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# Isoperimetric type inequalities for harmonic functions

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ABSTRACT

For $0 < p < +\infty$ let $h^p$ be the harmonic Hardy space and let $b^p$ be the harmonic Bergman space of harmonic functions on the open unit disk $\mathbb{U}$. Given $1 \leqslant p < +\infty$, denote by $\|\cdot\|_{b^p}$ and $\|\cdot\|_{h^p}$ the norms in the spaces $b^p$ and $h^p$, respectively. In this paper, we establish the harmonic $h^p$-analogue of the known isoperimetric type inequality $\|f\|_{b^{2p}} \leqslant \|f\|_{h^p}$, where $f$ is an arbitrary holomorphic function in the classical Hardy space $H^p$. We prove that for arbitrary $p > 1$, every function $f \in h^p$ satisfies the inequality

$$\|f\|_{b^{2p}} \leqslant a_p \|f\|_{h^p},$$

where $a_p > 1$ is a suitable constant depending only on $p$. Furthermore, by using the Carleman inequality in the form $\|f\|_{b^4} \leqslant \|f\|_{h^2}$ with $f \in H^2$, we prove the following refinement of the above inequality for $p = 2$

$$\|f\|_{b^4} \leqslant \sqrt[4]{1.5+\sqrt{2}}\,\|f\|_{h^2}.$$



## 1. Introduction

Throughout the paper we let $\mathbb{U} = \{z \in \mathbb{C}\colon |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$ be the unit circle in $\mathbb{C}$. The normalized area measure on $\mathbb{U}$ will be denoted by $d\sigma$. In terms of real (rectangular and polar) coordinates, we have

$$d\sigma = \frac{1}{\pi}\,dx\,dy = \frac{1}{\pi} r\,dr\,dt, \qquad z = x + iy = re^{it}.$$

Further, $dt/2\pi$ denotes the normalized Lebesgue measure on $\mathbb{T}$.

For $1 \leqslant p < +\infty$ let $L^p(\mathbb{U}, \sigma) = L^p$ denote the familiar Lebesgue space on $\mathbb{U}$ with respect to the measure $\sigma$. For such a $p$, the *harmonic Bergman space* $b^p$ is the space of all (complex-valued) harmonic functions $f$ on the disk $\mathbb{U}$ such that

$$\|f\|_{b^p} := \left(\int_{\mathbb{U}} \left|f(z)\right|^p d\sigma\right)^{1/p} < +\infty. \tag{1.1}$$

Recall that the *Bergman space* $A^p$ is the space of all holomorphic functions on $\mathbb{U}$ such that the integral in (1.1) is finite. We denote by $A_0^p$ the set of all functions $f \in A^p$ for which $f(0) = 0$.

The harmonic *Hardy space* $h^p$ is defined as the space of (complex-valued) harmonic functions $f$ on $\mathbb{U}$ such that

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$$\|f\|_{h^p} := \sup_{0\leqslant r<1}\left(\int_0^{2\pi}\left|f\left(re^{it}\right)\right|^p \frac{dt}{2\pi}\right)^{1/p} < \infty. \tag{1.2}$$

If $f \in h^p$ then by [1, Theorem 6.13] the *radial limit*

$$f\left(e^{it}\right) = \lim_{r\to 1^-} f\left(re^{it}\right)$$

exists for almost every $e^{it}$ in $\mathbb{T}$ and the boundary function $f(e^{it})$ is integrable on $\mathbb{T}$. It is well known that

$$\|f\|_{h^p}^p = \lim_{r\to 1^-}\int_0^{2\pi}\left|f\left(re^{it}\right)\right|^p \frac{dt}{2\pi} = \int_0^{2\pi}\left|f\left(e^{it}\right)\right|^p \frac{dt}{2\pi}.$$

The *Hardy space* $H^p$ equipped with the norm $\|\cdot\|_{h^p}$ defined above, consists of all holomorphic functions $f \in h^p$. For more information on Bergman and Hardy spaces, see the books [7] and [5].

The starting point of this paper is the well-known isoperimetric inequality for Jordan domains and isoperimetric inequality for minimal surfaces due to Carleman [4]. Among the other results, Carleman in [4] proved that for any smooth harmonic function $u$ on the closed disk $\overline{\mathbb{U}}$ we have

$$\int_{\mathbb{U}} e^{2u}\,dx\,dy \leqslant \frac{1}{4\pi}\left(\int_0^{2\pi} e^u\,dt\right)^2.$$

By using a similar approach as Carleman, Strebel in [10, Theorem 19.9, pp. 96–98] proved that if $f$ is in $H^1$

$$\int_{\mathbb{U}}\left|f(z)\right|^2 dx\,dy \leqslant \frac{1}{4\pi}\left(\int_{\mathbb{T}}\left|f\left(e^{it}\right)\right| dt\right)^2 \tag{1.3}$$

with "=" instead of "$\leqslant$" if and only if

$$f(z) = \frac{\alpha}{(1-az)^2},$$

where $|a| < 1$, $\alpha \in \mathbb{C}$. This inequality has been proved independently by Mateljević and Pavlović in [9]. More than one approach can be found in the expository papers by Gamelin and Khavinson [8] and Bénéteau and Khavinson [2] along with a brief history of the problem.

It is useful to observe that for our purposes the inequality (1.3) may be written in terms of the $A^2$ and $H^1$ norms as

$$\|f\|_{b^2} \leqslant \|f\|_{h^1}, \quad f \in H^1. \tag{1.4}$$

Further, Burbea [3] generalized the inequality (1.3) as

$$\frac{n-1}{\pi}\int_{\mathbb{U}}\left|f(z)\right|^{np}\left(1-|z|^2\right)^{n-2} dx\,dy \leqslant \left(\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(e^{it}\right)\right|^p dt\right)^n,$$

where $n \geqslant 2$ is a positive integer and $f \in H^p$ for some $0 < p < +\infty$.

Strebel in [10, Theorem 19.9, pp. 96–98] proved the previous inequality with $n = 2$ and arbitrary $0 < p < +\infty$. The same result is obtained by Vukotić [11, Theorem 1], where it is written in the form

$$\|f\|_{b^{2p}} \leqslant \|f\|_{h^p}, \quad f \in H^p. \tag{1.5}$$

Recently, Hang, Wang and Yang [6] extended the related isoperimetric type inequality for harmonic functions defined on the unit ball of $\mathbb{R}^n$ with $n \geqslant 3$.

Although it is impossible to establish the harmonic version of the inequality (1.3) (cf. Example 1.5), in this paper we prove its harmonic $h^p$-analogue for arbitrary $p > 1$ as follows.

**Theorem 1.1.** *Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Then every function $f \in h^p$ satisfies the inequality*

$$\|f\|_{b^{2p}} \leqslant a_p\|f\|_{h^p},$$

*where*

$$a_p = \begin{cases} (\frac{p}{p-1})^{1/p}, & \text{if } 1 < p \leqslant 2 \text{ and } f \text{ is real harmonic;} \\ \sqrt{2}(\frac{p}{p-1})^{1/p}, & \text{if } 1 < p \leqslant 2 \text{ and } f \text{ is complex harmonic;} \\ 1 + (\frac{q}{q-1})^{1/q}, & \text{if } p > 2 \text{ and } f \text{ is real harmonic;} \\ 2^{1-1/p}(1 + (\frac{q}{q-1})^{1/q}), & \text{if } p > 2 \text{ and } f \text{ is complex harmonic.} \end{cases}$$

Recall that the proof of Theorem 1.1 is based on the Riesz theorem [5, Theorem 4.1 and Exercise 5, p. 67]. Furthermore, by using the Carleman inequality in the form $\|f\|_{b^4} \leqslant \|f\|_{h^2}$ with $f \in H^2$, we give a direct proof of the refinement of the above inequality for $p = 2$ as follows.

**Theorem 1.2.** *Suppose $f$ is a nonzero function in the space $h^2$. Then $f$ belongs to $b^4$ and*

$$\|f\|_{b^4} \leqslant \sqrt[4]{1.5 + \sqrt{2}}\|f\|_{h^2}. \tag{1.6}$$

**Remark 1.3.** Notice that the constant $\sqrt[4]{(3 + 2\sqrt{2})/2} \approx 1.306563$ of Theorem 1.2 is less than the estimate $a_2 = \sqrt{2}$ from Theorem 1.1. Further, in terms of integrals, the inequality (1.6) can be written in the form

$$\int_{\mathbb{U}} |f(z)|^4 \, dx\, dy < b_2 \left( \int_{\mathbb{T}} |f(e^{it})|^2 \, dt \right)^2, \tag{1.7}$$

where $b_2 \leqslant (3 + 2\sqrt{2})/(8\pi)$.

**Remark 1.4.** Motivated by the inequality (1.5), the question arises whether we can replace $|f(z)|^4$ and $|f(e^{it})|^2$ in (1.7) by $|f(z)|^2$ and $|f(e^{it})|^1$, respectively. The answer is negative (see Example 1.5 below). Furthermore, it remains an open question whether the inequality (1.7) is sharp. The function from Example 1.6 shows that the best constant $b_2$ in the previous inequality is greater than or equal to $5/(8\pi)$.

**Example 1.5.** For $a \in \mathbb{C}$ with $|a| < 1$ let $f_a$ be the harmonic function on $\mathbb{U}$ defined as

$$f_a(z) = \frac{1 - |a|^2|z|^2}{|1 - za|^2}, \quad z \in \mathbb{U}.$$

Then for such $a$ we have $\int_{\mathbb{T}} |f_a(e^{it})|\, dt = 2\pi$, and hence $f_a \in h^1$. This together with $\int_{\mathbb{U}} |f_a(z)|^2 \, d\sigma \to \infty$ as $a \to 1$ establish the fact that the inequality of type (1.3) cannot be extended to the harmonic Hardy space $h^1$.

**Example 1.6.** As noticed previously, the inequality (1.3) is sharp. This is also true for the inequality (1.5) (see [11, Theorem 1]). The following example suggests that our inequality (1.7) could be sharp.

For $|a| < 1$, let $f_a$ be the function defined on $\mathbb{U}$ as

$$f_a(z) = \operatorname{Re} \frac{z}{1 - az}, \quad z \in \mathbb{U}.$$

Then as $a \to 1^-$, after integration in polar coordinates, we have

$$\int_{\mathbb{U}} |f_a(re^{it})|^4 \, r\, dr\, dt \Big/ \left( \frac{3 + 2\sqrt{2}}{8\pi} \left( \int_0^{2\pi} |f_a(e^{it})|^2 \, dt \right)^2 \right) \to \frac{5}{3 + 2\sqrt{2}} \approx 0.857864,$$

or, equivalently

$$\frac{\|f_a\|_{b^4}}{\|f_a\|_{h^2}} \to \sqrt[4]{2.5} \approx 1.257433 \quad \text{(cf. Theorem 1.2)}.$$

## 2. The general case

A function $s$ is said to be simple if its range is a finite set. A simple function with range in $\mathbb{R}$ always has the following representation

$$s(x) = \sum_{i=1}^{n} c_i \chi_{E_i},$$

where $c_i$ are distinct values of $s$ and $E_i = s^{-1}\{c_i\}$. If $E_i$ are measurable subsets of a measure space $(X, \mu)$, then the integral of $s$ over $X$ is defined by

$$\int_X s\, d\mu = \sum_{i=1}^{n} c_i \mu(E_i).$$

Let $f : X \to [0, +\infty]$ be a measurable function. Consider the set $S_f$ of all measurable simple functions $s$ such that $0 \leqslant s \leqslant f$. The integral of $f$ over $X$ is defined as

$$\int_X f\, d\mu = \sup_{s \in S_f} \int_X s\, d\mu.$$

**Lemma 2.1.** *Let $1 \leqslant p < \infty$, and let $u, v$ be real functions belonging to the measure space $(E, \mu)$, where $\mu(E) < \infty$. Then*

$$\|u\|_p + \|v\|_p \leqslant 2^{1-\min\{1/2,1/p\}} \|u + iv\|_p. \tag{2.1}$$

*The inequality is sharp. For $p \leqslant 2$, the equality is attained if $u = v$. If $p > 2$, then the equality is attained for $u = \chi_{E_1}$ and $v = \chi_{E_2}$, where $\mu(E_1) = \mu(E_2)$ and $E_1 \cap E_2 = \emptyset$.*

**Proof.** We will prove (2.1) applying the following sharp inequality

$$\left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p} \leqslant 2^{1-\min\{1/2,1/p\}} \left(\sum_{i=1}^{n} \left(u_i^2 + v_i^2\right)^{p/2}\right)^{1/p}, \tag{2.2}$$

where $u_i, v_i \in \mathbb{R}$, $i = 1, \ldots, n$. Assume for the moment that (2.2) is proved. Take the disjoint family of measurable sets $E_i$ such that $E = \bigcup_{i=1}^{n} E_i$ and take $s^u = \sum_{i=1}^{n} u^i \chi_{E_i}$, and $s^v = \sum_{i=1}^{n} v^i \chi_{E_i}$ such that the functions $s^u$ and $s^v$ satisfy $s^u \leqslant u$ and $s^v \leqslant v$. Then $s = |s^u + is^v|$ is a simple function and it satisfies the condition $s \leqslant |u + iv|$. It is enough to prove that

$$\left\|s^u\right\|_p + \left\|s^v\right\|_p \leqslant 2^{1-\min\{1/2,1/p\}} \left\|s^u + is^v\right\|_p. \tag{2.3}$$

Take

$$u_i = \left|u^i\right| \mu(E_i)$$

and

$$v_i = \left|v^i\right| \mu(E_i).$$

Now (2.2) coincides with (2.3). Let us prove (2.2). Put

$$A := \left(\sum_{i=1}^{n} \left(u_i^2 + v_i^2\right)^{p/2}\right)^{1/p},$$

$$B := \left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p}$$

and

$$C := \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}.$$

We will solve the extremal problem of finding the supremum of $f(u, v) := B + C \to \sup$, for $A = A_0$ i.e. for $(u, v) := (u_1, \ldots, u_n, v_1, \ldots, v_n) \in \{(u, v){:}\ A(u, v) = A_0\}$, $A_0 > 0$. It is clear that the set $K$ is compact in $\mathbb{R}^{2n}$. If $p = 1$, then our inequalities follow from the obvious inequality

$$|\alpha| + |\beta| \leqslant \sqrt{2}\sqrt{\alpha^2 + \beta^2}.$$

For $p > 1$, the function

$$f(u, v) = \left(\sum_{i=1}^{n} u_i^{2\frac{p}{2}}\right)^{1/p} + \left(\sum_{i=1}^{n} v_i^{2\frac{p}{2}}\right)^{1/p}$$

is continuous on $K$ and smooth on $K \setminus \{(u,v)\colon\ B=0 \vee C=0\}$, and therefore according to the Weierstrass theorem, $f(u,v)$ achieves its minimum and its maximum on $K$. If $B=0$ or $C=0$ then our inequality is trivial. Now assume that $B\neq 0$ and $C\neq 0$. The stationary points of the Lagrangian satisfy

$$B_{u_i}=\lambda A_{u_i},\quad i=1,\dots,n, \tag{2.4}$$

and

$$C_{v_i}=\lambda A_{v_i},\quad i=1,\dots,n. \tag{2.5}$$

Therefore,

$$\frac{u_i^{p-1}}{B^{p-1}}=\lambda\frac{u_i(u_i^2+v_i^2)^{p/2-1}}{A^{p-1}},\quad i=1,\dots,n, \tag{2.6}$$

$$\frac{v_i^{p-1}}{C^{p-1}}=\lambda\frac{v_i(u_i^2+v_i^2)^{p/2-1}}{A^{p-1}},\quad i=1,\dots,n. \tag{2.7}$$

Without loss of generality, we can suppose that $u_i\neq 0$ and $v_i\neq 0$ for $i\leqslant m\leqslant n$, and $u_i=0$ or $v_i=0$ for $i>m$. Then from (2.6) we obtain that there exists $t\neq 0$ such that

$$v_i=tu_i,\quad i=1,\dots,m. \tag{2.8}$$

Set

$$a^p:=\sum_{i=1}^m|u_i|^p, \tag{2.9}$$

$$b^p:=\sum_{i=m+1}^n|v_i|^p, \tag{2.10}$$

and

$$c^p:=\sum_{i=m+1}^n|u_i|^p. \tag{2.11}$$

Then (2.2) is equivalent to

$$\left(t^pa^p+b^p\right)^{\frac{1}{p}}+\left(a^p+c^p\right)^{\frac{1}{p}}\leqslant 2^{1-\min\{1/2,1/p\}}\left(\left(1+t^2\right)^{p/2}a^p+b^p+c^p\right)^{\frac{1}{p}}. \tag{2.12}$$

If $a=0$, then (2.12) is equivalent to

$$b+c\leqslant 2^{1-\min\{1/2,1/p\}}\left(b^p+c^p\right)^{\frac{1}{p}}.$$

The last inequality follows from the following well-known inequality

$$\frac{b+c}{2}\leqslant\left(\frac{b^p+c^p}{2}\right)^{\frac{1}{p}}. \tag{2.13}$$

Consider now the case when $a>0$. Dividing the inequality (2.12) by $a$, and substituting $x=b/a$ and $y=c/a$ into (2.12), we arrive to the inequality

$$\left(t^n+x^p\right)^{\frac{1}{p}}+\left(1+y^p\right)^{\frac{1}{p}}\leqslant 2^{1-\min\{1/2,1/p\}}\left(\left(1+t^2\right)^{\frac{p}{2}}+x^p+y^p\right)^{\frac{1}{p}}. \tag{2.14}$$

**The case $p\geqslant 2$.** Let $\varphi(s)=s^p$. Then $\varphi$ is a convex function and therefore, according to the inequality $x^p+1\leqslant(1+x^2)^{p/2}$ with $x\geqslant 0$ we obtain

$$\begin{aligned}\varphi\left(\frac{1}{2}\left(\left(t^p+x^p\right)^{1/p}+\left(1+y^p\right)^{1/p}\right)\right)&\leqslant\frac{1}{2}\left(\varphi\left(\left(t^p+x^p\right)^{1/p}\right)+\varphi\left(\left(1+y^p\right)^{1/p}\right)\right)\\&=\frac{1+t^p+x^p+y^p}{2}\\&\leqslant\frac{(1+t^2)^{p/2}+x^p+y^p}{2}.\end{aligned} \tag{2.15}$$

Now (2.12) follows from (2.15).

**The case $1 \leqslant p < 2$.** We again use the inequality

$$\varphi\left(\frac{1}{2}\left(\left(t^p + x^p\right)^{1/p} + \left(1 + y^p\right)^{1/p}\right)\right) \leqslant \frac{1 + t^p + x^p + y^p}{2}.$$

In this case it is enough to prove the inequality

$$\frac{1 + t^p + x^p + y^p}{2} \leqslant \frac{(1 + t^2)^{p/2} + x^p + y^p}{2^{p/2}}. \tag{2.16}$$

The inequality (2.16) follows from the inequality

$$\left(\frac{1 + t^p}{2}\right)^{\frac{2}{p}} \leqslant \frac{1 + t^2}{2}. \tag{2.17}$$

Taking now the convex function $\psi(s) = s^{\frac{2}{p}}$, we obtain

$$\psi\left(\frac{1}{2}\left(1 + t^p\right)\right) \leqslant \frac{1}{2}\left(\psi(1) + \psi\left(t^p\right)\right). \tag{2.18}$$

Then (2.18) coincides with (2.17). If $m = n$, then from (2.6) and (2.7) it follows that $t = 1$. This yields $u = v$. In this case (2.2) reduces to the equality for $p \leqslant 2$. On the other hand, taking $u_1 = 1$, $u_i = 0, i \neq 1$ and $v_2 = 1$ and $v_j = 0$ for $j \neq 2$, (2.2) becomes the equality for $p \geqslant 2$. The proof is now completed. $\square$

**Proof of Theorem 1.1.** Assume first that $1 < p \leqslant 2$. By the Riesz theorem [5, Theorem 4.1 and Exercise 5, p. 67], for the holomorphic function $F$ on $\mathbb{U}$ we have

$$\|F\|_{h^p} \leqslant \left(\frac{p}{p-1}\right)^{1/p} \|\operatorname{Re} F\|_{h^p}. \tag{2.19}$$

If $f$ is a real function, then $f = g + \bar{g}$ for some function $g$ holomorphic on $\mathbb{U}$. Therefore, by using the isoperimetric inequality (1.5) for holomorphic functions and (2.19), we have

$$\begin{aligned}
\|f\|_{2p}^{2p} &\leqslant 2^{2p} \|g\|_{2p}^{2p} \\
&\leqslant 2^{2p} \|g\|_{h^p}^{2p} \\
&\leqslant 2^{2p} \frac{p^2}{(p-1)^2} \|\operatorname{Re} g\|_{h^p}^{2p} \\
&\leqslant \frac{p^2}{(p-1)^2} \|f\|_{h^p}^{2p}.
\end{aligned}$$

Hence,

$$\|f\|_{2p} \leqslant \left(\frac{p}{p-1}\right)^{1/p} \|f\|_{h^p}. \tag{2.20}$$

Let us now consider a complex harmonic function $f$ such that

$$f = g + \bar{h} = u + iv.$$

From (2.20) and Lemma 2.1 we have

$$\begin{aligned}
\|f\|_{2p} &\leqslant \|u\|_{2p} + \|v\|_{2p} \\
&\leqslant \left(\frac{p}{p-1}\right)^{1/p} \left(\|u\|_{h^p} + \|v\|_{h^p}\right) \\
&\leqslant \sqrt{2}\left(\frac{p}{p-1}\right)^{1/p} \|f\|_{h^p}.
\end{aligned}$$

This establishes the case $p \leqslant 2$.

Assume now that $p \geqslant 2$. If $f$ is holomorphic, then from the proof of [5, Theorem 4.1] we obtain

$$\|\operatorname{Im} f\|_{h^p} \leqslant \left(\frac{q}{q-1}\right)^{1/q} \|\operatorname{Re} f\|_{h^p},$$

and therefore

$$\|f\|_{h^p} \leqslant \left(1 + \left(\frac{q}{q-1}\right)^{1/q}\right)\|\operatorname{Re} f\|_{h^p}. \tag{2.21}$$

Combining (2.21) with Lemma 2.1 we find that

$$\begin{aligned}\|f+\bar g\|_{2p} &\leqslant \big\|\operatorname{Re}(f+g)\big\|_{2p} + \big\|\operatorname{Re} i(f-g)\big\|_{2p} \\ &\leqslant \|f+g\|_{2p} + \big\|i(f-g)\big\|_{2p} \\ &\leqslant \left(1 + \left(\frac{q}{q-1}\right)^{1/q}\right)\big(\big\|\operatorname{Re}(f+g)\big\|_{h^p} + \big\|\operatorname{Re} i(f-g)\big\|_{h^p}\big) \\ &\leqslant 2^{1-1/p}\left(1 + \left(\frac{q}{q-1}\right)^{1/q}\right)\|f+\bar g\|_{h^p}.\end{aligned}$$

This completes the proof. □

## 3. A refinement of the case $p = 2$

In order to prove Theorem 1.2, we will need some auxiliary results.

**Lemma 3.1.** *For any complex number z the following equality holds*

$$\big|z\operatorname{Re}(z)\big| = \frac{\sqrt{2}}{2}|z|\sqrt{|z|^2 + \operatorname{Re}\big(z^2\big)}. \tag{3.1}$$

**Proof.** An easy calculation shows that

$$\big|z^2 + |z|^2\big| = \sqrt{2}|z|\sqrt{|z|^2 + \operatorname{Re}\big(z^2\big)}.$$

Now (3.1) follows from the previous identity and the identity $|z^2 + |z|^2| = 2|z\operatorname{Re}(z)|$. □

**Lemma 3.2.** *Let h be a function in $A_0^1$. Then*

$$\int_{\mathbb{U}} h(z)\,d\sigma = \int_{\mathbb{U}} \bar h(z)\,d\sigma = \int_{\mathbb{U}} \operatorname{Re}\big(h(z)\big)\,d\sigma = 0. \tag{3.2}$$

**Proof.** Since $h(0) = 0$, we can write $h(z) = \sum_{n=1}^{\infty} a_n z^n$. Therefore, the first two equalities in (3.2) immediately follow from the fact that $\int_0^{2\pi} e^{int}\,dt = 0$ for each $n = \pm 1, \pm 2, \dots$. From this and the identity $2\operatorname{Re}(h(z)) = h(z) + \bar h(z)$ it follows that $\int_{\mathbb{U}} \operatorname{Re}(h(z))\,d\sigma = 0$. □

**Lemma 3.3.** *If h is a function in the space $A_0^2$, then*

$$\int_{\mathbb{U}} \big|h(z)\big|^2\,d\sigma = 2\int_{\mathbb{U}} \big|\operatorname{Re}\big(h(z)\big)\big|^2\,d\sigma. \tag{3.3}$$

**Proof.** Since

$$\begin{aligned}\big|\operatorname{Re}\big(h(z)\big)\big|^2 &= \left|\frac{h(z)+\bar h(z)}{2}\right|^2 = \frac{1}{4}\big(h(z)+\bar h(z)\big)\big(\bar h(z)+h(z)\big) \\ &= \frac{1}{4}\big(2\big|h(z)\big|^2 + h^2(z) + \bar h^2(z)\big) = \frac{1}{2}\big|h(z)\big|^2 + \frac{1}{2}\operatorname{Re}\big(h^2(z)\big),\end{aligned}$$

by integrating this and applying the fact that by Lemma 3.2, $\int_{\mathbb{U}} \operatorname{Re}(h^2(z))\,d\sigma = 0$, we obtain (3.3). □

**Lemma 3.4.** *Let h be a function in the space $A_0^2$ that is not identically zero on $\mathbb{U}$. Then*

$$\int_{\mathbb{U}} \big|h(z)\operatorname{Re}\big(h(z)\big)\big|\,d\sigma < \frac{\sqrt{2}}{2}\int_{\mathbb{U}} \big|h(z)\big|^2\,d\sigma. \tag{3.4}$$

**Proof.** By the identity (3.1) of Lemma 3.1, we have

$$\left|h(z)\operatorname{Re}\big(h(z)\big)\right| = \frac{\sqrt{2}}{2}\left|h(z)\right|\sqrt{\left|h(z)\right|^2 + \operatorname{Re}\big(h^2(z)\big)}. \tag{3.5}$$

By applying the Cauchy–Schwarz inequality and the fact that by Lemma 3.2, $\int_{\mathbb{U}} \operatorname{Re}(h^2(z))\,d\sigma = 0$, we obtain

$$\begin{aligned}\left(\int_{\mathbb{U}} \left|h(z)\right|\sqrt{\left|h(z)\right|^2 + \operatorname{Re}\big(h^2(z)\big)}\,d\sigma\right)^2 &\leqslant \int_{\mathbb{U}} \left|h(z)\right|^2 d\sigma \left(\int_{\mathbb{U}} \left(\left|h(z)\right|^2 + \operatorname{Re}\big(h^2(z)\big)\right) d\sigma\right) \\ &= \left(\int_{\mathbb{U}} \left|h(z)\right|^2 d\sigma\right)^2 + \left(\int_{\mathbb{U}} \left|h(z)\right|^2 d\sigma\right)\left(\int_{\mathbb{U}} \operatorname{Re}\big(h^2(z)\big)\,d\sigma\right) \\ &= \left(\int_{\mathbb{U}} \left|h(z)\right|^2 d\sigma\right)^2.\end{aligned}$$

The above inequality and (3.5) immediately yield the desired inequality (3.4) with "$\leqslant$" instead of "$<$". In order to show the strict inequality, we first observe that the equality in the previously applied Cauchy–Schwarz inequality holds if and only if

$$\left|h^2(z)\right| = \lambda\sqrt{\left|h(z)\right|^2 + \operatorname{Re}\big(h^2(z)\big)} \tag{3.6}$$

for almost every $z \in \mathbb{U}$ and a nonnegative constant $\lambda$. If $\lambda = 0$ then obviously, we have $h \equiv 0$ on $\mathbb{U}$. If $\lambda > 0$ then (3.6) implies that $\operatorname{Re}(h^2(z)) = \frac{1-\lambda^2}{\lambda^2}|h^2(z)|$ for almost every $z \in \mathbb{U}$. Therefore, by the continuity of the functions $h^2$ and $\operatorname{Re}(h^2)$ on the disk $\mathbb{U}$, it follows that $\operatorname{Re}(h^2(z)) = \frac{1-\lambda^2}{\lambda^2}|h^2(z)|$ for each $z \in \mathbb{U}$. The last equality yields

$$\Delta\left|h^2(z)\right| = 4\left|h'(z)\right|^2 = 0$$

and hence

$$\operatorname{Re}\big(h^2(z)\big) = 0.$$

Thus, $h$ is a constant function on $\mathbb{U}$. Since $h(0) = 0$, we obtain $h \equiv 0$ on $\mathbb{U}$. This contradiction completes the proof. □

**Proof of Theorem 1.2.** Since the unit disk is a simply connected set, we have the representation $f = g + \bar{h}$, where $g$ and $h$ are holomorphic functions on the unit disk $\mathbb{U}$ such that $h(0) = 0$. Direct calculations yield

$$|f|^4 = |g|^4 + |h|^4 + 4|g|^2|h|^2 + 4\big(|g|^2 + |h|^2\big)\operatorname{Re}(hg) + 2\operatorname{Re}\big((hg)^2\big). \tag{3.7}$$

Suppose $g(z) = \sum_{n=0}^{\infty} a_n z^n$ and $h(z) = \sum_{m=1}^{\infty} b_m z^m$ are the Taylor expansions on $\mathbb{U}$ of functions $g$ and $h$, respectively. Since $f \in h^2$, we have $f(e^{it}) = \sum_{n=0}^{\infty} a_n e^{int} + \sum_{m=1}^{\infty} \bar{b}_m e^{-imt}$ for almost every $e^{it} \in \mathbb{T}$. This together with $|f|^2 = f\bar{f}$ and the orthogonality relation $\int_0^{2\pi} e^{ikt}\,dt = 0$ for $k = \pm 1, \pm 2, \ldots$, immediately yields

$$\|f\|_{h^2}^2 = \int_0^{2\pi} \left|f\big(e^{it}\big)\right|^2 \frac{dt}{2\pi} = \sum_{n=0}^{\infty} |a_n|^2 + \sum_{m=1}^{\infty} |b_m|^2 < +\infty.$$

Hence, both sums on the right hand side of the above equality are finite, and so the functions $h$ and $g$ belong to $H^2$. Therefore, according to (1.3), $g$ and $h$ are also in $A^4$. Thus, the previous equality yields

$$\begin{aligned}\|f\|_{h^2}^4 &= \left(\sum_{n=0}^{\infty} |a_n|^2 + \sum_{m=1}^{\infty} |b_m|^2\right)^2 \\ &= \left(\int_0^{2\pi} \left|g\big(e^{it}\big)\right|^2 \frac{dt}{2\pi} + \int_0^{2\pi} \left|h\big(e^{it}\big)\right|^2 \frac{dt}{2\pi}\right)^2 \\ &= \big(\|g\|_{h^2}^2 + \|h\|_{h^2}^2\big)^2.\end{aligned} \tag{3.8}$$

From this and the identity $gh = ((g+h)^2 - (g-h)^2)/4$ we see that $gh$ is in $A_0^2$. Therefore, all the terms on the right of (3.7) are integrable on $\mathbb{U}$. Therefore, we have $f \in b^4$ or equivalently,

$$\|f\|_{b^4}^4 = \int_{\mathbb{U}} \left|f(z)\right|^4 d\sigma < +\infty.$$

Applying the inequality (1.4) to the functions $g^2, h^2 \in H^1$, respectively, we immediately obtain

$$\int_{\mathbb{U}} \left|g(z)\right|^4 d\sigma = \left\|g^2\right\|_{b^2}^2 \leqslant \left\|g^2\right\|_{h^1}^2 = \|g\|_{h^2}^4, \tag{3.9}$$

$$\int_{\mathbb{U}} \left|h(z)\right|^4 d\sigma = \left\|h^2\right\|_{b^2}^2 \leqslant \left\|h^2\right\|_{h^1}^2 = \|h\|_{h^2}^4. \tag{3.10}$$

Since $gh \in A^2$, the Cauchy–Schwarz inequality together with inequalities (3.9) and (3.10) yield

$$\begin{aligned}\int_{\mathbb{U}} \left|g(z)\right|^2 \left|h(z)\right|^2 d\sigma &\leqslant \sqrt{\int_{\mathbb{U}} \left|g(z)\right|^4 d\sigma} \cdot \sqrt{\int_{\mathbb{U}} \left|h(z)\right|^4 d\sigma} \\ &= \left\|g^2\right\|_{b^2} \cdot \left\|h^2\right\|_{b^2} \\ &\leqslant \left\|g^2\right\|_{h^1} \cdot \left\|h^2\right\|_{h^1} \\ &= \|g\|_{h^2}^2 \cdot \|h\|_{h^2}^2. \end{aligned} \tag{3.11}$$

Using the facts that $h(0)g(0) = 0$, $gh \in A_0^2$, and applying Lemma 3.3 to the holomorphic function $gh$, the Cauchy–Schwarz inequality, and the estimates (3.3), (3.4) and (3.5), we obtain

$$\begin{aligned}\left|\int_{\mathbb{U}} \left(\left|h(z)\right|^2 + \left|g(z)\right|^2\right) \operatorname{Re}\left(h(z)g(z)\right) d\sigma\right| &\leqslant \left|\int_{\mathbb{U}} \left(\left|h(z)\right|^2 + \left|g(z)\right|^2\right)^2 d\sigma\right|^{1/2} \left|\int_{\mathbb{U}} \operatorname{Re}\left(h(z)g(z)\right)^2 d\sigma\right|^{1/2} \\ &= \left(\int_{\mathbb{U}} \left(\left|h(z)\right|^2 + \left|g(z)\right|^2\right)^2 d\sigma\right)^{1/2} \left(\int_{\mathbb{U}} \frac{1}{2}\left|h(z)\right|^2 \left|g(z)\right|^2 d\sigma\right)^{1/2} \\ &\leqslant \left(\int_{\mathbb{U}} \left(\left|h(z)\right|^2 + \left|g(z)\right|^2\right)^2 d\sigma\right)^{1/2} \left(\int_{\mathbb{U}} \frac{1}{8}\left(\left|h(z)\right|^2 + \left|g(z)\right|^2\right)^2 d\sigma\right)^{1/2} \\ &= \frac{\sqrt{2}}{4} \int_{\mathbb{U}} \left(\left|h(z)\right|^4 + 2\left|g(z)\right|^2 \left|h(z)\right|^2 + \left|g(z)\right|^4\right) d\sigma \\ &\leqslant \frac{\sqrt{2}}{4} \left(\|h\|_{h^2}^4 + 2\|g\|_{h^2}^2 \|h\|_{h^2}^2 + \|g\|_{h^2}^4\right). \end{aligned} \tag{3.12}$$

Furthermore, by Lemma 3.2,

$$\int_{\mathbb{U}} \operatorname{Re}\left(h^2(z)g^2(z)\right) d\sigma = 0. \tag{3.13}$$

Finally, after integration of (3.7) on the disk $\mathbb{U}$ by all the terms in the appropriate sum, and substituting the relations (3.8)–(3.13) into this, we immediately obtain

$$\begin{aligned}\|f\|_{b^4}^4 &\leqslant (1+\sqrt{2})\left(\|g\|_{h^2}^4 + \|h\|_{h^2}^4\right) + 2(2+\sqrt{2})\|g\|_{h^2}^2 \|h\|_{h^2}^2 \\ &\leqslant \frac{3+2\sqrt{2}}{2}\left(\|g\|_{h^2}^4 + 2\|g\|_{h^2}^2 \|h\|_{h^2}^2 + \|h\|_{h^2}^4\right) \\ &= \frac{3+2\sqrt{2}}{2}\|f\|_{h^2}^4. \end{aligned}$$

Recall that in the second inequality the inequality $a^4 + b^4 + 2\sqrt{2}a^2b^2 \leqslant \frac{\sqrt{2}+1}{2}(a^2+b^2)^2$ is applied for real numbers $a$ and $b$.

From the above, the inequality (1.6) clearly follows. The equality in the last inequality of (3.12) is attained if and only if $g = h$ almost everywhere on $\mathbb{U}$. Thus if the equality in (1.6) is attained, then it must be that $g = h$. Further, the equality in (3.4) is attained if and only if $g^2 \equiv 0$ on $\mathbb{U}$. This means that we have the strict inequality in (1.6) and the proof of Theorem 1.2 is completed. □

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## References


[1] S. Axler, P. Bourdon, W. Ramey, Harmonic Function Theory, Springer-Verlag, New York, 1992.
[2] C. Bénéteau, D. Khavinson, The isoperimetric inequality via approximation theory and free boundary problems, Comput. Methods Funct. Theory 6 (2006) 253–274.
[3] J. Burbea, Sharp inequalities for holomorphic functions, Illinois J. Math. 31 (1987) 248–264.
[4] T. Carleman, Zur Theorie der Minimalflächen, Math. Z. 9 (1921) 154–160.
[5] P.L. Duren, Theory of $H^p$ Spaces, Academic Press, New York, 1970, reprinted by Dover, Mineola, NY, 2000.
[6] F. Hang, X. Wang, X. Yan, Sharp integral inequalities for harmonic functions, Comm. Pure Appl. Math. 61 (2008) 54–95.
[7] H. Hedenmalm, B. Korenblum, K. Zhu, Theory of Bergman Spaces, Springer-Verlag, New York, 2000.
[8] T.W. Gamelin, D. Khavinson, The isoperimetric inequality and rational approximation, Amer. Math. Monthly 96 (1989) 18–30.
[9] M. Mateljević, M. Pavlović, New proofs of the isoperimetric inequality and some generalizations, J. Math. Anal. Appl. 98 (1984) 25–30.
[10] K. Strebel, Quadratic Differentials, Ergeb. Math. Grenzgeb. (3), vol. 5, Springer-Verlag, Berlin, 1984.
[11] D. Vukotić, The isoperimetric inequality and a theorem of Hardy and Littlewood, Amer. Math. Monthly 110 (2003) 532–536.