Remarks on eigenvalue problems involving the $p(x)$-Laplacian ☆

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Received 4 January 2008

Available online 11 June 2008

Submitted by V. Radulescu

Abstract

This paper deals with the eigenvalue problem involving the $p(x)$-Laplacian of the form

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = \lambda |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $p \in C^0(\overline{\Omega})$, $\inf_{x\in\Omega} p(x) > 1$, $q \in L^\infty(\Omega)$, $1 \leqslant q(x) \leqslant q(x) + \varepsilon < p^*(x)$ for $x \in \Omega$, ε is a positive constant, $p^*(x)$ is the Sobolev critical exponent. It is shown that for every $t > 0$, the problem has at least one sequence of solutions $\{(u_{n,t}, \lambda_{n,t})\}$ such that $\int_\Omega \frac{1}{p(x)}|\nabla u_{n,t}|^{p(x)} = t$ and $\lambda_{n,t} \to \infty$ as $n \to \infty$. The principal eigenvalues for the problem in several important cases are discussed especially. The similarities and the differences in the eigenvalue problem between the variable exponent case and the constant exponent case are exposed. Some known results on the eigenvalue problem are extended.
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Keywords: $p(x)$-Laplacian; Eigenvalue; Variable exponent Sobolev space; Constrained variational method

1. Introduction

In this paper we consider the eigenvalue problem of the form

$$\begin{cases} -\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = \lambda |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $p \in C^0(\overline{\Omega})$, $\inf_{x\in\Omega} p(x) > 1$, $q \in L^\infty(\Omega)$, $1 \leqslant q(x) \leqslant q(x) + \varepsilon < p^*(x)$ for a.e. $x \in \Omega$, ε is a positive constant and

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geqslant N. \end{cases}$$

Problem (1.1) involves the variable exponents $p(\cdot)$ and $q(\cdot)$. The variable exponent problems are interesting for some applications; see e.g. [23,32]. The study of various mathematical problems with variable exponent has been received considerable attention in recent years; for a survey see [3,6,22,33].

There are many essential differences between the variable exponent problems and the constant exponent problems. In the studies of the variable exponent problems many singular phenomena occurred and many special questions were raised. For example, Zhikov [37] has gave some examples of the Lavrintiev phenomenon for the variational problems with variable exponent. It is well known that in the constant exponent case the Lavrintiev phenomenon cannot occur. Zhikov's examples also show that, in general, smooth functions are not necessarily dense in the variable exponent Sobolev space, and the regularity for the variational problems and differential equations with variable exponent is a very complicated problem. For the study of the regularity for variable exponent problems see e.g. [1,11,12,18,37–39]. Kováčik and Rákosnik [26] have specially investigated the variable exponent Lebesgue–Sobolev spaces. In [26] many new questions, different from classical Lebesgue–Sobolev spaces, were raised, for example, it was proved (see [26, Theorem 2.10]) that when $p(\cdot)$ is continuous and nonconstant, space $L^{p(\cdot)}$ does not have the mean continuity property. Pick and Růžička [31] have given an example of a space $L^{p(x)}$ on which the Hardy–Littlewood maximal operator is not bounded. In recent years many researchers (see e.g. [3–7,10,15,19,22,24,25,33]) have studied the basic properties of variable exponent Lebesgue–Sobolev spaces and the boundedness of some classical operators in the variable exponent spaces, such as Hardy–Littlewood maximal operators, singular integrals, commutators and fractional integrals. These research results reflect the characteristics of the variable exponent problems very well.

The operator $-\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian which is a generalization of the p-Laplacian and possesses more complicated nonlinearities than the p-Laplacian, for example, it is inhomogeneous. From the study of the $p(x)$-Laplacian equations (see e.g. [2,3,8,16,17,28–30]) we can see that, the inhomogeneity is a main difficulty to generalize the results in the constant exponent case to the variable exponent case and it is also a main cause of the occurrence of some singular phenomena in the variable exponent problems. Jikov (=Zhikov), Kozlov and Oleinik [23] have studied the homogenization of differential equations and variational problems with variable exponent.

The aim of the present paper is to study eigenvalue problem (1.1). Put

$$\Lambda = \Lambda_{(p(\cdot),q(\cdot))} = \left\{ \lambda \in \mathbb{R} \mid \lambda \text{ is an eigenvalue for (1.1)} \right\},$$
$$\Lambda^+ = \Lambda^+_{(p(\cdot),q(\cdot))} = \left\{ \lambda \in \mathbb{R} \mid \lambda \text{ is a principal eigenvalue for (1.1)} \right\},$$

where λ is called a principal eigenvalue for (1.1) if there exists a nonnegative eigenfunction corresponding to λ. In this paper we will restrict ourselves to the subcritical case, i.e., the case when $q(x) + \varepsilon < p^*(x)$ for a.e. $x \in \Omega$. We will use the notations such as q_- and q_+, where $q_- = \text{ess inf}_{x \in \Omega} q(x)$ and $q_+ = \text{ess sup}_{x \in \Omega} q(x)$.

It is well known that problem (1.1) in the constant exponent case, i.e., when $p(\cdot) \equiv p$ (a constant) and $q(\cdot) \equiv q$ (a constant), has been studied sufficiently (see e.g. [21,27] and references therein). On the eigenvalue problems involving the $p(x)$-Laplacian, some interesting results have been obtained (see e.g. [2,8,13,14,17,28–30]), from which we can also see the differences between the variable and the constant exponent cases.

In [17] problem (1.1) with $q(x) = p(x)$ was studied and it was shown that, in this case, in general, $\inf \Lambda_{(p(\cdot),p(\cdot))} = 0$. This is a singular phenomenon, different from the constant exponent case, since when $q(\cdot) = p(\cdot) \equiv p$, $\inf \Lambda_{(p,p)}$ is the first eigenvalue which is positive. In [17] some sufficient conditions for $\inf \Lambda_{(p(\cdot),p(\cdot))} > 0$ were also given.

When $q_+ < p_-$, by Theorem 4.3 in [16], for every $\lambda > 0$, the energy functional I_λ corresponding to (1.1) is coercive and has a global minimizer which is nontrivial and nonnegative, and hence $\Lambda^+ = (0, \infty)$.

When $q_- > p_+$, by Theorem 4.7 in [16], for every $\lambda > 0$, the energy functional I_λ corresponding to (1.1) has a Mountain Pass type critical point which is nontrivial and nonnegative, and hence $\Lambda^+ = (0, \infty)$.

Mihăilescu and Rădulescu [29] have studied problem (1.1) under the basic assumption

$$1 < q_- < p_- < q_+ \tag{1.2}$$

and proved that there exists $\lambda_0 > 0$ such that any $\lambda \in (0, \lambda_0)$ is an eigenvalue for problem (1.1).

In [14] it was proved by using the sub–supersolution method that, roughly speaking (see Theorem 3.7 in the present paper for more details), if p is Lipschitz on $\overline{\Omega}$, $q \in C^0(\overline{\Omega})$ and there exists $x_0 \in \Omega$ such that $q(x_0) < p(x_0)$ (note that the assumption $q(x) < p^*(x)$ is needless), then Λ^+ is nonempty and connected, and $\inf \Lambda^+ = 0$. Moreover, for any $\lambda_1, \lambda_2 \in \Lambda^+$ with $\lambda_1 < \lambda_2$, there exist u_{λ_1} and u_{λ_2}, the positive eigenfunctions corresponding to λ_1 and λ_2, respectively, such that $u_{\lambda_1} < u_{\lambda_2}$ in Ω.

Alves and Souto [2] have studied the corresponding eigenvalue problems in $\mathbb{R}^N$ and obtained interesting results. In [8,13,28,30] some eigenvalue problems of other form, different from (1.1), have been studied.

For problem (1.1), when $p(\cdot) \equiv p$ (a constant) and $q(\cdot) \equiv q$ (a constant), there are only three cases: 1) $q = p$; 2) $q < p$; 3) $q > p$. However, in the variable exponent case the situation is more complicated. In the general case, for given $p(\cdot)$ and $q(\cdot)$, the sets $\Omega_0 = \{x \in \Omega : q(x) = p(x)\}$, $\Omega_- = \{x \in \Omega : q(x) < p(x)\}$ and $\Omega_+ = \{x \in \Omega : q(x) > p(x)\}$ can have all positive measure at the same time. In this paper we will consider the general case (note that condition (1.2) does not necessarily hold), also consider the case that $\Omega = \Omega_-$ (but $q_+ < p_-$ does not hold) and the case that $\Omega = \Omega_+$ (but $q_- > p_+$ does not hold). In particular, the main results of [17] and [29] are extended.

In Section 2, we study the eigenvalues for problem (1.1) by a constrained variational method. It is proved that for every $t > 0$, problem (1.1) has at least one sequence of solutions $\{(u_{n,t}, \lambda_{n,t})\}_{n=1}^{\infty}$ such that $\int_\Omega \frac{1}{p(x)} |\nabla u_{n,t}|^{p(x)} dx = t$ and $\lambda_{n,t} \to \infty$ as $n \to \infty$. In Section 3, we study the principal eigenvalues for problem (1.1) in the general case, in the case $\Omega = \Omega_-$ and in the case $\Omega = \Omega_+$, respectively. The similarities and the differences in problem (1.1) between the variable and the constant exponent cases are exposed. In the end of this paper it is pointed out that our main results on problem (1.1) can be generalized to the following problem that has more general form

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = \lambda f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

provided f satisfies appropriate conditions (see Theorem 3.8 in Section 3).

2. Eigenvalue problems

Let Ω be a bounded domain in $\mathbb{R}^N$. For a measurable function $q : \Omega \to \mathbb{R}$ and $E \subset \Omega$, define

$$q_-(E) = \operatorname*{ess\,inf}_{x \in E} q(x) \quad \text{and} \quad q_+(E) = \operatorname*{ess\,sup}_{x \in E} q(x),$$

and write $q_-(\Omega) = q_-$ and $q_+(\Omega) = q_+$ simply. Now let $p \in L^\infty(\Omega)$ satisfy condition $1 \leqslant p_- \leqslant p_+ < \infty$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u \,\middle|\, u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u|^{p(x)} dx < \infty \right\}$$

with the norm

$$|u|_{L^{p(\cdot)}(\Omega)} = |u|_{p(\cdot)} = \inf\left\{ \sigma > 0 \,\middle|\, \int_\Omega \left|\frac{u}{\sigma}\right|^{p(x)} dx \leqslant 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \,\middle|\, |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}$$

with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{1,p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)}.$$

Define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. $|\nabla u|_{p(\cdot)}$ is an equivalent norm on $W_0^{1,p(\cdot)}(\Omega)$. We refer to [3,6,10,15,19,22,26,33] for the elementary properties of the space $W^{1,p(x)}(\Omega)$.

Proposition 2.1. *(See [19].) Let $p, q \in C^0(\overline{\Omega})$ and $1 \leqslant q(x) < p^*(x)$ for all $x \in \overline{\Omega}$. Then there holds a compact imbedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Let $p \in C^0(\overline{\Omega})$ and $q \in L^\infty(\Omega)$. Writing $q \ll p^*$ we mean that there exists $\varepsilon > 0$ such that

$$q(x) + \varepsilon \leqslant p^*(x) \quad \text{for a.e. } x \in \Omega.$$

It is easy to see that, when $q \in C^0(\overline{\Omega})$, $q \ll p^*$ if and only if $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$. It is also easy to see that, for $q \in L^\infty(\Omega)$, $q \ll p^*$ if and only if there exists $r \in C^0(\overline{\Omega})$ such that $q \leqslant r \ll p^*$. Thus Proposition 2.1 can be extended as follows.

Proposition 2.2. *Let* $p \in C^0(\overline{\Omega})$ *and* $q \in L^\infty(\Omega)$ *with* $q_- \geqslant 1$. *If* $q \ll p^*$, *then there holds a compact imbedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

In what follows, it will be assumed that:

(p_0) $p \in C^0(\overline{\Omega})$ and $p_- > 1$.
(q_0) $q \in L^\infty(\Omega)$, $q_- \geqslant 1$ and $q \ll p^*$.

Let us consider problem (1.1). Below we write $X = W_0^{1,p(\cdot)}(\Omega)$ and $\|u\| = |\nabla u|_{p(\cdot)}$ for $u \in X$.

Definition 2.1. Let $\lambda \in \mathbb{R}$ and $u \in X$. (u, λ) is called a solution of problem (1.1) if

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx = \lambda \int_\Omega |u|^{q(x)-2} uv \, dx, \quad \forall v \in X.$$

If (u, λ) is a solution of (1.1) and $u \in X \setminus \{0\}$, as usual, we call λ and u an eigenvalue and an eigenfunction corresponding to λ for problem (1.1), respectively.

It is easy to see that, if (u, λ) is a solution of (1.1) and $u \in X \setminus \{0\}$, then

$$\lambda = \lambda(u) = \frac{\int_\Omega |\nabla u|^{p(x)} \, dx}{\int_\Omega |u|^{q(x)} \, dx}, \tag{2.1}$$

and hence $\lambda > 0$.

Set

$$\Lambda = \Lambda_{(p(\cdot), q(\cdot))} = \big\{ \lambda \in \mathbb{R} \mid \lambda \text{ is an eigenvalue for (1.1)} \big\}.$$

Define $J, \psi : X \to \mathbb{R}$ by

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx, \qquad \psi(u) = \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, dx, \quad \forall u \in X,$$

then J and ψ are even, $J, \psi \in C^1(X, R)$ and

$$\big(J'(u), v\big) = \int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx, \quad \forall u, v \in X,$$

$$\big(\psi'(u), v\big) = \int_\Omega |u|^{q(x)-2} uv \, dx, \quad \forall u, v \in X.$$

For any $t > 0$, define

$$M(t) = J^{-1}(t) = \big\{ u \in X \mid J(u) = t \big\}.$$

Then $M(t)$ is a C^1 submanifold of X because t is a regular value of J. Denote by ψ_t the restriction of ψ to $M(t)$:

$$\psi_t = \psi|_{M(t)} : M(t) \to \mathbb{R}.$$

Then ψ_t is a C^1 functional defined on $M(t)$.

It is well known (see e.g. [36, p. 292]) that, if u is a critical point of ψ_t on $M(t)$, then, by the Lagrange multiplier rule, (u, λ) is a solution of (1.1), where $\lambda = \lambda(u)$ is as in (2.1).

We know (see [16]) that $J' : X \to X^*$ is a monotone homeomorphism, and is of type (S_+), namely,

$$u_n \rightharpoonup u \quad \text{in } X \quad \text{and} \quad \varlimsup_{n \to \infty} \big(J'(u_n), u_n - u\big) \leqslant 0 \quad \Rightarrow \quad u_n \to u \quad \text{in } X,$$

where $u_n \rightharpoonup u$ and $u_n \to u$ denote the weak convergence and the strong convergence in X, respectively, the mappings J' and $(J')^{-1}$ are bounded, and $\psi' : X \to X^*$ is weakly-strongly continuous. From this, by a standard argument (see [13,21,34]), it follows that ψ_t satisfies the (P.S)$^+$ condition on $M(t)$.

Define

$$\Sigma = \left\{ A \subset X \setminus \{0\} \,\middle|\, A \text{ is compact and } -A = A \right\},$$
$$\Sigma_n = \left\{ A \in \Sigma \,\middle|\, \gamma(A) \geqslant n \right\}, \quad n = 1, 2, \dots,$$

where $\gamma(A)$ is the genus of A (see e.g. [36]), and

$$c_n(t) = \sup_{A \in \Sigma_n,\, A \subset M(t)} \inf_{u \in A} \psi(u), \quad n = 1, 2, \dots. \tag{2.2}$$

Obviously, $c_n(t) > 0$ and

$$c_1(t) \geqslant c_2(t) \geqslant \cdots \geqslant c_n(t) \geqslant c_{n+1}(t) \geqslant \cdots.$$

By the Ljusternik–Schnirelmann theory on C^1-manifolds (see [34]) we have the following

Theorem 2.1. *Let* (p$_0$) *and* (q$_0$) *hold. Then, for each* $t > 0$, *the following assertions hold.*

(1) *For each* $n = 1, 2, \dots$, $c_n(t)$ *is a critical value of* ψ_t *on* $M(t)$ *and the Ljusternik–Schnirelmann multiplicity result holds.*
(2) $c_n(t) \to 0$ *as* $n \to \infty$.

Define for $t > 0$ and $n = 1, 2, \dots$,

$$K_n(t) = \left\{ u \in M(t) \,\middle|\, u \text{ is a critical point of } \psi_t \text{ and } \psi_t(u) = c_n(t) \right\}, \tag{2.3}$$
$$\Lambda_n(t) = \left\{ \lambda(u) \,\middle|\, u \in K_n(t) \right\}, \quad \text{where } \lambda(u) \text{ is as in (2.1)}, \tag{2.4}$$
$$\mu_n(t) = \frac{t}{c_n(t)}. \tag{2.5}$$

By Theorem 2.1, for each $t > 0$ and $n = 1, 2, \dots$, $K_n(t) \neq \emptyset$, and for each $t > 0$, $\mu_n(t) \to +\infty$ as $n \to \infty$. As noted above, if $u \in K_n(t)$, then $(u, \lambda(u))$ is a solution of (1.1), where $\lambda(u)$ is as in (2.1). Thus $\Lambda_n(t) \subset \Lambda$ for each $t > 0$ and $n = 1, 2, \dots$.

Note that, in general, the set $\Lambda_n(t)$ needs not be a singleton. For brevity, we shall adopt the usual agreement about the notations for set-valued mappings. For example, an inequality $\Lambda_n(t) \leqslant C$ means that $\lambda \leqslant C$ for every $\lambda \in \Lambda_n(t)$, and a limit $\Lambda_n(1) \to +\infty$ as $n \to \infty$ means that given any $L > 0$, there exists $n_0 > 0$ such that $\lambda \geqslant L$ for all $\lambda \in \Lambda_n(1)$ provided $n \geqslant n_0$.

Now let any $t > 0$ be given. For any $\lambda \in \Lambda_n(t)$, there exists $u \in K_n(t)$ such that

$$\lambda = \lambda(u) = \frac{\int_\Omega |\nabla u|^{p(x)} \, dx}{\int_\Omega |u|^{q(x)} \, dx},$$

and consequently

$$\lambda \leqslant \frac{p_+ \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx}{q_- \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, dx} = \frac{p_+ t}{q_- c_n(t)} = \frac{p_+}{q_-} \mu_n(t),$$
$$\lambda \geqslant \frac{p_- \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx}{q_+ \int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, dx} = \frac{p_- t}{q_+ c_n(t)} = \frac{p_-}{q_+} \mu_n(t).$$

Thus we have that, for each $t > 0$ and $n = 1, 2, \dots$,

$$\frac{p_-}{q_+} \mu_n(t) \leqslant \Lambda_n(t) \leqslant \frac{p_+}{q_-} \mu_n(t). \tag{2.6}$$

In particular, since $\mu_n(t) \to +\infty$ as $n \to \infty$, we have that, for each $t > 0$, $\Lambda_n(t) \to +\infty$ as $n \to \infty$. Let us formulate these in Theorem 2.2.

Theorem 2.2. *Let* (p$_0$) *and* (q$_0$) *hold. Then, for each $t > 0$ and $n = 1, 2, \ldots$, the sets $K_n(t)$ and $\Lambda_n(t)$ are nonempty, $\Lambda_n(t) \subset \Lambda$, and for any $u \in K_n(t)$, $(u, \lambda(u))$ is a solution of* (1.1)*, where $\lambda(u)$ is as in* (2.1)*. For each $t > 0$, $\Lambda_n(t) \to +\infty$ as $n \to \infty$.*

In the following two remarks we point out two differences between the variable and the constant exponent cases.

Remark 2.1. In the case when $p(\cdot) \equiv p$ (a constant) and $q(\cdot) \equiv q$ (a constant), by (2.6), for each $t > 0$ and $n = 1, 2, \ldots$, the set $\Lambda_n(t) = \{\frac{p}{q} \mu_n(t)\}$ is a singleton. However, in the variable exponent case, we cannot guarantee that the set $\Lambda_n(t)$ must be a singleton.

Remark 2.2. It is well known that, in the case when $p(\cdot) \equiv p$ (a constant) and $q(\cdot) \equiv q$ (a constant), to consider problem (1.1) by the constrained variational method, thanks to the homogeneity of J, ψ, J' and ψ', it suffices to consider a constrained variational problem on $M(1) = \{u \in X \mid J(u) = 1\}$. In fact, if (u, λ) with $u \in M(1)$ is a solution of (1.1), then for given any $t > 0$, $t^{\frac{1}{p}} u \in M(t)$ and $(t^{\frac{1}{p}} u, t^{\frac{p-q}{p}} \lambda)$ is also a solution of (1.1). However this is not the case when $p(\cdot)$ and $q(\cdot)$ are variable exponents due to the inhomogeneity.

Now let us observe the dependence of $c_n(t)$, $\mu_n(t)$ and $\Lambda_n(t)$ on t. We first give a lemma as follows.

Lemma 2.1. *Let $t > 0$ be given. Then for any $u \in X \setminus \{0\}$, there exists a unique $s(u) > 0$ such that $s(u)u \in M(t)$. The function $s : X \setminus \{0\} \to (0, \infty)$, defined by $u \mapsto s(u)$, is continuously differentiable. The mapping $h_t : X \setminus \{0\} \to M(t)$, defined by $h_t(u) = s(u)u$, is continuously differentiable. For any $t_1 > 0$, the restriction of h_t on $M(t_1)$, denoted by $h_{(t_1,t)} = h_t|_{M(t_1)}$, is a C^1-homeomorphism of $M(t_1)$ with $M(t)$. When $t \to t_1$, $\|h_{(t_1,t)}(u) - u\| \to 0$ uniformly in $u \in M(t_1)$.*

Proof. Let $t > 0$ and $u \in X \setminus \{0\}$ be given. Define a function $\varphi : (0, \infty) \to (0, \infty)$ by $\varphi(s) = J(su) = \int_\Omega \frac{s^{p(x)}}{p(x)} |\nabla u|^{p(x)} \, dx$. It is clear that the function φ is continuous and strictly increasing, $\varphi(s) \to 0$ as $s \to 0$ and $\varphi(s) \to \infty$ as $s \to \infty$. Hence there exists a unique $s = s(u) > 0$ such that $\varphi(s(u)) = t$, i.e., $s(u)u \in M(t)$. Define $F : (X \setminus \{0\}) \times (0, \infty) \to (0, \infty)$ by $F(u, s) = J(su)$. Obviously $F \in C^1$. For any $u_0 \in X \setminus \{0\}$, letting $s_0 = s(u_0)$, then $F(u_0, s_0) = t$. It is easy to see that $\frac{\partial F}{\partial s}(u_0, s_0) = \int_\Omega s_0^{p(x)-1} |\nabla u_0|^{p(x)} \, dx \neq 0$. By the implicit function theorem, the function $s : X \setminus \{0\} \to (0, \infty)$, defined by $u \mapsto s(u)$, is continuously differentiable, and consequently, the mapping $h_t : X \setminus \{0\} \to M(t)$, defined by $h_t(u) = s(u)u$, is continuously differentiable. Now let $t_1 > 0$ and consider the mapping $h_{(t_1,t)} = h_t|_{M(t_1)} : M(t_1) \to M(t)$. It is obvious that the mapping $h_{(t_1,t)} : M(t_1) \to M(t)$ is a bijection. The mapping $h_{(t_1,t)}$ and its inverse mapping $h_{(t,t_1)}$ are of class C^1. Hence $h_{(t_1,t)} : M(t_1) \to M(t)$ is a C^1-homeomorphism. Let $u \in M(t_1)$ and $h_{(t_1,t)}(u) = s(u)u \in M(t)$. Then

$$t = J\big(s(u)u\big) = \big(s(u)\big)^{\overline{p}_u} J(u) = \big(s(u)\big)^{\overline{p}_u} t_1,$$

where $\overline{p}_u \in [p_-, p_+]$ is a constant depending on u. From this it follows that, when $t \to t_1$, $s(u) \to 1$ uniformly in $u \in M(t_1)$. Noting that $\|u\| \leqslant \max\{(p_+ t_1)^{\frac{1}{p_+}}, (p_+ t_1)^{\frac{1}{p_-}}\}$ for $u \in M(t_1)$, we have that, when $t \to t_1$,

$$\big\|h_{(t_1,t)}(u) - u\big\| = \big|s(u) - 1\big| \|u\| \to 0 \quad \text{uniformly in } u \in M(t_1).$$

The lemma is proved. □

Proposition 2.3. *For each fixed n, $c_n(t)$ and $\mu_n(t)$, as the functions of t, are continuous on $(0, \infty)$.*

Proof. Let any positive integer n be given. Take any $t_0 \in (0, \infty)$. Let us prove that $c_n(t)$, as a function of t, is continuous at $t = t_0$. To see this, let any $\varepsilon > 0$ be given. By Lemma 2.1 and the uniform continuity of ψ on every bounded set in X, there exists $\delta > 0$ small enough such that when $|t - t_0| < \delta$, $|\psi(h_{(t_0,t)}(u)) - \psi(u)| < \varepsilon$ for all $u \in M(t_0)$. By the definition of $c_n(t_0)$, there exists $A \subset M(t_0)$ such that $A \in \Sigma_n$ and $\inf_{u \in A} \psi(u) \geqslant c_n(t_0) - \varepsilon$. For any $t > 0$ with $|t - t_0| < \delta$, we have that $h_{(t_0,t)}(A) \subset M(t)$, $h_{(t_0,t)}(A) \in \Sigma_n$ and

$$\inf_{u \in h_{(t_0,t)}(A)} \psi(u) \geqslant \inf_{u \in A} \psi(u) - \varepsilon \geqslant c_n(t_0) - 2\varepsilon,$$

which shows that $c_n(t) \geqslant c_n(t_0) - 2\varepsilon$ provided $|t - t_0| < \delta$. Thus we have that $\underline{\lim}_{t \to t_0} c_n(t) \geqslant c_n(t_0)$. On the other hand, for each $t > 0$ with $|t - t_0| < \delta$, there exists $A_t \subset M(t)$ such that $A_t \in \Sigma_n$ and $\inf_{u \in A_t} \psi(u) \geqslant c_n(t) - \varepsilon$. Then $h_{(t,t_0)}(A_t) \subset M(t_0)$, $h_{(t,t_0)}(A_t) \in \Sigma_n$ and

$$\inf_{u \in h_{(t,t_0)}(A_t)} \psi(u) \geqslant \inf_{u \in A_t} \psi(u) - \varepsilon \geqslant c_n(t) - 2\varepsilon,$$

which shows that $c_n(t_0) \geqslant c_n(t) - 2\varepsilon$ provided $|t - t_0| < \delta$. Thus we have that $\overline{\lim}_{t \to t_0} c_n(t) \leqslant c_n(t_0)$. Hence $\lim_{t \to t_0} c_n(t) = c_n(t_0)$. This shows that the function $c_n(\cdot)$ is continuous on $(0, \infty)$. Since $\mu_n(t) = \frac{t}{c_n(t)}$, the function $\mu_n(\cdot)$ is also continuous on $(0, \infty)$. $\quad\square$

Proposition 2.4. *Let n be any positive integer. Then for each $t > 0$, the sets $K_n(t)$ and $\Lambda_n(t)$ are compact, and the set-valued mappings $K_n(\cdot)$ and $\Lambda_n(\cdot)$ are upper semicontinuous on $(0, \infty)$.*

Proof. Let any positive integer n be given. Then for each $t > 0$, since ψ_t satisfies the $(P.S)^+$ condition on $M(t)$, $K_n(t)$, the set of critical points of ψ_t with critical value $c_n(t)$, is compact. Take any $t_0 > 0$. Let us prove that the set-valued mapping $K_n(t)$ is upper semicontinuous at $t = t_0$. Arguing by contradiction, assume that this is not true. Then there exist an open neighborhood U of $K_n(t_0)$, $\{t_m\} \subset (0, \infty)$ and $\{u_m\} \subset X \setminus \{0\}$ such that $t_m \to t_0$ as $m \to \infty$, $u_m \in K_n(t_m)$ and $u_m \notin U$ for every m. Note that $u_m \in K_n(t_m)$ implies that $u_m \in M(t_m)$, $\psi(u_m) = c_n(t_m)$ and $J'(u_m) = \lambda_m \psi'(u_m)$ with $\lambda_m = \frac{\int_\Omega |\nabla u_m|^{p(x)} dx}{\int_\Omega |u_m|^{q(x)} dx}$. Obviously, $\{\|u_n\|\}$ and $\{\lambda_m\}$ are bounded. We may assume, taking a subsequence if necessary, that $u_n \rightharpoonup u_0$ in X and $\lambda_m \to \lambda_0$. Since ψ and ψ' are weakly-strongly continuous, we have that $\psi(u_m) \to \psi(u_0)$ and $\psi'(u_m) \to \psi'(u_0)$. By Proposition 2.3, $\psi(u_m) = c_n(t_m) \to c_n(t_0)$, so $\psi(u_0) = c_n(t_0)$. It is easy to see that $\lambda_0 > 0$. Because that $J'(u_m) = \lambda_m \psi'(u_m) \to \lambda_0 \psi'(u_0)$ and J' is of type (S_+), we have that $u_m \to u_0$ in X. Thus $J(u_m) \to J(u_0)$, $J'(u_m) \to J'(u_0)$ and $J'(u_0) = \lambda_0 \psi'(u_0)$. This shows that $u_0 \in K_n(t_0)$ which contradicts with $u_m \notin U$. Hence the set-valued mapping $K_n(\cdot)$ is upper semicontinuous on $(0, \infty)$. If we define a function $\lambda : X \setminus \{0\} \to (0, \infty)$ as in (2.1), then the function $\lambda(\cdot)$ is continuous, and $\Lambda_n(t) = \lambda(K_n(t))$ for every $t > 0$, namely, $\Lambda_n = \lambda \circ K_n$. From this it follows that the set-valued mapping $\Lambda_n(\cdot)$, as a composition of $K_n(\cdot)$ and $\lambda(\cdot)$, is also upper semicontinuous on $(0, \infty)$. $\quad\square$

Remark 2.3. We do not know whether for any n, the set-valued mapping $\Lambda_n(\cdot)$ (or $K_n(\cdot)$) has a continuous selection, i.e., there exists a continuous mapping $\lambda_n(\cdot)$ such that $\lambda_n(t) \in \Lambda_n(t)$ for every $t \in (0, \infty)$. In general, the relation between $c_n(t)$ and $\Lambda_n(t)$ is complicated. For the relation between critical values and eigenvalues in nonlinear minimax problems we refer to Tintarev [35].

3. Principal eigenvalues

Let us continue to use the notations as in Sections 1 and 2. Because the positive solutions of (1.1) possess special significance, the following definition is reasonable.

Definition 3.1. An eigenvalue $\lambda \in \Lambda$ for (1.1) is called principal if there exists a nonnegative eigenfunction corresponding to λ, i.e., if there exists a nonnegative $u \in X \setminus \{0\}$ such that (u, λ) is a solution of (1.1).

Remark 3.1. In the case when $p \in C^1(\overline{\Omega})$, by a strong maximum principle for the $p(x)$-Laplacian equations of [20], every nontrivial nonnegative solution u of (1.1) must be positive in Ω. Note that, in fact, from [20] we can see that the condition $p \in C^1(\overline{\Omega})$ can be replaced by a weaker condition that p is Lipschitz on $\overline{\Omega}$.

Define

$$\Lambda^+ = \Lambda^+_{(p(\cdot), q(\cdot))} = \{\lambda \in \mathbb{R} \mid \lambda \text{ is a principal eigenvalue for (1.1)}\}.$$

In this section, if no special explanation, it will always be assumed that (p_0) and (q_0) hold.

Proposition 3.1. $\Lambda_1(t) \subset \Lambda^+$ for every $t > 0$.

Proof. Let $t > 0$ and $\lambda \in \Lambda_1(t)$. Then there exists $u \in M(t)$ such that $\psi(u) = c_1(t) = \sup_{M(t)} \psi$ and $\lambda = \frac{\int_\Omega |\nabla u|^{p(x)} \, dx}{\int_\Omega |u|^{q(x)} \, dx}$. Put $v(x) = |u(x)|$ for $x \in \Omega$. Then $J(v) = J(u) = t$, $\psi(v) = \psi(u) = c_1(t)$, $\int_\Omega |\nabla v|^{p(x)} \, dx = \int_\Omega |\nabla u|^{p(x)} \, dx$ and $\int_\Omega |v|^{q(x)} \, dx = \int_\Omega |u|^{q(x)} \, dx$. Thus v is nonnegative and nontrivial, $v \in M(t)$, $v \in K_1(t)$ and $\lambda = \frac{\int_\Omega |\nabla v|^{p(x)} \, dx}{\int_\Omega |v|^{q(x)} \, dx}$. This shows that (v, λ) is a solution of (1.1) and so $\lambda \in \Lambda^+$. The proof is complete. $\square$

Theorem 3.1. *Assume the following condition is satisfied*:

(A$_1$) *There exist an open subset U of Ω and a compact subset E of U with positive measure $|E|$ such that $q_+(E) < p_-(\partial U)$.*

Then when $t \to 0$, $\mu_1(t) \to 0$ and $\Lambda_1(t) \to 0$, and consequently $\inf \Lambda^+ = 0$.

Proof. For any $\delta > 0$, define

$$U_\delta = \left\{ x \in U \mid \operatorname{dist}(x, \partial U) < \delta \right\}.$$

Then there exists $\delta > 0$ small enough such that $E \cap U_\delta = \emptyset$ and $q_+(E) < p_-(U_\delta)$. Putting $\varepsilon = p_-(U_\delta) - q_+(E)$, then $\varepsilon > 0$. Let $u_0 \in X$ be such that $u_0(x) = 0$ for $x \in \Omega \setminus U$ and $u_0(x) = 1$ for $x \in U \setminus U_\delta$. Given any $t > 0$ there exists a unique $s = s_t > 0$ such that $s u_0 \in M(t)$ and $s_t \to 0$ as $t \to 0$. Now let $t > 0$ be small enough such that $s = s_t \in (0, 1)$. Then

$$\mu_1(t) = \frac{t}{c_1(t)} \leqslant \frac{\int_\Omega \frac{s^{p(x)}}{p(x)} |\nabla u_0|^{p(x)} \, dx}{\int_\Omega \frac{s^{q(x)}}{q(x)} |u_0|^{q(x)} \, dx} \leqslant \frac{\int_{U_\delta} \frac{s^{p(x)}}{p(x)} |\nabla u_0|^{p(x)} \, dx}{\int_E \frac{s^{q(x)}}{q(x)} |u_0|^{q(x)} \, dx}$$

$$\leqslant \frac{s^{p_-(U_\delta)} \int_{U_\delta} \frac{1}{p(x)} |\nabla u_0|^{p(x)} \, dx}{s^{q_+(E)} \int_E \frac{1}{q(x)} |u_0|^{q(x)} \, dx} \leqslant s^\varepsilon \frac{\int_{U_\delta} \frac{1}{p(x)} |\nabla u_0|^{p(x)} \, dx}{\int_E \frac{1}{q(x)} |u_0|^{q(x)} \, dx}.$$

From this it follows that $\mu_1(t) \to 0$ as $t \to 0$. By (2.6), $\Lambda_1(t) \leqslant \frac{p_+}{q_-} \mu_1(t)$, and hence $\Lambda_1(t) \to 0$ as $t \to 0$. $\square$

Theorem 3.2. *Assume the following condition is satisfied*:

(A$_2$) *There exist an open subset U of Ω and a compact subset E of U with positive measure $|E|$ such that $q_-(E) > p_+(\partial U)$.*

Then when $t \to \infty$, $\mu_1(t) \to 0$ and $\Lambda_1(t) \to 0$, and consequently $\inf \Lambda^+ = 0$.

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1 and it is omitted here.

Remark 3.2. In [17] a special case of Theorems 3.1 and 3.2 when $q(\cdot) = p(\cdot)$ has been proved.

Remark 3.3. Condition (A$_1$) holds if the following condition is satisfied:

(A$_1$)$'$ There are $x_0 \in \Omega$ and an open neighborhood G such that $q \in C^0(G)$ and $q(x_0) < p(x_0)$.

Indeed, when (A$_1$)$'$ holds, there exists an open ball $B(x_0, r)$ such that $\overline{B(x_0, r)} \subset G$ and $q_+(\overline{B(x_0, r)}) < p_-(\overline{B(x_0, r)})$ and hence (A$_1$) holds by taking $U = B(x_0, r)$ and $E = \overline{B(x_0, \frac{r}{2})}$. Analogously we can see that condition (A$_2$) holds if the following condition is satisfied:

(A$_2$)$'$ There are $x_0 \in \Omega$ and an open neighborhood G such that $q \in C^0(G)$ and $q(x_0) > p(x_0)$.

Note that, when $q \in C^0(\overline{\Omega})$, condition (A$_1$)$'$ (respectively (A$_2$)$'$) is just the condition that there exists $x_0 \in \overline{\Omega}$ such that $q(x_0) < p(x_0)$ (respectively $q(x_0) > p(x_0)$).

Put

$$\mu_* = \inf \mu_1((0, \infty)), \qquad \mu^* = \sup \mu_1((0, \infty)),$$
$$\lambda_* = \inf \Lambda^+, \qquad \lambda^* = \sup \Lambda^+.$$

Theorems 3.1 and 3.2 show that, if either (A_1) or (A_2) holds, then $\mu_* = 0$. It is easy to see that, if (A_1) and (A_2) hold, then $\mu^* < \infty$. Note that $\mu_* = 0 \Rightarrow \lambda_* = 0$, and $\mu^* = \infty \Rightarrow \lambda^* = \infty$.

By Proposition 2.3, the function $\mu_1(\cdot)$ is continuous on $(0, \infty)$ and so $\mu_1((0, \infty))$, the image set of $\mu_1(\cdot)$, is connected. However we do not know whether $\Lambda_1((0, \infty))$, the image set of the set-valued function $\Lambda_1(\cdot)$, is connected. As noted in Introduction, Mihăilescu and Rădulescu [29, Theorem 2.1] have proved that, under the basic assumption (1.2) there exists $\lambda_0 > 0$ (small enough) such that any $\lambda \in (0, \lambda_0)$ is an eigenvalue for problem (1.1). The following theorem is a generalization of [29, Theorem 2.1] because condition (1.2) implies condition (A_1), and μ^* may be larger than λ_0 mentioned in [29, Theorem 2.1].

Theorem 3.3. *Let* (A_1) *hold. Then for any* $\lambda \in (0, \mu^*)$*, there holds* $\lambda \in \Lambda^+$ *and there exists a nonnegative eigenfunction* u_λ *corresponding to* λ *such that* u_λ *is a local minimizer of the energy functional* I_λ *associated to problem* (1.1) *and* $I_\lambda(u_\lambda) < 0$*, and moreover,* $J(u_\lambda) \to 0$ *as* $\lambda \to 0$*.*

Proof. Let (A_1) hold and any $\lambda \in (0, \mu^*)$ be given. Consider the energy functional

$$I_\lambda(u) = J(u) - \lambda \psi(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \lambda \int_\Omega \frac{1}{q(x)} |u|^{q(x)} dx, \quad \forall u \in X. \tag{3.1}$$

By Theorem 3.1, $\mu_1(t) \to 0$ as $t \to 0$ and so $\mu_* = 0$. Since $\mu_1((0, \infty))$ is connected and $\lambda \in (0, \mu^*)$, there exists $t_0 > 0$ such that $\lambda \leqslant \mu_1(t_0) = \frac{t_0}{c_1(t_0)}$. Put $D = \{u \in X : J(u) \leqslant t_0\}$. Then D is a bounded, convex and closed subset of X and $\partial D = J^{-1}(t_0) = M(t_0)$. Noting that $c_1(t_0) = \sup_{M(t_0)} \psi$, we have that, for any $u \in \partial D$,

$$I_\lambda(u) = J(u) - \lambda \psi(u) \geqslant t_0 - \lambda c_1(t_0) = t_0 \left(1 - \frac{\lambda c_1(t_0)}{t_0}\right) = t_0 \left(1 - \frac{\lambda}{\mu_1(t_0)}\right) \geqslant 0.$$

Because the functional $I_\lambda : D \to \mathbb{R}$ is (sequentially) weakly lower semicontinuous and D is (sequentially) weakly compact, there exists $u_0 \in D$ such that $I_\lambda(u_0) = \inf_{u \in D} I_\lambda(u)$. We claim that $I_\lambda(u_0) < 0$. Indeed, taking $t_1 \in (0, t_0)$ such that $\mu_1(t_1) < \lambda$, then there exists $v \in M(t_1) \subset D$ such that $\psi(v) = c_1(t_1)$ and consequently

$$I_\lambda(v) = J(v) - \lambda \psi(v) = t_1 - \lambda c_1(t_1) = t_1 \left(1 - \frac{\lambda c_1(t_1)}{t_1}\right) = t_1 \left(1 - \frac{\lambda}{\mu_1(t_1)}\right) < 0,$$

which shows $I_\lambda(u_0) < 0$. Putting $u_\lambda(x) = |u_0(x)|$ for $x \in \Omega$, then $I_\lambda(u_\lambda) = I_\lambda(u_0) = \inf_{u \in D} I_\lambda(u) < 0$, $u_\lambda \neq 0$ and $u_\lambda \in \text{int } D$. Thus u_λ is a local minimizer of I_λ and hence $I'_\lambda(u_\lambda) = 0$. This shows that (u_λ, λ) is a solution of (1.1) and $\lambda \in \Lambda^+$. It is easy to see that $J(u_\lambda) \to 0$ as $\lambda \to 0$. The proof is complete. $\quad\square$

Now let us consider the case that $q(x) < p(x)$ for a.e. $x \in \overline{\Omega}$. It is well known that, in the constant exponent case, when $q < p$, $\mu^* = \infty$ and for every $\lambda > 0$, the corresponding problem (1.1) has a unique positive solution u (see e.g. [9]). The following example shows that, in general, this is not the case when $p(\cdot)$ and $q(\cdot)$ are not constants.

Example 3.1. Let $\Omega = B(0, 2) := \{x \in \mathbb{R}^N \mid |x| < 2\} \subset \mathbb{R}^N$, $p(x) = 10 - |x|^2$ and $q(x) = 9 - |x|^2$. Then $p, q \in C^1(\overline{\Omega})$ and $q(x) < p(x)$ for all $x \in \overline{\Omega}$. Obviously, the condition (A_1) holds because condition $(A_1)'$ is satisfied. By Theorem 3.1,

$$\mu_1(t) \to 0 \quad \text{and} \quad \Lambda_1(t) \to 0 \quad \text{as } t \to 0.$$

Taking $U = \Omega = B(0, 2)$ and $E = \overline{B(0, 1)}$, then $q_-(E) = 9 - 1 = 8$ and $p_+(\partial U) = 10 - 2^2 = 6$. Thus $q_-(E) > p_+(\partial U)$. This shows that condition (A_2) holds. By Theorem 3.2,

$$\mu_1(t) \to 0 \quad \text{and} \quad \Lambda_1(t) \to 0 \quad \text{as } t \to \infty.$$

Thus, by the continuity of $\mu_1(\cdot)$ on $(0, \infty)$, $\mu_* = 0$ and $\mu^* < \infty$. Let $t^* \in (0, \infty)$ be such that $\mu_1(t^*) = \mu^*$. Take $t_1 > t^*$ large enough such that $\mu_1(t_1) < \frac{q_-}{p_+}\mu^* = \frac{5}{10}\mu^* = \frac{1}{2}\mu^*$. Then there exists a nonnegative $u_1 \in M(t_1)$ such that $\psi(u_1) = c_1(t_1)$. Putting $\lambda_1 = \frac{\int_\Omega |\nabla u_1|^{p(x)}\, dx}{\int_\Omega |u_1|^{q(x)}\, dx}$, then (u_1, λ_1) is a solution of problem (1.1) and $\lambda_1 \in \Lambda^+$. Noting that $\lambda_1 \leqslant \frac{p_+}{q_-}\mu_1(t_1) < \mu^*$, from the proof of Theorem 3.3 we can know that there exists a nonnegative eigenfunction u_2 corresponding to λ_1 such that $u_2 \in M(t_2)$ with $t_2 \in (0, t^*)$. Since $t_2 < t^* < t_1$, $u_1 \neq u_2$. Thus for $\lambda = \lambda_1$, problem (1.1) has two different positive solutions u_1 and u_2.

In [17] it was shown that when $q(\cdot) = p(\cdot)$, in general,

$$\lambda_{*,(p(\cdot),p(\cdot))} := \inf \Lambda^+_{(p(\cdot),p(\cdot))} = 0,$$

and in the case when $p(\cdot)$ satisfies some monotonicity conditions, $\lambda_{*,(p(\cdot),p(\cdot))} > 0$.

Theorem 3.4. *Suppose that $\lambda_{*,(p(\cdot),p(\cdot))} > 0$, and $q \ll p$, that is, there exists $\varepsilon > 0$ such that*

$$q(x) + \varepsilon \leqslant p(x) \quad \text{for a.e. } x \in \Omega. \tag{3.2}$$

Then for any $\lambda > 0$, problem (1.1) has a nontrivial nonnegative solution u_λ which is a global minimizer of the energy functional I_λ associated to problem (1.1) and $I_\lambda(u_\lambda) < 0$, and consequently $\Lambda^+ = (0, \infty)$.

Proof. Let any $\lambda > 0$ be given and I_λ be as in (3.1). Note that the condition $\lambda_{*,(p(\cdot),p(\cdot))} > 0$ implies that

$$\bar{\mu} := \inf_{u \in X \setminus \{0\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx}{\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\, dx} > 0. \tag{3.3}$$

From (3.2) it follows that there exists $R > 0$ large enough such that

$$\frac{\lambda |t|^{q(x)}}{q(x)} < \frac{\bar{\mu}|t|^{p(x)}}{2p(x)} \quad \text{for } |t| > R \text{ and a.e. } x \in \Omega. \tag{3.4}$$

By (3.3) and (3.4) we have that, for any $u \in X$,

$$
\begin{aligned}
I_\lambda(u) &= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\, dx \\
&= \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - \lambda \int_{|u(x)|>R} \frac{1}{q(x)}|u|^{q(x)}\, dx - \lambda \int_{|u(x)|\leqslant R} \frac{1}{q(x)}|u|^{q(x)}\, dx \\
&\geqslant \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - \frac{\bar{\mu}}{2}\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\, dx - C_1 \\
&\geqslant \frac{1}{2}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - C_1, \tag{3.5}
\end{aligned}
$$

where C_1 is a positive constant. (3.5) shows that I_λ is coercive, namely $I_\lambda(u) \to \infty$ as $\|u\| \to \infty$. Hence I_λ has a global minimizer u_0. It is obvious that (3.2) implies (A_1). By Theorem 3.3, we know that $I_\lambda(u_0) < 0$. Thus $u_0 \neq 0$. Putting $u_\lambda(x) = |u_0(x)|$ for $x \in \Omega$, then u_λ is a global minimizer of I_λ. Thus $\lambda \in \Lambda^+$, and consequently $\Lambda^+ = (0, \infty)$. The proof is complete. $\quad\square$

The following theorem, Theorem 3.5, is a generalization of Theorem 3.4.

Theorem 3.5. *Suppose that (A_1) holds and the following condition is satisfied:*

(A$_3$) *Given any $\delta > 0$, there exists a positive C_δ such that*

$$\int_\Omega \frac{1}{q(x)} |u|^{q(x)} \, dx \leqslant \delta \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \, dx + C_\delta \quad \text{for all } u \in X.$$

Then the assertions of Theorem 3.4 remain in force.

Proof. Let any $\lambda > 0$ be given. Taking $\delta = \frac{1}{2\lambda}$ and using condition (A$_3$), we can see that I_λ is coercive and hence I_λ has a global minimizer u_0. From (A$_1$) it follows that $I_\lambda(u_0) < 0$. Putting $u_\lambda(x) = |u_0(x)|$ for $x \in \Omega$, we can see that the assertions of Theorem 3.4 hold. $\square$

Remark 3.4. It is easy to see that, if the hypotheses of Theorem 3.4 are satisfied, then (A$_1$) and (A$_3$) hold. Hence Theorem 3.4 is a special case of Theorem 3.5. Another special case of Theorem 3.5 is the case when $q_+ < p_-$. It is clear that when $q_+ < p_-$, (A$_1$) and (A$_3$) hold.

Now let us turn to consider the case when $p(x) < q(x)$ for $x \in \overline{\Omega}$. It is well known that, in the constant exponent case, when $p < q$, $\mu_1(t) \to \infty$ as $t \to 0$, in particular, $\mu^* = \infty$. If we denote by (u_t, λ_t) with $t > 0$ the solutions of (1.1) such that $u_t \in M(t)$ and $u_t > 0$ in Ω, then, in the constant exponent case, $\lambda_t \to \infty$ as $t \to 0$. The following example shows that, in general, this is not the case when $p(\cdot)$ and $q(\cdot)$ are not constants.

Example 3.2. Let $\Omega = B(0, 2) \subset \mathbb{R}^N$ with $N = 2$, $p(x) = |x|^2 + 2$ and $q(x) = |x|^2 + 3$. Then $p, q \in C^1(\overline{\Omega})$ and $p(x) < q(x) < p^*(x) = \infty$ for all $x \in \overline{\Omega}$. Then condition (A$_2$) holds because condition (A$_2$)$'$ is satisfied. By Theorem 3.2, we have that

$$\mu_1(t) \to 0 \quad \text{and} \quad \Lambda_1(t) \to 0 \quad \text{as } t \to \infty,$$

which is the same as in the constant exponent case. Taking $U = \Omega = B(0, 2)$ and $E = \overline{B(0, 1)}$, then $q_+(E) = 4$ and $p_-(\partial U) = 6$. Thus $q_+(E) < p_-(\partial U)$, that is, condition (A$_1$) holds. By Theorem 3.1,

$$\mu_1(t) \to 0 \quad \text{and} \quad \Lambda_1(t) \to 0 \quad \text{as } t \to 0,$$

which is different from the constant exponent case. Thus $\mu_* = 0$, $\mu^* < \infty$ and problem (1.1) has a family of solutions $\{(u_t, \lambda_t) \mid t \in (0, \infty)\}$ such that $u_t \in M(t)$ and $u_t > 0$ in Ω for every $t > 0$, $\lambda_t \to 0$ as $t \to 0$, and $\lambda_t \to 0$ as $t \to \infty$. Let $t^* \in (0, \infty)$ be such that $\mu_1(t^*) = \mu^*$. Analogously to Example 3.1, we can see that, for some $\lambda \in \Lambda^+$, there are two different positive eigenfunctions corresponding to λ.

Under additional assumption $\lambda_{*, (p(\cdot), p(\cdot))} > 0$, we give the following theorem, Theorem 3.6, which is a generalization of the corresponding result in the constant exponent case (see e.g. [21]).

Theorem 3.6. *Suppose that $\lambda_{*, (p(\cdot), p(\cdot))} > 0$ and there exists $\varepsilon > 0$ such that*

$$p(x) + \varepsilon \leqslant q(x) \quad \text{for a.e. } x \in \Omega. \tag{3.6}$$

Then $\mu_1(t) \to 0$ as $t \to \infty$ and $\mu_1(t) \to \infty$ as $t \to 0$, and consequently $\mu_* = 0$, $\mu^* = \infty$, $\inf \Lambda^+ = 0$ and $\sup \Lambda^+ = \infty$.

Proof. It is obvious that (3.6) implies (A$_2$) and consequently, by Theorem 3.2, $\mu_1(t) \to 0$ as $t \to \infty$. This shows that $\mu_* = 0$ and $\inf \Lambda^+ = 0$. Now let us prove the assertion that $\mu_1(t) \to \infty$ as $t \to 0$. Suppose by contradiction that there exist a sequence $\{t_k\} \subset (0, 1)$ and a positive constant L such that $t_k \to 0$ as $k \to \infty$ and $\mu_1(t_k) \leqslant L$ for all k. Then there exist $\{u_k\} \subset X \setminus \{0\}$ and $\{\lambda_k\} \subset (0, \infty)$ such that for each k, $u_k \in M(t_k)$, $\psi(u_k) = c_1(t_k)$ and (u_k, λ_k) is a solution of (1.1). By (2.6),

$$\lambda_k \leqslant \frac{p_+}{q_-} \mu_n(t_k) \leqslant \frac{p_+}{q_-} L \quad \text{for all } k. \tag{3.7}$$

By the L^∞ regularity for the weak solutions of the $p(x)$-Laplacian equations (see e.g. [18]), we know that $u_k \in L^\infty(\Omega)$ and $|u_k|_{L^\infty(\Omega)}$ depends only on $\|u_k\|$, p_+, p_-, q_+, q_-, λ_k, Ω and N. Noting that $J(u_k) = t_k \to 0$ is equivalent to that $\|u_k\| \to 0$, we can see that $|u_k|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$. Let $\overline{\mu}$ is the positive constant defined by (3.3). It follows from (3.6) that there exists $\delta > 0$ small enough such that

$$\frac{|s|^{p(x)}}{p(x)} \geqslant \frac{2L|s|^{q(x)}}{\overline{\mu}q(x)} \quad \text{for } |s| \leqslant \delta \text{ and a.e. } x \in \Omega. \tag{3.8}$$

Take k_0 large enough such that $|u_{k_0}|_{L^\infty(\Omega)} \leqslant \delta$. Then

$$\mu_1(t_{k_0}) = \frac{t_{k_0}}{c_1(t_{k_0})} = \frac{\int_\Omega \frac{1}{p(x)}|\nabla u_{k_0}|^{p(x)}\,dx}{\int_\Omega \frac{1}{q(x)}|u_{k_0}|^{q(x)}\,dx} \geqslant \frac{\overline{\mu}\int_\Omega \frac{1}{p(x)}|u_{k_0}|^{p(x)}\,dx}{\int_\Omega \frac{1}{q(x)}|u_{k_0}|^{q(x)}\,dx} \geqslant \overline{\mu} \cdot \frac{2L}{\overline{\mu}} = 2L,$$

which contradicts with $\mu_1(t_k) \leqslant L$. Hence $\mu_1(t) \to \infty$ as $t \to 0$, and consequently, $\mu^* = \infty$ and $\sup \Lambda^+ = \infty$. The proof is complete. $\quad\square$

Remark 3.5. Let the assumptions of Theorem 3.6 hold and let any $\lambda > 0$ be given. It is easy to see that I_λ satisfies Mountain Pass Geometry. Indeed, since $\mu^* = \infty$, there exists $t_1 > 0$ such that $\mu_1(t_1) > \lambda$. Then for any $u \in M(t_1)$,

$$I_\lambda(u) = J(u) - \lambda\psi(u) \geqslant t_1 - \lambda c_1(t_1) = t_1\left(1 - \frac{\lambda}{\mu_1(t_1)}\right) := a > 0,$$

which shows that $I_\lambda|_{\partial U} \geqslant a > 0$, where $U = \{u \in X: J(u) < t_1\}$ is a bounded open set in X with boundary $\partial U = M(t_1)$. Since $\mu_1(t) \to 0$ as $t \to \infty$, there exists $t_2 > t_1$ such that $\mu_1(t_2) < \lambda$. Let $u_2 \in M(t_2)$ be such that $\psi(u_2) = c_1(t_2)$. Then $u_2 \notin \overline{U}$ and

$$I_\lambda(u_2) = J(u_2) - \lambda\psi(u_2) = t_2 - \lambda c_1(t_2) = t_2\left(1 - \frac{\lambda}{\mu_1(t_2)}\right) < 0.$$

Hence I_λ satisfies Mountain Pass Geometry. However, because of lack of (P.S) condition (we cannot prove that any $(P.S)_c$-sequence with $c > 0$ is bounded), it is not proved that for every $\lambda > 0$, I_λ has a nontrivial critical point though it is known that $\inf \Lambda^+ = 0$ and $\sup \Lambda^+ = \infty$.

In the present paper, in order to study problem (1.1) the variational method is used. In [14] the basic principles on the sub–supersolution method for $p(x)$-Laplacian equations have been established. It is well known that, the sub–supersolution method, when it is applicable, has some distinctive advantages. For example, it usually allows more flexible requirements on the growth conditions and can also give some order properties of the solutions. Applying Theorem 4.1 of [14] to problem (1.1) we have the following theorem, Theorem 3.7, in which the assumption $q \ll p^*$ is needless.

Theorem 3.7. *Suppose that p is Lipschitz on $\overline{\Omega}$, $p_- > 1$, $q \in C^0(\overline{\Omega})$, $q_- \geqslant 1$. If the condition (A$_1$) is satisfied, then the following assertions hold.*

(1) *If $u \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.1), then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$.*
(2) *Put*

$$\Lambda_\infty^+ = \left\{\lambda \in \Lambda^+ \mid \text{there exists a nonnegative eigenfunction } u \in L^\infty(\Omega) \text{ corresponding to } \lambda\right\}$$

(note that $\Lambda_\infty^+ = \Lambda^+$ if $q \ll p^*$). Then Λ_∞^+ is nonempty and connected, and $\inf \Lambda_\infty^+ = 0$. For any $\lambda_1, \lambda_2 \in \Lambda_\infty^+$ with $\lambda_1 < \lambda_2$, there exist u_{λ_1} and u_{λ_2}, the positive eigenfunctions corresponding to λ_1 and λ_2, respectively, such that $u_{\lambda_1} < u_{\lambda_2}$ in Ω.
(3) *For any $\lambda \in (0, \sup \Lambda_\infty^+)$, problem (1.1) has a positive solution u_λ which is a local minimizer of I_λ in the C^1 topology (in the case when $q \ll p^*$, u_λ is also a local minimizer of I_λ in the $W_0^{1,p(\cdot)}(\Omega)$ topology). Moreover, $\|u_\lambda\|_{C^1(\overline{\Omega})} \to 0$ as $\lambda \to 0$.*

Theorem 3.7 is a complement to Theorem 3.3.

In the end of this paper, let us point out that, the main results about problem (1.1), obtained in this paper, can be generalized to problem (1.3) provided f satisfies the appropriate conditions. We give the following theorem. The proof is omitted because it is similar to the previously-presented proof for the case when $f(x, u) = |u|^{q(x)-2}u$.

Theorem 3.8. *Let* (p_0) *hold. Suppose* f *satisfies the following conditions.*

(f_1) $f : \Omega \times R \to R$ *is a Carathéodory function and*

$$\left| f(x, t) \right| \leqslant C_1 + C_2 |t|^{q(x)-1} \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R},$$

where q satisfies (q_0), C_1 and C_2 are positive constants.
(f_2) $f(x, t)t > 0$ *for a.e.* $x \in \Omega$ *and* $t \neq 0$.
(f_3) $f(x, -t) = -f(x, t)$ *for a.e.* $x \in \Omega$ *and all* $t \in R$.

Then the following assertions hold, where $F(x, t) = \int_0^t f(x, s)\, ds$,

$$\psi(u) = \int_{\Omega} F(x, u)\, dx \quad \text{for } u \in X,$$

$$\lambda(u) = \frac{\int_{\Omega} |\nabla u|^{p(x)}\, dx}{\int_{\Omega} f(x, u) u\, dx},$$

and the meanings of other notations, as J, I_λ, $M(t)$, $c_n(t)$, $\mu_n(t)$, Λ, $K_n(t)$, $\Lambda_n(t)$ *and* Λ^+, *are similar to those defined for* $f(x, u) = |u|^{q(x)-2}u$ *above.*

(1) *For each* $t > 0$ *and* $n = 1, 2, \ldots$, $c_n(t)$ *is a critical value of* ψ_t *on* $M(t)$ *and the Ljusternik–Schnirelmann multiplicity result holds. Moreover,* $c_n(t) \to 0$ *as* $n \to \infty$.
(2) *For each* $t > 0$ *and* $n = 1, 2, \ldots$, *the sets* $K_n(t)$ *and* $\Lambda_n(t)$ *are nonempty,* $\Lambda_n(t) \subset \Lambda$, *and for any* $u \in K_n(t)$, $(u, \lambda(u))$ *is a solution of* (1.3). *For each* $t > 0$, $\Lambda_n(t) \to +\infty$ *as* $n \to \infty$.
(3) *For each fixed* n, $c_n(t)$ *and* $\mu_n(t)$, *as the functions of* t, *are continuous on* $(0, \infty)$.
(4) *For each* $t > 0$, *the sets* $K_n(t)$ *and* $\Lambda_n(t)$ *are compact, and the set-valued mappings* $K_n(\cdot)$ *and* $\Lambda_n(\cdot)$ *are upper semicontinuous on* $(0, \infty)$.
(5) *Assume the following condition* (A_1^f) *is satisfied*:
   (A_1^f) *There exist an open subset* U *of* Ω, *a compact subset* E *of* U *with positive measure* $|E|$, *and positive constants* r_0 *and* C, *such that* $1 < r_0 < p_-(\partial U)$ *and*

$$f(x, t) \geqslant C t^{r_0 - 1} \quad \text{for a.e. } x \in E \text{ and all } t \in (0, 1).$$

   Then when $t \to 0$, $\mu_1(t) \to 0$ *and* $\Lambda_1(t) \to 0$, *and consequently* $\inf \Lambda^+ = 0$. *Moreover, for any* $\lambda \in (0, \mu^*)$, *there holds* $\lambda \in \Lambda^+$ *and there exists a nonnegative eigenfunction* u_λ *corresponding to* λ *such that* u_λ *is a local minimizer of the energy functional* I_λ *associated to problem* (1.3) *and* $I_\lambda(u_\lambda) < 0$, *and furthermore, when* $\lambda \to 0$, $J(u_\lambda) \to 0$.
(6) *Suppose that* (A_1^f) *and* (A_3) *hold (note that this is the case if* $\lambda_{*,(p(\cdot), p(\cdot))} > 0$ *and* $q \ll p$). *Then for any* $\lambda > 0$, *problem* (1.3) *has a nontrivial nonnegative solution* u_λ *which is a global minimizer of the energy functional* I_λ *associated to problem* (1.3) *and* $I_\lambda(u_\lambda) < 0$, *and consequently* $\Lambda^+ = (0, \infty)$.

Acknowledgment

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