

# Structure relations for monic orthogonal polynomials in two discrete variables <sup>☆</sup>

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## Abstract

In this paper, extensions of several relations linking differences of bivariate discrete orthogonal polynomials and polynomials themselves are given, by using an appropriate vector–matrix notation. Three-term recurrence relations are presented for the partial differences of the monic polynomial solutions of admissible second order partial difference equation of hypergeometric type. Structure relations, difference representations as well as lowering and raising operators are obtained. Finally, expressions for all matrix coefficients appearing in these finite-type relations are explicitly presented for a finite set of Hahn and Kravchuk orthogonal polynomials.

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## 1. Introduction

Orthogonal polynomials of a discrete variable are employed in several areas of mathematics (coding theory, population dynamics, statistical analysis) and appear in a number of problems of theoretical and mathematical physics, group representation theory, or computational physics, that an exhaustive catalogue would simply be very lengthy and unwieldy [4,18].

In the theory of orthogonal polynomials in one variable (either continuous or discrete orthogonality) special attention has been received by the so-called structure and differential/difference relations [1,2,13,16,17]. In the classical continuous situation (Hermite, Laguerre, Jacobi and Bessel), if  $\{P_n(x)\}$  is a family of orthogonal polynomials satisfying the second-order linear differential equation of hypergeometric type [18]

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0,$$

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then, it is possible to obtain scalar constants  $E_n$ ,  $F_n$  and  $G_n \neq 0$  such that [2]

$$\sigma(x)P'_n(x) = E_n P_{n+1}(x) + F_n P_n(x) + G_n P_{n-1}(x).$$

This property constitutes a characterization of classical orthogonal polynomials. Furthermore, there exist some extensions to the semi-classical situation [14]. The general theory of the structure relation for Askey–Wilson polynomials was presented by Koornwinder in [9].

Also, in the classical discrete situation (Charlier, Meixner, Kravchuk and Hahn), if  $\{P_n(x)\}$  denotes a family of orthogonal polynomials satisfying the second-order difference equation of hypergeometric type [18]

$$\tilde{\sigma}(x)\Delta\nabla y(x) + \tilde{\tau}(x)\Delta y(x) + \tilde{\lambda}_n y(x) = 0,$$

then, it is possible to find scalar constants  $R_n$ ,  $S_n$ ,  $T_n$ ,  $\tilde{R}_n$ ,  $\tilde{S}_n$ , and  $\tilde{T}_n$  such that [7]

$$\begin{aligned}\tilde{\sigma}(x)\nabla P_n(x) &= R_n P_{n+1}(x) + S_n P_n(x) + T_n P_{n-1}(x), \\ [\tilde{\sigma}(x) + \tilde{\tau}(x)]\Delta P_n(x) &= \tilde{R}_n P_{n+1}(x) + \tilde{S}_n P_n(x) + \tilde{T}_n P_{n-1}(x).\end{aligned}$$

These finite-type relations [17] characterize classical discrete sequences of orthogonal polynomials and there exist generalizations to semi-classical situations.

In the multivariate continuous case, Lee [15] generalizes the Hahn–Sonine theorem to bivariate orthogonal polynomials and some analog characterizations of standard classical orthogonal polynomials are obtained.

In recent publications [19,20] we gave a general approach to study the orthogonal polynomial solutions of a second-order linear partial difference equation of hypergeometric type. Going on with this study, the aim of this paper is to present, under certain assumptions, finite-type relations (first and second structure relations) for these monic discrete orthogonal families and their partial differences. Also, ladder operators lowering or raising the degree but preserving the parameters of these families are built.

The present study is organized as follows. In Section 2, we begin by reviewing the study of an admissible second order partial difference equation of hypergeometric type done in [20] and its monic polynomial solutions. After the introduction of some notation, the orthogonality property of these monic polynomial sequences and their first order partial differences are presented. In Section 3, we establish general properties which constitute the main goal of this paper. In particular, we deduce  $\Delta$ -structure and  $\nabla$ -structure relations for monic bivariate discrete orthogonal polynomials and a second structure relation, i.e. a relation between a monic polynomial sequence and the sequence of their differences. These relations take a vector–matrix form. Finally, to illustrate these algebraic-difference equations, in Section 4, we present the explicit expression for the parameters involved in the aforementioned properties of monic bivariate Hahn and Kravchuk polynomials respectively, where the support of the pure point measures is a finite discrete set. We use *Mathematica 5* for doing our computations [26]. Moreover, relations between partial differences of Hahn and Kravchuk polynomials and the polynomials themselves are given, in a similar way as [18, Section 2.4.3].

## 2. Monic orthogonal polynomials in two discrete variables as solutions of an admissible second order partial difference equation of hypergeometric type

Problems involving linear partial differential equations of Mathematical Physics can be reduced to algebraic ones of a very much simpler structure by replacing the differentials by difference quotients on some mesh. Partial difference equations are difference equations that involve functions of two or more independent variables. In this context, the two-variable extension of the forward difference operator  $\Delta$  is defined as

$$\Delta_1 f(\mathbf{x}) = f(x+1, y) - f(x, y), \quad \Delta_2 f(\mathbf{x}) = f(x, y+1) - f(x, y),$$

where  $\mathbf{x} = (x, y)$  is a point of the 2D space, and the backward difference operator  $\nabla$  is

$$\nabla_1 f(\mathbf{x}) = f(x, y) - f(x-1, y), \quad \nabla_2 f(\mathbf{x}) = f(x, y) - f(x, y-1).$$

**Definition 2.1.** The equation

$$\begin{aligned}a_{11}(\mathbf{x})\Delta_1\nabla_1 u(\mathbf{x}) + a_{12}(\mathbf{x})\Delta_1\nabla_2 u(\mathbf{x}) + a_{21}(\mathbf{x})\Delta_2\nabla_1 u(\mathbf{x}) + a_{22}(\mathbf{x})\Delta_2\nabla_2 u(\mathbf{x}) + b_1(\mathbf{x})\Delta_1 u(\mathbf{x}) \\ + b_2(\mathbf{x})\Delta_2 u(\mathbf{x}) + \lambda u(\mathbf{x}) = 0,\end{aligned}\tag{1}$$

is said to be *hypergeometric* if the differences  $u_\alpha(\mathbf{x}) = \Delta_1^r \Delta_2^s u(\mathbf{x})$  of the solutions  $u = u(\mathbf{x})$  of (1) are also solutions of an equation of the same type as (1), where we denote by  $\alpha = (r, s)$  a multi-index of order  $|\alpha| = r + s$ .

**Definition 2.2.** An hypergeometric equation (1) is said to be *admissible* if there exists a sequence  $\{\lambda_n\}$  ( $n = 0, 1, \dots$ ) such that for  $\lambda = \lambda_n$ , there are precisely  $n + 1$  linearly independent solutions in the form of polynomials of total degree  $n$  and has no non-trivial solutions in the set of polynomials whose total degree is less than  $n$ .

This concept was introduced by Krall and Sheffer [12] in the case of second order partial differential equations and also by Y. Xu in [28, Section 2] in the case of second order partial difference equations, without the assumption of being an equation of hypergeometric type. Recently, in [8] all second order difference operators of several variables that have discrete orthogonal polynomials as eigenfunctions were characterized.

The definition of admissibility of Eq. (1) implies that all numbers

$$\lambda_0 = 0, \lambda_1, \lambda_2, \dots, \lambda_n, \dots,$$

are different ( $\lambda_m \neq \lambda_n, m \neq n$ ).

In a similar way as Suetin [22, p. 93], we can deduce for hypergeometric equations the following characterization of admissibility:

**Theorem 2.3.** An hypergeometric equation (1) is admissible if and only if it has the form

$$a_{11}(\mathbf{x})\Delta_1\nabla_1u(\mathbf{x}) + a_{12}(\mathbf{x})\Delta_1\nabla_2u(\mathbf{x}) + a_{21}(\mathbf{x})\Delta_2\nabla_1u(\mathbf{x}) + a_{22}(\mathbf{x})\Delta_2\nabla_2u(\mathbf{x}) + b_1(\mathbf{x})\Delta_1u(\mathbf{x}) + b_2(\mathbf{x})\Delta_2u(\mathbf{x}) - n((n-1)A + c)u(\mathbf{x}) = 0, \quad (2)$$

where the coefficients  $a_{ij}(\mathbf{x})$  and  $b_i(\mathbf{x})$  are

$$\begin{aligned} a_{11}(\mathbf{x}) &= x(Ax + B_1), & a_{22}(\mathbf{x}) &= y(Ay + B_2), \\ a_{12}(\mathbf{x}) &= y(Ax + B_3), & a_{21}(\mathbf{x}) &= x(Ay + B_4), \\ b_1(\mathbf{x}) &= cx + d_1, & b_2(\mathbf{x}) &= cy + d_2, \end{aligned}$$

where  $A, B_1, B_2, B_3, B_4, c, d_1, d_2$  are constants with  $c \neq 0$ .

In Section 4, we shall present two examples of the partial difference equation (2) where the constant  $A$  is different or equal to zero, whose polynomial solutions are Hahn or Kravchuk bivariate orthogonal polynomials, respectively.

In a similar way we can obtain a partial difference equation for the functions  $u^{(j)}(\mathbf{x}) = \Delta_j(u(\mathbf{x}))$  with  $j = 1, 2$

$$a_{11}^{(j)}(\mathbf{x})\Delta_1\nabla_1u^{(j)}(\mathbf{x}) + a_{12}^{(j)}(\mathbf{x})\Delta_1\nabla_2u^{(j)}(\mathbf{x}) + a_{21}^{(j)}(\mathbf{x})\Delta_2\nabla_1u^{(j)}(\mathbf{x}) + a_{22}^{(j)}(\mathbf{x})\Delta_2\nabla_2u^{(j)}(\mathbf{x}) + b_1^{(j)}(\mathbf{x})\Delta_1u^{(j)}(\mathbf{x}) + b_2^{(j)}(\mathbf{x})\Delta_2u^{(j)}(\mathbf{x}) + \mu^{(j)}u^{(j)}(\mathbf{x}) = 0, \quad (3)$$

where explicit relations for the coefficients  $a_{k\ell}^{(j)}, b_k^{(j)}, \mu^{(j)}$  were found [20] in terms of the coefficients of (2).

Let us introduce the following notation. Let  $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$  and  $|\alpha| = \alpha_1 + \alpha_2$ . Let  $\mathbf{x} = (x, y)$  and  $\mathbf{x}^n$  be the column vector, whose elements are arranged in the graded lexicographic order (see [6, p. 32])

$$\mathbf{x}^n = (x^{\alpha_1}y^{\alpha_2})_{|\alpha|=n}.$$

There exist various other orders for polynomials of several variables. For an alternative approach, see [5].

If we denote by  $\{P_\alpha^n\}_{|\alpha|=n}$  a sequence of monic polynomials where

$$P_\alpha^n(x, y) = x^{\alpha_1}y^{\alpha_2} + \text{terms of lower power},$$

in the space  $\Pi_n^2$  of all polynomials of degree at most  $n$  in two variables  $\mathbf{x} = (x, y)$  with real coefficients, we can write  $\hat{\mathbb{P}}_n$  as the (column) monic polynomial vector

$$\hat{\mathbb{P}}_n = \hat{\mathbb{P}}_n(x, y) = (P_\alpha^n(x, y))_{|\alpha|=n} = (P_{\alpha(1)}^n(x, y), \dots, P_{\alpha(n+1)}^n(x, y))^T \quad (4)$$

where  $\alpha^{(1)}, \dots, \alpha^{(n+1)}$  is the arrangement of elements in  $\{\alpha \in \mathbb{N}_0^2 \mid |\alpha| = n\}$  according to the graded lexicographic order, i.e.

$$\hat{\mathbb{P}}_n = (P_{n,0}^n(x, y), P_{n-1,1}^n(x, y), \dots, P_{1,n-1}^n(x, y), P_{0,n}^n(x, y))^T. \quad (5)$$

Using the vector notation, the monic polynomial  $\hat{\mathbb{P}}_n$  can be written as

$$\hat{\mathbb{P}}_n = \mathbf{x}^n + G_{n,n-1}\mathbf{x}^{n-1} + \dots + G_{n,0}\mathbf{x}^0.$$

Under certain boundary conditions (see [20]), the monic polynomial solutions of (2) are orthogonal with respect to a weight function  $\varrho(x, y)$  in a certain domain  $G \subset \mathbb{R}^2$  with

$$\varrho(x, y) = \kappa \prod_{i=y_0}^{y-1} \mathcal{G}_2(x, i) \prod_{j=x_0}^{x-1} \mathcal{G}_1(j, y_0), \quad (6)$$

where  $\kappa$  is a constant and

$$\begin{cases} \mathcal{G}_1(x, y) = \frac{\varphi_1(x, y)}{\varpi_1(x+1, y)} > 0, \\ \mathcal{G}_2(x, y) = \frac{\varphi_2(x, y)}{\varpi_2(x, y+1)} > 0, \end{cases} \quad (7)$$

being

$$\begin{aligned} \varpi_1(x, y) &= a_{11}(x, y) + a_{21}(x, y), & \varpi_2(x, y) &= a_{22}(x, y) + a_{12}(x, y), \\ \varphi_1(x, y) &= a_{11}(x, y) + a_{12}(x, y) + b_1(x, y), & \varphi_2(x, y) &= a_{22}(x, y) + a_{21}(x, y) + b_2(x, y). \end{aligned} \quad (8)$$

Let  $G$  be a certain discrete set, and  $\varrho(x, y)$  be a nonnegative function on  $G$ . This weight function  $\varrho(x, y)$  defines the linear functional

$$\mathcal{L}(P) = \sum_{(x,y) \in G} P(x, y) \varrho(x, y), \quad (9)$$

on the space of all polynomials in two variables if, of course, all such sums exist.

In what follows we consider the discrete domain

$$G = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a, 0 \leq y \leq \phi(x)\}, \quad (10)$$

where  $\phi(x)$  is determined [20] from the positivity of the functions  $\mathcal{G}_j$  defined in (7).

If  $\mathcal{L}(PQ) = 0$ , we say that the polynomials  $P$  and  $Q$  are mutually orthogonal with respect to  $\varrho(x, y)$  on the lattice set  $G$ .

Let us denote by  $\mathfrak{V}_n^2$  the space of orthogonal polynomials of total degree exactly  $n$ ,

$$\mathfrak{V}_n^2 = \{P \in \Pi_n^2: \mathcal{L}(PQ) = 0, \forall Q \in \Pi_{n-1}^2\}. \quad (11)$$

The dimension of  $\mathfrak{V}_n^2$  is  $n+1$  and the monic polynomials  $(P_\alpha^n(x, y))_{|\alpha|=n}$  constitute a basis of this subspace  $\mathfrak{V}_n^2$ , i.e. the elements of the polynomial vector  $\hat{\mathbb{P}}_n$  form a basis of the subspace  $\mathfrak{V}_n^2$ . Therefore, if  $Q \in \mathfrak{V}_m^2$ ,  $R \in \mathfrak{V}_n^2$ , with  $m \neq n$  implies  $\mathcal{L}(QR) = 0$ . As a consequence we have the orthogonality relation

$$\mathfrak{V}_m^2 \perp \mathfrak{V}_n^2$$

and we shall refer to  $\{\mathfrak{V}_n^2\}$  as the set of orthogonal vector spaces relative to  $\varrho(x, y)$  (see [12]).

**Definition 2.4.** Let  $\mathcal{L}$  be the linear functional defined in (9), and let  $\hat{\mathcal{L}}$  be the linear functional defined by the matrix

$$\hat{\mathcal{L}}[(A_{i,j}(x, y))] = (\mathcal{L}(A_{i,j}(x, y))), \quad (12)$$

for any polynomial matrix  $(A_{i,j}(x, y))$ . A sequence of monic polynomials  $\{P_\alpha^n(x, y) \in \Pi_n^2 \mid |\alpha| = n, n \in \mathbb{N}_0\}$ , is said to be *orthogonal* with respect to  $\hat{\mathcal{L}}$  or  $\{\hat{\mathbb{P}}_n\}_{n \geq 0}$  is an orthogonal family with respect  $\hat{\mathcal{L}}$  if

$$\hat{\mathcal{L}}[(\mathbf{x}^m \hat{\mathbb{P}}_n^T)] = \sum_{x=0}^a \sum_{y=0}^{\phi(x)} (\mathbf{x}^m \hat{\mathbb{P}}_n^T) \varrho(x, y) = 0, \quad n > m, \quad (13)$$

$$\hat{\mathcal{L}}[(\mathbf{x}^n \hat{\mathbb{P}}_n^T)] = \sum_{x=0}^a \sum_{y=0}^{\phi(x)} (\mathbf{x}^n \hat{\mathbb{P}}_n^T) \varrho(x, y) = H_n, \quad (14)$$

where  $H_n$  is a nonsingular matrix of size  $n + 1$ .

As a consequence of this definition we have

**Proposition 2.5.** Let  $\hat{\mathcal{L}}$  be the linear functional defined in (12) and let  $\hat{\mathbb{P}}_n$  be the monic orthogonal polynomial defined as above. Then

$$H_n = \hat{\mathcal{L}}[(\hat{\mathbb{P}}_n \hat{\mathbb{P}}_n^T)] \quad (15)$$

where  $H_n$  is defined in (14).

Moreover,

**Proposition 2.6.** Let  $\hat{\mathcal{L}}$  be the linear functional defined in (12) and let  $\hat{\mathbb{P}}_n$  be the monic orthogonal polynomial defined as above. Then  $\{\hat{\mathbb{P}}_0, \hat{\mathbb{P}}_1, \dots, \hat{\mathbb{P}}_n\}$  is a basis for  $\Pi_n^2$ .

Also,

**Lemma 2.7.** Let  $\hat{\mathcal{L}}$  be the linear functional defined in (12) and let  $\hat{\mathbb{P}}_n$  be a monic orthogonal polynomial with respect to  $\hat{\mathcal{L}}$ . Then  $\hat{\mathbb{P}}_n$  is uniquely determined by the matrix  $H_n$ .

Proceeding similarly, let us consider the orthogonality property of partial differences of the monic orthogonal family  $\hat{\mathbb{P}}_n$ .

From now on we make the assumption:

$$\Delta_1 P_{0,n}^n(x, y) = 0 \quad \text{and} \quad \Delta_2 P_{n,0}^n(x, y) = 0, \quad n \geq 1. \quad (16)$$

This property is the keystone to write the finite-type relations obtained in this paper in a similar way to the one variable case. For this purpose, let us denote by

$$\Delta_j \hat{\mathbb{P}}_n = \Delta_j \hat{\mathbb{P}}_n(x, y) = (\Delta_j P_{n,0}^n(x, y), \dots, \Delta_j P_{0,n}^n(x, y))^T, \quad j = 1, 2, \quad (17)$$

and let

$$\mathbb{Q}_n^{(x)} = (\Delta_1 P_{n+1,0}^{n+1}(x, y), \Delta_1 P_{n,1}^{n+1}(x, y), \dots, \Delta_1 P_{1,n}^{n+1}(x, y))^T, \quad (18)$$

and

$$\mathbb{Q}_n^{(y)} = (\Delta_2 P_{n,1}^{n+1}(x, y), \Delta_2 P_{n-1,2}^{n+1}(x, y), \dots, \Delta_2 P_{0,n+1}^{n+1}(x, y))^T, \quad (19)$$

where  $\hat{\mathbb{P}}_n = (P_{n,0}^n(x, y), P_{n-1,1}^n(x, y), \dots, P_{0,n}^n(x, y))^T$ .

From the assumption we have

$$\Delta_1 \hat{\mathbb{P}}_n = \begin{pmatrix} \mathbb{Q}_{n-1}^{(x)} \\ 0 \end{pmatrix}, \quad \Delta_2 \hat{\mathbb{P}}_n = \begin{pmatrix} 0 \\ \mathbb{Q}_{n-1}^{(y)} \end{pmatrix}.$$

Polynomials  $\mathbb{Q}_n^{(x)}$  and  $\mathbb{Q}_n^{(y)}$  are orthogonal with respect to the functionals  $\hat{\mathcal{L}}^{(1)}$  and  $\hat{\mathcal{L}}^{(2)}$  respectively, i.e. for  $n > k$

$$\hat{\mathcal{L}}^{(1)}[(\mathbb{Q}_k^{(x)} (\mathbb{Q}_n^{(x)})^T)] = \sum_{x=0}^{a-1} \sum_{y=0}^{\phi(x)-1} [\mathbb{Q}_k^{(x)} (\mathbb{Q}_n^{(x)})^T] \varpi_1(x+1, y) \varrho(x+1, y) = 0, \quad (20)$$

$$\hat{\mathcal{L}}^{(2)}[(\mathbb{Q}_k^{(y)} (\mathbb{Q}_n^{(y)})^T)] = \sum_{x=0}^{a-1} \sum_{y=0}^{\phi(x)-1} [\mathbb{Q}_k^{(y)} (\mathbb{Q}_n^{(y)})^T] \varpi_2(x, y+1) \varrho(x, y+1) = 0, \quad (21)$$

where the polynomials  $\varpi_j$  are defined in (8) and

$$\hat{\mathcal{L}}^{(j)}[(Q_n^{(x_j)}(Q_n^{(x_j)})^T)] = F_n^j, \quad (22)$$

where  $F_n^j$  are nonsingular matrices of size  $n + 1$ , for  $j = 1, 2$  and  $x_1 = x$ ,  $x_2 = y$ .

**Proposition 2.8.** For  $j = 1, 2$ , let  $\hat{\mathcal{L}}^{(j)}$  be the linear functionals defined as above and let  $Q_n^{(x_j)}$  be the partial difference derivatives of the monic orthogonal family  $\hat{\mathbb{P}}_n$  where  $x_1 = x$ ,  $x_2 = y$ . Then  $\{Q_0^{(x_j)}, \dots, Q_n^{(x_j)}\}$  is a basis of  $\Pi_n^2$ .

### 3. General properties

#### 3.1. The three term recurrence relations of monic orthogonal polynomials

The three term recurrence relation in several variables was first studied by Kowalski [10,11], by using a different vector notation as compared with [6,27], which is used in this paper.

Let  $\hat{\mathcal{L}}$  be the linear functional defined in (12). Using the vector notation introduced in (4), let  $\{\hat{\mathbb{P}}_n\}_{n \geq 0}$  be a sequence of monic orthogonal polynomials with respect to  $\hat{\mathcal{L}}$ .

Just as in the scalar case, the sequence  $\{\hat{\mathbb{P}}_n\}_{n \geq 0}$  satisfies three term recurrence relations which can be obtained in a similar way as in [6, p. 75], from the orthogonality of the family. These recurrences play a fundamental role in understanding the structure of orthogonal polynomials.

Let  $L_{n,j}$  be the matrices of size  $(n + 1) \times (n + 2)$  which are defined by

$$L_{n,1}\mathbf{x}^{n+1} = x\mathbf{x}^n, \quad L_{n,2}\mathbf{x}^{n+1} = y\mathbf{x}^n,$$

i.e.

$$L_{n,1} = \begin{pmatrix} 1 & & \circ & 0 \\ & \ddots & & \vdots \\ \circ & & 1 & 0 \end{pmatrix} \quad \text{and} \quad L_{n,2} = \begin{pmatrix} 0 & 1 & & \circ \\ \vdots & & \ddots & \\ 0 & \circ & & 1 \end{pmatrix}.$$

**Theorem 3.1.** For  $n \geq 0$ , there exist unique matrices  $L_{n,j}$  of size  $(n + 1) \times (n + 2)$ ,  $B_{n,j}$  of size  $(n + 1) \times (n + 1)$ , and  $C_{n,j}$  of size  $(n + 1) \times n$ , respectively, such that

$$x_j \hat{\mathbb{P}}_n = L_{n,j} \hat{\mathbb{P}}_{n+1} + B_{n,j} \hat{\mathbb{P}}_n + C_{n,j} \hat{\mathbb{P}}_{n-1}, \quad 1 \leq j \leq 2, \quad (23)$$

where  $x_1 = x$ ,  $x_2 = y$ , with the initial conditions  $\hat{\mathbb{P}}_{-1} = 0$  and  $\hat{\mathbb{P}}_0 = 1$ .

For  $n \geq 0$ , since

$$\text{rank}(L_{n,j}) = n + 1 = \text{rank}(C_{n+1,j}), \quad 1 \leq j \leq 2, \quad (24)$$

the columns of the joint matrices

$$L_n = (L_{n,1}^T, L_{n,2}^T)^T \quad \text{and} \quad C_n = (C_{n,1}^T, C_{n,2}^T)^T$$

of size  $(2n + 2) \times (n + 2)$  and  $(2n + 2) \times n$  respectively, are linearly independent, i.e.

$$\text{rank}(L_n) = n + 2, \quad \text{rank}(C_n) = n. \quad (25)$$

As a consequence, the matrix  $L_n$  has full rank and there exists a unique matrix  $D_n^\dagger$  of size  $(n + 2) \times (2n + 2)$ , called the *generalized inverse* of  $L_n$

$$D_n^\dagger = (D_{n,1} \mid D_{n,2}) = (L_n^T L_n)^{-1} L_n^T, \quad (26)$$

such that

$$D_n^\dagger L_n = I_{n+2}.$$

Using the left inverse  $D_n^\dagger$ , we can write a recursive formula for the monic orthogonal polynomials

$$\hat{\mathbb{P}}_{n+1} = D_n^\dagger \left[ \begin{pmatrix} x \\ y \end{pmatrix} \otimes I_{n+1} - B_n \right] \hat{\mathbb{P}}_n - D_n^\dagger C_n \hat{\mathbb{P}}_{n-1}, \quad n \geq 0, \quad (27)$$

using the notation  $B_n = (B_{n,1}^T, B_{n,2}^T)^T$ ,  $C_n = (C_{n,1}^T, C_{n,2}^T)^T$ , with the initial conditions  $\hat{\mathbb{P}}_{-1} = 0$ ,  $\hat{\mathbb{P}}_0 = 1$ , where  $\otimes$  denotes the Kronecker product and

$$D_n^\dagger = \begin{pmatrix} 1 & & & 0 & & \\ & 1/2 & & 0 & 1/2 & \\ & & \ddots & & & \ddots \\ & & & 1/2 & 0 & 1/2 \\ & & & & 0 & 1 \\ & & & & & & 1 \end{pmatrix}.$$

This recurrence (27) gives another presentation of [6, (3.2.10)].

### 3.2. The three term recurrence relations of the partial difference derivatives of monic orthogonal polynomials

In [15], the problem of characterizing orthogonal polynomials in two variables whose partial derivatives with respect to  $x$  are also orthogonal was studied.

In our situation, we obtain

**Theorem 3.2.** Assuming that  $\Delta_1 P_{0,n+1}^{n+1}(x, y) = 0$  and  $\Delta_2 P_{n+1,0}^{n+1}(x, y) = 0$ , for  $n \geq 0$ , there exist unique matrices  $W_{n,j}$  of size  $(n+1) \times (n+2)$ ,  $R_{n,j}$  of size  $(n+1) \times (n+1)$ , and  $U_{n,j}$  of size  $(n+1) \times n$ , such that

$$x \mathbb{Q}_n^{(x)} = W_{n,1} \mathbb{Q}_{n+1}^{(x)} + R_{n,1} \mathbb{Q}_n^{(x)} + U_{n,1} \mathbb{Q}_{n-1}^{(x)}, \quad (28)$$

$$y \mathbb{Q}_n^{(y)} = W_{n,2} \mathbb{Q}_{n+1}^{(y)} + R_{n,2} \mathbb{Q}_n^{(y)} + U_{n,2} \mathbb{Q}_{n-1}^{(y)}, \quad (29)$$

with the initial conditions  $\mathbb{Q}_{-1}^{(x)} = 0$ ,  $\mathbb{Q}_0^{(x)} = 1$ ,  $\mathbb{Q}_{-1}^{(y)} = 0$  and  $\mathbb{Q}_0^{(y)} = 1$ .

**Proof.** For  $j = 1, 2$ , the components of  $x_j \mathbb{Q}_n^{(x_j)}$  are polynomials of degree  $n+1$ , where  $x_1 = x$  and  $x_2 = y$ . Hence, they can be written as linear combinations of difference derivatives of orthogonal polynomials

$$x_j \mathbb{Q}_n^{(x_j)} = \sum_{\ell=0}^{n+1} M_{n,\ell}^{(j)} \mathbb{Q}_\ell^{(x_j)}.$$

Multiplying the above equation by  $(\mathbb{Q}_k^{(x_j)})^T$  from the right and applying the linear functional  $\hat{\mathcal{L}}^{(j)}$ , we get

$$\hat{\mathcal{L}}^{(j)}[(x_j (\mathbb{Q}_n^{(x_j)})(\mathbb{Q}_k^{(x_j)})^T)] = M_{n,k}^{(j)} F_k^j.$$

By orthogonality of  $\mathbb{Q}_n^{(x_j)}$  and using that  $F_k^j$  are nonsingular,  $M_{n,\ell}^{(j)} = 0$  for  $\ell \leq n-2$ . As a result, we obtain (28) and (29).  $\square$

### 3.3. Structure relations

Let  $\{\hat{\mathbb{P}}_n\}_{n \geq 0}$  be the family of monic orthogonal polynomials with respect to  $\hat{\mathcal{L}}$  and denote by

$$\nabla_j \hat{\mathbb{P}}_n = \nabla_j \hat{\mathbb{P}}_n(x, y) = (\nabla_j P_{n,0}^n(x, y), \dots, \nabla_j P_{0,n}^n(x, y))^T, \quad j = 1, 2. \quad (30)$$

Since the components of  $\varpi_1(x, y) \nabla_1 \hat{\mathbb{P}}_n + \varpi_2(x, y) \nabla_2 \hat{\mathbb{P}}_n$  are polynomials of degree  $n+1$ , we can write

$$\varpi_1(x, y) \nabla_1 \hat{\mathbb{P}}_n(x, y) + \varpi_2(x, y) \nabla_2 \hat{\mathbb{P}}_n(x, y) = \sum_{\ell=0}^{n+1} A_{n,\ell} \hat{\mathbb{P}}_\ell(x, y), \quad n \geq 1, \quad (31)$$

where  $\varpi_1(x, y)$  and  $\varpi_2(x, y)$  are defined in (8).

Polynomials  $\Delta_1 \hat{\mathbb{P}}_n(x, y)$  and  $\Delta_2 \hat{\mathbb{P}}_n(x, y)$  are orthogonal with respect to  $\varpi_1(x+1, y)\varrho(x+1, y)$  and  $\varpi_2(x, y+1)\varrho(x, y+1)$ , respectively, i.e. for  $n > k+1$

$$\sum_{x=0}^{a-1} \sum_{y=0}^{\phi(x)-1} [\mathbf{x}^k (\Delta_1 \hat{\mathbb{P}}_n)^T] \varpi_1(x+1, y) \varrho(x+1, y) = 0, \quad (32)$$

$$\sum_{x=0}^{a-1} \sum_{y=0}^{\phi(x)-1} [\mathbf{x}^k (\Delta_2 \hat{\mathbb{P}}_n)^T] \varpi_2(x, y+1) \varrho(x, y+1) = 0. \quad (33)$$

If we multiply Eq. (31) by  $\hat{\mathbb{P}}_k^T$  from the right and apply the above orthogonality relations and the orthogonality of  $\hat{\mathbb{P}}_k$ , we get  $0 = A_{n,k} H_k$ , for  $k < n-1$ , where the matrix  $H_k$  is defined in (13). By using that  $H_k$  is nonsingular, we conclude that  $A_{n,k} = 0$  for  $k < n-1$ .

Hence, we get the following equation, called  $\nabla$ -structure relation, that connects the first-order backward differences with the orthogonal polynomials,

**Theorem 3.3.** For  $n \geq 1$ , there exist unique and  $x, y$ -independent matrices  $G_n$ ,  $S_n$ , and  $T_n$  of sizes  $(n+1) \times (n+2)$ ,  $(n+1) \times (n+1)$ , and  $(n+1) \times n$ , respectively, such that

$$\varpi_1(x, y) \nabla_1 \hat{\mathbb{P}}_n + \varpi_2(x, y) \nabla_2 \hat{\mathbb{P}}_n = G_n \hat{\mathbb{P}}_{n+1} + S_n \hat{\mathbb{P}}_n + T_n \hat{\mathbb{P}}_{n-1}, \quad (34)$$

where  $\varpi_1(x, y)$  and  $\varpi_2(x, y)$  are defined in (8).

Moreover, following the same procedure we obtain the  $\Delta$ -structure relation given by

**Theorem 3.4.**

$$\varphi_1(x, y) \Delta_1 \hat{\mathbb{P}}_n + \varphi_2(x, y) \Delta_2 \hat{\mathbb{P}}_n = \tilde{G}_n \hat{\mathbb{P}}_{n+1} + \tilde{S}_n \hat{\mathbb{P}}_n + \tilde{T}_n \hat{\mathbb{P}}_{n-1}, \quad n \geq 1, \quad (35)$$

where  $\varphi_1(x, y)$  and  $\varphi_2(x, y)$  are defined in (8) and  $\tilde{G}_n$ ,  $\tilde{S}_n$  and  $\tilde{T}_n$  are  $x, y$ -independent matrices.

### 3.4. Raising and lowering operators for monic orthogonal polynomials of two discrete variables

In this section, by application of the above properties, we obtain operators lowering or raising the degree but preserving the initial family. Let us define

$$\mathbb{L}_n^- = \varpi_1(x, y) \nabla_1 + \varpi_2(x, y) \nabla_2 - \left( G_n D_n^\dagger \left[ \begin{pmatrix} x \\ y \end{pmatrix} \otimes I_{n+1} - B_n \right] + S_n \right) \mathbb{I},$$

$$\mathbb{J}_n = T_n - G_n D_n^\dagger C_n.$$

Then, using the relations (34) and (27), we have

$$\mathbb{L}_n^- \hat{\mathbb{P}}_n = \mathbb{J}_n \hat{\mathbb{P}}_{n-1}.$$

Hence,  $\mathbb{L}_n^-$  is a lowering or annihilation operator for  $\{\hat{\mathbb{P}}_n\}_{n \geq 0}$ .

Since the matrix  $C_n$  has full rank there exists an unique matrix  $C_n^\dagger$  of size  $n \times (2n+2)$ , called the *generalized inverse* of  $C_n$

$$C_n^\dagger = (C_n^T C_n)^{-1} C_n^T, \quad (36)$$

such that

$$C_n^\dagger C_n = I_n.$$

If we define

$$\mathbb{L}_n^+ = \varpi_1(x, y) \nabla_1 + \varpi_2(x, y) \nabla_2 - \left( T_n C_n^\dagger \left[ \begin{pmatrix} x \\ y \end{pmatrix} \otimes I_{n+1} - B_n \right] + S_n \right) \mathbb{I},$$

$$\mathbb{V}_n = G_n - T_n C_n^\dagger L_n,$$



then, using the relations (34) and (27), we have

$$\mathbb{L}_n^+ \hat{\mathbb{P}}_n = \mathbb{V}_n \hat{\mathbb{P}}_{n+1},$$

and, therefore,  $\mathbb{L}_n^+$  is a creation or raising operator for the monic family  $\{\hat{\mathbb{P}}_n\}_{n \geq 0}$ .

### 3.5. Derivative representations or second structure relations

We are looking for a finite-type relation between the monic orthogonal polynomial sequence  $\{\hat{\mathbb{P}}_n\}_{n \geq 0}$  and the sequence of partial differences  $\{\mathbb{Q}_n^{(x)}\}_{n \geq 0}$  (or  $\{\mathbb{Q}_n^{(y)}\}_{n \geq 0}$ ) and we get the following result.

**Theorem 3.5.** Assuming that  $\Delta_1 P_{0,n+1}^{n+1}(x, y) = 0$  and  $\Delta_2 P_{n+1,0}^{n+1}(x, y) = 0$ , for  $n \geq 1$ , there exist unique matrices  $V_{n,j}$  of size  $(n+1) \times (n+1)$ ,  $Y_{n,j}$  of size  $(n+1) \times n$ , and  $Z_{n,j}$  of size  $(n+1) \times (n-1)$ , such that

$$\hat{\mathbb{P}}_n = V_{n,1} \mathbb{Q}_n^{(x)} + Y_{n,1} \mathbb{Q}_{n-1}^{(x)} + Z_{n,1} \mathbb{Q}_{n-2}^{(x)}, \quad (37)$$

$$\hat{\mathbb{P}}_n = V_{n,2} \mathbb{Q}_n^{(y)} + Y_{n,2} \mathbb{Q}_{n-1}^{(y)} + Z_{n,2} \mathbb{Q}_{n-2}^{(y)}, \quad (38)$$

with the initial conditions  $\mathbb{Q}_{-1}^{(x)} = 0$ ,  $\mathbb{Q}_0^{(x)} = 1$ ,  $\mathbb{Q}_{-1}^{(y)} = 0$  and  $\mathbb{Q}_0^{(y)} = 1$ .

**Proof.** For  $n \geq 1$ , and  $j = 1, 2$ , we apply the operator  $\Delta_j$  to both sides of (23) to obtain:

$$\hat{\mathbb{P}}_n = L_{n,j} \Delta_j \hat{\mathbb{P}}_{n+1} + (B_{n,j} - I_{n+1}) \Delta_j \hat{\mathbb{P}}_n + C_{n,j} \Delta_j \hat{\mathbb{P}}_{n-1} - x_j \Delta_j \hat{\mathbb{P}}_n, \quad 1 \leq j \leq 2, \quad (39)$$

where  $x_1 = x$ ,  $x_2 = y$ . By using the notation (18) and the restriction  $\Delta_1 P_{0,n+1}^{n+1}(x, y) = 0$ , we can write

$$\hat{\mathbb{P}}_n = I_{n+1} \mathbb{Q}_n^{(x)} + \overrightarrow{(B_{n,1} - I_{n+1})} \mathbb{Q}_{n-1}^{(x)} + \overrightarrow{C_{n,1}} \mathbb{Q}_{n-2}^{(x)} - x \Delta_1 \hat{\mathbb{P}}_n,$$

where  $\overrightarrow{A}$  denotes  $A$  after removing the last column. Now, using

$$\Delta_1 \hat{\mathbb{P}}_n = \begin{pmatrix} \mathbb{Q}_{n-1}^{(x)} \\ 0 \end{pmatrix}, \quad (40)$$

and the recurrence relation (28) we obtain the relations (37), whose coefficients are given by

$$V_{n,1} = I_{n+1} - \begin{pmatrix} W_{n-1,1} \\ 0 \end{pmatrix}, \quad (41)$$

$$Y_{n,1} = \overrightarrow{B_{n,1} - I_{n+1}} - \begin{pmatrix} R_{n-1,1} \\ 0 \end{pmatrix}, \quad (42)$$

$$Z_{n,1} = \overrightarrow{C_{n,1}} - \begin{pmatrix} U_{n-1,1} \\ 0 \end{pmatrix}. \quad (43)$$

In a similar way, using  $\Delta_2 P_{n+1,0}^{n+1}(x, y) = 0$  we get the relations (38), where

$$V_{n,2} = I_{n+1} - \begin{pmatrix} 0 \\ W_{n-1,2} \end{pmatrix}, \quad (44)$$

$$Y_{n,2} = \overleftarrow{B_{n,2} - I_{n+1}} - \begin{pmatrix} 0 \\ R_{n-1,2} \end{pmatrix}, \quad (45)$$

$$Z_{n,2} = \overleftarrow{C_{n,2}} - \begin{pmatrix} 0 \\ U_{n-1,2} \end{pmatrix}, \quad (46)$$

where the notation  $\overleftarrow{A}$  means  $A$  after removing the first column.  $\square$

## 4. Examples

In this section we give two examples to illustrate our results. They are the monic bivariate Hahn and Kravchuk polynomials, introduced by Tratnik in [24], that are orthogonal with respect to pure point measures supported on finite sets. All these selected examples satisfy the assumption conditions (16).

#### 4.1. Monic Hahn orthogonal polynomials of two variables

The linear partial difference equation of hypergeometric type

$$\begin{aligned} & x(x - N - b - c - 3)\Delta_1 \nabla_1 u(x, y) + y(y - N - a - c - 3)\Delta_2 \nabla_2 u(x, y) + y(1 + a + x)\Delta_1 \nabla_2 u(x, y) \\ & + x(1 + b + y)\Delta_2 \nabla_1 u(x, y) + ((1 + a)(-N - 1) + x(a + b + c + 3))\Delta_1 u(x, y) \\ & + ((1 + b)(-N - 1) + (a + b + c + 3)y)\Delta_2 u(x, y) - (n + m)(n + m + a + b + c + 2)u(x, y) = 0, \end{aligned} \quad (47)$$

has as polynomial solutions the bivariate monic Hahn polynomials of total degree  $n + m$ , and they are defined as the generalized Kampé de Fériet hypergeometric series [3,21]

$$\begin{aligned} u(x, y) &= \hat{H}_{n,m}^{a,b,c}(x, y; N) \\ &= \frac{(a+1)_n(b+1)_m(-N-1)_{n+m}}{(n+m+a+b+c+2)_{n+m}} F_{1:2;2}^{1:1;1} \left( \begin{matrix} n+m+a+b+c+2 : -n, -x; -m, -y \\ -N-1 : a+1; b+1 \end{matrix} \middle| 1, 1 \right), \end{aligned} \quad (48)$$

where  $N \in \mathbb{N}$ ,  $a, b, c \in \mathbb{R}$  with  $a + b + c \geq -2$  and  $0 \leq n + m \leq N + 1$ .

In this case, the functions  $\mathcal{G}_1$  and  $\mathcal{G}_2$  defined in (7) are given by

$$\mathcal{G}_1(x, y) = \frac{(N+1-x-y)(1+a+x)}{(x+1)(N+c+1-x-y)}, \quad \mathcal{G}_2(x, y) = \frac{(N+1-x-y)(1+b+y)}{(y+1)(N+c+1-x-y)},$$

which are positive for  $c \geq 0$ . So, the discrete domain  $G$  where the polynomial solutions are orthogonal is

$$G = \{(x, y) \mid x \geq 0, y \geq 0, x + y \leq N + 1\}, \quad (49)$$

the weight function is given by

$$\varrho(x, y) = \varrho(x, y; a, b, c, N) = \frac{(a+1)_x(b+1)_y}{x!y!} \frac{(N+2-x-y)_{x+y}}{(N+c+2-x-y)_{x+y}}, \quad (50)$$

up to a constant factor.

As a result, they satisfy the following orthogonality relation

$$\sum_{x=0}^{N+1} \sum_{y=0}^{N+1-x} \hat{H}_{s_1, s_2}^{a,b,c}(x, y; N) \hat{H}_{m_1, m_2}^{a,b,c}(x, y; N) \varrho(x, y) = \hat{\Lambda}_S^{a,b,c} \delta_{S, M}, \quad (51)$$

where  $S = s_1 + s_2$ ,  $M = m_1 + m_2$ , with  $c \geq 0$ , i.e. when  $s_1 + s_2 < m_1 + m_2$ , that is, the polynomials  $\{\hat{H}_{m_1, m_2}^{a,b,c}(x, y; N)\}$  associated with the eigenvalue  $\lambda_M$  are orthogonal to all polynomials  $\{\hat{H}_{s_1, s_2}^{a,b,c}(x, y; N)\}$  associated with another eigenvalue  $\lambda_S$ , but they are not so for different polynomials of the same total degree.

It can be derived that the (column) vector of monic Hahn orthogonal polynomials

$$\hat{\mathbb{H}}_n = \hat{\mathbb{H}}_n^{a,b,c}(x, y; N) = (\hat{H}_{n,0}^{a,b,c}(x, y; N), \dots, \hat{H}_{n-i,i}^{a,b,c}(x, y; N), \dots, \hat{H}_{0,n}^{a,b,c}(x, y; N))^T$$

satisfies the three-term recurrence relations (23), where the coefficient matrices  $B_{n,j}$  and  $C_{n,j}$  are

$$B_{n,1} = \begin{pmatrix} b_{0,0} & 0 & & & & \bigcirc \\ b_{1,0} & b_{1,1} & & 0 & & \\ & \ddots & \ddots & \ddots & & \\ & & b_{n-1,n-2} & b_{n-1,n-1} & 0 & \\ \bigcirc & & & b_{n,n-1} & b_{n,n} & \end{pmatrix}, \quad (52)$$

where

$$\begin{aligned} b_{i,i} &= \frac{aN(2n+2+a+b+c) + (2n^2 - 2i^2 + 4n - 2i + 1)N + (2n - 2i + 1)(b+c)N}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ &+ \frac{(n-i)(b^2+c^2) - (n-1)a^2 + (2n-2i)bc + (1-i)a(b+c)}{(2n+1+a+b+c)(2n+3+a+b+c)} \end{aligned}$$

$$\begin{aligned}
& + \frac{(n^2 - i^2 + 5n - 6i + 1)(b + c) - (n^2 + i^2 - n + i - 2)a}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)} \\
& + \frac{(3n^2 - 5i^2 - 5i + 6n + 1)}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)}, \quad 0 \leq i \leq n, \\
b_{i+1,i} & = -\frac{(i + 1)(i + 1 + b)(5 + a + b + c + 2N)}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)}, \quad 0 \leq i \leq n - 1,
\end{aligned}$$

and

$$B_{n,2} = \begin{pmatrix} \tilde{b}_{0,0} & \tilde{b}_{0,1} & & & \bigcirc \\ 0 & \tilde{b}_{1,1} & \tilde{b}_{1,2} & & \\ & \ddots & \ddots & \ddots & \\ & & & \tilde{b}_{n-1,n-1} & \tilde{b}_{n-1,n} \\ \bigcirc & & & 0 & \tilde{b}_{n,n} \end{pmatrix}, \quad (53)$$

where

$$\begin{aligned}
\tilde{b}_{i,i} & = \frac{bN(2n + 2 + a + b + c) - (2i^2 - 4in - 2n - 2i - 1)N - (n - 1)b^2}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)} \\
& + \frac{(2i + 1)N(a + c) + ic^2 - (n - 1 + i)bc - (n - 1 - i)ab}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)} \\
& + \frac{2iac + ia^2 - (i^2 - 2in + n - 6i - 1)(a + c)}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)} \\
& + \frac{2ibc - (2n^2 + i^2 - 2in - i - 2)b - (2n^2 + 5i^2 - 10in - n - 5i - 1)}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)}, \quad 0 \leq i \leq n, \\
\tilde{b}_{i,i+1} & = -\frac{(n - i)(n - i + a)(5 + a + b + c + 2N)}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)}, \quad 0 \leq i \leq n - 1.
\end{aligned}$$

Moreover,

$$C_{n,1} = \begin{pmatrix} c_{0,0} & & & & \bigcirc \\ c_{1,0} & c_{1,1} & & & \\ c_{2,0} & c_{2,1} & & & \\ & \ddots & \ddots & \ddots & \\ & & & c_{n-1,n-1} & \\ \bigcirc & & c_{n,n-2} & c_{n,n-1} & \end{pmatrix}, \quad (54)$$

where

$$\begin{aligned}
c_{i,i} & = \frac{(n - i)(n - i + a)(n + 1 + i + b + c)(n + 1 + i + a + b + c)}{(2n + a + b + c)(2n + 1 + a + b + c)^2(2n + 2 + a + b + c)}(N - n + 2)(n + 3 + a + b + c + N), \\
& 0 \leq i \leq n - 1, \\
c_{i+1,i} & = -\frac{((2n^2 - 2i^2 + 2n - 4i - 2) + (2n - 1 - 2i)(b + c) + a(2n + 1 + a + b + c))}{(2n + a + b + c)(2n + 1 + a + b + c)^2(2n + 2 + a + b + c)} \\
& \quad \times (i + 1)(i + 1 + b)(N - n + 2)(n + 3 + a + b + c + N), \quad 0 \leq i \leq n - 1, \\
c_{i+2,i} & = \frac{(i + 1)(i + 2)(i + 1 + b)(i + 2 + b)(N - n + 2)(n + 3 + a + b + c + N)}{(2n + a + b + c)(2n + 1 + a + b + c)^2(2n + 2 + a + b + c)}, \\
& 0 \leq i \leq n - 2,
\end{aligned}$$

and

$$C_{n,2} = \begin{pmatrix} \tilde{c}_{0,0} & \tilde{c}_{0,1} & & & & \bigcirc \\ \tilde{c}_{1,0} & \tilde{c}_{1,1} & \tilde{c}_{1,2} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \tilde{c}_{n-2,n-3} & \tilde{c}_{n-2,n-2} & \tilde{c}_{n-2,n-1} & \\ & & & \tilde{c}_{n-1,n-2} & \tilde{c}_{n-1,n-1} & \\ \bigcirc & & & 0 & \tilde{c}_{n,n-1} & \end{pmatrix}, \quad (55)$$

where

$$\begin{aligned} \tilde{c}_{i,i} &= -\frac{((-2i^2 + 4in + 2n) + (2i + 1 + b)a + (2i + 1)c + (2n + 1 + b + c)b)}{(2n + a + b + c)(2n + 1 + a + b + c)^2(2n + 2 + a + b + c)} \\ &\quad \times (n - i)(n - i + a)(N - n + 2)(n + 3 + a + b + c + N), \quad 0 \leq i \leq n - 1, \\ \tilde{c}_{i,i+1} &= \frac{(n - i)(n - i - 1)(n - i + a)(n - 1 - i + a)(N - n + 2)(n + 3 + a + b + c + N)}{(2n + a + b + c)(2n + 1 + a + b + c)^2(2n + 2 + a + b + c)}, \\ &\quad 0 \leq i \leq n - 2, \\ \tilde{c}_{i+1,i} &= \frac{(i + 1)(i + 1 + b)(2n - i + a + c)(2n - 1 + a + b + c + N)}{(2n + a + b + c)(2n + 1 + a + b + c)^2(2n + 2 + a + b + c)}(N - n + 2)(2n - i + a + b + c), \\ &\quad 0 \leq i \leq n - 1. \end{aligned}$$

Using the notation introduced in (30), i.e.

$$\nabla_j \hat{\mathbb{H}}_n = (\nabla_j \hat{H}_{n,0}^{a,b,c}(x, y; N), \dots, \nabla_j \hat{H}_{n-i,i}^{a,b,c}(x, y; N), \dots, \nabla_j \hat{H}_{0,n}^{a,b,c}(x, y; N))^T, \quad j = 1, 2,$$

the  $\nabla$ -structure relation (34) can be written as

$$x(x + y - N - c - 2)\nabla_1 \hat{\mathbb{H}}_n + y(x + y - N - c - 2)\nabla_2 \hat{\mathbb{H}}_n = G_n \hat{\mathbb{H}}_{n+1} + S_n \hat{\mathbb{H}}_n + T_n \hat{\mathbb{H}}_{n-1}, \quad (56)$$

where the coefficient matrix  $G_n$  of size  $(n + 1) \times (n + 2)$  has the form

$$G_n = \begin{pmatrix} n & n & & & & \bigcirc \\ 0 & n & n & & & \\ & \ddots & \ddots & \ddots & & \\ \bigcirc & & & 0 & n & n \end{pmatrix},$$

the matrix  $S_n$  of size  $(n + 1) \times (n + 1)$  is tridiagonal

$$S_n = \begin{pmatrix} s_{0,0} & s_{0,1} & & & & \bigcirc \\ s_{1,0} & s_{1,1} & s_{1,2} & & & \\ & \ddots & \ddots & \ddots & & \\ & & s_{n-1,n-2} & s_{n-1,n-1} & s_{n-1,n} & \\ \bigcirc & & & s_{n,n-1} & s_{n,n} & \end{pmatrix}, \quad (57)$$

where

$$\begin{aligned} s_{i,i} &= \frac{(12i^2 - 4n - 12in - 8i^2n - 3n^2 + 8in^2 - 4i^2n^2 - 6n^3 + 4in^3 - 2n^4)}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)} \\ &\quad + \frac{(4i^2 - 6i + n - 4i^2n - 5n^2 + 6in^2 - 4n^3)a + (2in - 2n^2 + n - 2i)a^2}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)} \\ &\quad + \frac{(6i + 4i^2 - 5n - 8in - 4i^2n - n^2 + 2in^2 - 2n^3)b - (2n^2)ab - (2in + n - 2i)b^2}{(2n + 1 + a + b + c)(2n + 3 + a + b + c)} \end{aligned}$$

$$\begin{aligned}
& + \frac{(4i^2 - 9n - 4in - 4i^2n - 10n^2 + 4in^2 - 4n^3)c - (5n^2 - 2in + 5n + 2i)ac}{(2n+1+a+b+c)(2n+3+a+b+c)} \\
& + \frac{-(3n^2 + 2in + 7n - 2i)bc - 2nabc - ncb^2 - 2nac^2 - 2nbc^2 - (3n(n+2))c^2}{(2n+1+a+b+c)(2n+3+a+b+c)} \\
& + \frac{(n^2 + 6i^2 - 6in - n)N + (n^2 + 2i^2 - 2in - 3i + 2n)aN + (n-i)a^2N + ib^2N}{(2n+1+a+b+c)(2n+3+a+b+c)} \\
& + \frac{(n^2 + 2i^2 - 2in - n + 3i)bN - (n^2 + 2in + 2n - 2i^2)cN - (n-i)bcN}{(2n+1+a+b+c)(2n+3+a+b+c)} \\
& + \frac{nabN - nNc^2 - iacN - nca^2 - nc^3}{(2n+1+a+b+c)(2n+3+a+b+c)}, \quad 0 \leq i \leq n, \\
s_{i+1,i} &= \frac{(i+1)(i+1+b)(-(2n-2)(n+3+a+b+c) + (3+a+b+c)N)}{(2n+1+a+b+c)(2n+3+a+b+c)}, \quad 0 \leq i \leq n-1, \\
s_{i,i+1} &= \frac{(n-i)(n-i+a)(-(2n-2)(n+3+a+b+c) + (3+a+b+c)N)}{(2n+1+a+b+c)(2n+3+a+b+c)}, \quad 0 \leq i \leq n-1,
\end{aligned}$$

and the matrix  $T_n$  of size  $(n+1) \times n$  is tetradiagonal looks like

$$T_n = \begin{pmatrix} t_{0,0} & t_{0,1} & & & & \bigcirc \\ t_{1,0} & t_{1,1} & t_{1,2} & & & \\ t_{2,0} & t_{2,1} & & & & \\ & \ddots & \ddots & \ddots & & \\ & & & & t_{n-2,n-1} & \\ & & & t_{n-1,n-2} & t_{n-1,n-1} & \\ \bigcirc & & & t_{n,n-2} & t_{n,n-1} & \end{pmatrix}, \quad (58)$$

where

$$\begin{aligned}
t_{i,i} &= -\frac{(n-i)(n-i+a)(n+2+a+b+c)(N-n+2)(n+3+a+b+c+N)}{(2n+a+b+c)(2n+1+a+b+c)^2(2n+2+a+b+c)} \\
& \times ((n^2 + 3i^2 - 2in + 2i + 1) + a(n-i+c) + b(2i+1+c) + c(2n+1+c)), \quad 0 \leq i \leq n-1, \\
t_{i+1,i} &= -\frac{(i+1)(i+1+b)(n+2+a+b+c)(N-n+2)(n+3+a+b+c+N)}{(2n+a+b+c)(2n+1+a+b+c)^2(2n+2+a+b+c)} \\
& \times ((n^2 + 3i^2 - 4ni - 2n + 4i + 2) + c(2n+1+a+b+c) + (2n-1-2i)a + (i+1)b), \\
& 0 \leq i \leq n-1, \\
t_{i+2,i} &= -\frac{(i+1)(i+2)(i+1+b)(i+2+b)(n+2+a+b+c)}{(2n+a+b+c)(2n+1+a+b+c)^2(2n+2+a+b+c)}(N-n+2)(n+3+a+b+c+N), \\
& 0 \leq i \leq n-1, \\
t_{i,i+1} &= -\frac{(n-i)(n-i-1)(n-i+a)(n-i-1+a)}{(2n+a+b+c)(2n+1+a+b+c)^2(2n+2+a+b+c)}(n+2+a+b+c)(N-n+2) \\
& \times (n+3+a+b+c+N), \quad 0 \leq i \leq n-1.
\end{aligned}$$

Moreover, using the notation introduced in (17),

$$\Delta_j \hat{\mathbb{H}}_n = (\Delta_j \hat{H}_{n,0}^{a,b,c}(x, y; N), \dots, \Delta_j \hat{H}_{n-i,i}^{a,b,c}(x, y; N), \dots, \Delta_j \hat{H}_{0,n}^{a,b,c}(x, y; N))^T, \quad j = 1, 2,$$

the  $\Delta$ -structure relation (35) can be written as

$$(x+a+1)(x+y-N-1)\Delta_1 \hat{\mathbb{H}}_n + (y+b+1)(x+y-N-1)\Delta_2 \hat{\mathbb{H}}_n = \tilde{G}_n \hat{\mathbb{H}}_{n+1} + \tilde{S}_n \hat{\mathbb{H}}_n + \tilde{T}_n \hat{\mathbb{H}}_{n-1} \quad (59)$$

where  $\tilde{G}_n = G_n$ , and the matrix  $\tilde{S}_n$  of size  $(n+1) \times (n+1)$  is tridiagonal

$$\tilde{S}_n = \begin{pmatrix} \tilde{s}_{0,0} & \tilde{s}_{0,1} & & & & \bigcirc \\ \tilde{s}_{1,0} & \tilde{s}_{1,1} & \tilde{s}_{1,2} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \tilde{s}_{n-1,n-2} & \tilde{s}_{n-1,n-1} & \tilde{s}_{n-1,n} & \\ \bigcirc & & & \tilde{s}_{n,n-1} & \tilde{s}_{n,n} & \end{pmatrix}, \quad (60)$$

where

$$\begin{aligned} \tilde{s}_{i,i} = & \frac{(18i^2 - n - 18in + 8i^2n + 8n^2 - 8in^2 + 4i^2n^2 + 6n^3 - 4in^3 + 2n^4)}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ & + \frac{(-9i + 12i^2 + 8n - 16in + 4i^2n + 11n^2 - 6in^2 + 4n^3)a}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ & + \frac{(9i + 12i^2 - n - 8in + 4i^2n + 7n^2 - 2in^2 + 2n^3)b + (4n^2 + 4i^2 - 4in + 6n)ab}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ & + \frac{(2n-i)a^2b + (n+i)ab^2 + (4n^3 - 4in^2 + 4i^2n + 6n^2 - 12in + 12i^2 - 2n)c}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ & + \frac{(5n^2 + 4i^2 - 6in + 5n - 6i)ac + (2n-2i)ca^2 + (3n^2 + 4i^2 - 2in + 6i - n)bc}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ & + \frac{(2n^2 + 2i^2 - 2in - n)c^2 + (n-i)ac^2 + 2ib^2c + ibc^2 + (n^2 + 6i^2 - 6in - n)N}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ & + \frac{(n^2 + 2i^2 - 2in - 3i + 2n)aN + (n-i)a^2N + ib^2N(n^2 + 2i^2 - 2in - n + 3i)bN}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ & + \frac{(3n^2 + 2i^2 - 4in + 6n - 6i)a^2 + (n-i)a^3 + ib^3 + 2nabc + (n^2 + 2i^2 + 6i)b^2}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ & + \frac{nabN - (n^2 + 2in + 2n - 2i^2)cN - (n-i)bcN - nNc^2 - iacN}{(2n+1+a+b+c)(2n+3+a+b+c)}, \quad 0 \leq i \leq n, \\ \tilde{s}_{i+1,i} = & \frac{(2n^2 + 4n + 9 + a^2 + 3N + (b+c)(2n+6+b+c+N) + a(2(n+3+b+c)+N))}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ & \times (i+1)(i+1+b), \quad 0 \leq i \leq n-1, \\ \tilde{s}_{i,i+1} = & \frac{(2n^2 + 4n + 9 + a^2 + 3N + (b+c)(2n+6+b+c+N) + a(2(n+3+b+c)+N))}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ & \times (n-i)(n-i+a), \quad 0 \leq i \leq n-1, \end{aligned}$$

and for this bivariate Hahn polynomials, the matrix  $\tilde{T}_n$  of size  $(n+1) \times n$  verifies  $\tilde{T}_n = T_n$ .

Next, we give relations between  $\mathbb{Q}_n^{(x_j)}$  and another family of monic Hahn polynomials, extending the formulae given in [18, Section 2.4.3]. Using the notation introduced in (18) and (19), for  $n \geq 0$ , we have

$$\mathbb{Q}_n^{(x)}(x, y; a, b, c, N) = E_{n,1} \hat{\mathbb{H}}_n^{a+1,b,c+1}(x, y; N-1), \quad (61)$$

$$\mathbb{Q}_n^{(y)}(x, y; a, b, c, N) = E_{n,2} \hat{\mathbb{H}}_n^{a,b+1,c+1}(x, y; N-1), \quad (62)$$

where the diagonal matrices  $E_{n,j}$ ,  $j = 1, 2$  of size  $(n+1) \times (n+1)$  are

$$E_{n,1} = \begin{pmatrix} n+1 & & & \bigcirc \\ & n & & \\ & & \ddots & \\ & & & 1 \\ \bigcirc & & & & \end{pmatrix} \quad \text{and} \quad E_{n,2} = \begin{pmatrix} 1 & & & \bigcirc \\ & 2 & & \\ & & \ddots & \\ & & & n+1 \\ \bigcirc & & & & \end{pmatrix}. \quad (63)$$

Moreover, their differences constitute an orthogonal polynomial sequence. The coefficient matrices of the three-term relation for the differences (28) and (29) are

$$W_{n,1} = \begin{pmatrix} w_{0,0} & & & & \bigcirc \\ & w_{1,1} & & & \\ & & \ddots & & \\ & & & w_{n-1,n-1} & \\ \bigcirc & & & & w_{n,n} \end{pmatrix}, \quad w_{i,i} = \frac{n-i}{n+1-i}, \quad 0 \leq i \leq n, \quad (64)$$

and

$$W_{n,2} = \begin{pmatrix} \tilde{w}_{0,0} & & & & \bigcirc \\ & \tilde{w}_{1,1} & & & \\ & & \ddots & & \\ & & & \tilde{w}_{n-1,n-1} & \\ \bigcirc & & & 0 & \tilde{w}_{n,n} \end{pmatrix}, \quad \tilde{w}_{i,i} = \frac{i}{i+1}, \quad 0 \leq i \leq n. \quad (65)$$

Moreover,

$$R_{n,1} = \begin{pmatrix} r_{0,0} & & & & \bigcirc \\ r_{1,0} & r_{1,1} & & & \\ & \ddots & \ddots & & \\ & & r_{n-1,n-2} & r_{n-1,n-1} & \\ \bigcirc & & 0 & r_{n,n-1} & \end{pmatrix}, \quad (66)$$

where

$$r_{i,i} = \frac{aN(2n+3+a+b+c) + ((2n-2i)N - ia + n^2 - i^2 + 3n - 7i - 4)(b+c)}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ + \frac{2N(n-i)(2+n+i) + (n-1-i)(b^2+c^2) - (n^2+i^2+n+2i-2)a}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ + \frac{(2n-2(i+1))bc + (n^2-5i^2+2n-10i-3) - (n-1)a^2}{(2n+1+a+b+c)(2n+3+a+b+c)}, \quad 0 \leq i \leq n-1,$$

$$r_{i+1,i} = -\frac{(n-1-i)(i+1)(i+1+b)(5+a+b+c+2N)}{(n-i)(2n+1+a+b+c)(2n+3+a+b+c)}, \quad 0 \leq i \leq n-1,$$

and

$$R_{n,2} = \begin{pmatrix} \tilde{r}_{0,0} & \tilde{r}_{0,1} & & & \bigcirc \\ 0 & \tilde{r}_{1,1} & \tilde{r}_{1,2} & & \\ & \ddots & \ddots & \ddots & \\ & & \tilde{r}_{n-1,n-1} & \tilde{r}_{n-1,n} & \\ \bigcirc & & 0 & \tilde{r}_{n,n} & \end{pmatrix}, \quad (67)$$

where

$$\tilde{r}_{i,i} = \frac{bN(2n+3+a+b+c) + (2i+2)N(a+c) - (n-1)b^2 + i(a^2+c^2) + 2iac}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ + \frac{-(n-1-i)bc - (i^2-2in+2n-5i-2)(a+c) - (2n^2+i^2-2in+n-3)b}{(2n+1+a+b+c)(2n+3+a+b+c)} \\ + \frac{(2-2i^2+4n+4in)N - (4n^2+5i^2-10in-2n-2) - (n-i)ab}{(2n+1+a+b+c)(2n+3+a+b+c)}, \quad 0 \leq i \leq n,$$

$$\tilde{r}_{i,i+1} = -\frac{(n-1-i+a)(i+1)(n-1-i)(5+a+b+c+2N)}{(i+2)(2n+1+a+b+c)(2n+3+a+b+c)}, \quad 0 \leq i \leq n-1.$$

Finally,

$$U_{n,1} = \begin{pmatrix} u_{0,0} & & & & \bigcirc \\ u_{1,0} & u_{1,1} & & & \\ u_{2,0} & u_{2,1} & u_{2,2} & & \\ \ddots & \ddots & \ddots & & \\ \bigcirc & u_{n-1,n-3} & u_{n-1,n-2} & u_{n-1,n-1} & \\ & & u_{n,n-2} & u_{n,n-1} & \end{pmatrix}, \quad (68)$$

where

$$\begin{aligned} u_{i,i} &= \frac{(n-i)(n-i+a)(n+1+i+b+c)(N-n+2)(n+3+a+b+c+N)}{(2n+a+b+c)(2n+1+a+b+c)^2(2n+2+a+b+c)}(n+2+i+a+b+c), \\ 0 \leq i \leq n-1, \\ u_{i+1,i} &= -\frac{(i+1)(i+1+b)(N-n+2)(n+3+a+b+c+N)}{(2n+a+b+c)(2n+1+a+b+c)^2(2n+2+a+b+c)} \\ &\quad \times (2(n-1-i)(n+i+2+b+c) + a(2n+2+a+b+c)), \quad 0 \leq i \leq n-1, \\ u_{i+2,i} &= -\frac{(-12-17i-3i^2+11n+10in-n^2)(i+1+b)(i+2+b)(N-n+2)}{12(2n+a+b+c)(2n+1+a+b+c)^2(2n+2+a+b+c)}(n+3+a+b+c+N), \\ 0 \leq i \leq n-2, \end{aligned}$$

and

$$U_{n,2} = \begin{pmatrix} \tilde{u}_{0,0} & \tilde{u}_{0,1} & & & & \bigcirc \\ \tilde{u}_{1,0} & \tilde{u}_{1,1} & \tilde{u}_{1,2} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \tilde{u}_{n-2,n-3} & \tilde{u}_{n-2,n-2} & \tilde{u}_{n-2,n-1} & \\ \bigcirc & & & \tilde{u}_{n-1,n-2} & \tilde{u}_{n-1,n-1} & \\ & & & 0 & \tilde{u}_{n,n-1} & \end{pmatrix}, \quad (69)$$

where

$$\begin{aligned} \tilde{u}_{i,i} &= -\frac{(n-1-i)(n-1-i+a)(N-n+2)(n+3+a+b+c+N)}{(2n+a+b+c)(2n+1+a+b+c)^2(2n+2+a+b+c)} \\ &\quad \times (a(2i+2+b) + (2i+2)(2n-i+c) + b(2n+2+b+c)), \quad 0 \leq i \leq n-1, \\ \tilde{u}_{i,i+1} &= \frac{(-3i^2+6(n-2)(n-1)-i(4n+1))(n-1-i+a)(n-2-i+a)(N-n+2)}{12(2n+a+b+c)(2n+1+a+b+c)^2(2n+2+a+b+c)} \\ &\quad \times (n+3+a+b+c+N), \quad 0 \leq i \leq n-1, \\ \tilde{u}_{i+1,i} &= \frac{(i+2)(i+2+b)(2n-i-1+a+c)(N-n+2)(2n-i+a+b+c)}{(2n+a+b+c)(2n+1+a+b+c)^2(2n+2+a+b+c)}(n+3+a+b+c+N), \\ 0 \leq i \leq n-1. \end{aligned}$$

#### 4.1.1. Limit relations between Hahn and Appell's polynomials

Monic Appell polynomials of the second kind are defined [3,22,25]

$$\hat{B}_{n,m}^{a,b,c}(x,y) = (-1)^{n+m} \frac{(a)_n(b)_m}{(n+m+a+b+c-1)_{n+m}} F_{0;1;1}^{1;1;1} \left( \begin{matrix} n+m+a+b+c-1 : -n, -m \\ - : a; b \end{matrix} \middle| x, y \right), \quad (70)$$

where the parameters  $a, b, c$  are positive. They are orthogonal on the triangular domain

$$G = \{(x, y) : x > 0, y > 0, x + y < 1\},$$



with respect to the weight function

$$\rho(x, y) = x^{a-1} y^{b-1} (1 - x - y)^{c-1},$$

i.e.

$$\iint_G \hat{B}_{n,m}^{a,b,c}(x, y) \hat{B}_{k,s}^{a,b,c}(x, y) \rho(x, y) dx dy = 0$$

for  $k + s < n + m$ .

These Appell polynomials can be deduced from the Hahn polynomials defined by (48) by dividing by  $(N + 1)_{n+m}$  and letting  $N \rightarrow \infty$ :

$$\lim_{N \rightarrow \infty} \frac{\hat{H}_{n,m}^{a,b,c}(Nx, Ny; N)}{(N + 1)_{n+m}} = \hat{B}_{n,m}^{a+1,b+1,c+1}(x, y). \quad (71)$$

#### 4.2. Monic Kravchuk orthogonal polynomials of two variables

The linear partial difference equation

$$\begin{aligned} (p_1 - 1)x\Delta_1\nabla_1 u(x, y) + (p_2 - 1)y\Delta_2\nabla_2 u(x, y) + p_1 y\Delta_1\nabla_2 u(x, y) + p_2 x\Delta_2\nabla_1 u(x, y) \\ + (x - Np_1)\Delta_1 u(x, y) + (y - Np_2)\Delta_2 u(x, y) - (n_1 + n_2)u(x, y) = 0, \end{aligned} \quad (72)$$

has as a solution the bivariate monic Kravchuk polynomials of total degree  $n_1 + n_2$ , defined by means of the generalized Kampé de Fériet hypergeometric series [23,24]

$$\hat{K}_{n_1, n_2}^{p_1, p_2}(x, y; N) = (-1)^{n_1 + n_2} p_1^{n_1} p_2^{n_2} (N - n_1 - n_2 + 1)_{n_1 + n_2} F_{1;0;0}^{0;2;2} \left( \begin{matrix} - : -n_1, -x; -n_2, -y \\ -N : -; - \end{matrix} \middle| \frac{1}{p_1}, \frac{1}{p_2} \right), \quad (73)$$

where  $N$  is a non-negative integer and  $p_1, p_2$  are real parameters satisfying

$$p_1 > 0, \quad p_2 > 0, \quad 0 < p_1 + p_2 < 1.$$

In this case, the functions  $\mathcal{G}_1$  and  $\mathcal{G}_2$  defined in (7) are given by

$$\mathcal{G}_1(x, y) = \frac{p_1(N - x - y)}{(x + 1)(1 - p_1 - p_2)}, \quad \mathcal{G}_2(x, y) = \frac{p_2(N - x - y)}{(y + 1)(1 - p_1 - p_2)},$$

the discrete domain is

$$G = \{(x, y) \mid x \geq 0, y \geq 0, 0 \leq x + y \leq N\}, \quad (74)$$

and the weight function is

$$\varrho(x, y) = \varrho(x, y; p_1, p_2, N) = \frac{p_1^x (1 - p_1 - p_2)^{-x-y} p_2^y N!}{x!(N - x - y)!y!}, \quad (75)$$

up to a constant factor, which coincides with the trinomial distribution [23,29].

These polynomials satisfy the following orthogonality relation

$$\sum_{x=0}^N \sum_{y=0}^{N-x} \hat{K}_{m_1, m_2}^{p_1, p_2}(x, y; N) \hat{K}_{s_1, s_2}^{p_1, p_2}(x, y; N) \varrho(x, y) = \Lambda_S \delta_{S, M}, \quad (76)$$

where

$$\begin{aligned} \Lambda_S = (-1)^{N+S} (N - S + 1)_S p_1^{s_1} p_2^{s_2} (p_1 + p_2 - 1)^{-N} s_1! s_2! (p_1 - 1)^{s_1} (p_2 - 1)^{s_2} \\ \times {}_2F_1 \left( \begin{matrix} -s_2, -s_1 \\ 1 \end{matrix} \middle| \frac{p_1 p_2}{(p_1 - 1)(p_2 - 1)} \right), \end{aligned}$$

with  $S = s_1 + s_2$ ,  $M = m_1 + m_2$  and  $S, M \leq N$ .

The monic bivariate Kravchuk polynomials given by (73) follow from the monic bivariate Hahn polynomials defined by (48) by taking  $a = \frac{cp_1}{1-p_1-p_2}$ ,  $b = \frac{cp_2}{1-p_1-p_2}$  and letting  $c \rightarrow \infty$  in the following way:

$$\lim_{c \rightarrow \infty} \hat{H}_{n,m}^{\frac{cp_1}{1-p_1-p_2}, \frac{cp_2}{1-p_1-p_2}, c}(x, y; N) = \hat{K}_{n,m}^{p_1, p_2}(x, y; N+1). \quad (77)$$

Moreover, these monic bivariate Kravchuk polynomials defined by Tratnik [24] (see [20, p. 743])  $\{\hat{K}_{m_1, m_2}^{p_1, p_2}(x, y; N)\}$  and  $\{K_{n_1, n_2}^{p_1, p_2}(x, y; N)\}$  form a biorthogonal polynomial family.

Using the notation introduced in (4), (18) and (19), the (column) vector of monic orthogonal polynomials is defined as

$$\mathbb{K}_n = \hat{\mathbb{K}}_n^{p_1, p_2}(x, y; N) = (\hat{K}_{n,0}^{p_1, p_2}(x, y; N), \dots, \hat{K}_{n-i,i}^{p_1, p_2}(x, y; N), \dots, \hat{K}_{0,n}^{p_1, p_2}(x, y; N))^T, \quad (78)$$

and, for  $n \geq 0$ , the coefficient matrices of the three-term relation (23) are

$$B_{n,1} = \begin{pmatrix} b_{0,0} & & & & \bigcirc \\ b_{1,0} & b_{1,1} & & & \\ & \ddots & \ddots & & \\ & & b_{n-1,n-2} & b_{n-1,n-1} & \\ \bigcirc & & & b_{n,n-1} & b_{n,n} \end{pmatrix}, \quad (79)$$

where

$$b_{i,i} = (n-i) + (N-(2n-i))p_1, \quad 0 \leq i \leq n,$$

$$b_{i+1,i} = -(i+1)p_2, \quad 0 \leq i \leq n-1,$$

and

$$B_{n,2} = \begin{pmatrix} \tilde{b}_{0,0} & \tilde{b}_{0,1} & & & \bigcirc \\ 0 & \tilde{b}_{1,1} & \tilde{b}_{1,2} & & \\ & \ddots & \ddots & \ddots & \\ & & \tilde{b}_{n-1,n-1} & \tilde{b}_{n-1,n} & \\ \bigcirc & & & 0 & \tilde{b}_{n,n} \end{pmatrix}, \quad (80)$$

where

$$\tilde{b}_{i,i} = i + (N-i-2)p_2, \quad 0 \leq i \leq n,$$

$$\tilde{b}_{i,i+1} = -(n-i)p_1, \quad 0 \leq i \leq n-1.$$

Moreover,

$$C_{n,1} = \begin{pmatrix} c_{0,0} & & & & \bigcirc \\ c_{1,0} & c_{1,1} & & & \\ & c_{2,1} & c_{2,2} & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-1,n-2} & c_{n-1,n-1} & \\ \bigcirc & & & c_{n,n-1} & \end{pmatrix}, \quad (81)$$

where

$$c_{i,i} = -(n-i)(-n+1+N)(-1+p_1)p_1, \quad 0 \leq i \leq n-1,$$

$$c_{i+1,i} = -(i+1)(-n+1+N)p_1p_2, \quad 0 \leq i \leq n-1,$$

and

$$C_{n,2} = \begin{pmatrix} \tilde{c}_{0,0} & & & & \bigcirc \\ \tilde{c}_{1,0} & \tilde{c}_{1,1} & & & \\ & \ddots & \ddots & & \\ & & \tilde{c}_{n-2,n-3} & \tilde{c}_{n-2,n-2} & \\ & & & \tilde{c}_{n-1,n-2} & \tilde{c}_{n-1,n-1} \\ \bigcirc & & & 0 & \tilde{c}_{n,n-1} \end{pmatrix}, \quad (82)$$

where

$$\begin{aligned}\tilde{c}_{i,i} &= -(n-i)(-n+1+N)p_1p_2, \quad 0 \leq i \leq n-1, \\ \tilde{c}_{i+1,i} &= -(i+1)(-n+1+N)(-1+p_2)p_2, \quad 0 \leq i \leq n-1.\end{aligned}$$

It is easy to verify that the following representation holds

$$\mathbb{Q}_n^{(x)}(x, y; N) = E_{n,1} \hat{\mathbb{K}}_n^{p_1,p_2}(x, y; N-1), \quad (83)$$

$$\mathbb{Q}_n^{(y)}(x, y; N) = E_{n,2} \hat{\mathbb{K}}_n^{p_1,p_2}(x, y; N-1), \quad (84)$$

where the diagonal matrices  $E_{n,j}$ ,  $j = 1, 2$  of size  $(n+1) \times (n+1)$  are defined in (63), extending the formulae given in [18, Section 2.4.3].

Using the notation

$$\nabla_j \hat{\mathbb{K}}_n = (\nabla_j \hat{K}_{n,0}^{p_1,p_2}(x, y; N), \dots, \nabla_j \hat{K}_{n-i,i}^{p_1,p_2}(x, y; N), \dots, \nabla_j \hat{K}_{0,n}^{p_1,p_2}(x, y; N))^T, \quad j = 1, 2,$$

and

$$\Delta_j \hat{\mathbb{K}}_n = (\Delta_j \hat{K}_{n,0}^{p_1,p_2}(x, y; N), \dots, \Delta_j \hat{K}_{n-i,i}^{p_1,p_2}(x, y; N), \dots, \Delta_j \hat{K}_{0,n}^{p_1,p_2}(x, y; N))^T, \quad j = 1, 2,$$

the matrices  $G_n$  and  $\tilde{G}_n$  in the structure relations (34) and (35) are identically zero, i.e.

$$x \nabla_1 \hat{\mathbb{K}}_n + y \nabla_2 \hat{\mathbb{K}}_n = S_n \hat{\mathbb{K}}_n + T_n \hat{\mathbb{K}}_{n-1}, \quad (85)$$

$$p_1(x+y-N) \Delta_1 \hat{\mathbb{K}}_n + p_2(x+y-N) \Delta_2 \hat{\mathbb{K}}_n = \tilde{S}_n \hat{\mathbb{K}}_n + \tilde{T}_n \hat{\mathbb{K}}_{n-1}, \quad (86)$$

and the entries of the matrices  $S_n$ ,  $T_n$ ,  $\tilde{S}_n$  defined in (57), (58) and (60) respectively, are

$$\begin{aligned}s_{i,j} &= 0, \quad i \neq j, \quad s_{i,i} = n, \\ t_{i,i} &= (n-i)(1-n+N)p_1, \quad t_{i+1,i} = i(1-n+N)p_2, \\ \tilde{s}_{i,i+1} &= (n-i)p_1, \quad \tilde{s}_{i,i} = (n-i)p_1 + ip_2, \quad \tilde{s}_{i+1,i} = (i+1)p_2.\end{aligned}$$

The entries of the matrix

$$\tilde{T}_n = \begin{pmatrix} \tilde{t}_{0,0} & & & & \bigcirc \\ \tilde{t}_{1,0} & \tilde{t}_{1,1} & & & \\ & \ddots & \ddots & & \\ & & \tilde{t}_{n-1,n-2} & \tilde{t}_{n-1,n-1} & \\ \bigcirc & & & \tilde{t}_{n,n-1} & \end{pmatrix}, \quad (87)$$

are

$$\begin{aligned}\tilde{t}_{i,i} &= (i-n)(1-n+N)(1-p_1-p_2)p_1, \quad 0 \leq i \leq n-1, \\ \tilde{t}_{i+1,i} &= -(i+1)(1-n+N)(1-p_1-p_2)p_2, \quad 0 \leq i \leq n-1.\end{aligned}$$

Moreover, the matrices  $Z_{n,1}$  and  $Z_{n,2}$  in the relations (37) and (38) are identically zero with

$$\begin{aligned}\hat{\mathbb{K}}_n &= V_{n,1} \mathbb{Q}_n^{(x)} + Y_{n,1} \mathbb{Q}_{n-1}^{(x)}, \\ \hat{\mathbb{K}}_n &= V_{n,2} \mathbb{Q}_n^{(y)} + Y_{n,2} \mathbb{Q}_{n-1}^{(y)},\end{aligned} \quad (88)$$

with  $V_{n,j} = E_{n,j}^{-1}$ ,  $j = 1, 2$ , where the diagonal matrices  $E_{n,j}$  are defined in (63) and

$$Y_{n,1} = \begin{pmatrix} \frac{-p_1}{n} & -p_1 & & & \bigcirc \\ & \ddots & \ddots & & \\ & & \frac{-(n-1)p_2}{2} & -p_1 & \\ \bigcirc & & & -np_2 & \end{pmatrix}, \quad Y_{n,2} = \begin{pmatrix} -np_1 & \frac{-(n-1)p_1}{2} & & & \bigcirc \\ -p_2 & \ddots & \ddots & & \\ & & \ddots & -p_2 & \frac{-p_1}{n} \\ \bigcirc & & & -p_2 & \end{pmatrix}. \quad (89)$$

## 5. Final remarks

A special case of the monic bivariate Kravchuk polynomials are the monic Meixner polynomials [23,24]

$$\hat{M}_{n_1, n_2}^{\gamma, a_1, a_2}(x, y) = \frac{a_1^{n_1} a_2^{n_2}}{(a_1 + a_2 - 1)^{n_1 + n_2}} (\gamma)_{n_1 + n_2} F_{1:0;0}^{0:2;2} \left( \begin{matrix} - : -n_1, -x; -n_2, -y \\ \gamma : -; - \end{matrix} \middle| \frac{a_1 + a_2 - 1}{a_1}, \frac{a_1 + a_2 - 1}{a_2} \right), \quad (90)$$

where  $a_1, a_2$  are non-zero real parameters satisfying

$$|a_1| + |a_2| < 1, \quad (91)$$

and the variable  $\gamma$  is arbitrary, including negative integers. They are related by means

$$\hat{M}_{n_1, n_2}^{-N, \frac{a_1}{a_1 + a_2 - 1}, \frac{a_2}{a_1 + a_2 - 1}}(x, y) = \hat{K}_{n_1, n_2}^{a_1, a_2}(x, y; N). \quad (92)$$

We have omitted results on monic bivariate Meixner polynomials because of (92) and lack of space.

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