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An improved estimate of PSWF approximation and approximation by Mathieu functions ☆

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ABSTRACT

In this paper, an error estimate of spectral approximations by prolate spheroidal wave functions (PSWFs) with explicit dependence on the bandwidth parameter and optimal order of convergence is derived, which improves the existing result in [Chen et al., Spectral methods based on prolate spheroidal wave functions for hyperbolic PDEs, SIAM J. Numer. Anal. 43 (5) (2005) 1912–1933]. The underlying argument is applied to analyze spectral approximations of periodic functions by Mathieu functions, which leads to new estimates featured with explicit dependence on the intrinsic parameter.

1. Introduction

The PSWFs are a family of special functions arisen from two different contexts. On the one hand, they are eigenfunctions of a Sturm–Liouville equation associated with the method of separation-of-variables for solving elliptic problems in spheroidal geometry (see, e.g., [8]):

$$\big((1-x^2)\big(\psi_n^c\big)'\big)' + \big(\chi_n^c - c^2x^2\big)\psi_n^c = 0, \quad x \in (-1,1),\ c \geqslant 0, \tag{1.1}$$

where ψ_n^c is referred to as the PSWF of degree n (and of order 0), and χ_n^c is the corresponding eigenvalue. On the other hand, they appear in the study of time-frequency concentration problem (see a series of papers by Slepian et al. [20,13,19]). As a remarkable coincidence, the PSWFs are the eigenfunctions of an integral equation:

$$\lambda_n^c \psi_n^c(x) = \int_{-1}^{1} e^{\mathrm{i}cxt}\psi_n^c(t)\,dt, \quad x \in (-1,1),\ c > 0, \tag{1.2}$$

where c is the bandwidth parameter. The PSWFs have been proven to be an optimal tool for approximating bandlimited functions (see, e.g., [26,25,18,15]). Moreover, being the eigen-system of a singular Sturm–Liouville problem, they serve as an ideal orthogonal basis for spectral methods for partial differential equations (PDEs). Indeed, spectral approximations based on PSWFs enjoy some advantages over the polynomial counterparts (see, e.g., [5,3,12,24,27]). In addition, they have been used in the wavelet methods (see, e.g., [22,21,23]).

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An interesting but challenging issue on approximation by PSWFs is to analyze how the approximation errors depend on the bandwidth parameter c. The first estimate in Sobolev spaces was derived in Chen et al. [5] (note: Boyd [3] presented a similar result without proof) under the condition $q_N = c/\sqrt{\chi_N^c} < 1$ (see (2.7) below). This result predicts that a spectral accuracy can be achieved when $c < \sqrt{\chi_N^c}$, or roughly, $c < \pi N/2$ (cf. [5]). However, the estimate (2.7) is suboptimal and conservative at least for fixed bandwidth c, since the expected convergence order is $O(N^{-s})$ instead of $O(N^{-2s/3})$, where s is the regularity index of the associated Sobolev space. Moreover, many numerical evidences have demonstrated that a much better accuracy can be attained when $c \leqslant \kappa N$ for certain κ marginally less than $\pi/2$. Theoretically, it would be interesting to find such a κ (as large as possible) that guarantees an optimal order of convergence. This will be one of the main objectives of this paper.

A second purpose of this paper is to apply the argument to analyze spectral approximation by (angular) Mathieu functions. This family of special functions originates from the method of separation-of-variables for solving the Helmholtz equation in elliptic domains [14], and has many applications in physics and engineering (see, e.g., [16] and the references therein). In addition, the Mathieu functions form a natural basis for spectral-Galerkin methods for scattering problems in elliptic domains (see, e.g., [7]). Spectral approximations in Sobolev spaces were first established in [17]. However, the explicit dependence of the errors on the underlying parameter has not been investigated. Interestingly, the accuracy of approximating periodic functions can oftentimes be improved by using Mathieu functions when compared with that by Fourier series. This situation is reminiscent to the comparison between the PSWFs and Legendre polynomials.

The rest of the paper is organized as follows. We present in Section 2 the improved estimate of PSWF approximation, and provide some numerical results to support the theoretical results. In Section 3, we analyze spectral approximations by Mathieu functions in Sobolev spaces.

2. An improved estimate of approximation by PSWFs

This section is devoted to error estimate of spectral approximation by truncated PSWF series in Sobolev spaces.

We first make some necessary preparations. Let $L^2(I)$ with $I := (-1, 1)$ be the space of square integrable functions with the norm $\|\cdot\|$, and let $H^s(I)$ with integer $s \geqslant 0$ be the Sobolev space with the norm $\|\cdot\|_s$ as in Adams [2].

Recall that the PSWFs $\{\psi_n^c(x)\}_{n=0}^{\infty}$, defined by (1.1) or (1.2), form a complete orthonormal system in $L^2(I)$ with $I := (-1, 1)$, namely,

$$\int_{-1}^{1} \psi_n^c(x)\psi_m^c(x)\,dx = \delta_{mn}, \quad \forall c \geqslant 0, \tag{2.1}$$

where δ_{mn} is the Kronecker symbol. If $c = 0$, the PSWFs coincide with the (normalized) Legendre polynomials, denoted by $L_k(x)$, which satisfy

$$\begin{aligned} &xL_k(x) = a_k L_{k-1}(x) + a_{k+1} L_{k+1}(x), \quad k \geqslant 1,\ x \in I; \\ &L_0(x) = \frac{1}{\sqrt{2}}, \qquad L_1(x) = \sqrt{\frac{3}{2}}x \quad \text{with } a_k = \frac{k}{\sqrt{(2k-1)(2k+1)}}. \end{aligned} \tag{2.2}$$

We expand ψ_n^c in terms of Legendre polynomials as

$$\psi_n^c(x) = \sum_{k=0}^{\infty} \beta_k^n L_k(x) \quad \text{with } \beta_k^n := \beta_k^n(c) = \int_{-1}^{1} L_k(x)\psi_n^c(x)\,dx, \tag{2.3}$$

and find from (1.1) and (2.1)–(2.2) that

$$\beta_{k+2}^n = \frac{1}{f(k+2)}\left(\frac{1}{q_n^2}\left(1 - \frac{k(k+1)}{\chi_n^c}\right) - g(k)\right)\beta_k^n - \frac{f(k)}{f(k+2)}\beta_{k-2}^n, \quad k \geqslant 2, \tag{2.4}$$

where

$$f(k) = \frac{k(k-1)}{(2k-1)\sqrt{(2k-3)(2k+1)}}, \qquad g(k) = \frac{2k(k+1)-1}{(2k-1)(2k+3)}, \qquad q_n = \frac{c}{\sqrt{\chi_n^c}}. \tag{2.5}$$

In what follows, we shall use the following inequality (see Lemma 2.2 in [24]):

$$n(n+1) < \chi_n^c < n(n+1) + c^2, \quad n \geqslant 0,\ c > 0. \tag{2.6}$$

The following estimate was stated in [5,3].

Theorem 2.1. *Let $u \in H^s(I)$ with $s \geqslant 0$, and let $q_N = c/\sqrt{\chi_N^c} < 1$. Then the $(N+1)$th coefficient $\hat{u}_N^c = \int_{-1}^1 u(x)\psi_N^c(x)\,dx$ of the PSWF expansion decays like*

$$\left|\hat{u}_N^c\right| \leqslant D\left(N^{-2s/3}\|u\|_s + (q_N)^{\delta N}\|u\|\right), \tag{2.7}$$

where D and δ are positive constants independent of u, N and c.

As the best L^2-approximation error is dominated by $|\hat{u}_N^c|$, this estimate shows that spectral accuracy can be achieved if $q_N < 1$ or roughly $c < \pi N/2$ (cf. [5]). However, when c is small or fixed, the decay rate should be of the same order as the Legendre approximation, that is, $O(N^{-s})$. Indeed, ample numerical results (see, e.g., [5,3,24]) showed that when $c < \kappa N$ with κ being marginally less than $\pi/2$, the PSWF approximations behave like or usually outperform the Legendre approximations. Hence, it is necessary to find such a κ (as large as possible) that can guarantee an optimal order of convergence. We also point out a set of optimal estimates was derived in [24], but the bandwidth parameter c is implicitly built in the Sobolev-type norm characterized by the Sturm–Liouville operator.

To improve the estimate (2.7), we first recall the following result on the upper bound of $|\beta_k^n|$ (see Lemmas A.1 and A.2 in [5]).

Lemma 2.1. *For any $c > 0$, and any positive integer m satisfying*

$$2m(2m+1) < \frac{\ln 2}{2}\chi_n^c, \tag{2.8}$$

we have

$$\begin{aligned}\left|\beta_k^n\right| &\leqslant D\left(\frac{2}{q_n}\right)^{2[k/2]}\max\left\{\left|\beta_0^n\right|, \left|\beta_1^n\right|\right\}\\ &\leqslant Dm^{-1/2}\left(\frac{2}{q_n}\right)^{2[k/2]} q_n^{2m}\exp\left(\frac{8m^3}{3\chi_n^c}\right), \quad 0 \leqslant k \leqslant 2m,\end{aligned} \tag{2.9}$$

where $[a]$ is the largest integer $\leqslant a$ and D is a positive constant independent of m, n and c.

Hereafter, we use ∂_x^k to denote the ordinary derivative d^k/dx^k, whenever no confusion may cause. The main approximation result to be derived is stated below.

Theorem 2.2. *Let $c > 0$, and $q_N = c/\sqrt{\chi_N^c}$. Given a positive constant $q_* < 1$, if*

$$q_N \leqslant \frac{q_*}{\sqrt[6]{2}} \approx 0.8909 q_*, \tag{2.10}$$

and

$$u \in B^s(I) := \left\{u\colon \left(1-x^2\right)^{k/2}\partial_x^k u \in L^2(I),\ 0 \leqslant k \leqslant s\right\} \tag{2.11}$$

with integer $s \geqslant 0$, we have the estimate

$$\left|\hat{u}_N^c\right| \leqslant D\left(N^{-s}\left\|\left(1-x^2\right)^{s/2}\partial_x^s u\right\| + (q_*)^{\delta N}\|u\|\right), \tag{2.12}$$

where D and δ are positive constants independent of u, N and c.

Proof. The proof is based on an argument similar to that for Theorem 2.1 (cf. [5]), but the analysis is subtler.

Let M be a positive integer to be specified later, and denote by $u_M(x)$ the truncated Legendre series:

$$u_M(x) = \sum_{k=0}^{M} a_k L_k(x) \quad \text{with } a_k = \int_{-1}^{1} u(x) L_k(x)\,dx. \tag{2.13}$$

Rewrite $\hat{u}_N^c$ as

$$\hat{u}_N^c = \int_{-1}^{1}\left(u(x) - u_M(x)\right)\psi_N^c(x)\,dx + \int_{-1}^{1} u_M(x)\psi_N^c(x)\,dx. \tag{2.14}$$

Using the Legendre approximation results (see, e.g., [4,10]), we obtain the estimate for the first term:

$$\left|\int_{-1}^{1}(u-u_M)\psi_N^c\,dx\right| \leqslant \|u-u_M\|\big\|\psi_N^c\big\| \leqslant DM^{-s}\big\|\big(1-x^2\big)^{s/2}\partial_x^s u\big\|. \tag{2.15}$$

Next, we obtain from Lemma 2.1 that for any m satisfying (2.8),

$$\begin{aligned}\left|\int_{-1}^{1}u_M\psi_N^c\,dx\right| &\leqslant \sum_{k=0}^{M}\big|a_k\beta_k^N\big| \leqslant \left(\sum_{k=0}^{M}a_k^2\right)^{\frac12}\left(\sum_{k=0}^{M}\big(\beta_k^N\big)^2\right)^{\frac12}\\ &\leqslant Dm^{-1/2}q_N^{2m}\exp\left(\frac{8m^3}{3\chi_N^c}\right)\left(\sum_{k=0}^{M}\left(\frac{2}{q_N}\right)^{4[\frac{k}{2}]}\right)^{\frac12}\|u\|\\ &\leqslant D\sqrt{\frac{M}{m}}\left(\frac{2}{q_N}\right)^{M}q_N^{2m}\exp\left(\frac{8m^3}{3\chi_N^c}\right)\|u\|\\ &\leqslant D\sqrt{\frac{M}{m}}\left\{p_N\left(\frac{2}{q_N}\right)^{\frac{M}{2m}}\right\}^{2m}\|u\|,\end{aligned} \tag{2.16}$$

where in the last step, we used the fact

$$q_N^{2m}\exp\left(\frac{8m^3}{3\chi_N^c}\right) \leqslant q_N^{2m}\exp\left(\frac{4m^2(2m+1)}{3\chi_N^c}\right) := p_N^{2m} \quad \text{with } p_N = q_N\exp\left(\frac{2m(2m+1)}{3\chi_N^c}\right). \tag{2.17}$$

The rest of the proof is to determine m and M so as to derive an optimal estimate of the upper bound in (2.16). Firstly, given $0<q_*<1$, we choose m so that $p_N \leqslant q_*$, that is,

$$2m(2m+1) \leqslant 3\chi_N^c \ln\frac{q_*}{q_N}, \tag{2.18}$$

so in view of (2.8), it suffices to require

$$\frac{\ln 2}{2}\chi_N^c \leqslant 3\chi_N^c\ln\frac{q_*}{q_N} \quad\Leftrightarrow\quad q_N \leqslant \frac{q_*}{\sqrt[6]{2}}. \tag{2.19}$$

The second requirement is to ensure there exists a constant $0<\gamma<1$ such that

$$p_N\left(\frac{2}{q_N}\right)^{\frac{M}{2m}} = p_N^{1-\gamma} \quad\Leftrightarrow\quad \frac{1}{\gamma}\frac{M}{2m} = \frac{\ln\frac{1}{p_N}}{\ln\frac{2}{q_N}} = \frac{-\frac{2m(2m+1)}{3\chi_N^c}+\ln\frac{1}{q_N}}{\ln 2+\ln\frac{1}{q_N}} := h_N. \tag{2.20}$$

It is clear that $h_N<1$, so we require that $h_* \leqslant h_N$ for some constant $0<h_*<1$, that is,

$$\frac{2m(2m+1)}{3\chi_N^c} \leqslant (1-h_*)\ln\frac{1}{q_N} - h_*\ln 2, \quad \text{for all } 0 \leqslant q_N \leqslant q_*/\sqrt[6]{2}. \tag{2.21}$$

Hence, we make it meet (2.8) and require that

$$\frac{\ln 2}{6} \leqslant (1-h_*)\ln\frac{1}{q_N} - h_*\ln 2 \quad\Rightarrow\quad q_N \leqslant 2^{\frac{h_*+1/6}{h_*-1}}. \tag{2.22}$$

It is important to notice that if $q_N \leqslant q_*/\sqrt[6]{2}$, we can always find $h_* \in (0,1)$ to ensure (2.21). In other words, (2.20) holds, and we have

$$\left(\frac{2}{q_N}\right)^{\frac{M}{2m}} = p_N^{-\gamma} \quad\text{and}\quad \gamma h_* \leqslant \frac{M}{2m} \leqslant \gamma. \tag{2.23}$$

We choose m to be the largest integer satisfying (2.8), and choose M of the same order of m, that is,

$$2m = \left[\sqrt{\frac{\ln 2}{2}\chi_N^c}\right] \overset{(2.6)}{=} O(N) \quad\text{and}\quad M = \gamma(1+h_*)m = O(N). \tag{2.24}$$

Thus, using (2.16), (2.20), (2.24) and the fact $p_N \leqslant q_*$, leads to

$$\left|\int_{-1}^{1}u_M\psi_N^c\,dx\right| \leqslant Dp_N^{(1-\gamma)\sqrt{\frac{\ln 2}{2}\chi_N^c}}\|u\| \leqslant D(p_N)^{\delta N}\|u\| \leqslant D(q_*)^{\delta N}\|u\|. \tag{2.25}$$

Finally, the estimate (2.12) follows from (2.14), (2.15) and (2.25). □

Table 2.1
Samples of κ and q_N.

$\kappa=0.4$		$\kappa=0.5$		$\kappa=0.6$		$\kappa=0.8$		$\kappa=1.0$	
$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N
0.371	0.386	0.447	0.472	0.515	0.553	0.625	0.695	0.707	0.810

$\kappa=1.178$		$\kappa=1.2$		$\kappa=1.3$		$\kappa=1.5$		$\kappa=\pi/2$	
$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N	$\underline{\kappa}$	q_N
0.762	0.8909	0.768	0.899	0.793	0.935	0.832	0.989	0.844	1.0

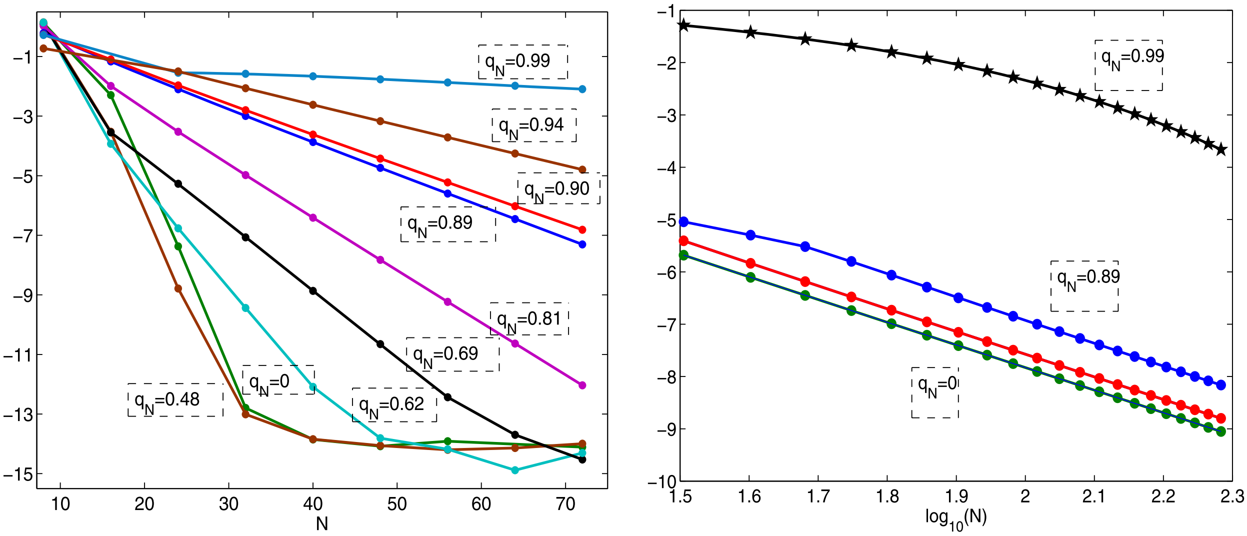

Fig. 2.1. Left: $\log_{10}(|\hat{u}_N^c|)$ against $N \in [8, 80]$ with $c = \kappa N$ (or $c = q_N\sqrt{\chi_N^c}$), where $\kappa = 0$, 0.5, 0.7, 0.8, 1, 1.178, 1.2, 1.3, 1.5, and correspondingly, $q_N = 0$, 0.48, 0.62, 0.69, 0.81, 0.89, 0.90, 0.94, 0.99. Right: $\log_{10}(|\hat{u}_N^c|)$ against $\log_{10} N$ with $N \in [32, 196]$. Here, $\kappa = 0$, 0.5, 1.178, 1.5, and $q_N = 0$, 0.61, 0.89, 0.99.

Remark 2.1. It is seen that under the condition (2.10), the optimal order of convergence can be recovered. In practice, it is more informative to express (2.10) in terms of the relation between c and N. Setting $\kappa = c/N$ with κ being a positive constant, we find from (2.6) that

$$1 + \frac{1}{N} < \frac{\kappa^2}{q_N^2} < 1 + \frac{1}{N} + \kappa^2, \quad \forall N \geqslant 1, \tag{2.26}$$

which implies that for $N \gg 1$,

$$\underline{\kappa} := \frac{\kappa}{\sqrt{1+\kappa^2}} \leqslant q_N < \kappa. \tag{2.27}$$

Hence, q_N falls into the interval $[\underline{\kappa}, \kappa)$, but it seems difficult to obtain an explicit asymptotic expression between q_N and κ. Here, we just provide in Table 2.1 some samples of κ and the corresponding numerical approximations of q_N with $N = 128$.

We find from Table 2.1 that q_N lies between $\underline{\kappa}$ and κ. Roughly speaking, if $c = \kappa N$ with $\kappa < 1.178$, the condition (2.10) is met. We also notice that when $\kappa = \pi/2$, we have $q_N \approx 1$, which numerically confirms the asymptotic estimate of q_N for $c = \pi/2N$ (see, e.g., [5]). □

Next, we present some numerical results to support the estimate in Theorem 2.2, and refer the readers to other numerical tests in [5,3,24], which are consistent with this theoretical result as well.

In this first example, we test PSWF approximation of the analytic function: $u(x) = \sin(3\pi x)\exp(2x)$. We plot in Fig. 2.1 (left) $\log_{10}|\hat{u}_N^c|$ against N. We observe an exponential decay of $|\hat{u}_N^c|$ for all choices of c that meet the condition (2.10), and the fast growth of $|\hat{u}_N^c|$ (or divergence of the truncated PSWF series), when q_N gets close to or approaches 1.

In the second test, we purposely choose $u(x) = (x-1)^b e^{\sin x}$ with $b = 7/5$, which has a finite regularity in the space $B^s(I)$ (cf. (2.11)) with $s = 1 + 2b + \varepsilon$ and $\varepsilon > 0$. As predicted by Theorem 2.2, $|\hat{u}_N^c|$ decays like $O(N^{\varepsilon-(1+2b)})$ if q_N satisfies (2.10). We plot in Fig. 2.1 (right) $\log_{10}|\hat{u}_N^c|$ against $\log_{10} N$, and find that the slopes are slightly smaller than the theoretical prediction.

In the end of this section, we briefly discuss the exponential convergence of PSWF approximations of analytic functions on $[-1,1]$. Suppose that u is analytic on and within a closed contour $\mathcal{C}$ that contains $[-1,1]$. It follows from Lemma 4.2 of [9] (with a different normalization of the Legendre polynomials and making use of the Stirling formula in Lemma 2.3 of [9]) that the Legendre coefficient a_k in (2.13) decays exponentially:

$$|a_k| \leqslant \frac{C_\rho}{(2\rho)^k}, \quad k \gg 1, \tag{2.28}$$

where ρ is the distance of $\mathcal{C}$ to the interval $[-1,1]$, and C_ρ is a positive constant independent of k. In view of this, we update the estimate in (2.15) as

$$\begin{aligned}\left|\int_{-1}^{1}(u-u_M)\psi_N^c\,dx\right| &\leqslant \sum_{k=M+1}^{\infty}|a_k|\int_{-1}^{1}\left|L_k(x)\right|\left|\psi_N^c(x)\right|dx \leqslant \sum_{k=M+1}^{\infty}|a_k|\|L_k\|\left\|\psi_N^c\right\| \\ &\leqslant \sum_{k=M+1}^{\infty}|a_k| \leqslant C_\rho\sum_{M+1}^{\infty}\frac{1}{(2\rho)^k} = \frac{C_\rho}{2\rho-1}\frac{1}{(2\rho)^M},\end{aligned} \tag{2.29}$$

if $\rho > 1/2$. Therefore, under the above assumption of analyticity and under the conditions of Theorem 2.2, $|\hat{u}_N^c|$ decays exponentially.

As pointed out in [6], it is standard to assume that u is analytic on and within an ellipse with foci $x = \pm 1$:

$$\mathcal{E}_r = \left\{z = \left(re^{\mathrm{i}\theta} + r^{-1}e^{-\mathrm{i}\theta}\right):\ \theta \in [0, 2\pi)\right\}, \quad r > 1,$$

where $\mathrm{i} = \sqrt{-1}$ is the complex unit. In this case, the distance of $\mathcal{E}_r$ to $[-1,1]$ is $\rho = (r + r^{-1})/2 - 1$. In this case, a relatively sharper estimate of the Legendre coefficients might be derived, and we refer to the book of Davis [6] for the details.

3. Approximation by Mathieu functions

The Mathieu functions have found their applications in various areas (see, e.g., [16,11,7]). Some spectral approximation results in Sobolev spaces were first derived in [17]. However, to the best of our knowledge, the issue on the dependence of approximation errors on the intrinsic parameter has not been investigated by far. In this section, we apply the previous argument for the PSWFs to establish similar estimates for Mathieu functions.

The (angular) Mathieu equation:

$$\partial_\theta^2\Phi + \left(a - \frac{\rho^2}{2}\cos 2\theta\right)\Phi = 0, \quad \theta \in [-\pi, \pi),\ \rho \geqslant 0, \tag{3.1}$$

appears in the method of separation-of-variables for solving the Helmholtz equation in elliptic coordinates. Here, a is the separation constant, and ρ is a constant related to the wave number (cf. [14]). The Mathieu equation (3.1) admits two families of linearly independent periodic solutions (eigenfunctions), that is, the even "cosine-elliptic" and odd "sine-elliptic" functions:

$$\Phi_n(\theta;\rho) = \mathrm{ce}_n(\theta;\rho) \quad \text{or} \quad \mathrm{se}_{n+1}(\theta;\rho), \quad n = 0, 1, \ldots, \tag{3.2}$$

with the corresponding eigenvalues denoted by $a_n(\rho)$ and $b_{n+1}(\rho)$, respectively. In the analysis, it is more convenient to rewrite (3.1) in the Sturm–Liouville form:

$$\left(-\partial_\theta^2 + \rho^2\cos^2\theta\right)\Phi_n = \lambda_n\Phi_n, \tag{3.3}$$

where

$$\lambda_n = \begin{cases} a_n(\rho) + \rho^2/2 := \lambda_n^c, & \text{if } \Phi_n = \mathrm{ce}_n, \\ b_n(\rho) + \rho^2/2 := \lambda_n^s, & \text{if } \Phi_n = \mathrm{se}_n. \end{cases} \tag{3.4}$$

We collect below some relevant properties of the Mathieu functions (see, e.g., [14]).

- If $\rho = 0$, we have $\mathrm{ce}_n = \cos n\theta$ and $\mathrm{se}_n = \sin n\theta$. Like the trigonometric basis, $\{\mathrm{ce}_n, \mathrm{se}_{n+1}\}$ are 2π-periodic, mutually orthogonal and form a complete orthogonal system in $L^2(0, 2\pi)$. In what follows, we assume that they are orthonormal with respect to the inner product

$$(u, v) = \frac{1}{\pi}\int_{-\pi}^{\pi} u(\theta)\bar{v}(\theta)\,d\theta,$$

where $\bar{v}$ is the complex conjugate of v.

- The eigenvalues are all real, positive, distinct and of the order

$$0 < \lambda_0^c < \lambda_1^s < \lambda_1^c < \cdots < \lambda_n^c < \lambda_n^s < \cdots. \tag{3.5}$$

Moreover, by Lemma 3.1 of [17],

$$n^2 < \lambda_n < n^2 + \rho^2 \quad \text{with } \lambda_n = \lambda_n^c \text{ or } \lambda_n^s. \tag{3.6}$$

Similar to the analysis of PSWF approximations, it is necessary to study the decay property of the coefficients of the trigonometric expansions:

$$\mathrm{ce}_n(\theta;\rho) = \frac{A_0^{(n)}}{2} + \sum_{k=1}^{\infty} A_k^{(n)} \cos k\theta; \qquad \mathrm{se}_n(\theta;\rho) = \sum_{k=1}^{\infty} B_k^{(n)} \sin k\theta,$$

where

$$A_k^{(n)} = \frac{2}{\pi}\int_0^{\pi} \mathrm{ce}_n(\theta,\rho)\cos k\theta\, d\theta, \quad k \geqslant 0,$$
$$B_k^{(n)} = \frac{2}{\pi}\int_0^{\pi} \mathrm{se}_n(\theta,\rho)\sin k\theta\, d\theta, \quad k \geqslant 1. \tag{3.7}$$

One finds from (3.3) the recurrence formulas:

$$\begin{aligned}&\big(4\lambda_n^c - 2\rho^2 - 4k^2\big)A_k^{(n)} - \rho^2\big(A_{k-2}^{(n)} + A_{k+2}^{(n)}\big) = 0, \quad k \geqslant 1,\\ &\big(2\lambda_n^c - \rho^2\big)A_0^{(n)} = \rho^2 A_2^{(n)}, \qquad A_{-1}^{(n)} = A_1^{(n)},\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\big(4\lambda_n^s - 2\rho^2 - 4k^2\big)B_k^{(n)} - \rho^2\big(B_{k-2}^{(n)} + B_{k+2}^{(n)}\big) = 0, \quad k \geqslant 1,\\ &B_{-1}^{(n)} = -B_1^{(n)}, \qquad B_0^{(n)} = 0.\end{aligned} \tag{3.9}$$

We have the following upper bounds for $|A_k^{(n)}|$ and $|B_{k+1}^{(n)}|$ with $k = 0, 1$.

Lemma 3.1. *For any $\rho > 0$, let $\eta_n = \rho/\sqrt{\lambda_n}$, and let m be a positive integer satisfying*

$$m^2 < \frac{\ln 2}{8}\lambda_n. \tag{3.10}$$

Then we have

$$\big|Z_k^{(n)}\big| \leqslant D m^{-1/4}\eta_n^{2m} \exp\left(\frac{8m^3}{3\lambda_n}\right), \quad k = 0, 1, \tag{3.11}$$

where

$$\big\{\lambda_n, Z_k^{(n)}\big\} = \begin{cases}\{\lambda_n^c, A_k^{(n)}\}, & \textit{for } \mathrm{ce}_n,\\ \{\lambda_n^s, B_{k+1}^{(n)}\}, & \textit{for } \mathrm{se}_n,\end{cases} \tag{3.12}$$

and D is a positive constant independent of n, m and c.

Proof. The main idea for the proof is similar to that of Lemma A.1 in [5] for the PSWFs.

We first consider the estimate of $|A_k^{(n)}|$ with $k = 0, 1$. Define

$$\mu_k^{(n)} = \int_{-\pi}^{\pi} \cos^k\theta\, \mathrm{ce}_n(\theta;\rho)\, d\theta. \tag{3.13}$$

Our starting point is to show that under the condition (3.10),

$$\big|\mu_0^{(n)}\big| \leqslant \eta_n^{2m}\big|\mu_{2m}^{(n)}\big| \prod_{k=0}^{m-1}\left(1 - \frac{4k^2}{\lambda_n^c}\right)^{-1}. \tag{3.14}$$

Indeed, multiplying (3.3) with $\Phi_n = \mathrm{ce}_n$ by $\cos^{2k}\theta$ $(0 \leqslant k \leqslant m)$ and integrating the resulting equation over $(0, 2\pi)$, leads to

$$\begin{aligned}&\rho^2 \mu_{2k+2}^{(n)} = 2k(2k-1)\mu_{2k-2}^{(n)} + \left(\lambda_n^c - 4k^2\right)\mu_{2k}^{(n)}, \quad 1 \leqslant k \leqslant m,\\ &\mu_0^{(n)} = \eta_n^2 \mu_2^{(n)}.\end{aligned} \tag{3.15}$$

In view of (3.10), we find that $\mu_2^{(n)}, \ldots, \mu_{2m}^{(n)}$ have the same sign as $\mu_0^{(n)}$, so it follows that

$$\rho^2 \left|\mu_{2k+2}^{(n)}\right| \geqslant \left(\lambda_n^c - 4k^2\right)\left|\mu_{2k}^{(n)}\right| \quad \Rightarrow \quad \left|\mu_{2k}^{(n)}\right| \leqslant \eta_n^2 \left|\mu_{2k+2}^{(n)}\right| \left(1 - \frac{4k^2}{\lambda_n^c}\right)^{-1}. \tag{3.16}$$

Then, an induction yields (3.14).

It is clear that if $0 \leqslant x \leqslant \ln 2/2$, we have $1 - x \geqslant e^{-2x}$. Therefore, under (3.10),

$$1 - \frac{4k^2}{\lambda_n^c} \geqslant \exp\left(-\frac{8k^2}{\lambda_n^c}\right), \quad 0 \leqslant k \leqslant m,$$

which implies

$$\prod_{k=0}^{m-1}\left(1 - \frac{4k^2}{\lambda_n^c}\right)^{-1} \leqslant \exp\left(\frac{8\sum_{k=0}^{m-1} k^2}{\lambda_n^c}\right) \leqslant D \exp\left(\frac{8m^3}{3\lambda_n^c}\right). \tag{3.17}$$

Next, using (3.14), the property of the Beta function and the Stirling formula (cf. [1]), we obtain that

$$\left|\mu_{2m}^{(n)}\right| \leqslant \left\|\cos^{2m}\theta\right\| \|\mathrm{ce}_n\| \leqslant D\left(\int_0^1 t^{4m}\left(1 - t^2\right)^{-1/2} dt\right)^{1/2} \leqslant D\sqrt{\frac{\Gamma(2m+1)}{\Gamma(2m+3/2)}} \leqslant \frac{D}{\sqrt[4]{m}}. \tag{3.18}$$

Hence, by (3.14), (3.17) and (3.18),

$$\left|\mu_0^{(n)}\right| \leqslant D m^{-1/4} \eta_n^{2m} \exp\left(\frac{8m^3}{3\lambda_n^c}\right). \tag{3.19}$$

Since $\mu_0^{(n)} = \pi A_0^{(n)}$, we derive the upper bound for $|A_0^{(n)}|$.

Now, we turn to the estimate for $|A_1^{(n)}|$. Similar to (3.15), we multiply (3.3) by $\cos^{2k-1}\theta$ $(1 \leqslant k \leqslant m)$ rather than $\cos^{2k}\theta$ $(1 \leqslant k \leqslant m)$, to derive

$$\begin{aligned}&\rho^2 \mu_{2k+1}^{(n)} = (2k-1)(2k-2)\mu_{2k-3}^{(n)} + \left(\lambda_n^c - (2k-1)^2\right)\mu_{2k-1}^{(n)}, \quad 2 \leqslant k \leqslant m-1,\\ &\rho^2 \mu_3^{(n)} = \left(\lambda_n^c - 1\right)\mu_1^{(n)}.\end{aligned} \tag{3.20}$$

Under (3.10), the derivation of the upper bound for $|A_1^n|$ is essentially the same as above.

To estimate $|B_1^{(n)}|$ and $|B_2^{(n)}|$, we redefine

$$\mu_k^{(n)} = -\int_{-\pi}^{\pi} \partial_\theta\left(\cos^k\theta\right) \mathrm{se}_n(\theta;\rho)\, d\theta = k\int_{-\pi}^{\pi} \cos^{k-1}\theta \sin\theta\, \mathrm{se}_n(\theta;\rho)\, d\theta.$$

Replacing the testing functions $\cos^{2k}\theta$ and $\cos^{2k-1}\theta$ in the derivation of (3.15) and (3.20) by $\partial_\theta(\cos^{2k}\theta)$ and $\partial_\theta(\cos^{2k-1}\theta)$, respectively, we derive the recurrence formulas:

$$\begin{aligned}&\rho^2 \mu_{2k+2}^{(n)} = \frac{k+1}{k}\left\{2k(2k-1)\mu_{2k-2}^{(n)} + \left(\lambda_n^s - 4k^2\right)\mu_{2k}^{(n)}\right\}, \quad 2 \leqslant k \leqslant m-1,\\ &\rho^2 \mu_4^{(n)} = 2\left(\lambda_n^c - 4\right)\mu_2^{(n)},\end{aligned}$$

and

$$\begin{aligned}&\rho^2 \mu_{2k+1}^{(n)} = \frac{2k+1}{2k-1}\left\{(2k-1)(2k-2)\mu_{2k-3}^{(n)} + \left(\lambda_n^s - (2k-1)^2\right)\mu_{2k-1}^{(n)}\right\}, \quad 2 \leqslant k \leqslant m-1,\\ &\rho^2 \mu_3^{(n)} = 3\left(\lambda_n^s - 1\right)\mu_1^{(n)}.\end{aligned}$$

The rest of the analysis is similar to that for $|A_0^{(n)}|$. $\square$

It is essential to prove the following property of the expansion coefficients $A_k^{(n)}$ and $B_k^{(n)}$ in (3.7).

Lemma 3.2. *Let $\rho > 0$, and let m be a positive integer such that*

$$4 < \lambda_n^c - \frac{3\rho^2}{4} \quad \text{and} \quad m^2 < \lambda_n^c - \frac{\rho^2}{2}, \tag{3.21}$$

then for all $0 \leqslant k \leqslant m$,

$$\left|A_k^{(n)}\right| \leqslant s_n^{[k/2]} \max\left\{\left|A_0^{(n)}\right|, \left|A_1^{(n)}\right|\right\}, \tag{3.22}$$

where

$$s_n = 2d_n^2 + \sqrt{1 + 4d_n^4} \quad \text{with } d_n = \frac{\sqrt{\lambda_n^c - \rho^2/2}}{\rho}. \tag{3.23}$$

Under the same conditions, the above estimate holds for $B_k^{(n)}$ for all $1 \leqslant k \leqslant m$ with $B_1^{(n)}$, $B_2^{(n)}$ and λ_n^s in place of $A_0^{(n)}$, $A_1^{(n)}$ and λ_n^c, but d_n being replaced by $\sqrt{\lambda_n^s - \rho^2/4}/\rho$.

Proof. We first prove (3.22) with even k by induction. It is trivial for $k = 0$. For $k = 2$, we obtain from (3.8) and (3.21) that

$$\left|A_2^{(n)}\right| = \frac{2\lambda_n^c - \rho^2}{\rho^2}\left|A_0^{(n)}\right| = 2d_n^2\left|A_0^{(n)}\right| \leqslant s_n\left|A_0^{(n)}\right|,$$

which verifies (3.22) with $k = 2$. Assuming that the upper bound holds for even $k, k-2 (k \geqslant 2)$, we derive again from (3.8) and (3.21) that

$$\begin{aligned}\left|A_{k+2}^{(n)}\right| &\leqslant \frac{4\lambda_n^c - 2\rho^2 - 4k^2}{\rho^2}\left|A_k^{(n)}\right| + \left|A_{k-2}^{(n)}\right| \leqslant 4d_n^2\left|A_k^{(n)}\right| + \left|A_{k-2}^{(n)}\right| \\ &\leqslant s_n^{1+k/2}\left(4d_n^2 s_n^{-1} + s_n^{-2}\right)\left|A_0^{(n)}\right| = s_n^{1+k/2}\left|A_0^{(n)}\right|.\end{aligned} \tag{3.24}$$

Notice that the identify is due to that s_n is the positive root of the quadratic equation: $x^2 - 4d_n^2 x - 1 = 0$.

For odd k, we find from (3.8) and (3.21) with $k = 3$ that

$$\left|A_3^{(n)}\right| = \frac{4\lambda_n^c - 3\rho^2 - 4}{\rho^2}\left|A_1^{(n)}\right| \leqslant \frac{4\lambda_n^c - 2\rho^2}{\rho^2}\left|A_1^{(n)}\right| \leqslant s_n\left|A_1^{(n)}\right|.$$

The rest of the induction remains the same as the previous case.

We now turn to dealing with $B_k^{(n)}$. Observe from (3.8) and (3.9) that $A_k^{(n)}$ and $B_k^{(n)}$ share the same recurrence formula when $k > 1$, so it suffices to check the induction base. By (3.9),

$$\left|B_3^{(n)}\right| = \frac{4\lambda_n^s - \rho^2 - 4}{\rho^2}\left|B_1^{(n)}\right| \leqslant s_n\left|B_1^{(n)}\right|,$$

where in s_n, $d_n = \sqrt{\lambda_n^s - \rho^2/4}/\rho$. Similar analysis can be applied to $B_2^{(n)}$ and $B_4^{(n)}$. Therefore, the desired estimate follows. □

With the above preparation, we derive the following result on the decay property of the coefficients $A_k^{(n)}$ and $B_k^{(n)}$ in (3.7).

Theorem 3.1. *For any $\rho > 0$, assume that m is a positive integer satisfying (3.10) and $\eta_n = \rho/\sqrt{\lambda_n} < 1$. Then*

$$\left|Z_k^{(n)}\right| \leqslant D m^{-1/4} \eta_n^{2m} \exp\left(\frac{8m^3}{3\lambda_n}\right)\left(\frac{2+\sqrt{5}}{\eta_n^2}\right)^{[k/2]}, \quad 0 \leqslant k \leqslant m,\ n \geqslant 1, \tag{3.25}$$

where λ_n and $Z_k^{(n)}$ are the same as in (3.12), and D is a positive constant independent of m, n and c.

Proof. We only consider the estimate of $A_k^{(n)}$, since the proof for $B_{k+1}^{(n)}$ is similar.

Observe that if m satisfies (3.10) and $\eta_n < 1$, the condition (3.21) is met. Moreover, we have

$$d_n < \frac{1}{\eta_n} \quad \Rightarrow \quad s_n \leqslant \frac{2 + \sqrt{\eta_n^4 + 4}}{\eta_n^2} < \frac{2+\sqrt{5}}{\eta_n^2}.$$

Therefore, using Lemmas 3.1 and 3.2 leads to the desired result. □

Now, we are ready to estimate approximation of periodic functions by truncated Mathieu series. The error essentially depends on the decay property of the expansion coefficients:

$$\hat{v}_n^c := \hat{v}_n^c(\rho) = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\theta)\,\mathrm{ce}_n(\theta;\rho)\,d\theta; \qquad \hat{v}_n^s := \hat{v}_n^s(\rho) = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\theta)\,\mathrm{se}_n(\theta;\rho)\,d\theta. \tag{3.26}$$

For convenience, we decompose v into even and odd parts as

$$v(\theta) = \frac{v(\theta)+v(-\theta)}{2} + \frac{v(\theta)-v(-\theta)}{2} := v^e(\theta) + v^o(\theta),$$

so we have

$$\hat{v}_n^c := \hat{v}_n^c(\rho) = \frac{2}{\pi}\int_0^{\pi} v^e(\theta)\,\mathrm{ce}_n(\theta;\rho)\,d\theta; \qquad \hat{v}_n^s := \hat{v}_n^s(\rho) = \frac{2}{\pi}\int_0^{\pi} v^o(\theta)\,\mathrm{se}_n(\theta;\rho)\,d\theta. \tag{3.27}$$

For real $r > 0$, we use $H_p^r(-\pi,\pi)$ to denote the 2π-periodic Sobolev space with the semi-norm denoted by $|\cdot|_{H_p^r(-\pi,\pi)}$ as usual (cf. [4]).

Theorem 3.2. *Let $\rho > 0$, $\eta_N = \rho/\sqrt{\lambda_N}$ and $0 < \eta_* < 1$ be a given constant. For any $v \in H_p^r(-\pi,\pi)$ with $r \geqslant 0$,*

- *if $\eta_N \leqslant \eta_*$, then*

$$|\hat{z}_N| \leqslant D\big(N^{-2r/3}|v|_{H_p^r(-\pi,\pi)} + N^{1/6}(\eta_*)^{\sigma N^{2/3}}\|v\|_{L^2(-\pi,\pi)}\big); \tag{3.28}$$

- *if $\eta_N \leqslant \eta_*/\sqrt[6]{2} \approx 0.8909\eta_*$, then*

$$|\hat{z}_N| \leqslant D\big(N^{-r}|v|_{H_p^r(-\pi,\pi)} + N^{1/4}(\eta_*)^{\sigma N}\|v\|_{L^2(-\pi,\pi)}\big), \tag{3.29}$$

where

$$\{\lambda_N, \hat{z}_N\} = \begin{cases} \{\lambda_N^c, \hat{v}_N^c\}, & \text{for } \mathrm{ce}_N, \\ \{\lambda_N^s, \hat{v}_N^s\}, & \text{for } \mathrm{se}_N, \end{cases} \tag{3.30}$$

and D and σ are positive generic constants independent of N, v and ρ.

Proof. It suffices to prove the estimate of v_N^c, since the same argument can be applied to estimate v_N^s. Given a cut-off M, denote

$$v_M^e(\theta) = \frac{\nu_0}{2} + \sum_{k=1}^{M} \nu_k \cos k\theta \quad \text{where } \nu_k = \frac{2}{\pi}\int_0^{\pi} v^e(\theta)\cos k\theta\,d\theta,$$

and rewrite $\hat{v}_N^c$ as

$$\hat{v}_N^c = \frac{2}{\pi}\int_0^{\pi}\big(v^e(\theta) - v_M^e(\theta)\big)\,\mathrm{ce}_N(\theta;\rho)\,d\theta + \frac{2}{\pi}\int_0^{\pi} v_M^e(\theta)\mathrm{ce}_N(\theta;\rho)\,d\theta. \tag{3.31}$$

Using the standard Fourier approximations (see, e.g., [4]) leads to

$$\begin{aligned}\left|\int_0^{\pi}\big(v^e - v_M^e\big)\,\mathrm{ce}_N\,d\theta\right| &\leqslant D\big\|v^e - v_M^e\big\|_{L^2(0,\pi)}\|\mathrm{ce}_N\|_{L^2(0,\pi)} \\ &\leqslant DM^{-r}\big|v^e\big|_{H_p^r(0,\pi)} \leqslant DM^{-r}|v|_{H_p^r(-\pi,\pi)}.\end{aligned} \tag{3.32}$$

Table 3.1
Samples of ι and η_N.

$\iota = 0.4$		$\iota = 0.5$		$\kappa = 0.6$		$\iota = 0.8$		$\iota = 1.0$	
$\underline{\iota}$	η_N	$\underline{\iota}$	η_N	$\underline{\iota}$	η_N	$\underline{\iota}$	η_N	$\underline{\iota}$	η_N
0.371	0.385	0.447	0.471	0.515	0.551	0.625	0.693	0.707	0.984

$\iota = 1.1$		$\iota = 1.1832$		$\iota = 1.2$		$\iota = 1.4$		$\iota = \pi/2$	
$\underline{\iota}$	η_N	$\underline{\iota}$	η_N	$\underline{\iota}$	η_N	$\underline{\iota}$	η_N	$\underline{\iota}$	η_N
0.740	0.856	0.764	0.8909	0.768	0.897	0.814	0.963	0.844	0.999

Now, we deal with the second term at the right-hand side of (3.31). We derive from (3.7) and Theorem 3.1 that

$$\begin{aligned}\frac{2}{\pi}\left|\int_0^{\pi} v_M^e \,\mathrm{ce}_N \,d\theta\right| &\leqslant \sum_{k=0}^{M}\left|\nu_k A_k^{(N)}\right| \leqslant \left(\sum_{k=0}^{M}(\nu_k)^2\right)^{\frac{1}{2}}\left(\sum_{k=0}^{M}\left(A_k^{(N)}\right)^2\right)^{\frac{1}{2}}\\ &\leqslant D m^{-1/4}\eta_N^{2m}\exp\left(\frac{8m^3}{3\lambda_N^c}\right)\left(\sum_{k=0}^{M}\left(\frac{2+\sqrt{5}}{\eta_N^2}\right)^{2[k/2]}\right)^{1/2}\left\|v^e\right\|_{L^2(0,\pi)}\\ &\leqslant D\left(\frac{M}{m}\right)^{1/4} M^{1/4}\left(\frac{\sqrt{2+\sqrt{5}}}{\eta_N}\right)^{M}\eta_N^{2m}\exp\left(\frac{8m^3}{3\lambda_N^c}\right)\|v\|_{L^2(-\pi,\pi)}.\end{aligned} \tag{3.33}$$

We choose m so that the exponential factor is uniformly bounded, that is,

$$m = O\left(\left(\lambda_N^c\right)^{1/3}\right) \overset{(3.6)}{=} O\left(N^{2/3}\right),$$

which verifies the condition (3.10). Consequently,

$$\begin{aligned}\left|\int_0^{\pi} v_M^e \,\mathrm{ce}_N \,d\theta\right| &\leqslant D N^{1/6}\left\{\eta_N\left(\frac{\sqrt{2+\sqrt{5}}}{\eta_N}\right)^{\frac{M}{2m}}\right\}^{2m}\|v\|_{L^2(-\pi,\pi)}\\ &= D N^{1/6}\eta_N^{2(1-\gamma)m}\|v\|_{L^2(-\pi,\pi)},\end{aligned} \tag{3.34}$$

where the constant $0<\gamma<1$. The existence of γ can be proved in the same way as for (2.20). Therefore, we take $M = O(m) = O(N^{2/3})$ and (3.28) follows.

Now we turn to the proof of (3.29). We obtain from (3.33) that

$$\left|\int_0^{\pi} v_M^e \,\mathrm{ce}_N \,d\theta\right| \leqslant D\left(\frac{M}{m}\right)^{1/4} M^{1/4}\left(p_N\left(\frac{\sqrt{2+\sqrt{5}}}{\eta_N}\right)^{\frac{M}{2m}}\right)^{2m}\|v\|_{L^2(-\pi,\pi)}, \tag{3.35}$$

where $p_N = \eta_N \exp(\frac{4m^2}{3\lambda_N^c})$. Following the same lines as for the derivation of (2.25), we can show that if $\eta_N \leqslant \eta_*/\sqrt[6]{2}$, we can choose $m = O(N)$ and $M = O(N)$ such that there exists a constant $0<\bar{\gamma}<1$ such that

$$\left|\int_0^{2\pi} v_M^e \,\mathrm{ce}_N \,d\theta\right| \leqslant D N^{1/4} p_N^{2(1-\bar{\gamma})}\|v\|_{L^2(-\pi,\pi)} \leqslant D N^{1/4}(\eta_*)^{\delta N}\|v\|_{L^2(-\pi,\pi)}. \tag{3.36}$$

Thus, a combination of (3.31), (3.32), and (3.36) leads to (3.29). □

Remark 3.1. Like in Remark 2.1, it is more desirable to quantify the condition of $\eta_N = \rho/\sqrt{\lambda_N}$ by using $\iota = \rho/N$. In view of (3.6), we have

$$1 < \frac{\iota^2}{\eta_N^2} < 1+\iota^2 \quad\Rightarrow\quad \underline{\iota} := \frac{\iota}{\sqrt{1+\iota^2}} < \eta_N < \iota.$$

However, it is difficult to find the explicit relation between η_N and ι. We just provide some samples computed with $N = 128$. (See Table 3.1.)

Indeed, we find that η_N lies in the interval $(\underline{\iota}, \iota)$, and to meet the condition for (3.29), we may choose $\rho = \iota N$ roughly with $\iota < 1.1832$. □

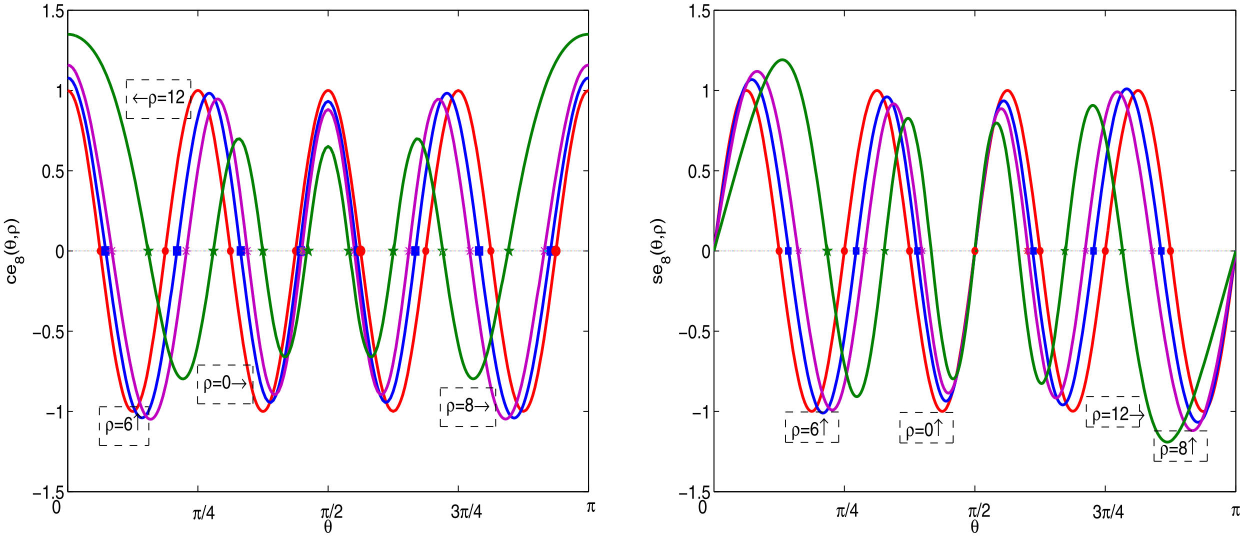

Fig. 3.1. Graphs of $\mathrm{ce}_8(\theta;\rho)$ (left) and $\mathrm{se}_8(\theta;\rho)$ (right) with $\theta \in [0,\pi]$ and $\rho = 0, 6, 8, 12$.

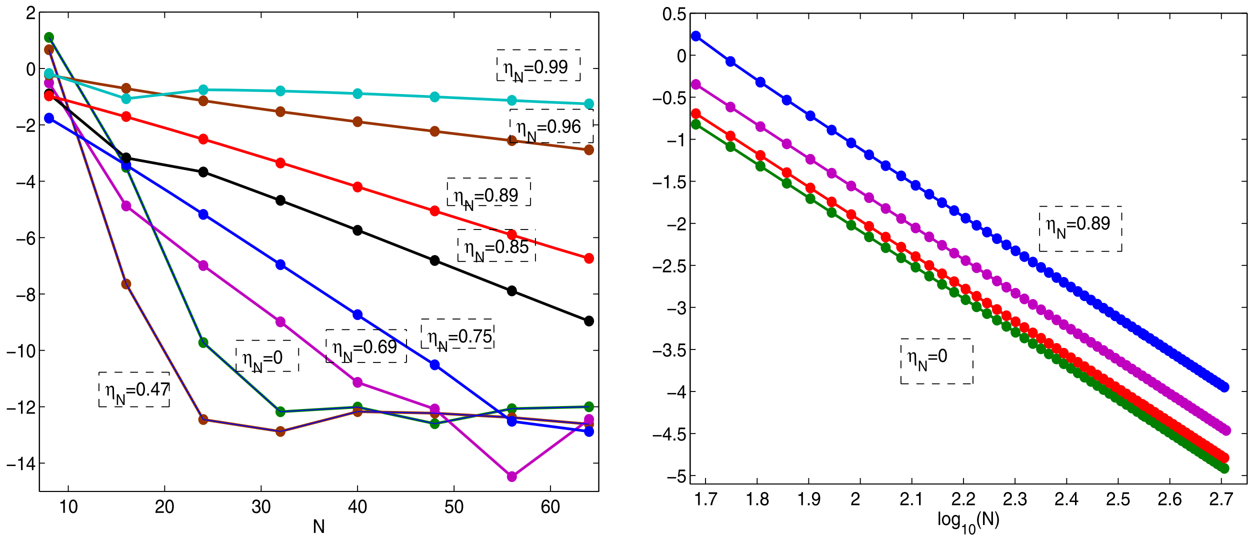

Fig. 3.2. Left: $\log_{10}(|\hat{z}_N|)$ against $N \in [8, 64]$ with $\rho = \iota N$ (or $\rho = \eta_N\sqrt{\lambda_N}$), where $\iota = 0, 0.5, 0.8, 0.9, 1.1832, 1.2, 1.4, 1.5$, and correspondingly, $\eta_N = 0, 0.47, 0.69, 0.75, 0.85, 0.89, 0.96, 0.99$. Right: $\log_{10}(|\hat{v}_N^c|)$ against $\log_{10} N$ with $N \in [48, 512]$. Here, $\iota = 0, 0.5, 0.9, 1.1832$ and $\eta_N = 0, 0.47, 0.75, 0.89$.

Finally, we present some numerical results to support the estimates in Theorem 3.2. First, we plot in Fig. 3.1 the Mathieu functions $\mathrm{ce}_8(\theta;\rho)$ (left) and $\mathrm{se}_8(\theta;\rho)$ (right) with $\theta \in [0,\pi]$ and $\rho = 0, 6, 8, 12$.

Compared with their trigonometric counterparts $\cos(8\theta)$ and $\sin(8\theta)$, the Mathieu functions oscillate non-uniformly and their zeros move towards $\theta = \pi/2$ (symmetric about $\theta = \pi/2$) as ρ increases. It is anticipated that the use of Mathieu functions with a suitable tuning parameter ρ may lead to a better approximation than the Fourier basis. This situation is reminiscent to approximation by PSWFs and Legendre polynomials (cf. Section 2).

Next, we consider two typical examples. In the first example, we test the analytic periodic function: $v(\theta) = \exp(7\sin\theta)$. We plot in Fig. 3.2 (left) $\log_{10}(|\hat{z}_N|)$ (with $|\hat{z}_N| = \max\{|\hat{v}_N^c|, |\hat{v}_N^s|\}$) against various N with $\eta_N = 0, 0.47, 0.69, 0.75, 0.85, 0.89, 0.96, 0.99$. Indeed, we find that when η_N meets the condition for (3.29), an exponential decay of $|\hat{z}_N|$ is observed. Moreover, a suitable choice of ρ may lead to a faster decay than the Fourier expansion.

In the second example, we test a function with finite periodicity: $v(\theta) = (\theta^2 - \pi^2)^3 e^\theta$. It is anticipated that the decay rate for $|\hat{v}_N^c|$ (resp. $|\hat{v}_N^s|$) should be $O(N^{-4})$ (resp. $O(N^{-5})$). In Fig. 3.2 (right), we plot $\log_{10}(|\hat{v}_N^c|)$ against $\log_{10} N$ and find that the slopes are about -3.99, as expected.

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