



# Fermionic approach to soliton equations

Metin Ünal

Department of Mathematics, University of Uşak, Uşak, Turkey

## ARTICLE INFO

### Article history:

Received 3 September 2010  
Available online 9 March 2011  
Submitted by Goong Chen

### Keywords:

Soliton equations  
Fermion particles

## ABSTRACT

In this paper we exploit the algebraic structure of the soliton equations and find solutions in terms of fermion particles. We show how determinants arise naturally in the fermionic approach to soliton equations. We write the  $\tau$ -function for charged free fermions in terms of determinants. Examples of how to get soliton, rational and dromion solutions from  $\tau$ -functions for the various soliton equations are given.

© 2011 Elsevier Inc. All rights reserved.

## 1. Introduction

In recent years solutions to soliton equations in soliton theory have been given in many ways such as by means of Grammians, Wronskians, Pfaffians, etc. The diversity of expressing solutions reflects the richness of algebraic structures which the soliton equations possess in common. It is Sato [1] that unveiled the structures by means of the method of algebraic analysis in the study of the Kadomtsev–Petviashvili (KP) hierarchy [2,3]. Among the variety of soliton equations, the KP hierarchy is the most widely studied one.

Here we study the algebraic structure of soliton equations and find solutions in terms of fermion particles [4]. These particles can either be charged or neutral [5], and they can have one component structure or they can have more than one, depending on the structure of the equation. In this paper we are interested in charged fermions. An example of their fermionic structure is shown in the table below for KP and DS equations.

Fermions	1 component	2 component
charged ( $\psi_l$ )	KP	DS

We write the  $\tau$ -function [6] for charged free fermions in terms of determinants in the following form

$$\tau_\psi = \det(A) \det(A^{-1} + V),$$

where  $A$  is a constant matrix and  $V$  is a matrix with the entries of charged fermions. These are explained in detail in the later sections.

This paper is organized as follows. In Section 2, we introduce some properties of pfaffians. In Section 3, we recall some results from [7] and apply *Wick's theorem* to compute the expectation values of fermions. We give time evolution to the fermions via Hamiltonian in order to use fermion particles with time variable. Next we introduce a polynomial  $\tau(\underline{x}, g)$  function in (6), where  $\underline{x}$  is time variable and  $g$  represents fermions. As a new result the  $\tau(\underline{x}, g)$  function in terms of Schur function give rational solutions of the KP equation. In Section 4 we derive similar results to Section 3 for the free fermions with 2 components. In Section 5, we derive a new general formulae for charged fermions, from which the rational solutions

E-mail address: drmetintr@yahoo.co.uk.

and soliton solutions for the KP hierarchy can be obtained. Examples of how to get the soliton and dromion solutions to various soliton equations, from  $\tau$ -functions are also given.

## 2. Pfaffians

Let

$$A = (a_{ij})$$

be a  $n \times n$  skew-symmetric matrix (i.e.  $a_{ij} = -a_{ji}$  and consequently  $a_{ii} = 0$  for  $i, j = 1, 2, \dots, n$ ). It is known that if  $n$  is odd, then  $\det(A)$  is zero, but if  $n$  is even  $\det(A)$  is a perfect square of a polynomial in the entries  $a_{ij}$ , called the *pfaffian* of  $A$  and denoted by  $\text{Pf}(A)$ . Roughly speaking, a pfaffian is the square root of the determinant of a skew-symmetric matrix. To be precise, for even  $n$

$$\text{Pf}(A) = \sum_{\sigma} \epsilon(\sigma) a_{\sigma(1), \sigma(2)} \cdots a_{\sigma(n-1), \sigma(n)},$$

where  $\sigma$  runs over the permutations of  $\{1, \dots, n\}$  such that

$$\begin{aligned} \sigma(1) < \sigma(2), \quad \sigma(3) < \sigma(4), \quad \dots, \quad \sigma(n-1) < \sigma(n), \\ \sigma(1) < \sigma(3) < \dots < \sigma(n-1), \end{aligned}$$

and  $\epsilon(\sigma)$  ( $= \pm 1$ ) is the parity of this permutation.

A classical notation for the pfaffian of  $A$  [4] is

$$\text{Pf}(A) = (1, 2, \dots, n),$$

where  $(i, j) = a_{ij}$ . One expansion rule for pfaffians is given by

$$(1, 2, \dots, n) = \sum_{i=2}^n (-1)^i (1, i)(2, 3, \dots, \hat{i}, \dots, n),$$

where  $\hat{i}$  indicates that the index underneath should be deleted. For example for  $n = 4$  we can write the pfaffian representation as

$$(1, 2, 3, 4) = (1, 2)(3, 4) - (1, 3)(2, 4) + (1, 4)(2, 3).$$

(See [8] for more information on pfaffians.)

## 3. Charged free fermions

We recall some results from [7]. Let  $\mathbf{A}$  be an associative algebra over  $\mathbb{C}$  with generators  $\psi_i, \psi_i^*$  ( $i \in \mathbb{Z}$ ), satisfying the anti-commutator relations

$$[\psi_i, \psi_j]_+ = 0, \quad [\psi_i, \psi_j^*]_+ = \delta_{ij}, \quad [\psi_i^*, \psi_j^*]_+ = 0, \quad (1)$$

where  $[X, Y]_+ = XY + YX$ . The generators  $\psi_i, \psi_i^*$  ( $i \in \mathbb{Z}$ ) will be referred to as *free fermions*.

Here  $\langle \rangle$  denotes a linear form on  $\mathbf{A}$ , called the (*vacuum*) *expectation value*, defined as follows. For  $a \in \mathbb{C}$  or quadratic in free fermions

$$\begin{aligned} \langle a \rangle &= a, & \langle \psi_i \rangle &= \langle \psi_i^* \rangle = 0 \quad (i \in \mathbb{Z}), \\ \langle \psi_i \psi_j \rangle &= 0, & \langle \psi_i^* \psi_j^* \rangle &= 0 \quad (i, j \in \mathbb{Z}), \\ \langle \psi_i \psi_j^* \rangle &= \begin{cases} \delta_{i,j} & (i < 0), \\ 0 & (\text{otherwise}), \end{cases} & \langle \psi_j^* \psi_i \rangle &= \begin{cases} \delta_{i,j} & (i \geq 0), \\ 0 & (\text{otherwise}). \end{cases} \end{aligned} \quad (2)$$

For a general product  $w_1 \cdots w_r$  of free fermions  $w_i$ , we apply *Wick's theorem* to compute the expectation values

$$\langle w_1 \cdots w_r \rangle = \begin{cases} 0 & (r \text{ odd}), \\ \sum_{\sigma} \text{sgn } \sigma \langle w_{\sigma(1)} w_{\sigma(2)} \rangle \cdots \langle w_{\sigma(r-1)} w_{\sigma(r)} \rangle & (r \text{ even}), \end{cases}$$

where  $\sigma$  runs over the permutations such that  $\sigma(1) < \sigma(2), \dots, \sigma(r-1) < \sigma(r)$  and  $\sigma(1) < \sigma(3), \dots, \sigma(r-1)$ . We see that this theorem gives the expectation value of the general product of free fermions  $w_1 \cdots w_r$  in terms of a pfaffian [8]. Therefore, Wick's theorem can be expressed in terms of pfaffians in the following way

$$\langle w_1 \cdots w_r \rangle = \begin{cases} 0 & (r \text{ odd}), \\ \text{Pf}(\langle w_i w_j \rangle) & (r \text{ even}). \end{cases}$$

**Theorem 3.1.** From Wick's theorem we write the following expectation value

$$\langle \psi_{i_1} \cdots \psi_{i_r} \psi_{j_1}^* \cdots \psi_{j_s}^* \rangle = \begin{cases} (-1)^{\frac{1}{2}r(r+1)} \det(\langle \psi_{i_p} \psi_{j_q}^* \rangle), & r = s, \\ 0, & r \neq s, \end{cases}$$

where  $p = 1, \dots, r$  and  $q = 1, \dots, s$ .

**Proof.**

$$(\langle \psi_{i_1} \cdots \psi_{i_r} \psi_{j_1}^* \cdots \psi_{j_s}^* \rangle)^2 = \det \begin{pmatrix} \langle \psi_{i_k} \psi_{i_l} \rangle & \langle \psi_{i_k} \psi_{j_n}^* \rangle \\ -\langle \psi_{i_l} \psi_{j_m}^* \rangle & \langle \psi_{j_m}^* \psi_{j_n}^* \rangle \end{pmatrix},$$

where  $k, l = 1, \dots, r$  and  $m, n = 1, \dots, s$ . Using the definition of the expectation value in (2)

$$(\langle \psi_{i_1} \cdots \psi_{i_r} \psi_{j_1}^* \cdots \psi_{j_s}^* \rangle)^2 = \det \begin{pmatrix} 0 & \langle \psi_{i_k} \psi_{j_n}^* \rangle \\ -\langle \psi_{i_l}^* \psi_{j_m} \rangle & 0 \end{pmatrix} = \begin{cases} \det(\langle \psi_{i_p} \psi_{j_q}^* \rangle)^2, & r = s, \\ 0, & r \neq s, \end{cases}$$

hence

$$\langle \psi_{i_1} \cdots \psi_{i_r} \psi_{j_1}^* \cdots \psi_{j_s}^* \rangle = \begin{cases} \pm \det(\langle \psi_{i_p} \psi_{j_q}^* \rangle), & r = s, \\ 0, & r \neq s. \end{cases}$$

Looking at the term  $\langle \psi_{i_1} \psi_{j_1}^* \rangle \cdots \langle \psi_{i_r} \psi_{j_s}^* \rangle$  on each side we see that the sign is  $(-1)^{\frac{1}{2}r(r+1)}$ .  $\square$

It is convenient to use the generating functions for free fermions  $\psi_i, \psi_j^*$ , defined as follows

$$\psi(p) := \sum_{i \in \mathbb{Z}} \psi_i p^i, \quad \psi^*(q) := \sum_{j \in \mathbb{Z}} \psi_j^* q^{-j}. \quad (3)$$

**Theorem 3.2.** The expectation values to the generating functions  $\psi(p), \psi^*(q)$  are given by

$$\langle \psi(p_i) \psi^*(q_j) \rangle = \frac{q_j}{p_i - q_j}, \quad \langle \psi^*(q_j) \psi(p_i) \rangle = -\frac{q_j}{p_i - q_j}.$$

**Proof.** From the definition (3)

$$\langle \psi(p_i) \psi^*(q_j) \rangle = \sum_{m, n \in \mathbb{Z}} \langle \psi_m \psi_n^* \rangle p_i^m q_j^{-n} = \sum_{m, n < 0} \delta_{m, n} p_i^m q_j^{-n} = \sum_{m=1}^{\infty} \left( \frac{q_j}{p_i} \right)^m = \frac{q_j}{p_i - q_j}$$

and similarly

$$\langle \psi^*(q_j) \psi(p_i) \rangle = \sum_{m, n \in \mathbb{Z}} \langle \psi_n^* \psi_m \rangle p_i^m q_j^{-n} = \sum_{m, n \geq 0} \delta_{m, n} p_i^m q_j^{-n} = \sum_{m=0}^{\infty} \left( \frac{p_i}{q_j} \right)^m = -\frac{q_j}{p_i - q_j}. \quad \square$$

The time evolution for the free fermions are given by the following Hamiltonian

$$H(\underline{x}) = \sum_{n \geq 1} \left( x_n \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+n}^* \right),$$

where  $\underline{x}$  is the time variable.

Note first of all that

$$[H(\underline{x}), \psi(p)] = \sum_{n \geq 1, k \in \mathbb{Z}} x_n \left[ \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+n}^*, \psi_k \right] p^k = \sum_{n \geq 1, k \in \mathbb{Z}} x_n p^k \left( \sum_{i \in \mathbb{Z}} (\psi_i \psi_{i+n}^* \psi_k - \psi_k \psi_i \psi_{i+n}^*) \right).$$

Then using (1) we have

$$\begin{aligned} [H(\underline{x}), \psi(p)] &= \sum_{n \geq 1, k \in \mathbb{Z}} x_n p^k \left( \sum_{i \in \mathbb{Z}} \delta_{k, i+n} \psi_i \right) \\ &= \sum_{n \geq 1, k \in \mathbb{Z}} x_n p^k \psi_{k-n} = \sum_{n \geq 1} x_n p^n \psi(p) \\ &= \xi(\underline{x}, p) \psi(p), \end{aligned}$$

where  $\xi(\underline{x}, p) := \sum_{n \geq 1} x_n p^n$ . Then

$$\psi(p, \underline{x}) = e^{H(\underline{x})} \psi(p) e^{-H(\underline{x})} = e^{(\text{ad } H(\underline{x}))} \psi(p) = \left( 1 + \text{ad } H(\underline{x}) + \frac{1}{2} (\text{ad } H(\underline{x}))^2 + \dots \right) \psi(p),$$

where  $(\text{ad } H(\underline{x}))X = [H(\underline{x}), X]$  and so  $(\text{ad } H(\underline{x}))^j \psi(p) = (\xi(\underline{x}, p))^j \psi(p)$ , hence

$$\psi(p, \underline{x}) = e^{\xi(\underline{x}, p)} \psi(p). \quad (4)$$

Similarly,  $[H(\underline{x}), \psi^*(q)] = -\xi(\underline{x}, q) \psi^*(q)$  and so

$$\psi^*(q, \underline{x}) = e^{H(\underline{x})} \psi^*(q) e^{-H(\underline{x})} = e^{-\xi(\underline{x}, q)} \psi^*(q). \quad (5)$$

Later we will use  $e^{\xi(\underline{x}, p)}$  and  $e^{-\xi(\underline{x}, q)}$  to generate polynomials.

Next we call a polynomial  $\tau(x)$  a  $\tau$ -function if it is representable in the following form for some  $g$ :

$$\tau_l(\underline{x}, g) = \langle l | g(\underline{x}) | l \rangle := \langle \Psi_l^* g(\underline{x}) \Psi_l \rangle \quad (6)$$

for each  $l \in \mathbb{Z}$ , where

$$\Psi_i^* = \begin{cases} \psi_{-1} \cdots \psi_i, & i < 0, \\ 1, & i = 0, \\ \psi_0^* \cdots \psi_{i-1}^*, & i > 0, \end{cases} \quad \Psi_i = \begin{cases} \psi_i^* \cdots \psi_{-1}^*, & i < 0, \\ 1, & i = 0, \\ \psi_{i-1} \cdots \psi_0, & i > 0. \end{cases}$$

For example, we take

$$g = \psi(p_1) \cdots \psi(p_r) \psi^*(q_1) \cdots \psi^*(q_s) \quad (7)$$

for some  $r, s$ . Then for  $r = s$ ,

$$\tau_0(\underline{x}, g) = \langle 0 | g(\underline{x}) | 0 \rangle = e^{\sum_{i=1}^r \xi(\underline{x}, p_i) - \xi(\underline{x}, q_i)} \det(\langle \psi(p_i) \psi^*(q_j) \rangle)$$

and then, using Theorem 3.2, we get

$$\tau_0(\underline{x}, g) = \det \left( e^{\xi(\underline{x}, p_i) - \xi(\underline{x}, q_j)} \frac{q_j}{p_i - q_j} \right). \quad (8)$$

Next we wish to express the  $\tau$ -function  $\tau_1$ . From (6)

$$\begin{aligned} \tau_1(\underline{x}, g) &= \langle 1 | g(\underline{x}) | 1 \rangle = \langle \Psi_1^* g(\underline{x}) \Psi_1 \rangle = \langle \psi_0^* g(\underline{x}) \psi_0 \rangle \\ &= e^{\sum_{i=1}^r \xi(\underline{x}, p_i) - \xi(\underline{x}, q_i)} \langle \psi_0^* \psi(p_1) \cdots \psi(p_r) \psi^*(q_1) \cdots \psi^*(q_r) \psi_0 \rangle \\ &= e^{\sum_{i=1}^r \xi(\underline{x}, p_i) - \xi(\underline{x}, q_i)} \begin{vmatrix} -\langle \psi^*(q_1) \psi_0 \rangle & \cdots & -\langle \psi^*(q_r) \psi_0 \rangle & -\langle \psi_0^* \psi_0 \rangle \\ \langle \psi(p_1) \psi^*(q_1) \rangle & \cdots & \langle \psi(p_1) \psi^*(q_r) \rangle & -\langle \psi_0^* \psi(p_1) \rangle \\ \vdots & \ddots & \vdots & \vdots \\ \langle \psi(p_r) \psi^*(q_1) \rangle & \cdots & \langle \psi(p_r) \psi^*(q_r) \rangle & -\langle \psi_0^* \psi(p_r) \rangle \end{vmatrix} \end{aligned}$$

and using Theorem 3.2, we get

$$\tau_1(\underline{x}, g) = e^{\sum_{i=1}^r \xi(\underline{x}, p_i) - \xi(\underline{x}, q_i)} \begin{vmatrix} -1 & \cdots & -1 & -1 \\ \frac{q_1}{p_1 - q_1} & \cdots & \frac{q_r}{p_1 - q_r} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ \frac{q_1}{p_r - q_1} & \cdots & \frac{q_r}{p_r - q_r} & -1 \end{vmatrix} \quad (9)$$

$$= (-1)^r \det \left( e^{\xi(\underline{x}, p_i) - \xi(\underline{x}, q_j)} \frac{p_i}{p_i - q_j} \right). \quad (10)$$

**Theorem 3.3.** For all  $g \in A$ ,

$$\tau_l(\underline{x}, g) = \tau_{l-m}(\underline{x}, \iota_m(g(\underline{x}))), \quad (11)$$

where  $\iota_l(\psi_i) = \psi_{i-l}$  and  $\iota_l(\psi_i^*) = \psi_{i-l}^*$ .

**Proof.** In order to proof the theorem, we first need to show

$$\tau_1(\underline{x}, g) = \tau_0(\underline{x}, \iota_1(g)), \quad (12)$$

where  $g$  is given by (7). Using the definition of fermions from (3)

$$\iota_1(\psi(p)) = \sum_{i \in \mathbb{Z}} \iota_1(\psi_i) p^i = \sum_{i \in \mathbb{Z}} \psi_{i-1} p^i = p^1 \psi(p)$$

and

$$\iota_1(\psi^*(q)) = \sum_{j \in \mathbb{Z}} \iota_1(\psi_j^*) q^{-j} = \sum_{j \in \mathbb{Z}} \psi_{j-1}^* q^{-j} = q^{-1} \psi^*(q),$$

and hence

$$\iota_1(g) = \frac{\prod_{i=1}^r p_i}{\prod_{i=1}^s q_i} g.$$

Using these results we can show the equality in (12):

$$\begin{aligned} \tau_0(\underline{x}, \iota_1(g)) &= \frac{\prod_{i=1}^r p_i}{\prod_{i=1}^s q_i} \tau_0(\underline{x}, g) \\ &= \frac{\prod_{i=1}^r p_i}{\prod_{i=1}^s q_i} \det \left( e^{\xi(\underline{x}, p_i) - \xi(\underline{x}, q_j)} \frac{q_j}{p_i - q_j} \right) \\ &= \det \left( e^{\xi(\underline{x}, p_i) - \xi(\underline{x}, q_j)} \frac{p_i}{p_i - q_j} \right) \\ &= \tau_1(\underline{x}, g). \end{aligned}$$

Then, clearly

$$\begin{aligned} \tau_l(\underline{x}, g) &= \tau_{l-1}(\underline{x}, \iota_1(g)) = \tau_{l-2}(\underline{x}, \iota_1(\iota_1(g))) = \tau_{l-2}(\underline{x}, \iota_2(g)) \cdots \\ &= \tau_{l-l}(\underline{x}, \iota_l(g)) = \tau_0(\underline{x}, \iota_l(g)) \end{aligned}$$

and

$$\tau_l(\underline{x}, g) = \tau_{l-m}(\underline{x}, \iota_m(g)) = \tau_0(\underline{x}, \iota_{l-m}(\iota_m(g))) = \tau_0(\underline{x}, \iota_l(g)).$$

Every product of free fermions is the coefficient of some power of  $p_1, \dots, p_r, q_1, \dots, q_r$  in  $g$  defined in (7). By expanding both sides of (11) with respect to these parameters, the full result follows. Hence the result is proved for  $g$  given by (7).  $\square$

Now we wish to express the  $\tau$ -function  $\tau_0$  in terms of Schur functions [9,10]. In general a Schur function  $S_\lambda$ , where  $\lambda = (\alpha_1, \dots, \alpha_r \mid \beta_1, \dots, \beta_r)$ , is defined by

$$S_\lambda = \det(S_{(\alpha_i \mid \beta_j)}), \quad (13)$$

where

$$S_{(\alpha \mid \beta)} = \sum_{k=0}^{\beta} (-1)^k h_{\alpha+1+k}(x) e_{\beta-k}(x),$$

where  $h_i(x)$  and  $e_j(x)$  are the complete and elementary symmetric functions, respectively.

The element  $g$  as given by (7) can be written as

$$g = \sum_{\substack{i_1, \dots, i_n \in \mathbb{Z} \\ j_1, \dots, j_n \in \mathbb{Z}}} p_1^{i_1} \cdots p_n^{i_n} q_1^{-j_1} \cdots q_n^{-j_n} g',$$

where

$$g' = \psi_{i_1} \cdots \psi_{i_n} \psi_{j_1}^* \cdots \psi_{j_n}^*.$$

Then (8) can be used as a generating function to determine  $\tau_0(\underline{x}; g')$  by looking at the coefficients of  $p_1^{i_1} \cdots p_n^{i_n} q_1^{-j_1} \cdots q_n^{-j_n}$ , where  $i_1 > i_2 > \cdots > i_n, j_1 > j_2 > \cdots > j_n \in \mathbb{Z}$ . Next we expand the entries of the determinant in (8) in the following way:

$$e^{\xi(\underline{x}, p_i)} = \sum_{i=0}^{\infty} h_i(x) p^i, \quad e^{-\xi(\underline{x}, q_j)} = \sum_{j=0}^{\infty} (-1)^j e_j(x) q^j \quad (14)$$

and

$$\frac{q_j}{p_i - q_j} = - \sum_{k=0}^{\infty} \left( \frac{p_i}{q_j} \right)^k.$$

Thus the  $(i, j)$ th entry in (8) can be written as

$$\begin{aligned} e^{\sum_{i=1}^r (\xi(\underline{x}, p_i) - \xi(\underline{x}, q_i))} \frac{q_j}{p_i - q_j} &= \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+1} h_i(x) e_j(x) p^{i+k} q^{j-k} \\ &= \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{k-j+1} h_{i-k}(x) e_{k-j}(x) p^i q^{-j}, \end{aligned} \quad (15)$$

where  $h_n(x) = e_n(x) = 0$  for  $n < 0$  and  $h_0(x) = e_0(x) = 1$ .

**Theorem 3.4.** The coefficients of  $p_1^{i_1} \cdots p_n^{i_n} q_1^{-j_1} \cdots q_n^{-j_n}$  in (15) can be expressed in terms of Schur function in the following form:

$$\sum_{k=0}^{\infty} (-1)^{k-j+1} h_{i-k}(x) e_{k-j}(x) = (-1)^{-j-1} S_{(i|-j-1)} = (-1)^{-j-1} S_{(i+1, 1-j-1)}.$$

Hence (8) can be written as

$$\tau_0(\underline{x}, g) = \det \left( \sum_{i,j=0}^{\infty} (-1)^{-j-1} S_{(i|-j-1)} p^i q^{-j} \right) \quad (16)$$

and

$$\tau_0(\underline{x}, g') = (-1)^{-j_1 - \cdots - j_n - n} \det(S_{(i|-j-1)}) = (-1)^{-j_1 - \cdots - j_n - n} \det(S_{(i+1, 1-j-1)}).$$

Similarly  $\tau_1(\underline{x}, g)$  can be written from (16) by using Theorem 3.3 as

$$\tau_1(\underline{x}, g) = \det \left( \sum_{i,j=0}^{\infty} (-1)^{-j} S_{(i-1|-j)} p^i q^{-j} \right)$$

and

$$\tau_1(\underline{x}, g') = (-1)^{-j_1 - \cdots - j_n} \det(S_{(i-1|-j)}) = (-1)^{-j_1 - \cdots - j_n} \det(S_{(i, 1-j)}).$$

In general

$$\tau_l(\underline{x}, g) = \det \left( \sum_{i,j=0}^{\infty} (-1)^{-j+l-1} S_{(i-l|-j+l-1)} p^i q^{-j} \right)$$

and

$$\tau_l(\underline{x}, g') = (-1)^{-j_1 - \cdots - j_n + nl - n} \det(S_{(i-l|-j+l-1)}) = (-1)^{-j_1 - \cdots - j_n + nl - n} \det(S_{(i-l+1, 1-j+l-1)}).$$

Hence, by (13), each  $\tau_l$  is a Schur function. These give rational solutions of the KP equation, where  $u = 2\partial_x^2(\log \tau)$ .

#### 4. Charged free fermions with 2 components

In this section we consider free fermions with 2 components. Consider free fermions  $\psi_n^{(j)}$ ,  $\psi_n^{(j)*}$  indexed by  $n \in \mathbb{Z}$  and  $j = 1, 2$ , satisfying the anti-commutator relations

$$[\psi_m^{(j)}, \psi_n^{(k)}]_+ = 0, \quad [\psi_m^{(j)*}, \psi_n^{(k)*}]_+ = 0, \quad [\psi_m^{(j)}, \psi_n^{(k)*}]_+ = \delta_{jk} \delta_{mn}, \quad (17)$$

where  $[X, Y]_+ = XY + YX$ . Such fermions are obtainable by renumbering the fermions of a single component. For example, the simplest choice is

$$\begin{aligned} \psi_n^{(1)} &= \psi_{2n}, & \psi_n^{(2)} &= \psi_{2n+1}, \\ \psi_n^{(1)*} &= \psi_{2n}^*, & \psi_n^{(2)*} &= \psi_{2n+1}^*. \end{aligned} \quad (18)$$

Fixing the renumbering (18), we identify the vacuum expectation values for the 2 component fermions with the single component fermions. The time evolution for the 2 component fermions are induced by the following Hamiltonian

$$H(\underline{x}^{(1)}, \underline{x}^{(2)}) = \sum_{\substack{i \geq 1 \\ n \in \mathbb{Z} \\ j=1,2}} x_i^{(j)} \psi_n^{(j)} \psi_{i+n}^{(j)*},$$

where the time variables are  $\underline{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \dots)$  ( $j = 1, 2$ ).

**Theorem 4.1.** *The expectation values of 2 component free fermions are given by*

$$\langle \psi^{(h)}(p_i^{(h)}) \psi^{(h)*}(q_j^{(h)}) \rangle = \frac{q_j^{(h)}}{p_i^{(h)} - q_j^{(h)}}, \quad \langle \psi^{(h)*}(q_j^{(h)}) \psi^{(h)}(p_i^{(h)}) \rangle = -\frac{q_j^{(h)}}{p_i^{(h)} - q_j^{(h)}}$$

and

$$\begin{aligned} \langle \psi^{(h)}(p_i^{(h)}) \psi^{(k)*}(q_j^{(k)}) \rangle &= \langle \psi^{(h)*}(q_j^{(h)}) \psi^{(k)}(p_i^{(k)}) \rangle = 0, \\ \langle \psi^{(h)}(p_i^{(h)}) \psi^{(h)}(p_j^{(h)}) \rangle &= \langle \psi^{(h)*}(q_i^{(h)}) \psi^{(h)*}(q_j^{(h)}) \rangle = 0, \end{aligned}$$

where  $h, k = 1, 2$ .

Note first of all that

$$\begin{aligned} [H(\underline{x}^{(1)}, \underline{x}^{(2)}), \psi^{(j)}(p^{(j)})] &= \sum_{i \geq 1, k \in \mathbb{Z}} x_i^{(j)} \left[ \sum_{n \in \mathbb{Z}} \psi_n^{(j)} \psi_{i+n}^{(j)*}, \psi_k^{(j)} \right] p^{(j)k} \\ &= \sum_{i \geq 1, k \in \mathbb{Z}} x_i^{(j)} p^{(j)k} \left( \sum_{n \in \mathbb{Z}} (\psi_n^{(j)} \psi_{i+n}^{(j)*} \psi_k^{(j)} - \psi_k^{(j)} \psi_n^{(j)} \psi_{i+n}^{(j)*}) \right) \end{aligned}$$

then using (17), we have

$$\begin{aligned} [H(\underline{x}^{(1)}, \underline{x}^{(2)}), \psi^{(j)}(p^{(j)})] &= \sum_{i \geq 1, k \in \mathbb{Z}} x_i^{(j)} p^{(j)k} \left( \sum_{n \in \mathbb{Z}} \delta_{k, i+n} \psi_n^{(j)} \right) \\ &= \sum_{i \geq 1, k \in \mathbb{Z}} x_i^{(j)} p^{(j)k} \psi_{k-i}^{(j)} = \sum_{i \geq 1} x_i^{(j)} p^{(j)i} \psi^{(j)}(p^{(j)}) \\ &= \xi(\underline{x}^{(j)}, p^{(j)}) \psi^{(j)}(p^{(j)}), \end{aligned}$$

where  $\xi(\underline{x}^{(j)}, p^{(j)}) := \sum_{i \geq 1} x_i^{(j)} p^{(j)i}$ . Then the time evolution

$$\begin{aligned} \psi^{(h)}(p^{(h)}, \underline{x}^{(1)}, \underline{x}^{(2)}) &= e^{H(\underline{x}^{(1)}, \underline{x}^{(2)})} \psi^{(j)}(p^{(j)}) e^{-H(\underline{x}^{(1)}, \underline{x}^{(2)})} = e^{(\text{ad } H(\underline{x}^{(1)}, \underline{x}^{(2)}))} \psi^{(j)}(p^{(j)}) \\ &= \left( 1 + \text{ad } H(\underline{x}^{(1)}, \underline{x}^{(2)}) + \frac{1}{2} (\text{ad } H(\underline{x}^{(1)}, \underline{x}^{(2)}))^2 + \dots \right) \psi^{(j)}(p^{(j)}), \end{aligned}$$

where  $(\text{ad } H(\underline{x}^{(1)}, \underline{x}^{(2)}))X = [H(\underline{x}^{(1)}, \underline{x}^{(2)}), X]$  and so  $(\text{ad } H(\underline{x}^{(1)}, \underline{x}^{(2)}))^m \psi^{(j)}(p^{(j)}) = (\xi(\underline{x}^{(j)}, p^{(j)}))^m \psi^{(j)}(p^{(j)})$ . Hence

$$\psi^{(j)}(p^{(j)}, \underline{x}^{(1)}, \underline{x}^{(2)}) = e^{\xi(\underline{x}^{(j)}, p^{(j)})} \psi^{(j)}(p^{(j)}). \quad (19)$$

Similarly,  $[H(\underline{x}^{(1)}, \underline{x}^{(2)}), \psi^{(j)*}(q^{(j)})] = -\xi(\underline{x}^{(j)}, q^{(j)}) \psi^{(j)*}(q^{(j)})$  and the time evolution for

$$\psi^{(j)*}(q^{(j)}, \underline{x}^{(1)}, \underline{x}^{(2)}) = e^{H(\underline{x}^{(1)}, \underline{x}^{(2)})} \psi^{(j)*}(q^{(j)}) e^{-H(\underline{x}^{(1)}, \underline{x}^{(2)})} = e^{-\xi(\underline{x}^{(j)}, q^{(j)})} \psi^{(j)*}(q^{(j)}).$$

The  $\tau$ -functions for the 2 component free fermions with the total charge  $l_1 + l_2$ , we define in the following form for some  $g = g^{(1)} g^{(2)}$ :

$$\tau_{l_1, l_2; l_3}(\underline{x}^{(1)}, \underline{x}^{(2)}) = \langle l_1, l_2 | e^{H(\underline{x}^{(1)}, \underline{x}^{(2)})} g | l_2 - l_3, l_1 + l_3 \rangle = \langle \psi_{l_1}^{(1)*} \psi_{l_2}^{(2)*} e^{H(\underline{x}^{(1)}, \underline{x}^{(2)})} g \psi_{l_2-l_3}^{(2)} \psi_{l_1+l_3}^{(1)} \rangle, \quad (20)$$

where

$$\psi_l^{(i)*} = \begin{cases} \psi_{-1}^{(i)} \cdots \psi_l^{(i)}, & l < 0, \\ 1, & l = 0, \\ \psi_0^{(i)*} \cdots \psi_{l-1}^{(i)*}, & l > 0, \end{cases} \quad \psi_l^{(i)} = \begin{cases} \psi_l^{(i)*} \cdots \psi_{-1}^{(i)*}, & l < 0, \\ 1, & l = 0, \\ \psi_{l-1}^{(i)} \cdots \psi_0^{(i)}, & l > 0. \end{cases}$$

For example we take

$$g^{(j)} = \psi^{(j)}(p_1^{(j)}) \cdots \psi^{(j)}(p_r^{(j)}) \psi^{(j)*}(q_1^{(j)}) \cdots \psi^{(j)*}(q_s^{(j)}) \quad (21)$$

for some  $r, s$ . Then for  $r = s$ ,

$$\tau_{0,0;0}(\underline{x}^{(1)}, \underline{x}^{(2)}) = \langle 0, 0 | g(\underline{x}^{(1)}, \underline{x}^{(2)}) | 0, 0 \rangle = e^{\sum_{j=1,2}^r \xi(\underline{x}^{(j)}, p_i^{(j)}) - \xi(\underline{x}^{(j)}, q_i^{(j)})} \langle g \rangle$$

and, using Theorem 4.1, we get

$$\begin{aligned} \tau_{0,0;0}(\underline{x}^{(1)}, \underline{x}^{(2)}) &= e^{\sum_{j=1,2}^r \xi(\underline{x}^{(j)}, p_i^{(j)}) - \xi(\underline{x}^{(j)}, q_i^{(j)})} \det((\psi^{(1)}(p_i^{(1)}) \psi^{(1)*}(q_j^{(1)}))) \det((\psi^{(2)}(p_i^{(2)}) \psi^{(2)*}(q_j^{(2)}))) \\ &= (-1)^{r(r-1)} \det\left(e^{\xi(\underline{x}^{(1)}, p_i^{(1)}) - \xi(\underline{x}^{(1)}, q_j^{(1)})} \frac{q_j^{(1)}}{p_i^{(1)} - q_j^{(1)}}\right) \det\left(e^{\xi(\underline{x}^{(2)}, p_i^{(2)}) - \xi(\underline{x}^{(2)}, q_j^{(2)})} \frac{q_j^{(2)}}{p_i^{(2)} - q_j^{(2)}}\right). \end{aligned}$$

## 5. Charged free fermions in general

In this section, we wish to express  $g$  more generally in the following form

$$g = e^{\sum_{i,j=1}^N a_{ij} \psi^i \psi^{j*}}, \quad (22)$$

where the  $\psi^i, \psi^{j*}$  ( $i, j = 1, \dots, N$ ) can be either one-component or two-component fermions. For example, in this chapter we will take  $\psi^i = \psi(p_i)$  for the one-component case and  $\psi^i = \psi^{(1)}(p^{(1)})$  or  $\psi^i = \psi^{(2)}(p^{(2)})$  for the two-component case. Then the  $\tau$ -function  $\tau_0$  is

$$\begin{aligned} \tau_0 = \langle g(\underline{x}) \rangle &= 1 + \sum_{i_1, j_1=1}^N a_{i_1 j_1} \langle \psi^{i_1} \psi^{j_1*} \rangle + \sum_{i_1, j_1, i_2, j_2=1}^N a_{i_1 j_1} a_{i_2 j_2} \langle \psi^{i_1} \psi^{j_1*} \psi^{i_2} \psi^{j_2*} \rangle + \dots \\ &+ \sum_{i_1, j_1, \dots, i_N, j_N=1}^N a_{i_1 j_1} \cdots a_{i_N j_N} \langle \psi^{i_1} \psi^{j_1*} \cdots \psi^{i_N} \psi^{j_N*} \rangle. \end{aligned} \quad (23)$$

If we give the following expectation values

$$\langle \psi^i \psi^j \rangle := \langle \psi^{i*} \psi^{j*} \rangle := 0 \quad (24)$$

and using Wick's theorem, the  $\tau$ -function in (23) can be written in the  $(N \times N)$  determinantal structure in the following form:

$$\tau_0 = \det(I + AV), \quad (25)$$

where  $I$  is the identity matrix,  $A$  is a constant matrix with the entries  $A = [a_{ji}]$ , and  $V$  is a matrix with the entries of expectation values of quadratic free fermions  $V = [\langle \psi^i \psi^{j*} \rangle]$ . See Appendix A for the proof.

Next we give a general  $M$  order  $\tau$ -function, from which the higher order  $\tau$ -functions can be obtained. This formula can be written in the following form

$$\tau_M = \langle \psi^{i_1*} \cdots \psi^{i_M*} g \psi^{i_1} \cdots \psi^{i_M} \rangle = \begin{vmatrix} X & Y \\ G & Z \end{vmatrix}, \quad (26)$$

where

$$\begin{aligned} X &= \begin{pmatrix} \langle \psi^{i_1} \psi^{1*} \rangle & \cdots & \langle \psi^{i_1} \psi^{N*} \rangle \\ \vdots & \ddots & \vdots \\ \langle \psi^{i_M} \psi^{1*} \rangle & \cdots & \langle \psi^{i_M} \psi^{N*} \rangle \end{pmatrix}, & Y &= \begin{pmatrix} \langle \psi^{i_1} \psi^{i_1*} \rangle & \cdots & \langle \psi^{i_1} \psi^{i_M*} \rangle \\ \vdots & \ddots & \vdots \\ \langle \psi^{i_M} \psi^{i_1*} \rangle & \cdots & \langle \psi^{i_M} \psi^{i_M*} \rangle \end{pmatrix}, \\ Z &= \begin{pmatrix} \langle \psi^1 \psi^{i_1*} \rangle & \cdots & \langle \psi^1 \psi^{i_M*} \rangle \\ \vdots & \ddots & \vdots \\ \langle \psi^N \psi^{i_1*} \rangle & \cdots & \langle \psi^N \psi^{i_M*} \rangle \end{pmatrix}, \end{aligned}$$

and while  $g$  can be given either as

$$g = \psi^{1*} \cdots \psi^{N*} \psi^1 \cdots \psi^N \quad (27)$$



or as in (22), then the matrix  $G$  can be either  $G = (e^{\xi(x, p_i) - \xi(x, q_j)} \frac{q_j}{p_i - q_j})$  or  $G = I + AV$ , respectively. Thus,  $G$  depends on the choice of  $g$ . For example, for  $M = 1$ ,  $N = r$  and  $g$  is given by (27), we have

$$\tau_1 = \langle \psi^{i_1*} g(x) \psi^{i_1} \rangle = \begin{vmatrix} \langle \psi^{i_1} \psi^{1*} \rangle & \dots & \langle \psi^{i_1} \psi^{r*} \rangle & \langle \psi^{i_1} \psi^{i_1*} \rangle \\ & & & \langle \psi^1 \psi^{i_1*} \rangle \\ & G & & \vdots \\ & & & \langle \psi^r \psi^{i_1*} \rangle \end{vmatrix}.$$

If we choose  $\psi^{i_1} = \psi_0$ ,  $\psi^{i_1*} = \psi_0^*$  and  $\psi^j = \psi(p_j)$ ,  $\psi^{j*} = \psi^*(q_j)$  ( $j = 1, \dots, r$ ), then we recover the  $\tau_1$ -function in (9).

In order to get the soliton solution for the 1-component case, we choose the constants  $a_{ij} = 0$  ( $i \neq j$ ). From (23)

$$\begin{aligned} \tau &= 1 + \sum_{i_1=1}^N a_{i_1 i_1} \langle \psi^{i_1} \psi^{i_1*} \rangle + \sum_{i_1 < i_2=1}^N a_{i_1 i_1} a_{i_2 i_2} \langle \psi^{i_1} \psi^{i_1*} \psi^{i_2} \psi^{i_2*} \rangle + \dots \\ &+ \sum_{i_1 < i_2 < \dots < i_N=1}^N a_{i_1 i_1} \dots a_{i_N i_N} \langle \psi^{i_1} \psi^{i_1*} \dots \psi^{i_N} \psi^{i_N*} \rangle \end{aligned}$$

gives rise to the  $N$ -soliton solution.

For example, for  $N = 2$  from (23) we have the following solution

$$\begin{aligned} \tau_0 &= 1 + \sum_{i_1, j_1=1}^2 a_{i_1 j_1} \langle \psi^{i_1} \psi^{j_1*} \rangle + \sum_{i_1, j_1, i_2, j_2=1}^2 a_{i_1 j_1} a_{i_2 j_2} \langle \psi^{i_1} \psi^{j_1*} \psi^{i_2} \psi^{j_2*} \rangle \\ &= \begin{vmatrix} 1 + a_{11} \langle \psi^1 \psi^{1*} \rangle + a_{21} \langle \psi^2 \psi^{1*} \rangle & a_{11} \langle \psi^1 \psi^{2*} \rangle + a_{21} \langle \psi^2 \psi^{2*} \rangle \\ a_{12} \langle \psi^1 \psi^{1*} \rangle + a_{22} \langle \psi^2 \psi^{1*} \rangle & 1 + a_{12} \langle \psi^1 \psi^{2*} \rangle + a_{22} \langle \psi^2 \psi^{2*} \rangle \end{vmatrix} \end{aligned}$$

and this can be written as in (25),  $\tau = \det(I_2 + A_2 V_2)$  where  $I_2 = \text{diag}(1, 1)$ ,  $A_2 = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$  and  $V_2 = \begin{pmatrix} \langle \psi^1 \psi^{1*} \rangle & \langle \psi^1 \psi^{2*} \rangle \\ \langle \psi^2 \psi^{1*} \rangle & \langle \psi^2 \psi^{2*} \rangle \end{pmatrix}$ .

In general the 2-soliton solution can be written from (28) by choosing  $a_{12} = a_{21} = 0$  (the reason for this choice will be explained in Example 5.2)

$$\tau = \begin{vmatrix} 1 + a_{11} \langle \psi^1 \psi^{1*} \rangle & a_{11} \langle \psi^1 \psi^{2*} \rangle \\ a_{22} \langle \psi^2 \psi^{1*} \rangle & 1 + a_{22} \langle \psi^2 \psi^{2*} \rangle \end{vmatrix}. \quad (28)$$

**Example 5.1.** As an example for the one component fermions, we put  $\psi^1 = \psi(p_1)$ ,  $\psi^{1*} = \psi^*(q_1)$  and  $\psi^2 = \psi(p_2)$ ,  $\psi^{2*} = \psi^*(q_2)$  in (28). The  $\tau$ -function is

$$\begin{aligned} \tau &= 1 + a_{11} \langle \psi(p_1) \psi^*(q_1) \rangle + a_{22} \langle \psi(p_2) \psi^*(q_2) \rangle + a_{11} a_{22} (\langle \psi(p_1) \psi^*(q_1) \rangle \langle \psi(p_2) \psi^*(q_2) \rangle \\ &- \langle \psi(p_1) \psi^*(q_2) \rangle \langle \psi(p_2) \psi^*(q_1) \rangle). \end{aligned}$$

Taking the expectation values and choosing  $a_{11} = \frac{p_1 - q_1}{q_1}$ ,  $a_{22} = \frac{p_2 - q_2}{q_2}$  we get

$$\tau = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2},$$

where

$$\eta_i = \xi(p_i, x) - \xi(q_i, x), \quad i = 1, 2 \quad \text{and} \quad A_{12} = \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)}.$$

Hence  $u = 2\partial_x^2(\log \tau)$  gives the 2-soliton solution [11] for the KP equation

$$(u_x + 6uu_x + u_{xxx})_x + 3u_{yy} = 0.$$

**Example 5.2.** Here we give an example for the resonant soliton solution for the KP equation. The 4-soliton solution for the KP equation arises from the following choice

$$g = e^{\sum_{k=1}^4 a_{kk} \psi(p_k) \psi^*(q_k)}.$$

After we make the following choice of the parameters, namely we substitute  $p_2 = p_1$ , then  $p_3 = p_2$ ,  $p_4 = p_2$ ,  $q_4 = q_2$ , and  $q_3 = q_1$  then the  $\tau$ -function

$$\begin{aligned} \tau &= 1 + a_{11} \langle \psi(p_1) \psi^*(q_1) \rangle + a_{22} \langle \psi(p_1) \psi^*(q_2) \rangle + a_{33} \langle \psi(p_2) \psi^*(q_1) \rangle + a_{44} \langle \psi(p_2) \psi^*(q_2) \rangle \\ &+ a_{11} a_{44} \langle \psi(p_1) \psi^*(q_1) \psi(p_2) \psi^*(q_2) \rangle + a_{22} a_{33} \langle \psi(p_1) \psi^*(q_2) \psi(p_2) \psi^*(q_1) \rangle. \end{aligned}$$

Taking the expectation values and choosing  $a_{11} = \frac{p_1 - q_1}{q_1}$ ,  $a_{22} = \frac{p_1 - q_2}{q_2}$ ,  $a_{33} = \frac{p_2 - q_1}{q_1}$ ,  $a_{44} = \frac{p_2 - q_2}{q_2}$  we get

$$\tau = 1 + e^{\xi(p_1, x) - \xi(q_1, x)} + e^{\xi(p_1, x) - \xi(q_2, x)} + e^{\xi(p_2, x) - \xi(q_1, x)} + e^{\xi(p_2, x) - \xi(q_2, x)} + A_{12} e^{\xi(p_1, x) - \xi(q_1, x) + \xi(p_2, x) - \xi(q_2, x)},$$

where

$$A_{12} = \frac{(p_2 - p_1)(q_1 - q_2)(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(p_2 - q_1)(p_1 - q_1)(p_2 - q_2)}.$$

This gives rise to the resonant 4-soliton solution of the KP equation.

In order to get the  $\tau$ -function for the 2-soliton solution, as in the previous example, here we choose  $a_{22} = a_{33} = 0$ . The reason for this choice is that this makes the 2nd and 3rd terms vanish in the  $\tau$ -function. Hence the resonant behaviour that gives rise to a solitoff [12] vanishes and we get the 2-soliton solution.

**Example 5.3.** For the two component free fermions, we take  $a_{11} = a_{22} = 0$  and put  $\psi^1 = \psi^{(1)}(p^{(1)})$ ,  $\psi^{1*} = \psi^{(1)*}(q^{(1)})$  and  $\psi^2 = \psi^{(2)}(p^{(2)})$ ,  $\psi^{2*} = \psi^{(2)*}(q^{(2)})$  in (28). Then the  $\tau$ -function  $\tau_1$  from (28) is

$$\tau_1 = \begin{vmatrix} 1 + a_{21} \langle \psi^2 \psi^{1*} \rangle & a_{21} \langle \psi^2 \psi^{2*} \rangle \\ a_{12} \langle \psi^1 \psi^{1*} \rangle & 1 + a_{12} \langle \psi^1 \psi^{2*} \rangle \end{vmatrix}. \quad (29)$$

Taking the expectation values and choosing  $a_{12} = \frac{p^{(1)} - q^{(1)}}{q^{(1)}}$ ,  $a_{21} = \frac{q^{(2)} - p^{(2)}}{q^{(2)}}$ , we get the  $\tau_1$ -function for the two component KP-hierarchy (DS equations)

$$\tau_1 = 1 + e^{\eta^{(1)} + \eta^{(2)}},$$

where

$$\eta^{(j)} = \xi(p^{(j)}, x) - \xi(q^{(j)}, x), \quad j = 1, 2. \quad (30)$$

Hence  $u = \frac{\tau_3}{\tau_1}$  gives the 1-soliton solution [7] to the DS equations, where  $\tau_3$  is the bordered determinant of (29).

**Example 5.4.** In order to get the 1-dromion solution for the DS equations we define the following functions [7] (see (20))

$$\begin{aligned} \tau_1(z) &= \langle l_1, l_2 - 1 | e^{H(z)} g | l_2 - l - 1, l_1 + l \rangle, \\ \tau_2(z) &= \langle l_1 + 1, l_2 - 2 | e^{H(z)} g | l_2 - l - 1, l_1 + l \rangle, \\ \tau_3(z) &= \langle l_1 - 1, l_2 | e^{H(z)} g | l_2 - l - 1, l_1 + l \rangle, \end{aligned} \quad (31)$$

then the DS equations can be written in the following bilinear equations

$$\begin{aligned} D_2^{(2)} \tau_2 \cdot \tau_1 + D_1^{(2)^2} \tau_1 \cdot \tau_2 &= 0, \\ D_2^{(1)} \tau_1 \cdot \tau_2 + D_1^{(1)^2} \tau_2 \cdot \tau_1 &= 0, \\ D_2^{(2)} \tau_1 \cdot \tau_3 + D_1^{(2)^2} \tau_3 \cdot \tau_1 &= 0, \\ D_2^{(1)} \tau_3 \cdot \tau_1 + D_1^{(1)^2} \tau_1 \cdot \tau_3 &= 0, \\ D_1^{(1)} D_1^{(2)} \tau_1 \cdot \tau_1 - 2 \tau_2 \cdot \tau_3 &= 0. \end{aligned} \quad (32)$$

The  $\tau$ -functions in (31) provide a method for generating solutions of the two-component KP hierarchy of bilinear equations. In what follows we take the simplest choice  $l = l_1 = l_2 - 1 = 0$ . Hence we have

$$\tau_1(z) = \langle g(z) \rangle, \quad \tau_2(z) = \langle \psi_0^{(1)*} \psi_{-1}^{(2)} g(z) \rangle, \quad \tau_3(z) = \langle \psi_{-1}^{(1)} \psi_0^{(2)*} g(z) \rangle,$$

where  $g(z) = e^{H(z)} g e^{-H(z)}$ . The  $\tau_1$ -function is as in (28)

$$\tau_1 = \begin{vmatrix} 1 + a_{11} \langle \psi^1 \psi^{1*} \rangle + a_{21} \langle \psi^2 \psi^{1*} \rangle & a_{11} \langle \psi^1 \psi^{2*} \rangle + a_{21} \langle \psi^2 \psi^{2*} \rangle \\ a_{12} \langle \psi^1 \psi^{1*} \rangle + a_{22} \langle \psi^2 \psi^{1*} \rangle & 1 + a_{12} \langle \psi^1 \psi^{2*} \rangle + a_{22} \langle \psi^2 \psi^{2*} \rangle \end{vmatrix},$$

where we choose  $\psi^1 = \psi^{(1)}(p^{(1)})$ ,  $\psi^{1*} = \psi^{(1)*}(q^{(1)})$ ,  $\psi^2 = \psi^{(2)}(p^{(2)})$ ,  $\psi^{2*} = \psi^{(2)*}(q^{(2)})$  and  $a_{11} = \frac{p^{(1)} - q^{(1)}}{q^{(1)}}$ ,  $a_{22} = \frac{p^{(2)} - q^{(2)}}{q^{(2)}}$ ,  $a_{12} = \frac{p^{(1)} - q^{(2)}}{q^{(2)}}$ ,  $a_{21} = \frac{p^{(2)} - q^{(1)}}{q^{(1)}}$ . Taking the expectation values, we get

$$\tau_1 = 1 + e^{\eta^{(1)}} + e^{\eta^{(2)}} + A_{12} e^{\eta^{(1)} + \eta^{(2)}},$$

where

$$\eta^{(j)} = \xi(p^{(j)}, z) - \xi(q^{(j)}, z), \quad j = 1, 2 \quad \text{and} \quad A_{12} = \frac{(p^{(2)} - p^{(1)})(q^{(2)} - q^{(1)})}{(p^{(1)} - q^{(1)})(p^{(2)} - q^{(2)})}.$$

The  $\tau_2$ -function can be written from (26) in the following form:

$$\tau_2 = \begin{vmatrix} \langle \psi^{i_1} \psi^{1*} \rangle & \langle \psi^{i_1} \psi^{2*} \rangle & \langle \psi^{i_1} \psi^{i_1*} \rangle \\ 1 + a_{11} \langle \psi^1 \psi^{1*} \rangle + a_{21} \langle \psi^2 \psi^{1*} \rangle & a_{11} \langle \psi^1 \psi^{2*} \rangle + a_{21} \langle \psi^2 \psi^{2*} \rangle & \langle \psi^1 \psi^{i_1*} \rangle \\ a_{12} \langle \psi^1 \psi^{1*} \rangle + a_{22} \langle \psi^2 \psi^{1*} \rangle & 1 + a_{12} \langle \psi^1 \psi^{2*} \rangle + a_{22} \langle \psi^2 \psi^{2*} \rangle & \langle \psi^2 \psi^{i_1*} \rangle \end{vmatrix},$$

where we choose  $\psi^1 = \psi^{(1)}(p^{(1)})$ ,  $\psi^{1*} = \psi^{(1)*}(q^{(1)})$ ,  $\psi^2 = \psi^{(2)}(p^{(2)})$ ,  $\psi^{2*} = \psi^{(2)*}(q^{(2)})$  and  $\psi^{i_1*} = \psi_0^{(1)*}$ ,  $\psi^{i_1} = \psi_{-1}^{(2)}$ ,  $a_{12} = \frac{p^{(1)} - q^{(2)}}{q^{(2)}}$ . Then the  $\tau_2$ -function is

$$\tau_2 = \frac{q^{(1)}(p^{(1)} - q^{(2)})}{p^{(1)} - q^{(1)}} e^{2\xi(p^{(1)}, z) - \xi(q^{(1)}, z) - \xi(q^{(2)}, z)}.$$

Similarly the  $\tau_3$ -function can be written from (26) in the following form

$$\tau_3 = \begin{vmatrix} \langle \psi^{i_1} \psi^{1*} \rangle & \langle \psi^{i_1} \psi^{2*} \rangle & \langle \psi^{i_1} \psi^{i_1*} \rangle \\ 1 + a_{11} \langle \psi^1 \psi^{1*} \rangle + a_{21} \langle \psi^2 \psi^{1*} \rangle & a_{11} \langle \psi^1 \psi^{2*} \rangle + a_{21} \langle \psi^2 \psi^{2*} \rangle & \langle \psi^1 \psi^{i_1*} \rangle \\ a_{12} \langle \psi^1 \psi^{1*} \rangle + a_{22} \langle \psi^2 \psi^{1*} \rangle & 1 + a_{12} \langle \psi^1 \psi^{2*} \rangle + a_{22} \langle \psi^2 \psi^{2*} \rangle & \langle \psi^2 \psi^{i_1*} \rangle \end{vmatrix},$$

where we choose  $\psi^1 = \psi^{(1)}(p^{(1)})$ ,  $\psi^{1*} = \psi^{(1)*}(q^{(1)})$ ,  $\psi^2 = \psi^{(2)}(p^{(2)})$ ,  $\psi^{2*} = \psi^{(2)*}(q^{(2)})$  and  $\psi^{i_1*} = \psi_0^{(2)*}$ ,  $\psi^{i_1} = \psi_{-1}^{(1)}$ ,  $a_{21} = \frac{p^{(2)} - q^{(1)}}{q^{(1)}}$ . Then the  $\tau_3$ -function becomes

$$\tau_3 = \frac{q^{(2)}(p^{(2)} - q^{(1)})}{p^{(2)} - q^{(2)}} e^{2\xi(p^{(2)}, z) - \xi(q^{(2)}, z) - \xi(q^{(1)}, z)}.$$

The transformations  $P = \frac{\tau_2}{\tau_1}$ ,  $Q = \frac{\tau_3}{\tau_1}$ ,  $U = \log \tau_1$  yield the bilinear equations (32) to the following DS equations

$$\begin{aligned} -iP_t + P_{xx} + P_{yy} + 2(U_{xx} + U_{yy})P &= 0, \\ iQ_t + Q_{xx} + Q_{yy} + 2(U_{xx} + U_{yy})Q &= 0, \\ QP &= 4U_{xy}. \end{aligned}$$

Hence  $Q = \frac{\tau_3}{\tau_1}$  is the 1-dromion solution [7] for the DS equations.

## 6. Conclusion

In this paper, we have elucidated the role of determinants in determining new solutions by using charged fermion particles. We have presented soliton and dromion solutions to the KP and DS equations by using the  $\tau$  function in terms of determinants. We derived new general formulae (26) for charged fermions, from which higher order  $\tau$ -functions and hence soliton solutions and dromion solutions for the KP hierarchy can be obtained.

## Acknowledgments

The author expresses his sincere thanks for the reviewer's suggestions and comments.

## Appendix A

To show that the expression given in (25) is valid, we take  $W = AV$  (where  $W$  is an  $(N \times N)$  matrix with the entries  $W = [w_{ij}]$ ) and expand in the following form [13]:

$$\tau_0 = \det(I + AV) = \det(I + W) = 1 + \sum_{i=1}^N w_{ii} + \sum_{i < j=1}^N \begin{vmatrix} w_{ii} & w_{ij} \\ w_{ji} & w_{jj} \end{vmatrix} + \cdots + |W|, \quad (33)$$

where  $w_{ij} = a_{ki} \langle \psi^k \psi^{j*} \rangle$ . Hence the expression in (23) is reduced to the expansion in (33) after eliminating some terms according to the definition given in (24).

## References

- [1] M. Sato, An introduction to Sato theory, RIMS Kyoto Univ. 439 (1981) 30.
- [2] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Transformation groups for soliton equations IV. A new hierarchy of soliton equations of KP-type, Phys. D 4 (1982) 343–365.
- [3] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, Operator approach to the Kadomtsev–Pevashvilli equation—Transformation groups for soliton equations III, J. Phys. Soc. Japan 50 (1981) 3806–3812.
- [4] E.R. Caianiello, Combinatorics and Renormalization in Quantum Field Theory, Benjamin, Reading, MA, 1973.
- [5] Y. You, Polynomial solutions of the BKP hierarchy and projective representations of symmetric groups, 1989.
- [6] E. Date, M. Kashiwara, T. Miwa, Transformation groups for soliton equations II. Vertex operators and tau functions, Proc. Japan Acad. Ser. A 57 (1981) 387–392.
- [7] M. Jimbo, T. Miwa, Solitons and infinite dimensional Lie algebras, RIMS Kyoto Univ. 19 (1983) 943–1001.
- [8] M. Ünal, Application of the Pfaffian technique to the KR and mNVN equations, J. Math. Anal. Appl. 362 (2010) 224–230.
- [9] I.M. Macdonald, Symmetric Functions and Hall Polynomials, Oxford University Press, Oxford, 1979.
- [10] J.J.C. Nimmo, Hall–Littlewood symmetric functions and the BKP equation, J. Phys. A: Math. Gen. 23 (1990) 751–760.
- [11] N.C. Freeman, J.J.C. Nimmo, Soliton solutions of the Korteweg–de Vries and the Kadomtsev–Petviashvili equations: The Wronskian technique, Proc. R. Soc. Lond. Ser. A 389 (1983) 319–329.
- [12] C.R. Gilson, Resonant behaviour in the Davey–Stewartson equation, Phys. Lett. A 161 (1992) 423–428.
- [13] A. Aitken, Determinants and Matrices, University Mathematical Texts, Oliver and Boyd, Interscience, 1956.