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Weak and point-wise convergence in $C(K)$ for filter convergence

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To Isaac Namioka with respect and admiration on the occasion of his 80th birthday

Abstract

We study those filters $\mathcal{F}$ on $\mathbb{N}$ for which weak $\mathcal{F}$-convergence of bounded sequences in $C(K)$ is equivalent to point-wise $\mathcal{F}$-convergence. We show that it is sufficient to require this property only for $C[0,1]$ and that the filter-analogue of the Rainwater extremal test theorem arises from it. There are ultrafilters which do not have this property and under the continuum hypothesis there are ultrafilters which have it. This implies that the validity of the Lebesgue dominated convergence theorem for $\mathcal{F}$-convergence is more restrictive than the property which we study.

Keywords: Measurable functions; Filter convergence; Dominated convergence theorem for filters; Extremal test for weak convergence; Banach space

1. Introduction

Every theorem of classical Analysis, Functional Analysis or of Measure Theory that states a property of sequences leads to a class of filters for which this theorem is valid. Sometimes such a class of filters is trivial (say, all filters or the filters with a countable base), but in several cases this approach leads to a new class of filters, and the characterization of this class can be very non-trivial task. Among such non-trivial examples are *Lebesgue filters* (for which the Lebesgue dominated convergence theorem is valid), *Egorov filters* which correspond to the Egorov theorem on almost uniform convergence [4], *Schur filters* (for which weak convergence in ℓ_1 coincides with the strong one) [1] and those filters $\mathcal{F}$ for which every weakly $\mathcal{F}$ convergent sequence has a norm-bounded subsequence [3].

One of the reasons to study such questions is that they bring a new light to the classical results. Say, one can ask to what extent the Lebesgue dominated convergence theorem for functions on $[0,1]$ implies the same theorem for other measure spaces. This question is not a well-posed mathematical problem: if both the theorems are true, how one can see that one of them is not deducible from the other one? But if one looks at these theorems in a general setting when the ordinary convergence of sequences is substituted by a filter convergence, the problem makes sense. This problem and some other related ones are studied below.

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The structure of the paper is as follows: in the next section we show at first that if one considers the dominated convergence theorem for Borel functions on a compact, the continuity of the functions involved in the definition of the filter property we study is not a restriction (Theorem 2.6). Then we apply a bit of Banach space theory to demonstrate that the dominated convergence theorem for continuous functions from $C(K)$ with respect to regular Borel measures (which is equivalent to coincidence of weak convergence in $C(K)$ with point-wise convergence for bounded sequences) can be deduced from the same theorem only for **metric** compacts K (Theorem 2.10). Afterwards by a pure measure-theoretic argument we reduce this theorem to $[0, 1]$ equipped with the standard Lebesgue measure λ (Theorem 3.9). In the last section we show that for some ultrafilters even the simplest $([0, 1], \lambda)$ particular case of the dominated convergence theorem fails to be true, and that under the continuum hypothesis there are free ultrafilters for which it **is** true. The latter, together with [4, Corollary 2.16] (which states that the dominated convergence theorem for general finite measure spaces cannot be true for a free ultrafilter) demonstrates that the general dominated convergence theorem does not follow from the simplified $([0, 1], \lambda)$ version.

Recall that a *filter* $\mathcal{F}$ on $\mathbb{N}$ is a non-empty collection of subsets of $\mathbb{N}$ satisfying the following axioms: $\emptyset \notin \mathcal{F}$; if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$; and for every $A \in \mathcal{F}$ if $B \supset A$ then $B \in \mathcal{F}$.

A sequence (x_n), $n \in \mathbb{N}$, in a topological space X is said to be *$\mathcal{F}$-convergent* to x (and we write $x = \mathcal{F}\text{-}\lim x_n$ or $x_n \to_{\mathcal{F}} x$) if for every neighborhood U of x the set $\{n \in \mathbb{N}\colon x_n \in U\}$ belongs to $\mathcal{F}$.

In particular if one takes as $\mathcal{F}$ the filter of sets with finite complements (the *Fréchet filter*), then $\mathcal{F}$-convergence coincides with the ordinary one.

A point $x \in X$ is said to be *a cluster point of the sequence* (x_n) *with respect to* $\mathcal{F}$ if x belongs to the closure of $\{x_n\}_{n\in A}$ for every $A \in \mathcal{F}$.

The natural ordering on the set of filters on $\mathbb{N}$ is defined as follows: $\mathcal{F}_1 \succ \mathcal{F}_2$ if $\mathcal{F}_1 \supset \mathcal{F}_2$. If G is a centered collection of subsets (i.e. all finite intersections of elements of G are non-empty), then there is a filter containing all elements of G. The smallest filter, containing all elements of G is called *the filter generated by* G.

If $x = \mathcal{F}_2\text{-}\lim x_n$, then $x = \mathcal{F}_1\text{-}\lim x_n$ for every $\mathcal{F}_1 \succ \mathcal{F}_2$. If y is not a cluster point of the sequence (x_n) with respect to $\mathcal{F}_2$ and $\mathcal{F}_1 \succ \mathcal{F}_2$, then y is not a cluster point of the sequence (x_n) with respect to $\mathcal{F}_1$.

Let $\mathcal{F}$ be a filter. A collection of subsets $G \subset \mathcal{F}$ is called *the base of* $\mathcal{F}$ if for every $A \in \mathcal{F}$ there is a $B \in G$ such that $B \subset A$. The simplest filters are those with a countable base.

A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be *free* if it dominates the Fréchet filter. Below when we say "filter" we mean a free filter on $\mathbb{N}$. In particular every ordinary convergent sequence will be automatically $\mathcal{F}$-convergent.

A maximal in the natural ordering filter is called an *ultrafilter*. The Zorn lemma implies that every filter is dominated by an ultrafilter. A filter $\mathcal{F}$ on $\mathbb{N}$ is an ultrafilter if and only if for every $A \subset \mathbb{N}$ either A or $\mathbb{N} \setminus A$ belongs to $\mathcal{F}$. More about filters, ultrafilters and their applications one can find in every modern General Topology textbook, for example in [6].

All over the text below, unless it is specified otherwise, we use the following notations: K stands for a compact set, X is a Banach space, $\mathcal{F}$ is a filter on $\mathbb{N}$, B_X is the closed unit ball of X, δ_x is the delta measure ($\delta_x(A) = 1$ if $x \in A$, otherwise $\delta_x(A) = 0$), $\mathrm{ex}(A)$ is the set of extreme points of A. All the spaces, functionals and operators are assumed to be over the field of reals. The word "measure" means for us a countably additive finite measure. For the standard Banach space terminology we refer to [5].

2. Basic definitions. Rainwater filters. Reduction to metric compacts

Definition 2.1. Let K be a compact, and μ be a regular Borel measure on K. A filter $\mathcal{F}$ is said to be *C-Lebesgue with respect to μ on K* (has the *C-Lebesgue property with respect to μ on K*) if for every uniformly bounded sequence of functions $f_n \in C(K)$ which is point-wise $\mathcal{F}$-convergent to 0, the $\mathcal{F}\text{-}\lim \int_K f_n \, d\mu$ equals 0.

Definition 2.2. A filter $\mathcal{F}$ is said to be *C-Lebesgue* (has the *C-Lebesgue property*) if for every compact K and regular Borel measure μ on it, $\mathcal{F}$ has the C-Lebesgue property with respect to μ on K.

Remark 2.3. In Definition 2.1 it is sufficient to consider only functions taking values from $[0, 1]$. This follows from the uniform boundedness in connection with formula $f_n = f_n^+ - f_n^-$, where $f_n^+ = \max\{f_n, 0\}$ and $f_n^- = f_n^+ - f_n$ are positive functions. In fact, for arbitrary sequence (f_n) with $\sup_n \|f_n\| = C < \infty$ we can write $f_n = C(g_n - h_n)$, where $g_n = \frac{1}{C} f_n^+$, $h_n = \frac{1}{C} f_n^-$, and apply the definition to (g_n) and (h_n) to get the required property of (f_n).

Remark 2.4. Hahn–Jordan decomposition theorem says that each regular Borel measure μ is a linear combination of two probability regular measures. Therefore, in Definition 2.2 it is sufficient to consider probability measures μ (i.e. nonnegative with $\mu(K) = 1$).

Lemma 2.5. *Let K be a compact, μ be a regular Borel probability measure. The following properties of a filter $\mathcal{F}$ are equivalent*:

(1) *For every sequence of Borel measurable sets $A_n \subset K$ the point-wise $\mathcal{F}$-convergence to 0 of χ_{A_n} implies that*

$$\mathcal{F}\text{-}\lim \mu(A_n) = 0.$$

(2) *For every uniformly bounded sequence of Borel measurable functions $f_n : K \to \mathbb{R}$ which is point-wise $\mathcal{F}$-convergent to 0, the $\mathcal{F}\text{-}\lim \int_K f_n \, d\mu$ equals 0.*

Proof. We only have to prove that (1) $\Rightarrow$ (2). Let $f_n : K \to [0,1]$ be $\mathcal{F}$ point-wise converging to 0 sequence of Borel measurable functions. For an $\varepsilon > 0$ define $A_n = \{x \in K\colon f_n(x) \geqslant \varepsilon/2\}$. Since $f_n(x) \geqslant \frac{\varepsilon}{2}\chi_{A_n}$ it follows that the sequence of χ_{A_n} is point-wise $\mathcal{F}$-convergent to 0. So by (1) there is $I \in \mathcal{F}$ such that for each $m \in I$ it is true that $\mu(A_n) < \varepsilon/2$. Finally,

$$\int\limits_K f_m \, d\mu = \int\limits_{K \setminus A_m} f_m \, d\mu + \int\limits_{A_m} f_m \, d\mu < \varepsilon/2 + \mu(A_m) < \varepsilon. \qquad \square$$

Theorem 2.6. *The following properties of a filter $\mathcal{F}$ are equivalent*:

(1) *$\mathcal{F}$ is C-Lebesgue.*
(2) *For every compact K, every regular Borel probability measure μ on K and for every sequence of Borel measurable sets $A_n \subset K$ the point-wise $\mathcal{F}$-convergence to 0 of χ_{A_n} implies that*

$$\mathcal{F}\text{-}\lim \mu(A_n) = 0.$$

(3) *For every compact K, every regular Borel probability measure μ on K and for every uniformly bounded sequence of Borel measurable functions $f_n : K \to \mathbb{R}$ which is point-wise $\mathcal{F}$-convergent to 0, the $\mathcal{F}\text{-}\lim \int_K f_n \, d\mu$ equals 0.*

Proof. (1) $\Rightarrow$ (2). For any $\varepsilon > 0$ approximate A_n and its complement $K \setminus A_n$ from inside with closed sets B_n and C_n to make $|\mu(A_n) - \mu(B_n)| < \varepsilon_n/2$, $|\mu(K \setminus A_n) - \mu(C_n)| < \varepsilon_n/2$, where $\varepsilon_n > 0$ and $\sum_n \varepsilon_n \leqslant \varepsilon$. Consider $G = \bigcap_n (B_n \sqcup C_n)$ and the functions f_n—restrictions to G of χ_{B_n}. We have $\mu(K \setminus G) < \varepsilon$, $f_n \in C(G)$ and point-wise $\mathcal{F}$-converge to 0. By the definition of C-Lebesgue filters

$$\mathcal{F}\text{-}\lim \int\limits_G f_n \, d\mu = 0. \tag{2.1}$$

Also we have

$$\int\limits_K \chi_{A_n} \, d\mu = \int\limits_G \chi_{A_n} \, d\mu + \int\limits_{K \setminus G} \chi_{A_n} \, d\mu < \int\limits_G f_n \, d\mu + \varepsilon.$$

Since ε is arbitrary, Eq. (2.1) gives (2). The implication (2) $\Rightarrow$ (3) is already proved in the previous lemma, and the remaining implication (3) $\Rightarrow$ (1) is obvious. $\square$

Remark 2.7. The above theorem is still true if we restrict the filter properties (1)–(3) of Theorem 2.6 to metric compacts. The proof does not change.

Recall that the members of $C(K)^*$ act like integration via regular Borel measures on K. The connection between weak $\mathcal{F}$-convergence of bounded sequences in $C(K)$ and the C-Lebesgue property of $\mathcal{F}$ is obvious:

Proposition 2.8. *$\mathcal{F}$ is C-Lebesgue if and only if for every compact K each bounded point-wise $\mathcal{F}$-convergent to* 0 *sequence of $f_n \in C(K)$ is weakly $\mathcal{F}$-convergent to* 0.

The famous Rainwater extremal test for weak convergence is also related to the dominated convergence theorem and will be useful for us.

Rainwater's theorem. *(See [2, Chapter IX].) Let X be a Banach space and (x_n) be a bounded sequence in X. In order that (x_n) weakly converge to $x \in X$ it is both necessary and sufficient that* $\lim_n x^* x_n = x^* x$ *for each extreme point x^* of B_{X^*}.*

Its particular case is the criterium of weak convergence in $C(K)$ and the main part in its proof is based on the Lebesgue's dominated convergence theorem.

Definition 2.9. $\mathcal{F}$ is said to be a *Rainwater filter* (has the *Rainwater property*) if for every Banach space X and for every bounded sequence $(x_n) \subset X$ $\mathcal{F}$-convergence to 0 of $x^* x_n$ for all $x^* \in \mathrm{ex}(B_{X^*})$ implies that (x_n) weak $\mathcal{F}$-converges to 0.

Theorem 2.10. *The following conditions are equivalent.*

(1) *$\mathcal{F}$ is a C-Lebesgue filter.*
(2) *For every metric compact K and μ on K $\mathcal{F}$ is C-Lebesgue with respect to μ on K.*
(3) *For every separable X and for every bounded sequence $(x_n) \subset X$ (x_n) weakly $\mathcal{F}$-converge to* 0 *provided* $\mathcal{F}\text{-}\lim x^* x_n = 0$ *for each* $x^* \in \mathrm{ex}(B_{X^*})$.
(4) *$\mathcal{F}$ is a Rainwater filter.*

Proof. (4) $\Rightarrow$ (1). Consider $X = C(K)$. Each extreme point x^* of B_{X^*} is represented by a measure either of the form δ_t or of the form $-\delta_t$, $t \in K$; the action of δ_t on $f \in X$ is $\delta_t(f) = f(t)$. Thus the condition $\mathcal{F}\text{-}\lim x^* x_n = 0$ for extreme points of B_{X^*} is equivalent to the point-wise $\mathcal{F}$-convergence. Now Proposition 2.8 gives us the C-Lebesgue property of $\mathcal{F}$.

The implication (1) $\Rightarrow$ (2) is obvious. To check implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) let us follow the proof of the Rainwater theorem [2, Chapter IX, pp. 155–156].

(2) $\Rightarrow$ (3). Let X be a separable Banach space and $(x_n) \subset X$ be such a bounded sequence that $\mathcal{F}\text{-}\lim x^* x_n = 0$ for each extreme point x^* of B_{X^*}. Then B_{X^*} is metrizable convex weak* compact and we can use the Choquet integral representation theorem. It says that for each point x^* of B_{X^*} there is a regular probability measure μ on $(B_{X^*}, (\text{weak}^*))$ which is concentrated on $\mathrm{ex}(B_{X^*})$ such that

$$a(x^*) = \int\limits_{B_{X^*}} a(y^*)\, d\mu(y^*) = \int\limits_{\mathrm{ex}(B_{X^*})} a(y^*)\, d\mu(y^*)$$

for each affine continuous function a. In particular we can take for a any $x \in X$ considering it as a functional on X^*. The set of extreme points of B_{X^*} is a Borel set. Applying Theorem 2.6 (the implication (1) $\Rightarrow$ (3)) in the view of Remark 2.7 for Borel measurable functions $f_n(y^*) = x_n(y^*) \chi_{\mathrm{ex}(B_{X^*})}(y^*)$, $y^* \in B_{X^*}$, we obtain

$$\begin{aligned} 0 &= \int\limits_{\mathrm{ex}(B_{X^*})} \mathcal{F}\text{-}\lim x_n(y^*)\, d\mu(y^*) = \int\limits_{B_{X^*}} \mathcal{F}\text{-}\lim f_n(y^*)\, d\mu(y^*) \\ &= \mathcal{F}\text{-}\lim \int\limits_{B_{X^*}} f_n(y^*)\, d\mu(y^*) = \mathcal{F}\text{-}\lim \int\limits_{\mathrm{ex}(B_{X^*})} x_n(y^*)\, d\mu(y^*) \\ &= \mathcal{F}\text{-}\lim x_n(x^*). \end{aligned}$$

This means that 0 is the weak $\mathcal{F}$-limit of (x_n).

(3) $\Rightarrow$ (4). Let X be a general Banach space and let $(x_n) \subset X$ be such that $\mathcal{F}\text{-}\lim x^* x_n = 0$ for every extreme point x^* of B_{X^*}. Denote by X_0 the closed linear span of (x_n). X_0 is separable. Let us show that $\mathcal{F}\text{-}\lim x^* x_n = 0$ for

every extreme point x^* of $B_{X_0^*}$. Take any $y^* \in \mathrm{ex}(B_{X_0^*})$ and let $HB(y^*) = \{x^* \in B_{X^*}: x^*|_{X_0} = y^*\}$. $HB(y^*)$ is non-empty, convex, weak* compact, and is extremal subset of B_{X^*}. So $HB(y^*)$ contains some extreme point z^* of B_{X^*} and

$$\mathcal{F}\text{-}\lim z^* x_n = \mathcal{F}\text{-}\lim y^* x_n = 0.$$

Now, since X_0 is separable, we know that (x_n) weakly $\mathcal{F}$-converges to 0 in X_0 and hence in X as well. □

3. Reduction of the general case to the case of [0, 1]

The aim of this section is to show that the C-Lebesgue property follows from the C-Lebesgue property with respect to the Lebesgue measure λ on [0, 1]. Thanks to Remark 2.4 and Theorem 2.10 we may consider only metric compacts and regular probability measures.

Lemma 3.1. *Let K be a metric compact, μ be a purely atomic measure on K and $\mathcal{F}$ be an arbitrary filter. Then $\mathcal{F}$ is C-Lebesgue with respect to μ on K.*

Proof. Since K is a metric compact, all atoms of μ are equivalent to one-point sets $\{t_k\}$ and $\mu = \sum_{k=1}^{\infty} a_k \delta_{t_k}$, $a_k \in \mathbb{R}$. Let (f_n), $0 \leqslant f_n \leqslant 1$, be a sequence of continuous functions on K point-wise $\mathcal{F}$-convergent to 0. Let us fix an $\varepsilon > 0$ and choose N big enough to make $\sum_{k=N+1}^{\infty} a_k < \varepsilon/2$. Take some $\varepsilon_k > 0$ such that $\sum_{k=1}^{N} \varepsilon_k < \varepsilon/2$. Due to the point-wise $\mathcal{F}$-convergence of (f_n), $(f_n(t_k))_{n\in\mathbb{N}}$ $\mathcal{F}$-converge to 0 for every fixed $k \in \mathbb{N}$. So we can find for each k an element $I_k \in \mathcal{F}$ such that $f_n(t_k) < \varepsilon_k$ for all $n \in I_k$. Put $I = \bigcap_{k=1}^{N} I_k \in \mathcal{F}$. Then, for all $n \in I$ we have

$$\int_K f_n \, d\mu = \sum_{k=1}^{\infty} f_n(t_k) a_k < \sum_{k=1}^{N} f_n(t_k) + \varepsilon/2 < \varepsilon.$$

Therefore, $\mathcal{F}$ is C-Lebesgue with respect to μ on K. □

Let K be a metric compact with non-atomic probability measure μ on it.

Definition 3.2. A subset $K_1 \subset K$ is said to be *(d,q)-disconnected subset of K with μ* if it is represented as a finite union $K_1 = \bigsqcup_{n=1}^{N} B_n$ of disjoint closed subsets $B_n \subset K$ (called *parts of* K_1) such that for all $n = 1, 2, \ldots, N$

(1) $\mathrm{diam}\, B_n < d$,
(2) $0 < \mu(B_n) < d$, and
(3) $\mu(K) > \mu(K_1) > \mu(K) - q$.

Proposition 3.3. *For every $d, q > 0$ there is a (d,q)-disconnected subset K_1 of K with μ. Moreover, if K is represented as a finite union of closed sets C_n, then K_1 can be constructed in such a way, that every part of K_1 is a subset of some C_m.*

Proof. Cover K by the closed balls $B_K(x, d/2)$ and choose a finite subcovering $U_1 = B_K(x_1, d/2)$, $U_2 = B_K(x_2, d/2)$, $\ldots$, $U_N = B_K(x_N, d/2)$. Then make this covering disjoint by introducing

$$A_1 = U_1, \quad A_2 = U_2 \setminus A_1, \quad \ldots, \quad A_N = U_N \setminus A_{N-1}.$$

After that, consider the measures of the sets. The condition that μ is non-atomic allows us to divide each A_n with $\mu(A_n) \geqslant d$ into finite number of parts $\widetilde{A}_k$ each of $\mu(\widetilde{A}_k) < d$. So we can reckon that $\mu(A_n) < d$ for $n = 1, \ldots, N$. Since μ is regular we can approximate each A_n by a closed subset $B_n \subset A_N$ with

$$\mu(A_n) < \mu(B_n) + q/N.$$

Consider now B_n with $\mu(B_n) > 0$. First two conditions of Definition 3.2 are fulfilled for them. If $\mu(K) = \sum_{i=1}^{N} \mu(B_i)$ then we throw away one of the B_n with $0 < \mu(B_n) < q$ or its open subset $E \subset B_n$ with $0 < \mu(E) < q$ in the case of $d \geqslant q$. Now taking for K_1 the union of obtained B_i we satisfy the third condition of Definition 3.2.

To obtain the "moreover" part of the claim simply consider for the parts of K_1 all μ-positive sets of the form $B_n \cap C_m$, $n = 1, 2, \dots, N$, $m = 1, 2, \dots, M$. $\square$

Let G be some closed subset of [0, 1] with Lebesgue measure λ on it.

Definition 3.4. We call (K_1, G_1, f_1) a (d, q)-*triplet for* K *and* G if

(1) $K_1 = \bigsqcup_{n=1}^{N} B_n$ is (d, q)-disconnected subset of K with μ,
(2) $G_1 = \bigsqcup_{n=1}^{N} D_n$ is (d, q)-disconnected subset of G with λ,
(3) $\mu(B_n) = \lambda(D_n)$ for all $n = 1, 2, \dots, N$,
(4) $f_1 : G_1 \to K_1$ is a continuous step function which maps every D_n into a single point of corresponding B_n.

Proposition 3.5. *For every* $d, q > 0$ *there is* (K_1, G_1, f_1)*—a* (d, q)*-triplet for* K *and* G*. Moreover, for every* $d_2, q_2 > 0$ *there is a* (d_2, q_2)*-triplet* (K_2, G_2, f_2) *for* K_1 *and* G_1 *such that for every* $t \in G_2$ $f_1(t)$ *and* $f_2(t)$ *belong to the same part of* K_1 *and hence* $\rho(f_1(t), f_2(t)) < d$.

Proof. First, Proposition 3.3 gives us $K_1 = \bigsqcup_{i=1}^{n_1} B_{i,1}$ a (d, q)-disconnected subset of K with μ. The third condition of (d, q)-disconnectedness and Remark 2.4 allow us to find disjoint segments $D_{i,1} \subset [0, 1]$ with lengths $\lambda(D_{i,1}) = \mu(B_{i,1})$, $i = 1, 2, \dots, n_1$. This obviously makes $G_1 = \bigsqcup_{i=1}^{n_1} D_{i,1}$ to be (d, q)-disconnected subset of G with λ. In each $B_{i,1}$ choose $x_{i,1}$ and define continuous step function $f_1 : G_1 \to K_1$ such that $f_1(D_{i,1}) = x_i$, $i = 1, 2, \dots, n_1$. The (d, q)-triplet (K_1, G_1, f_1) for K and G is constructed.

Next, we apply Proposition 3.3 for K_1, d_2 and q_2. We obtain $K_2 = \bigsqcup_{i=1}^{n_2} B_{i,2}$ where $B_{i,2}$, $i = 1, 2, \dots, n_2$, are such that for each i there is $n(i) \in \mathbb{N}$ that $B_{i,2} \subset B_{n(i),1}$. We can find disjoint segments $D_{i,2} \subset G_1$, such that $D_{i,2} \subset D_{n(i),1}$, with lengths $\lambda(D_{i,2}) = \mu(B_{i,2})$, $i = 1, 2, \dots, n_2$. Choose $x_{i,2} \in B_{i,2}$, denote $G_2 = \bigsqcup_{i=1}^{n_2} D_{i,2}$ and define $f_2 : G_2 \to K$ such that $f_2(D_{i,2}) = x_{i,2}$. Then (K_2, G_2, f_2) is the (d_2, q_2)-triplet for K_1 and G_1. Let us evaluate the distance between $f_1(t)$ and $f_2(t)$ for $t \in G_2$. Let $t \in D_{i,2} \subset D_{n(i),1}$, then $f_1(t), f_2(t) \in B_{n(i),1}$ and $\rho(f_1(t), f_2(t)) \leqslant \operatorname{diam} B_{n(i),1} < d$. $\square$

Lemma 3.6. *For every metric compact* K*, every* $\varepsilon > 0$*, and every non-atomic Borel measure* μ *on* K *there are a closed subset* $G \subset [0, 1]$*, a compact* $K_\varepsilon \subset K$ *and a homeomorphism* $s : G \to K_\varepsilon$ *such that* $\mu(K \setminus K_\varepsilon) < \varepsilon$ *and* $\mu(s(A)) = \lambda(A)$ *for any Borel subset* $A \subset G$*, where* λ *is the Lebesgue measure.*

Proof. Let us fix $\varepsilon > 0$, and choose $q_k > 0$ such that $\sum_{k=1}^{\infty} q_k < \varepsilon$ and $d_k = 2^{-k}$. We construct K_ε, G and s using step-by-step approximation.

We approximate with (d_k, q_k) triples using Proposition 3.5. First, we find some (K_1, G_1, f_1) (d_1, q_1)-triplet for K and [0, 1]. Then, (K_2, G_2, f_2) (d_2, q_2)-triplet for K_1, G_1 such that $\rho(f_1(t), f_2(t)) < d_1$ for every $t \in G_2$. On the ith step Proposition 3.5 gives us (K_i, G_i, f_i) (d_i, q_i)-triplet for K_{i-1}, G_{i-1} such that $\rho(f_{i-1}(t), f_i(t)) < d_{i-1}$ for every $t \in G_i$.

Proceeding in this way we obtain the sequences of embedded compacts K_i, G_i and functions f_i. Consider $K_\varepsilon = \bigcap_i K_i$ and $G = \bigcap_i G_i$. Using inequality (3) of Definition 3.2 for K_i we have

$$\mu(K \setminus K_\varepsilon) \leqslant \sum_{i=1}^{\infty} q_k < \varepsilon.$$

Look at (f_i) as a sequence of continuous maps acting from G to $K_i \subset K$. It is a Cauchy sequence:

$$\rho\bigl(f_n(t), f_{n+m}(t)\bigr) \leqslant \sum_{k=n+1}^{n+m} \frac{1}{2^k} \to 0 \quad \text{as } n, m \to \infty.$$

Take for desired continuous $s : G \to K_\varepsilon$ the uniform limit of f_n. Since G is compact, to check that s is homeomorphism is to show that s is bijective.

Injectivity: let $t_1, t_2 \in G$ and $t_1 \neq t_2$. Since diameters of the parts of G_i tend to 0 as i tends to infinity, we can find M big enough that on the Mth step t_1 and t_2 turn out to be in the different parts. Thus $f_M(t_1)$ and $f_M(t_2)$ belong to different closed parts A and B of K_M. Then for all $k \in \mathbb{N}$ $f_{M+k}(t_1) \in A$, $f_{M+k}(t_2) \in B$, so $s(t_1) \in A$, $s(t_2) \in B$, i.e. $s(t_1) \neq s(t_2)$.

Surjectivity: let us check that the range of s is dense in K_ε. We know that the values of each f_i form a finite d_i-net of K_i (property (4) of Definition 3.4 and (1) of Definition 3.2). Thus, for each $\delta > 0$ and $x \in K_\varepsilon$ we can find $N \in \mathbb{N}$ and $t \in G$ such that $\sum_{k=N}^{\infty} d_k < \delta$ and for each $n \geqslant N$, $\rho(f_n(t), x) < d_n$, hence $\rho(s(t), x) < \delta$.

Note, that $\lambda(G) = \lambda(\bigcap_{n=1}^{\infty} G_n) = \lim_{n\to\infty} \lambda(G_n) = \lim_{n\to\infty} \mu(K_n) = \mu(K_\varepsilon)$. By the same reason if D is a part of some G_i and B is the corresponding part of K_i, then $\lambda(G \cap D) = \mu(K_\varepsilon \cap B)$. Since $s(G \cap D) = K_\varepsilon \cap B$ we have that $\mu \circ s = \lambda$ on the collection

$$\Phi = \bigcup \{D \cap G\colon\ D \text{ is a part of some } G_i\} \cup \{G\} \cup \{\emptyset\}$$

of sets. Φ forms a unital semiring that generates the Borel σ-algebra of G, and thus $\mu \circ s$ has a unique extension to the σ-algebra of Borel subsets of G, hence coincides with λ. □

Lemma 3.7. *Let $G \subset [0,1]$ be closed, and let $(g_n) \subset C(G)$, $0 \leqslant g_n \leqslant 1$, be a point-wise $\mathcal{F}$-convergent to 0 sequence. Then there is a sequence $(f_n) \subset C[0,1]$, $0 \leqslant f_n \leqslant 1$, point-wise $\mathcal{F}$-convergent to 0 on $[0,1]$ such that $f_n|_G = g_n$.*

Proof. Without loss of generality we may assume that 0 and 1 belong to G. Represent $[0,1] \setminus G$ as at most countable union of disjoint open intervals:

$$[0,1] \setminus G = \bigsqcup_{k=1}^{N} (t_k, \bar{t}_k), \quad N \in \mathbb{N} \cup \{\infty\}.$$

Now, we extend g_n to f_n on $[0,1]$ by linear interpolation: define $f_n(t) = g_n(t)$ for $t \in G$ and for every $t \in \bigsqcup_{i=1}^{N} (t_i, \bar{t}_i)$, $t = \alpha t_k + (1-\alpha)\bar{t}_k$, $\alpha \in [0,1]$ put $f_n(t) = \alpha g_n(t_k) + (1-\alpha) g(\bar{t}_k)$. Then $f_n \in C[0,1]$ and for any $t \in (t_k, \bar{t}_k)$

$$\mathcal{F}\text{-}\lim f_n(t) \leqslant \mathcal{F}\text{-}\lim \bigl[g_n(t_k) + g_n(\bar{t}_k)\bigr] = 0. \qquad \square$$

Corollary 3.8. *If $\mathcal{F}$ is C-Lebesgue with respect to λ on $[0,1]$ then $\mathcal{F}$ is C-Lebesgue with respect to λ on every closed subset $G \subset [0,1]$.*

Proof. For every sequence $(g_n) \subset C(G)$, $0 \leqslant g_n \leqslant 1$, point-wise $\mathcal{F}$-convergent to 0 there is $(f_n) \subset C[0,1]$, $0 \leqslant f_n \leqslant 1$, point-wise $\mathcal{F}$-convergent to 0 on $[0,1]$ such that $f_n|_G = g_n$. So by the C-Lebesgue property with respect to λ on $[0,1]$ we have

$$0 = \mathcal{F}\text{-}\lim \int_{[0,1]} f_n \, d\lambda \geqslant \mathcal{F}\text{-}\lim \int_G g_n \, d\mu \geqslant 0. \qquad \square$$

Theorem 3.9. *$\mathcal{F}$ is C-Lebesgue if and only if for every bounded sequence $f_n \in C[0,1]$ its point-wise $\mathcal{F}$-convergence to 0 implies that*

$$\mathcal{F}\text{-}\lim \int_{[0,1]} f_n \, d\lambda = 0,$$

where λ is the Lebesgue measure on $[0,1]$.

Proof. In view of Theorem 2.10 we need to show that $\mathcal{F}$ is C-Lebesgue with respect to every regular Borel probability measure μ on every metric compact K. Lemma 3.1 gives us the C-Lebesgue property for every purely atomic μ on K. For non-atomic μ Lemma 3.6 with Corollary 3.8 for every $\varepsilon > 0$ provides us with compact $K_\varepsilon \subset K$ such that $\mu(K \setminus K_\varepsilon) < \varepsilon$ and $\mathcal{F}$ is C-Lebesgue with respect to μ on K_ε. So, for a given point-wise $\mathcal{F}$ convergent to 0 sequence of $h_n \in C(K)$, $0 \leqslant h_n \leqslant 1$, we have $\mathcal{F}\text{-}\lim \int_{K_\varepsilon} h_n \, d\mu = 0$, and the assertion immediately follows from the estimation

$$\int_K h_n \, d\mu = \int_{K_\varepsilon} h_n \, d\mu + \int_{K \setminus K_\varepsilon} h_n \, d\mu < \int_{K_\varepsilon} h_n \, d\mu + \varepsilon. \qquad \square$$

4. Ultrafilters and the C-Lebesgue property

Let us compare the C-Lebesgue property with the following Lebesgue property, studied in [4]:

Definition 4.1. A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be a *Lebesgue filter* (has the *Lebesgue property*) if the following statement takes place: for every finite measure space (Ω, Σ, μ), for every point-wise $\mathcal{F}$-convergent to 0 sequence of measurable functions f_n on Ω if $|f_n|$ are dominated by a fixed integrable function $g \in L_1(\Omega, \Sigma, \mu)$ then $\int_\Omega f_n \, d\mu \to_{\mathcal{F}} 0$.

Evidently the Lebesgue property implies the C-Lebesgue property. So, as examples of C-Lebesgue filters one can take all Lebesgue filters. Among them there are many filters of rather complicated structure, say all filters generated by a matrix summability method (in particular, the statistical convergence filter), but also all filters with a countable base. We will need the last fact, so let us state it.

Proposition 4.2. *(See [4].) Every filter with a countable base is C-Lebesgue.*

It is also proved in [4] that all ultrafilters do not have the Lebesgue property. As for the C-Lebesgue property, we have the following result.

Theorem 4.3. *There is a filter $\mathcal{F}$ which does not have the C-Lebesgue property. Moreover, there is an ultrafilter which does not have the C-Lebesgue property.*

Proof. Let us show that there is a sequence of sets $A_n \subset [0,1]$ with $\lambda(A_n) = 1/2$ such that 0 is a point-wise cluster point of the sequence (χ_{A_n}). This will guarantee the existence of some filter $\mathcal{F}$ having 0 point-wise $\mathcal{F}$-limit of χ_{A_n} and $\mathcal{F}\text{-}\lim \lambda(A_n) = 1/2$ which according to Theorem 2.6 means that $\mathcal{F}$ is not C-Lebesgue. Obviously every filter dominating this $\mathcal{F}$ is not C-Lebesgue, so in particular every ultrafilter dominating $\mathcal{F}$ is not C-Lebesgue as well.

In order to do this consider the collection V of all sets of the form $\coprod_{k=1}^n (a_k, b_k)$, where $n \in \mathbb{N}$ $a_k, b_k \in \mathbb{Q} \cap [0,1]$ and $\sum_{k=1}^n |b_k - a_k| = \frac{1}{2}$. The collection V is countable, so let us enumerate it as A_n, $n = 1, 2, \ldots$. Evidently for every finite set $T = \{t_j\}_1^m \subset [0,1]$ there is an $A \in V$, such that $T \cap A = \emptyset$. This means that 0 is a point-wise cluster point of the sequence (χ_{A_n}). $\square$

Denote by $\widetilde{\mathbb{N}}$ the set of all free ultrafilters $\mathcal{U}$ on $\mathbb{N}$, equipped with the topology defined by means of its base $\{\widetilde{A}\colon A \subset \mathbb{N}\}$, where $\widetilde{A} = \{\mathcal{U} \in \widetilde{\mathbb{N}}\colon A \in \mathcal{U}\}$. Remark, that in this topology the basic open sets $\widetilde{A}$ are at the same time closed. $\widetilde{\mathbb{N}}$ can be identified with $\beta\mathbb{N} \setminus \mathbb{N}$ where $\beta\mathbb{N}$ denotes the Stone–Čech compactification of $\mathbb{N}$.

Applying all permutations to the set of indices of the A_n from Theorem 4.3, one can easily show that the set of ultrafilters without the C-Lebesgue property is dense in $\widetilde{\mathbb{N}}$.

Our next goal is to show that if one assumes the continuum hypothesis then there is a free C-Lebesgue ultrafilter. This means that the C-Lebesgue property is strictly weaker then the Lebesgue property.

Recall that a subset of $\mathbb{N}$ is called *stationary with respect to a filter $\mathcal{F}$* (or just *$\mathcal{F}$-stationary*) if it has non-empty intersection with each element of the filter. Denote the collection of all $\mathcal{F}$-stationary sets by $\mathcal{F}^*$. For $I \in \mathcal{F}^*$ we call the collection of sets $\{A \cap I\colon A \in \mathcal{F}\}$ *the trace of $\mathcal{F}$ on I* (which is evidently a filter on I), and by $\mathcal{F}(I)$ we denote the filter on $\mathbb{N}$ generated by the trace of $\mathcal{F}$ on I. Clearly $\mathcal{F}(I)$ dominates $\mathcal{F}$. Any subset of $\mathbb{N}$ is either an element of $\mathcal{F}$ or the complement of an element of $\mathcal{F}$ or the set and its complement are both $\mathcal{F}$-stationary sets. $\mathcal{F}^*$ is precisely the union of all ultrafilters dominating $\mathcal{F}$. $\mathcal{F}^*$ is a filter base if and only if it is equal to $\mathcal{F}$ and $\mathcal{F}$ is an ultrafilter.

Theorem 4.4. *(See [1].) Let X be topological space, $x_n, x \in X$ and let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then the following conditions are equivalent*:

(1) *x_n is $\mathcal{F}$-convergent to x*;
(2) *x_n is $\mathcal{F}(I)$-convergent to x for every $I \in \mathcal{F}^*$*;
(3) *x is a cluster point of $(x_n)_{n \in I}$ for every $I \in \mathcal{F}^*$.*

Proof. Implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are evident. Let us prove that (3) $\Rightarrow$ (1). Suppose the sequence (x_n) does not $\mathcal{F}$-converge to x. Then there is such a neighborhood U of x that in each $A \in \mathcal{F}$ there is a $j \in A$ such that $x_j \notin U$. Consequently $I = \{j \in \mathbb{N}\colon x_j \notin U\}$ is stationary and x is not a cluster point of $(x_n)_{n\in I}$. □

Remark 4.5. If $\mathcal{F}$ is a C-Lebesgue filter, then for every $I \in \mathcal{F}^*$ the filter $\mathcal{F}(I)$ is C-Lebesgue.

Let us introduce one more concept of technical character:

Definition 4.6. A filter $\mathcal{F}$ on $\mathbb{N}$ is said to be *strongly C-Lebesgue* if for every sequence $(f_n) \subset B_{C[0,1]}$ whose corresponding sequence of integrals does not $\mathcal{F}$-converge to 0 there is a point $t \in [0, 1]$ at which 0 is not a cluster point of $(f_n(t))$ with respect to $\mathcal{F}$.

It is evident that every filter which dominates a strongly C-Lebesgue filter is a strongly C-Lebesgue filter itself and that a strongly C-Lebesgue filter is necessarily a C-Lebesgue filter.

Theorem 4.7. *Under the assumption of continuum hypothesis there is a strongly C-Lebesgue filter, and consequently there exists a C-Lebesgue ultrafilter.*

Proof. Choose a countable dense subset of functions $F \subset B_{C[0,1]}$. It is sufficient to construct a filter $\mathcal{F}$ which satisfies the conditions of Definition 4.6 for every sequence $(f_n) \subset F$: in such a case by small perturbation argument $\mathcal{F}$ will be strongly C-Lebesgue.

In order to do this denote by ω_1 the set of all ordinals which are finite or countable. Let us enumerate as $F_\alpha = (f_{\alpha,n})_{n=1}^\infty$, $\alpha \in \omega_1$ all the sequences of functions from F (at this point we use the continuum hypothesis). Now let us construct the family $\mathcal{F}_\alpha$, $\alpha \in \omega_1$ of filters on $\mathbb{N}$ with countable base using the following recurrent procedure. Take as $\mathcal{F}_1$ the Fréchet filter. If $\mathcal{F}_\alpha$ is already constructed we look at the sequence $a_n = \int_0^1 f_{\alpha,n}(\tau)\,d\tau$. If $\mathcal{F}_\alpha\text{-}\lim a_n = 0$, then we put $\mathcal{F}_{\alpha+1} := \mathcal{F}_\alpha$ in the opposite case of $\mathcal{F}_\alpha\text{-}\lim a_n \neq 0$ since $\mathcal{F}_\alpha$ has a countable base (and by this reason is a C-Lebesgue filter) we know that the sequence of functions $f_{\alpha,n}$ at some point $t = t_\alpha$ does not $\mathcal{F}_\alpha$ converge to 0. By Theorem 4.4 there is an $I = I_\alpha \in \mathcal{F}_\alpha^*$ such that 0 is not a cluster point of $(f_{\alpha,n}(t))_{n\in I}$. In this case we take as $\mathcal{F}_{\alpha+1}$ the filter, generated by $\mathcal{F}_\alpha \cup \{I_\alpha\}$. If β is a limiting ordinal, and for all $\alpha < \beta$ the filters $\mathcal{F}_\alpha$ are already constructed, put $\mathcal{F}_\beta = \bigcup_{\alpha<\beta} \mathcal{F}_\alpha$.

Finally define $\mathcal{F} = \bigcup_{\alpha\in\omega_1} \mathcal{F}_\alpha$. Let us prove that $\mathcal{F}$ is the filter we need. Take an arbitrary sequence of functions $f_n \in F$ such that corresponding sequence of integrals does not $\mathcal{F}$-converge to 0. Select $\alpha \in \omega_1$ for which $(f_n) = F_\alpha$. Since $\mathcal{F} \succ \mathcal{F}_\alpha$, we know that the sequence of integrals of f_n does not $\mathcal{F}_\alpha$-converge to 0. Then according to our construction there is a $t = t_\alpha$ at which 0 is not a cluster point of $(f_n(t))$ with respect to $\mathcal{F}_{\alpha+1}$. But then 0 is not a cluster point of $(f_n(t))$ with respect to $\mathcal{F}$ as well.

To get a strongly C-Lebesgue ultrafilter it is sufficient to take arbitrary ultrafilter dominating $\mathcal{F}$. □

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