(x, 0)} g(x, t; y, s) w^p(y, s) \sigma(dy) ds,$$

for $(x, t) \in \Omega$. We will prove that T maps S into itself. Let $w \in S$. We have

$$\lambda h(x, t) + H(x, t) = \lambda - \frac{\lambda}{G(x, 0)} \int_{-\infty}^0 g(x, t; 0, s) ds + \frac{1}{G(x, 0)} \int_D g(x, t; y, 0) u_0(y) dy.$$

The functions $\int_{-\infty}^0 g(\cdot, \cdot; 0, s) ds$ and $\int_D g(\cdot, \cdot; y, 0) u_0(y) dy$ are continuous solutions of the heat equation on Ω , and $\frac{1}{G(x, 0)}$ is continuous on D . Hence $\lambda h + H$ is continuous on Ω . Moreover, by Proposition 2.4, the function

$$(x, t) \rightarrow \int_0^t \int_D \frac{G(y, 0)}{G(x, 0)} g(x, t; y, s) w^p(y, s) \sigma(dy) ds$$

is continuous on Ω . Therefore $Tw \in C(\Omega)$. On the other hand, by (2.1), Proposition 2.3 and (2.2), we have, for $(x, t) \in \Omega$,

$$0 \leq Tw(x, t) \leq \lambda + \|v_0\|_\infty + (3\lambda)^p 2C_0 \|\sigma\|.$$

Since $p > 1$, for $0 < \lambda < \frac{1}{3}(6C_0\|\sigma\|)^{-\frac{1}{p-1}}$ and $\|v_0\|_\infty \leq \lambda$, we obtain

$$0 \leq Tw(x, t) \leq 3\lambda,$$

and so $T(S) \subset S$. Moreover, for w_1 and w_2 in S , we have, by (2.1) and (2.2),

$$\begin{aligned} \|Tw_1 - Tw_2\|_\infty &\leq 2C_0\|\sigma\| \|w_1^p - w_2^p\|_\infty \\ &\leq p(3\lambda)^{p-1} 2C_0\|\sigma\| \|w_1 - w_2\|_\infty. \end{aligned}$$

It follows that, for $0 < \lambda < \frac{1}{6}(2pC_0\|\sigma\|)^{-\frac{1}{p-1}}$,

$$\|Tw_1 - Tw_2\|_\infty \leq \frac{1}{2} \|w_1 - w_2\|_\infty.$$

Thus, for $0 < \lambda < \lambda_0 = \frac{1}{3}(6pC_0\|\sigma\|)^{-\frac{1}{p-1}}$ and $\|v_0\|_\infty \leq \lambda$, the operator T is a $\frac{1}{2}$ -Lipschitz mapping from S into itself. By the well-known fixed point theorem there exists a unique $w \in S$ such that $Tw = w$. \square

Proof of Corollary 1.4. Let u be the solution of the parabolic problem (\mathcal{P}) obtained in Theorem 1.2. The function $w(x, t) = \frac{u(x, t)}{G(x, 0)} \in C(\Omega)$ and satisfies the integral equation:

$$w(x, t) = \lambda h(x, t) + H(x, t) + \int_0^t \int_D \frac{G(y, 0)}{G(x, 0)} g(x, t; y, s) w^p(y, s) \sigma(dy) ds \quad (3.5)$$

with

$$0 \leq w(x, t) \leq 3\lambda, \quad (3.6)$$

for all $(x, t) \in \Omega$. Clearly by (2.1) and the Dini-theorem, we have

$$\lim_{t \rightarrow \infty} h(x, t) = \lim_{t \rightarrow \infty} \frac{1}{G(x, 0)} \int_0^t g(x, s; 0, 0) ds = 1,$$

locally uniformly on D , and

$$H(x, t) \leq \frac{1}{G(x, 0)} \frac{\lambda_0}{4\pi^{n/2} t^{n/2}} \int_D G(y, 0) dy$$

which yields

$$\lim_{t \rightarrow \infty} H(x, t) = 0,$$

locally uniformly on D . On the other hand, by Proposition 2.4, the function

$$F(x, t) = \int_0^t \int_D \frac{G(y, 0)}{G(x, 0)} g(x, t; y, s) w^p(y, s) \sigma(dy) ds,$$

is continuous on Ω and $\{F(\cdot, t), t > 0\}$ are equi-continuous on D and uniformly bounded by the constant $2C_0\|\sigma\|(3\lambda)^p$. By the Ascoli-theorem there exists a sequence $(t_k)_k$, $t_k \rightarrow \infty$, such that $F(x, t_k)$ converges locally uniformly on D , as $k \rightarrow \infty$, to a continuous function. Hence $w(x, t_k)$

converges locally uniformly on D , as $k \rightarrow \infty$, to a function $w_\infty \in C(D)$. From (3.5) and (3.6), we have

$$w(x, t_k) = \lambda h(x, t_k) + H(x, t_k) + \int_0^{t_k} \int_D \frac{G(y, 0)}{G(x, 0)} g(x, s; y, 0) w^p(y, t_k - s) \sigma(dy) ds$$

with

$$0 \leq w(x, t_k) \leq 3\lambda.$$

By taking the limit as $k \rightarrow \infty$ and using the dominated convergence theorem, we obtain that w_∞ satisfies

$$w_\infty(x) = \lambda + \int_D \frac{G(y, 0)}{G(x, 0)} G(x, y) w_\infty^p(y) \sigma(dy)$$

and

$$\lambda \leq w_\infty(x) \leq 3\lambda, \quad \forall x \in D.$$

The function $u_\infty(x) = w_\infty(x)G(x, 0) \in C(D \setminus \{0\})$ and satisfies

$$u_\infty(x) = \lambda G(x, 0) + \int_D G(x, y) u_\infty^p(y) \mu(dy)$$

with

$$\lambda G(x, 0) \leq u_\infty(x) \leq 3\lambda G(x, 0).$$

Hence u_∞ is a solution of the elliptic problem (E). \square

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