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Additive comparisons of stopping values and supremum values for finite stage multiparameter stochastic processes

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ABSTRACT

This paper concerns the optimal stopping problem for discrete time multiparameter stochastic processes with the index set $\mathbf{N}^d$. In the classical optimal stopping problems, the comparisons between the expected reward of a player with complete foresight and the expected reward of a player using nonanticipating stop rules, known as prophet inequalities, have been studied by many authors. Ratio comparisons between these values in the case of multiparameter optimal stopping problems are studied by Krengel and Sucheston (1981) [9] and Tanaka (2007, 2006) [14,15]. In this paper an additive comparison in the case of finite stage multiparameter optimal stopping problems is given.

1. Introduction

Let $d \geqslant 1$ and $\mathbf{N}$ be the set of all nonnegative integers. In this paper we consider stochastic processes indexed by $\mathbf{N}^d$, which is equipped with the following partial order: for $z = (z_1, z_2, \ldots, z_d)$, $w = (w_1, w_2, \ldots, w_d) \in \mathbf{N}^d$, $z \leqslant w$ if and only if $z_i \leqslant w_i$ for all i, and $z < w$ if and only if $z \leqslant w$, $z \neq w$. Let e_i be the element for which the ith coordinate is 1 and all other coordinates are 0, and any $t \in \mathbf{N}^d$ be fixed. We set $|z| = \sum_{i=1}^d z_i$ for $z = (z_1, z_2, \ldots, z_d)$, $I = \{z \in \mathbf{N}^d\colon\ z \leqslant t\}$, and $I_k = \{z \in I\colon\ |z| \leqslant k\}$ for $k \in \mathbf{N}$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a family $\{\mathcal{F}_z,\ z \in I\}$ of sub σ-fields of $\mathcal{F}$ which satisfies the following conditions: $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and if $z \leqslant w$, then $\mathcal{F}_z \subseteq \mathcal{F}_w$.

An $\{\mathcal{F}_z\}$-stopping point is a random variable T taking values in I such that $\{T \leqslant z\} \in \mathcal{F}_z$ for all $z \in I$.

A tactic is a family $(\{\sigma(n),\ 0 \leqslant n \leqslant |t|\}, \tau)$ which satisfies the following conditions: $\sigma(0) = 0$, $\sigma(n)$ is an $\{\mathcal{F}_z\}$-stopping point for all n, $\sigma(n+1) \in d(\sigma(n))$ P-a.e., $\sigma(n+1)$ is $\mathcal{F}_{\sigma(n)}$-measurable for all n, and τ is an $\{\mathcal{F}_{\sigma(n)},\ 0 \leqslant n \leqslant |t|\}$-stopping time, where $d(z)$ is the set of all direct successors of z, and for a stopping point T, $\mathcal{F}_T = \{A \in \mathcal{F}\colon\ A \cap \{T = z\} \in \mathcal{F}_z$ for all $z \in I\}$.

A stopping point T is said to be accessible if there exists a tactic $(\{\sigma(n)\}, \tau)$ such that $T = \sigma(\tau)$. We denote the set of all accessible stopping points taking values in I by A.

Let $\{X(z),\ z \in I\}$ be an $\{\mathcal{F}_z\}$-adapted integrable stochastic process. Then the multiparameter optimal stopping problem is to find a stopping point $T^* \in A$ (a tactic $(\{\sigma^*(n)\}, \tau^*)$) such that

$$V\bigl[\bigl\{X(z),\ z \in I\bigr\}\bigr] := E\bigl[X\bigl(T^*\bigr)\bigr] = \sup_{T \in A} E\bigl[X(T)\bigr] \quad \Bigl(E\bigl[X\bigl(\sigma^*\bigl(\tau^*\bigr)\bigr)\bigr] = \sup_{(\{\sigma(n)\}, \tau)} E\bigl[X\bigl(\sigma(\tau)\bigr)\bigr]\Bigr).$$

The discrete time multiparameter optimal stopping problems have been studied by many authors, for example, Cairoli and Dalang [2], Krengel and Sucheston [9], Lawler and Vanderbei [10], Mandelbaum [11], Mandelbaum and Vanderbei [12]

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and Mazziotto [13]. We refer to [2] for the formulation and the terminology of the discrete time multiparameter optimal stopping problems.

In the case of the finite index set I, we can apply the backward induction method to this problem (for example, Cairoli and Dalang [2]).

Now in this paper we shall compare the expected reward of a player with complete foresight $E[\max_{z\in I} X(z)]$ and the expected reward of a player using stopping points $\sup_{T\in A} E[X(T)]$.

Prophet inequalities have been studied by many authors, for example, Hill [3,4], Hill and Kertz [5–7], Krengel and Sucheston [8] in the case of one-parameter optimal stopping problems, and Krengel and Sucheston [9], Tanaka [14] in the case of multiparameter optimal stopping problems. Especially, [7] contains very nice introduction to prophet theory for one-parameter optimal stopping problems.

In one-parameter optimal stopping theory, the following inequality has been well known: let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, P)$ be a probability space and B be the set of all $\{\mathcal{F}_n\}$-stopping times.

Theorem 1.1. *(See [6,7].) If $\{X(n)\}$ is a finite (or an infinite) sequence of independent random variables taking values in $[a,b]$, then*

$$E\Big[\sup_n X(n)\Big] - \sup_{\tau\in B} E\big[X(\tau)\big] \leqslant \frac{1}{4}(b-a),$$

and $\frac{1}{4}(b-a)$ is the best possible bound.

In the multiparameter optimal stopping theory, Krengel and Sucheston [9] and Tanaka [14,15] have given ration comparisons between $E[\max_{z\in I} X(z)]$ and $\sup_{T\in A} E[X(T)]$.

In this paper, we shall give an additive comparison between $E[\max_{z\in I} X(z)]$ and $\sup_{T\in A} E[X(T)]$. That is, we shall extend the inequality in Theorem 1.1 in the case of one-parameter optimal stopping problems to the case of general multiparameter optimal stopping problems with stopping points taking values in a finite index set, and study a universal bound which depends on the finite index set.

2. Preliminaries

In this section, we develop the discussion by using balayage or dilation technique introduced by Boshuizen [1] and Hill and Kertz [6]. Throughout this paper we use the notation $a \vee b = \max\{a,b\}$ and $\bigvee_\lambda a_\lambda = \max\{a_\lambda, \lambda\}$.

Definition 2.1. (See [1,6].) For a $[0,1]$-valued random variable Y and $0 \leqslant a < b \leqslant 1$, let Y_a^b denote a random variable with the following distribution

$$P\big(Y_a^b \in B\big) = P(Y \in B) \quad \text{if } P\big(Y \in B \cap [a,b]\big) = 0 \text{ and } B \in \mathcal{B}\big([0,1]\big),$$
$$P\big(Y_a^b = a\big) = \frac{1}{b-a} \int_{Y\in[a,b]} (b-Y)\,dP,$$
$$P\big(Y_a^b = b\big) = \frac{1}{b-a} \int_{Y\in[a,b]} (Y-a)\,dP,$$

where $\mathcal{B}([0,1])$ is a topological Borel field of $[0,1]$.

Now it will be assumed that Y_a^b exists, even random variables which are independent of other given random variables. Indeed, we can construct such a random variable by enlarging the probability space by means of taking product spaces.

Lemma 2.1. *(See [1,6].) Let Y be a $[0,1]$-valued random variable and $0 \leqslant a < b \leqslant 1$. Then we have the following:*

(1) $E[Y] = E[Y_a^b]$.
(2) *If X is independent of both Y and Y_a^b, then $E[X \vee Y] \leqslant E[X \vee Y_a^b]$.*

In the remainder of this section and the next section, let t and I be mentioned in Section 1, and let $\{X(z),\ z \in I\}$ be an $\{\mathcal{F}_z,\ z\in I\}$-adapted, $[0,1]$-valued stochastic process whose elements are mutually independent.

For any $\{\mathcal{F}_z,\ z \in I\}$-adapted stochastic process, $\{Y(z),\ z\in I\}$, we set

$$D\big(\{Y(z),\ z\in I\}\big) := E\Big[\bigvee_{z\in I} Y(z)\Big] - V\big[\{Y(z),\ z\in I\}\big].$$

Proposition 2.1. *For any t and any k ($0 < k \leqslant |t|$), we set $\mu := \bigvee_{r \in d(0)} V[\{X(z),\ z \in I_k,\ z \geqslant r\}]$. Then we have*

$$D\big(\big\{X(z),\ z \in I_k\big\}\big) \leqslant D\big(\big\{\mu, \big\{X(z),\ z \in I_k - \{0\}\big\}\big\}\big).$$

Proof. By the backward induction method, we have

$$V\big[\big\{X(z),\ z \in I_k\big\}\big] = \mu + E\big[\big(X(0) - \mu\big)^+\big]$$

and

$$\mu = V\big[\big\{\mu, \big\{X(z),\ z \in I_k - \{0\}\big\}\big\}\big],$$

which follow

$$V\big[\big\{X(z),\ z \in I_k\big\}\big] = V\big[\big\{\mu, \big\{X(z),\ z \in I_k - \{0\}\big\}\big\}\big] + E\big[\big(X(0) - \mu\big)^+\big]. \tag{1}$$

Also we have

$$E\Bigg[\bigvee_{z \in I_k} X(z)\Bigg] \leqslant E\Bigg[\mu \vee \bigvee_{z \in I_k - \{0\}} X(z)\Bigg] + E\big[\big(X(0) - \mu\big)^+\big]. \tag{2}$$

By (1) and (2), we have the conclusion. $\square$

Proposition 2.2. *Let $k \geqslant 2$, μ be defined in Proposition 2.1, and t_j ($j = 1, 2, \ldots, \ell$) be all elements such that $|z| = k$ and $z \in I_k$. Let $\mathbf{Q} := \{Q(t_j),\ j = 1, 2, \ldots, \ell\}$ and $\mathbf{Y} := \{Y(t_j - e_i),\ j = 1, 2, \ldots, \ell,\ i = 1, 2, \ldots, d,\ t_j - e_i \in I_k\}$ be sequences of mutually independent random variables defined by*

$$Q(t_j) = \big(X(t_j)\big)_0^1 \quad \text{and} \quad Y(t_j - e_i) = \big(X(t_j - e_i)\big)^1_{\bigvee_{r \in d(t_j - e_i)} E[X(r)]},$$

which are independent of $\{X(z),\ z \in I_{k-2}\}$. Then we have

$$D\big(\big\{\mu, \big\{X(z),\ z \in I_k - \{0\}\big\}\big\}\big) \leqslant D\big(\big\{\mu, \big\{X(z),\ z \in I_{k-2} - \{0\}\big\}, \mathbf{Y}, \mathbf{Q}\big\}\big).$$

Proof. By the backward induction method and the definition of $Q(t_j)$ and $Y(t_j - e_i)$, we have, for i, j ($t_j - e_i \in I_k$),

$$\begin{aligned} &V\big[\big\{X(t_j - e_i), \big\{X(r),\ r \in d(t_j - e_i)\big\}\big\}\big] = V\big[\big\{Y(t_j - e_i), \big\{Q(r),\ r \in d(t_j - e_i)\big\}\big\}\big], \\ &V\big[\big\{\mu, \big\{X(z),\ z \in I_k - \{0\}\big\}\big\}\big] = V\big[\big\{\mu, \big\{X(z),\ z \in I_{k-2} - \{0\}\big\}, \mathbf{Y}, \mathbf{Q}\big\}\big]. \end{aligned} \tag{3}$$

Let s_j ($j = 1, 2, \ldots, m$) be all elements such that $|z| = k - 1$ and $z \in I_k$. By Lemma 2.1, we have

$$\begin{aligned} E\Bigg[\mu \vee \bigvee_{z \in I_k - \{0\}} X(z)\Bigg] &= E\Bigg[\mu \vee \bigvee_{z \in I_{k-1} - \{0\}} X(z) \vee \bigvee_{j = 2, \ldots, \ell} X(t_j) \vee X(t_1)\Bigg] \\ &\leqslant E\Bigg[\mu \vee \bigvee_{z \in I_{k-1} - \{0\}} X(z) \vee \bigvee_{j = 2, \ldots, \ell} X(t_j) \vee Q(t_1)\Bigg] \\ &= E\Bigg[\mu \vee \bigvee_{z \in I_{k-1} - \{0\}} X(z) \vee \bigvee_{j = 3, \ldots, \ell} X(t_j) \vee X(t_2) \vee Q(t_1)\Bigg] \\ &\leqslant E\Bigg[\mu \vee \bigvee_{z \in I_{k-1} - \{0\}} X(z) \vee \bigvee_{j = 3, \ldots, \ell} X(t_j) \vee Q(t_2) \vee Q(t_1)\Bigg] \\ &\ \ \vdots \\ &\leqslant E\Bigg[\mu \vee \bigvee_{z \in I_{k-1} - \{0\}} X(z) \vee \bigvee_{j = 1, 2, \ldots, \ell} Q(t_j)\Bigg] \\ &= E\Bigg[\mu \vee \bigvee_{z \in I_{k-2} - \{0\}} X(z) \vee \bigvee_{j = 1, 2, \ldots, \ell} Q(t_j) \vee \bigvee_{|z| = k-1} X(z)\Bigg] \\ &= E\Bigg[\mu \vee \bigvee_{z \in I_{k-2} - \{0\}} X(z) \vee \bigvee_{j = 1, 2, \ldots, \ell} Q(t_j) \vee \bigvee_{j = 2, \ldots, m} X(s_j) \vee X(s_1)\Bigg] \end{aligned}$$

$$\leqslant E\Big[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z) \vee \bigvee_{j=1,2,\ldots,\ell} Q(t_j) \vee \bigvee_{j=2,\ldots,m} X(s_j) \vee Y(s_1)\Big]$$

$$= E\Big[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z) \vee \bigvee_{j=1,2,\ldots,\ell} Q(t_j) \vee \bigvee_{j=3,\ldots,m} X(s_j) \vee X(s_2) \vee Y(s_1)\Big]$$

$$\leqslant E\Big[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z) \vee \bigvee_{j=1,2,\ldots,\ell} Q(t_j) \vee \bigvee_{j=3,\ldots,m} X(s_j) \vee Y(s_2) \vee Y(s_1)\Big]$$

$$\vdots$$

$$= E\Big[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z) \vee \bigvee_{j=1,2,\ldots,\ell} Q(t_j) \vee \bigvee_{j=1,2,\ldots,m} Y(s_j)\Big]. \tag{4}$$

By (3) and (4), we have the conclusion. □

Proposition 2.3. *Let $k \geqslant 2$, μ be defined in Proposition 2.1, and $\mathbf{Y}$ be defined in Proposition 2.2. Let z_v $(v = 1, 2, \ldots, \alpha)$ be all elements such that $|z| = k-1$ and $z \in I_k$. Let $\mathbf{W} := \{W(z_v),\ v = 1, 2, \ldots, \alpha\}$ be a sequence of mutually independent random variables defined by*

$$P\big(W(z_v) = 1\big) = V\big[\big\{Y(z_v), \big\{Q(r),\ r \in d(z_v)\big\}\big\}\big],$$
$$P\big(W(z_v) = 0\big) = 1 - V\big[\big\{Y(z_v), \big\{Q(r),\ r \in d(z_v)\big\}\big\}\big],$$

which are independent of $\{X(z),\ z \in I_{k-2} - \{0\}\}$.

For each v, we take $t^{z_v} \in d(z_v) \cap I_k$ such that $\bigvee_{r\in d(z_v)} E[X(r)] = E[X(t^{z_v})]$ and set $\{t^{z_{v^i}}\colon\ i = 1, 2, \ldots, \beta\} := \{t^{z_v}\colon\ v = 1, 2, \ldots, \alpha\}$, and $S_i := \{z_v\colon \bigvee_{r\in d(z_v)} E[X(r)] = E[X(t^{z_{v^i}})]\} =: \{z_{v^i_j}\colon\ j = 1, 2, \ldots, p_i\}$. Also we set $R := \{z \in I\colon\ |z| = k\} - \{t^{z_v}\colon\ v = 1, 2, \ldots, \alpha\}$ and $p := \prod_{r\in R}(1 - E[X(r)])$ if $R \neq \emptyset$ and $p := 1$ otherwise.

Suppose that $V[\{X(z_v), \{X(r)\ (r \in d(z_v))\}\}] < 1$ for all $v = 1, 2, \ldots, \alpha$. Then there exist constants C_1, C_2 such that

$$D\big(\big\{\mu, \big\{X(z),\ z \in I_{k-2} - \{0\}\big\}, \mathbf{Y}, \mathbf{Q}\big\}\big) = C_1 D\big(\big\{\mu, \big\{X(z),\ z \in I_{k-2} - \{0\}\big\}, \mathbf{W}\big\}\big) + C_2.$$

Proof. By the backward induction method, we have, for z_v $(v = 1, 2, \ldots, \alpha)$,

$$E\big[W(z_v)\big] = V\big[\big\{Y(z_v), \big\{Q(r),\ r \in d(z_v)\big\}\big\}\big],$$
$$E\big[Q(r)\big] = E\big[X(r)\big] \leqslant \mu \quad \big(r \in d(z_v)\big),$$
$$V\big[\big\{\mu, \big\{X(z),\ z \in I_{k-2} - \{0\}\big\}, \mathbf{Y}, \mathbf{Q}\big\}\big] = V\big[\big\{\mu, \big\{X(z),\ z \in I_{k-2} - \{0\}\big\}, \mathbf{W}\big\}\big] = \mu. \tag{5}$$

We set

$$A_1 := \mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z) \vee \Big(\bigvee_{v=1,2,\ldots,\alpha} Y(z_v) \vee Q\big(t^{z_v}\big)\Big),$$
$$B_1 := \bigvee_{r\in R} Q(r),$$

and then we have

$$E\Big[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z) \vee \bigvee_{|z|=k-1,\, z\in I_k} Y(z) \vee \bigvee_{|z|=k,\, z\in I_k} Q(z)\Big]$$
$$= E\Big[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z) \vee \Big(\bigvee_{v=1,2,\ldots,\alpha} Y(z_v) \vee Q\big(t^{z_v}\big)\Big) \vee \bigvee_{r\in R} Q(r)\Big]$$
$$= E[A_1] + E\big[(B_1 - A_1)^+\big]$$
$$= E[A_1] + P(B_1 = 1)E\big[(1 - A_1)^+ \bigm| B_1 = 1\big] + P(B_1 = 0)E\big[(0 - A_1)^+ \bigm| B_1 = 0\big]$$
$$= E[A_1] + P(B_1 = 1)\big(1 - E[A_1]\big).$$

Moreover we obtain $P(B_1 = 1) = 1 - \prod_{r\in R}(1 - E[X(r)]) = 1 - p$.

From the fact that $\bigcup_{i=1}^{\beta} S_i = \{z_v\colon\ v = 1, 2, \ldots, \alpha\}$ and $\{S_i,\ i = 1, 2, \ldots, \beta\}$ is disjoint, we have

$$E[A_1] = E\left[\mu \vee \bigvee_{z \in I_{k-2}-\{0\}} X(z) \vee \left(\bigvee_{i=1,2,\ldots,\beta} \left\{ Q\left(t^{z_{v^i}}\right) \vee \left\{ \bigvee_{j=1,2,\ldots,p_i} Y(z_{v^i_j}) \right\}\right\}\right)\right].$$

At first we shall estimate the term $E[A_1]$. We set

$$A_2 := \mu \vee \bigvee_{z \in I_{k-2}-\{0\}} X(z) \vee \left(\bigvee_{i=2,3,\ldots,\beta} \left\{ Q\left(t^{z_{v^i}}\right) \vee \left\{ \bigvee_{j=1,2,\ldots,p_i} Y(z_{v^i_j}) \right\}\right\}\right),$$

$$q_1\left(t^{z_{v^1}}\right) := E\left[X\left(t^{z_{v^1}}\right)\right] = P\left(Q\left(t^{z_{v^1}}\right) = 1\right),$$

$$v(z) := V\left[\left\{X(z), \left\{X(r)\ \left(r \in d(z)\right)\right\}\right\}\right] = V\left[\left\{Y(z), \left\{Q(r)\ \left(r \in d(z)\right)\right\}\right\}\right] \quad \text{for } z\ \left(|z| = k-1\right).$$

Then we have

$$\begin{aligned}
E[A_1] &= E\left[A_2 \vee \left\{ Q\left(t^{z_{v^1}}\right) \vee \left\{ \bigvee_{j=1,2,\ldots,p_1} Y(z_{v^1_j}) \right\}\right\}\right] \\
&= P\left(Q\left(t^{z_{v^1}}\right) = 0\right) E\left[A_2 \vee \left\{ \bigvee_{j=1,2,\ldots,p_1} Y(z_{v^1_j}) \right\}\right] + P\left(Q\left(t^{z_{v^1}}\right) = 1\right) \\
&= \left(1 - q_1\left(t^{z_{v^1}}\right)\right) E\left[A_2 \vee \left\{ \bigvee_{j=1,2,\ldots,p_1} Y(z_{v^1_j}) \right\}\right] + q_1\left(t^{z_{v^1}}\right).
\end{aligned}$$

Next we shall estimate the term $E[A_2 \vee \{\bigvee_{j=1,2,\ldots,p_1} Y(z_{v^1_j})\}]$.

$$\begin{aligned}
E\left[A_2 \vee \left\{ \bigvee_{j=1,2,\ldots,p_1} Y(z_{v^1_j}) \right\}\right] &= E\left[A_2 \vee \left\{ \bigvee_{j=2,3,\ldots,p_1} Y(z_{v^1_j}) \right\} \vee Y(z_{v^1_1})\right] \\
&= P\left(Y(z_{v^1_1}) = E\left[X\left(t^{z_{v^1}}\right)\right]\right) E\left[A_2 \vee \left\{ \bigvee_{j=2,3,\ldots,p_1} Y(z_{v^1_j}) \right\} \vee E\left[X\left(t^{z_{v^1}}\right)\right]\right] \\
&\quad + P\left(Y(z_{v^1_1}) = 1\right) E\left[A_2 \vee \left\{ \bigvee_{j=2,3,\ldots,p_1} Y(z_{v^1_j}) \right\} \vee 1\right] \\
&\quad + P\left(Y(z_{v^1_1}) = X(z_{v^1_1})\right) E\left[A_2 \vee \left\{ \bigvee_{j=2,3,\ldots,p_1} Y(z_{v^1_j}) \right\} \vee X(z_{v^1_1})\right] \\
&= P\left(Y(z_{v^1_1}) = 1\right) + \left(1 - P\left(Y(z_{v^1_1}) = 1\right)\right) E\left[A_2 \vee \left\{ \bigvee_{j=2,3,\ldots,p_1} Y(z_{v^1_j}) \right\}\right] \\
&= \left(v(z_{v^1_1}) - q_1\left(t^{z_{v^1}}\right) + \left(1 - v(z_{v^1_1})\right) E\left[A_2 \vee \left\{ \bigvee_{j=2,3,\ldots,p_1} Y(z_{v^1_j}) \right\}\right]\right) \Big/ \left(1 - q_1\left(t^{z_{v^1}}\right)\right).
\end{aligned}$$

By repeating this calculation to the term $E[A_2 \vee \{\bigvee_{j=2,3,\ldots,p_1} Y(z_{v^1_j})\}]$, we obtain

$$E\left[A_2 \vee \left\{ \bigvee_{j=2,3,\ldots,p_1} Y(z_{v^1_j}) \right\}\right] = \left(v(z_{v^1_2}) - q_1\left(t^{z_{v^1}}\right) + \left(1 - v(z_{v^1_2})\right) E\left[A_2 \vee \left\{ \bigvee_{j=3,4,\ldots,p_1} Y(z_{v^1_j}) \right\}\right]\right) \Big/ \left(1 - q_1\left(t^{z_{v^1}}\right)\right).$$

Then we have

$$\begin{aligned}
&E\left[A_2 \vee \left\{ \bigvee_{j=1,2,\ldots,p_1} Y(z_{v^1_j}) \right\}\right] \\
&= \left\{ \left(v(z_{v^1_1}) - q_1\left(t^{z_{v^1}}\right)\right)\left(v(z_{v^1_2}) - q_1\left(t^{z_{v^1}}\right)\right) + \left(1 - v(z_{v^1_1})\right)\left(1 - q_1\left(t^{z_{v^1}}\right)\right) \right. \\
&\quad \left. + \left(1 - v(z_{v^1_2})\right)\left(1 - v(z_{v^1_1})\right) E\left[A_2 \vee \left\{ \bigvee_{j=3,4,\ldots,p_1} Y(z_{v^1_j}) \right\}\right]\right\} \Big/ \left(1 - q_1\left(t^{z_{v^1}}\right)\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \Big\{ \big(v(z_{v_1^1}) - q_1\big(t^{z_{v^1}}\big)\big)\big(1 - q_1\big(t^{z_{v^1}}\big)\big)^2 + \big(1 - v(z_{v_1^1})\big)\big(1 - q_1\big(t^{z_{v^1}}\big)\big)\big(v(z_{v_2^1}) - q_1\big(t^{z_{v^1}}\big)\big) \\
&\quad + \big(1 - v(z_{v_2^1})\big)\big(1 - v(z_{v_1^1})\big)\big(v(z_{v_3^1}) - q_1\big(t^{z_{v^1}}\big)\big) \\
&\quad + \big(1 - v(z_{v_3^1})\big)\big(1 - v(z_{v_2^1})\big)\big(1 - v(z_{v_1^1})\big)E\Big[A_2 \vee \Big\{ \bigvee_{j=4,5,\ldots,p_1} Y(z_{v_j^1}) \Big\}\Big]\Big\} / \big(1 - q_1\big(t^{z_{v^1}}\big)\big)^3 \\
&\quad \vdots \\
&= \big\{ \big(v(z_{v_1^1}) - q_1\big(t^{z_{v^1}}\big)\big)\big(1 - q_1\big(t^{z_{v^1}}\big)\big)^{p_1 - 1} \\
&\quad + \big(v(z_{v_2^1}) - q_1\big(t^{z_{v^1}}\big)\big)\big(1 - v(z_{v_1^1})\big)\big(1 - q_1\big(t^{z_{v^1}}\big)\big)^{p_1 - 2} \\
&\quad + \big(v(z_{v_3^1}) - q_1\big(t^{z_{v^1}}\big)\big)\big(1 - v(z_{v_2^1})\big)\big(1 - v(z_{v_1^1})\big)\big(1 - q_1\big(t^{z_{v^1}}\big)\big)^{p_1 - 3} \\
&\quad \vdots \\
&\quad + \big(v(z_{v_{p_1-1}^1}) - q_1\big(t^{z_{v^1}}\big)\big)\big(1 - v(z_{v_{p_1-2}^1})\big) \cdots \big(1 - v(z_{v_1^1})\big)\big(1 - q_1\big(t^{z_{v^1}}\big)\big) \\
&\quad + \big(v(z_{v_{p_1}^1}) - q_1\big(t^{z_{v^1}}\big)\big)\big(1 - v(z_{v_{p_1-1}^1})\big) \cdots \big(1 - v(z_{v_1^1})\big)\big\} / \big(1 - q_1\big(t^{z_{v^1}}\big)\big)^{p_1} \\
&\quad + \big\{\big(1 - v(z_{v_{p_1}^1})\big)\big(1 - v(z_{v_{p_1-1}^1})\big) \cdots \big(1 - v(z_{v_1^1})\big)E[A_2]\big\} / \big(1 - q_1\big(t^{z_{v^1}}\big)\big)^{p_1}.
\end{aligned}
$$

Now we set

$$
\begin{aligned}
& q_i := q_i\big(t^{z_{v^i}}\big) := E\big[X\big(t^{z_{v^i}}\big)\big] = P\big(Q\big(t^{z_{v^i}}\big) = 1\big), \\
& v_v := v(z_v), \\
& v_j^i := v(z_{v_j^i}), \\
& F_i := \big(v_1^i - q_i\big)(1 - q_i)^{p_i - 1} \\
&\qquad + \big(v_2^i - q_i\big)\big(1 - v_1^i\big)(1 - q_i)^{p_i - 2} \\
&\qquad + \big(v_3^i - q_i\big)\big(1 - v_2^i\big)\big(1 - v_1^i\big)(1 - q_i)^{p_i - 3} \\
&\qquad \vdots \\
&\qquad + \big(v_{p_i - 1}^i - q_i\big)\big(1 - v_{p_1 - 2}^i\big) \cdots \big(1 - v_1^i\big)(1 - q_i) \\
&\qquad + \big(v_{p_i}^i - q_i\big)\big(1 - v_{p_i - 1}^i\big) \cdots \big(1 - v_1^i\big), \\
& G_i := \big(1 - v_{pi}^i\big)\big(1 - v_{p_i - 1}^i\big) \cdots \big(1 - v_1^i\big) / (1 - q_i)^{p_i}, \\
& A_i := \mu \vee \bigvee_{z \in I_{k-2} - \{0\}} X(z) \vee \Big(\bigvee_{\delta = i, i+1, \ldots, \beta} \Big\{ Q\big(t^{z_{v^\delta}}\big) \vee \Big\{ \bigvee_{j = 1, 2, \ldots, p_\delta} Y(z_{v_j^\delta}) \Big\}\Big\}\Big), \\
& H_\beta := (1 - q_1)F_1 + (1 - q_1)(1 - q_2)F_2 G_1 \\
&\qquad + (1 - q_1)(1 - q_2)(1 - q_3)F_3 G_1 G_2 \\
&\qquad \vdots \\
&\qquad + (1 - q_1)(1 - q_2) \cdots (1 - q_\beta)F_\beta G_1 G_2 \cdots G_{\beta - 1} \\
&\qquad + q_1 + (1 - q_1)q_2 G_1 + (1 - q_1)(1 - q_2)q_3 G_1 G_2 \\
&\qquad + (1 - q_1)(1 - q_2) \cdots (1 - q_{\beta - 1})q_\beta G_1 G_2 \cdots G_{\beta - 1}, \\
& K_\beta := (1 - q_1)(1 - q_2) \cdots (1 - q_\beta)G_1 G_2 \cdots G_\beta, \\
& L_\alpha := (1 - v_1)(1 - v_2) \cdots (1 - v_{\alpha - 1})v_\alpha + \\
&\qquad \vdots \\
&\qquad + (1 - v_1)(1 - v_2)v_3 + (1 - v_1)v_2 + v_1, \\
& M_\alpha := (1 - v_1)(1 - v_2) \cdots (1 - v_\alpha).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
E[A_1] &= (1-q_1)E\left[A_2 \vee \left\{\bigvee_{j=1,2,\ldots,p_1} Y(z_{v_j^1})\right\}\right] + q_1 \\
&= (1-q_1)\big(F_1 + G_1 E[A_2]\big) + q_1 \\
&= (1-q_1)F_1 + (1-q_1)G_1\big\{(1-q_2)\big(F_2 + G_2 E[A_3]\big) + q_2\big\} + q_1 \\
&= (1-q_1)F_1 + (1-q_1)(1-q_2)F_2G_1 + (1-q_1)q_2G_1 + q_1 \\
&\quad + (1-q_1)(1-q_2)G_1G_2\big\{(1-q_3)\big(F_3 + G_3E[A_4]\big) + q_3\big\} \\
&\quad \vdots \\
&= (1-q_1)F_1 + (1-q_1)(1-q_2)F_2G_1 \\
&\quad + (1-q_1)(1-q_2)(1-q_3)F_3G_1G_2 \\
&\quad \vdots \\
&\quad + (1-q_1)(1-q_2)\cdots(1-q_\beta)F_\beta G_1G_2\cdots G_{\beta-1} \\
&\quad + q_1 + (1-q_1)q_2G_1 + (1-q_1)(1-q_2)q_3G_1G_2 \\
&\quad \vdots \\
&\quad + (1-q_1)(1-q_2)\cdots(1-q_{\beta-1})q_\beta G_1G_2\cdots G_{\beta-1} \\
&\quad + (1-q_1)(1-q_2)\cdots(1-q_\beta)G_1G_2\cdots G_\beta E\left[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z)\right] \\
&= H_\beta + K_\beta E\left[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z)\right],
\end{aligned}
$$

and then

$$
\begin{aligned}
&E\left[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z) \vee \bigvee_{|z|=k-1,\, z\in I_k} Y(z) \vee \bigvee_{|z|=k,\, z\in I_k} Q(z)\right] \\
&\quad = p\left(H_\beta + K_\beta E\left[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z)\right]\right) + (1-p),
\end{aligned}
\tag{6}
$$

and

$$
E\left[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z) \vee \bigvee_{v=1,2,\ldots,\alpha} W(z_v)\right] = L_\alpha + M_\alpha E\left[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z)\right]. \tag{7}
$$

Hence we obtain, from (5), (6) and (7),

$$
\begin{aligned}
&D\big(\big\{\mu, \big\{X(z),\ z\in I_{k-2}-\{0\}\big\}, \mathbf{Y}, \mathbf{Q}\big\}\big) \\
&\quad = p\left(H_\beta + K_\beta E\left[\mu \vee \bigvee_{z\in I_{k-2}-\{0\}} X(z)\right]\right) + (1-p) - \mu \\
&\quad = p\left(H_\beta + K_\beta \frac{E[\mu \vee \bigvee_z X(z) \vee \bigvee_v W(z_v)] - L_\alpha}{M_\alpha}\right) + (1-p) - \mu \\
&\quad = \frac{pK_\beta}{M_\alpha} D\big(\big\{\mu, \big\{X(z),\ z\in I_{k-2}-\{0\}\big\}, \mathbf{W}\big\}\big) + \left(\frac{pK_\beta}{M_\alpha} - 1\right)\mu + p\left(H_\beta - \frac{L_\alpha K_\beta}{M_\alpha} - 1\right) + 1.
\end{aligned}
$$

By setting

$$
\begin{aligned}
C_1 &= \frac{pK_\beta}{M_\alpha}, \\
C_2 &= \left(\frac{pK_\beta}{M_\alpha} - 1\right)\mu + p\left(H_\beta - \frac{L_\alpha K_\beta}{M_\alpha} - 1\right) + 1,
\end{aligned}
$$

we have the assertion. □

3. Prophet inequality

In this section, we discuss the additive comparisons of optimal stopping values and supremum values for finite stage multiparameter stochastic processes.

Let $t \in \mathbf{N}^d$ and I, given in Section 1, be fixed. We find the best possible bound δ such that

$$D\big(\big\{X(z),\ z \in I\big\}\big) := E\bigg[\bigvee_{z\in I} X(z)\bigg] - V\big[\big\{X(z),\ z \in I\big\}\big] \leqslant \delta$$

for any $\{\mathcal{F}_z,\ z \in I\}$-adapted, $[0,1]$-valued stochastic processes $\{X(z),\ z \in I\}$ whose elements are mutually independent. For this purpose, we give an optimization problem to determine the best possible bound δ of prophet inequality.

We set $\mu := \bigvee_{r\in d(0)} V[\{X(z),\ z \in I,\ z \geqslant r\}]$ for $\{X(z),\ z \in I\}$.

Proposition 3.1. *For any processes $\{X(z),\ z \in I\}$ satisfying $\mu = 1$, we have $D(\{X(z),\ z \in I\}) = 0$.*

Proof. By the proof of Proposition 2.1, we obtain $V[\{1\ (z=0), \{X(z),\ z \in I - \{0\}\}\}] = 1$. Therefore, by Proposition 2.1, we have

$$\begin{aligned} D\big(\big\{X(z),\ z \in I\big\}\big) &\leqslant D\big(\big\{1\ (z=0), \big\{X(z),\ z \in I - \{0\}\big\}\big\}\big) \\ &= E\bigg[1 \vee \bigvee_{z\in I-\{0\}} X(z)\bigg] - V\big[\big\{1\ (z=0), \big\{X(z),\ z \in I - \{0\}\big\}\big\}\big] \\ &= 0. \qquad \square \end{aligned}$$

For the sake of simplicity of discussion in the remainder of this section, we restrict the class of the processes $\{X(z),\ z \in I\}$ satisfying the above conditions and $\mu < 1$, because of using Proposition 2.3 repeatedly.

The sequence of constants $C_1^{(|t|)}, C_1^{(|t|-1)}, \ldots, C_1^{(2)}, C_2^{(|t|)}, C_2^{(|t|-1)}, \ldots, C_2^{(2)}$ and the sequence of stochastic processes $\mathbf{Q}^{(|t|)}, \mathbf{Q}^{(|t|-1)}, \ldots, \mathbf{Q}^{(2)}, \mathbf{Y}^{(|t|)}, \mathbf{Y}^{(|t|-1)}, \ldots, \mathbf{Y}^{(2)}, \mathbf{W}^{(|t|)}, \mathbf{W}^{(|t|-1)}, \ldots, \mathbf{W}^{(2)}$ are defined by the following: To begin with, applying Propositions 2.1, 2.2 and 2.3 to $\{X(z),\ z \in I\}$ and defining $\mathbf{Q}$, $\mathbf{Y}$, $\mathbf{W}$ and C_1, C_2 in Propositions 2.2 and 2.3 by $\mathbf{Q}^{(|t|)}$, $\mathbf{Y}^{(|t|)}$, $\mathbf{W}^{(|t|)}$ and $C_1^{(|t|)}$, $C_2^{(|t|)}$ respectively, we have

$$\begin{aligned} D\big(\big\{X(z),\ z \in I\big\}\big) &\leqslant D\big(\big\{\mu, \big\{X(z),\ z \in I - \{0\}\big\}\big\}\big) \\ &\leqslant D\big(\big\{\mu, \big\{X(z),\ z \in I_{|t|-2} - \{0\}\big\}, \mathbf{Y}^{(|t|)}, \mathbf{Q}^{(|t|)}\big\}\big) \\ &= C_1^{(|t|)} D\big(\big\{\mu, \big\{X(z),\ z \in I_{|t|-2} - \{0\}\big\}, \mathbf{W}^{(|t|)}\big\}\big) + C_2^{(|t|)}. \end{aligned}$$

For the process $\{\mu, \{X(z),\ z \in I_{|t|-2} - \{0\}\}, \mathbf{W}^{(|t|)}\}$, we obtain

$$\mu = \bigvee_{r\in d(0)} V\big[\big\{\big\{X(z),\ z \in I_{|t|-2} - \{0\},\ z \geqslant r\big\}, \big\{W^{(|t|)}(z),\ z \in I_{|t|-1},\ |z| = |t|-1,\ z \geqslant r\big\}\big\}\big].$$

Next, applying Propositions 2.1, 2.2 and 2.3 to $\{\mu, \{X(z),\ z \in I_{|t|-2} - \{0\}\}, \mathbf{W}^{(|t|)}\}$ and defining $\mathbf{Q}$, $\mathbf{Y}$, $\mathbf{W}$ and C_1, C_2 in Propositions 2.2 and 2.3 by $\mathbf{Q}^{(|t|-1)}$, $\mathbf{Y}^{(|t|-1)}$, $\mathbf{W}^{(|t|-1)}$ and $C_1^{(|t|-1)}$, $C_2^{(|t|-1)}$ respectively, we have

$$D\big(\big\{\mu, \big\{X(z),\ z \in I_{|t|-2} - \{0\}\big\}, \mathbf{W}^{(|t|)}\big\}\big) \leqslant C_1^{(|t|-1)} D\big(\big\{\mu, \big\{X(z),\ z \in I_{|t|-3} - \{0\}\big\}, \mathbf{W}^{(|t|-1)}\big\}\big) + C_2^{(|t|-1)}.$$

By the same way as above, for the process $\{\mu, \{X(z),\ z \in I_{|t|-3} - \{0\}\}, \mathbf{W}^{(|t|-1)}\}$, we obtain

$$\mu = \bigvee_{r\in d(0)} V\big[\big\{\big\{X(z),\ z \in I_{|t|-3} - \{0\},\ z \geqslant r\big\}, \big\{W^{(|t|-1)}(z),\ z \in I_{|t|-2},\ |z| = |t|-2,\ z \geqslant r\big\}\big\}\big].$$

Applying Propositions 2.1, 2.2 and 2.3 to this process and defining $\mathbf{Q}$, $\mathbf{Y}$, $\mathbf{W}$ and C_1, C_2 in Propositions 2.2 and 2.3 by $\mathbf{Q}^{(|t|-2)}$, $\mathbf{Y}^{(|t|-2)}$, $\mathbf{W}^{(|t|-2)}$ and $C_1^{(|t|-2)}$, $C_2^{(|t|-2)}$ respectively, we have

$$D\big(\big\{\mu, \big\{X(z),\ z \in I_{|t|-3} - \{0\}\big\}, \mathbf{W}^{(|t|-1)}\big\}\big) \leqslant C_1^{(|t|-2)} D\big(\big\{\mu, \big\{X(z),\ z \in I_{|t|-4} - \{0\}\big\}, \mathbf{W}^{(|t|-2)}\big\}\big) + C_2^{(|t|-2)}.$$

By repeatedly using Propositions 2.1, 2.2 and 2.3, we obtain

$$\begin{aligned} D\big(\big\{X(z),\ z \in I\big\}\big) &\leqslant C_1^{(|t|)} D\big(\big\{\mu, \big\{X(z),\ z \in I_{|t|-2} - \{0\}\big\}, \mathbf{W}^{(|t|)}\big\}\big) + C_2^{(|t|)} \\ &\leqslant C_1^{(|t|)} C_1^{(|t|-1)} D\big(\big\{\mu, \big\{X(z),\ z \in I_{|t|-3} - \{0\}\big\}, \mathbf{W}^{(|t|-1)}\big\}\big) + C_1^{(|t|)} C_2^{(|t|-1)} + C_2^{(|t|)} \end{aligned}$$

$$\begin{aligned}
&\leqslant C_1^{(|t|)} C_1^{(|t|-1)} C_1^{(|t|-2)} D\big(\big\{\mu, \big\{X(z),\ z \in I_{|t|-4} - \{0\}\big\}, \mathbf{W}^{(|t|-2)}\big\}\big) \\
&\quad + C_1^{(|t|)} C_1^{(|t|-1)} C_2^{(|t|-2)} + C_1^{(|t|)} C_2^{(|t|-1)} + C_2^{(|t|)} \\
&\quad \vdots \\
&\leqslant C_1^{(|t|)} C_1^{(|t|-1)} \cdots C_1^{(2)} D\big(\big\{\mu, \mathbf{W}^{(2)}\big\}\big) \\
&\quad + C_1^{(|t|)} C_1^{(|t|-1)} \cdots C_1^{(3)} C_2^{(2)} \\
&\quad + C_1^{(|t|)} C_1^{(|t|-1)} \cdots C_1^{(4)} C_2^{(3)} \\
&\quad \vdots \\
&\quad + C_1^{(|t|)} C_1^{(|t|-1)} C_2^{(|t|-2)} + C_1^{(|t|)} C_2^{(|t|-1)} + C_2^{(|t|)}.
\end{aligned}$$

And also we have

$$\begin{aligned}
D\big(\big\{\mu, \mathbf{W}^{(2)}\big\}\big) &= E\Big[\mu \vee \bigvee_{r \in d(0)} W^{(2)}(r)\Big] - V\big[\big\{\mu, \mathbf{W}^{(2)}\big\}\big] \\
&= P\Big(\bigvee_{r \in d(0)} W^{(2)}(r) = 1\Big)(\mu \vee 1) + P\Big(\bigvee_{r \in d(0)} W^{(2)}(r) = 0\Big)(\mu \vee 0) - \mu \\
&= (1-\mu)\Big(1 - \prod_{r \in d(0)} \big(1 - \mu_1(r)\big)\Big),
\end{aligned}$$

where we set $\mu_1(r) = E[W^{(2)}(r)]$.

We note that all constants $C_i^{(j)}$ $(i = 1, 2,\ j = |t|, |t-1|, \ldots, 2)$ depend on the process $\{X(z),\ z \in I\}$, and therefore these constants are handled as a variable.

Then we have the following result.

Proposition 3.2. *The best possible bound δ such that*

$$D\big(\big\{X(z),\ z \in I\big\}\big) := E\Big[\bigvee_{z \in I} X(z)\Big] - V\big[\big\{X(z),\ z \in I\big\}\big] \leqslant \delta$$

for any $\{\mathcal{F}_z,\ z \in I\}$-adapted, $[0, 1]$-valued stochastic processes $\{X(z),\ z \in I\}$ whose elements are mutually independent and satisfying $\mu < 1$, is determined by the following nonlinear programming problem:

$$\begin{aligned}
\textit{maximize}\quad & C_1^{(|t|)} C_1^{(|t|-1)} \cdots C_1^{(2)} (1-\mu)\Big(1 - \prod_{r \in d(0)} \big(1 - \mu_1(r)\big)\Big) \\
& + C_1^{(|t|)} C_1^{(|t|-1)} \cdots C_1^{(3)} C_2^{(2)} \\
& + C_1^{(|t|)} C_1^{(|t|-1)} \cdots C_1^{(4)} C_2^{(3)} \\
& \vdots \\
& + C_1^{(|t|)} C_1^{(|t|-1)} C_2^{(|t|-2)} + C_1^{(|t|)} C_2^{(|t|-1)} + C_2^{(|t|)} \\
\textit{subject to}\quad & \mu = \bigvee_{r \in d(0)} \mu_1(r), \\
& 0 \leqslant \mu_1(r) < 1 \quad \big(r \in d(0)\big), \\
& C_1^{(k)}, C_2^{(k)}\ \big(k = 2, 3, \ldots, |t|\big) \textit{ are defined in Propositions } 2.1, 2.2 \textit{ and } 2.3.
\end{aligned}$$

Finally we state an example.

Example 3.1. We consider the case $d = 2$, $t = e_1 + e_2$.

We set $\mu(e_1 + e_2) := E[X(e_1 + e_2)]$, $\mu(e_1) := V[\{X(e_1), X(e_1 + e_2)\}]$, $\mu(e_2) := V[\{X(e_2), X(e_1 + e_2)\}]$, $\mu := \max\{\mu(e_1), \mu(e_2)\}$.

In the case that $\mu(e_1 + e_2) = 1$, that is, $X(e_1 + e_2) = 1$ a.e., we have

$$\mu(e_1) = \mu(e_2) = \mu = 1,$$

and then

$$\begin{aligned}
&D\big(\{X(0), X(e_1), X(e_2), X(e_1+e_2)\}\big)\\
&= E\big[X(0) \vee X(e_1) \vee X(e_2) \vee X(e_1+e_2)\big] - V\big[\{X(0), X(e_1), X(e_2), X(e_1+e_2)\}\big]\\
&= 0.
\end{aligned}$$

We assume that $\mu(e_1+e_2) < 1$. By Propositions 2.1 and 2.2, we have

$$\begin{aligned}
&D\big(\{X(0), X(e_1), X(e_2), X(e_1+e_2)\}\big)\\
&\leqslant E\big[\mu \vee Y(e_1) \vee Y(e_2) \vee Q(e_1+e_2)\big] - V\big[\{\mu, \{Y(e_1), Y(e_2)\}, \{Q(e_1+e_2)\}\}\big]\\
&= E\big[\mu \vee Y(e_1) \vee Y(e_2) \vee Q(e_1+e_2)\big] - \mu\\
&= P\big(Q(e_1+e_2)=1\big)E\big[\mu \vee Y(e_1) \vee Y(e_2) \vee 1\big] + P\big(Q(e_1+e_2)=0\big)E\big[\mu \vee Y(e_1) \vee Y(e_2) \vee 0\big] - \mu\\
&= E\big[X(e_1+e_2)\big] + \big(1 - E\big[X(e_1+e_2)\big]\big)\\
&\quad \times \big\{P\big(Y(e_1) = E\big[X(e_1+e_2)\big]\big)E\big[\mu \vee E\big[X(e_1+e_2)\big] \vee Y(e_2)\big] + P\big(Y(e_1)=1\big)E\big[\mu \vee 1 \vee Y(e_2)\big]\\
&\quad + P\big(Y(e_1) = X(e_1)\big)E\big[\mu \vee X(e_1) \vee Y(e_2)\big]\big\}\\
&= E\big[X(e_1+e_2)\big] + E\big[\big(X(e_1) - E\big[X(e_1+e_2)\big]\big)^+\big] + \big(1 - V\big[\{X(e_1), X(e_1+e_2)\}\big]\big)\\
&\quad \times \left\{\frac{E[(X(e_2) - E[X(e_1+e_2)])^+] + \mu(1 - E[X(e_1+e_2)] - E[(X(e_2) - E[X(e_1+e_2)])^+])}{1 - E[X(e_1+e_2)]}\right\}\\
&= \mu(e_1) + \big(1-\mu(e_1)\big)\frac{\mu(e_2) - \mu(e_1+e_2) + \mu(1-\mu(e_2))}{1-\mu(e_1+e_2)} - \mu
\end{aligned}$$

and consider the following nonlinear programming problem:

$$\begin{aligned}
\text{maximize}\quad & \mu(e_1) + \big(1-\mu(e_1)\big)\frac{\mu(e_2) - \mu(e_1+e_2) + \mu(1-\mu(e_2))}{1-\mu(e_1+e_2)} - \mu\\
\text{subject to}\quad & \mu = \max\{\mu(e_1), \mu(e_2)\},\\
& \mu(e_1+e_2) \leqslant \mu(e_1), \qquad \mu(e_1+e_2) \leqslant \mu(e_2),\\
& 0 \leqslant \mu(e_1+e_2) < 1,\\
& 0 \leqslant \mu(e_1), \mu(e_2) \leqslant 1.
\end{aligned}$$

The optimal value of this problem is approximately 0.38 and therefore 0.38 is a universal bound.

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