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Pullback attractors for non-autonomous reaction–diffusion equations with dynamical boundary conditions

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ABSTRACT

In this paper we prove the existence and uniqueness of a weak solution for a non-autonomous reaction–diffusion model with dynamical boundary conditions. After that, a continuous dependence result is established via an energy method, including in particular some compactness properties. Finally, the precedent results are used in order to ensure the existence of minimal pullback attractors in the frameworks of universes of fixed bounded sets and that given by a tempered growth condition. The relation among these families is also discussed.

1. Introduction and setting of the problem

Partial differential equations with dynamical boundary conditions arise for example in hydrodynamics and the heat transfer theory. For instance, they allow to model heat flow inside the considered domain subject to nonlinear heating or cooling at the boundary, or heat transfer in a solid in contact with a moving fluid, in thermoelasticity, heat transfer in two mediums, etc. This type of problems has been studied by many authors (e.g., cf. [1,2,7,8,10,12,13,15] and the references therein).

Several approaches have been used for these problems, like the theory of semigroups, with Bessel potential and Besov spaces, and of course the variational setting as well. Some questions addressed concerning these models are the local and global existence of solutions or blow-up phenomena. Namely, in [2] the critical exponents allowed in the nonlinearities such that the problem is well-posed are studied.

Another question is the study of these problems under the introduction of singular perturbations. For instance, in [15] the behavior of solutions of a singularly perturbed model (damped wave equation) when the introduced parameter goes to zero and the relation with the limit problem is analyzed.

A different sort of question, with a great variety of results, is the long-time behavior of the (global) solutions. For an autonomous model, the existence of a global attractor is, for instance, studied in [9], although the nonlinearity is the same in the domain and in the boundary (see also [17]). For a non-autonomous reaction–diffusion equation and using the approach of skew-product formulation, the existence of a uniform attractor is established in [16]. But to our knowledge, there does not seem to be in the literature any study of the existence of pullback attractors for non-autonomous dynamical systems associated to this kind of problems (up to the stochastic framework, e.g., cf. [5]).

Let us introduce the model we will be involved with in this paper. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary $\partial\Omega$.

We consider the following problem for a non-autonomous reaction–diffusion equation with dynamical boundary condition,

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$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \kappa u + f(u) = h(t) & \text{in } \Omega \times (\tau, \infty), \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial \vec{n}} + g(u) = \rho(t) & \text{on } \partial\Omega \times (\tau, \infty), \\ u(x, \tau) = u_\tau(x), & \text{for } x \in \Omega, \\ u(x, \tau) = \psi_\tau(x), & \text{for } x \in \partial\Omega, \end{cases} \tag{1}$$

where $\vec{n}$ is the outer normal to $\partial\Omega$, $\tau \in \mathbb{R}$ is an initial time, and

$$\kappa > 0, \qquad u_\tau \in L^2(\Omega), \qquad \psi_\tau \in L^2(\partial\Omega), \tag{2}$$

$$h \in L^2_{loc}\big(\mathbb{R}; L^2(\Omega)\big), \qquad \rho \in L^2_{loc}\big(\mathbb{R}; L^2(\partial\Omega)\big), \tag{3}$$

are given.

We also assume that the functions f and $g \in C(\mathbb{R})$ are given, and satisfy that there exist constants $p \geqslant 2$, $q \geqslant 2$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$, and $l > 0$, such that

$$\alpha_1 |s|^p - \beta \leqslant f(s)s \leqslant \alpha_2 |s|^p + \beta, \quad \text{for all } s \in \mathbb{R}, \tag{4}$$

$$\alpha_1 |s|^q - \beta \leqslant g(s)s \leqslant \alpha_2 |s|^q + \beta, \quad \text{for all } s \in \mathbb{R}, \tag{5}$$

$$\big(f(s) - f(r)\big)(s - r) \geqslant -l(s - r)^2, \quad \text{for all } s, r \in \mathbb{R}, \tag{6}$$

and

$$\big(g(s) - g(r)\big)(s - r) \geqslant -l(s - r)^2, \quad \text{for all } s, r \in \mathbb{R}. \tag{7}$$

It is easy to see from (4) and (5) that there exists a constant $C > 0$, such that

$$\big|f(s)\big| \leqslant C\big(1 + |s|^{p-1}\big), \qquad \big|g(s)\big| \leqslant C\big(1 + |s|^{q-1}\big), \quad \text{for all } s \in \mathbb{R}. \tag{8}$$

Remark 1. If u is regular enough, then a compatibility condition for problem (1) is that ψ_τ must coincide with the restriction to $\partial\Omega$ of u_τ, and therefore the fourth equation in (1) is omitted. Nevertheless, this equation seems necessary for the concept of weak solution (see below).

Remark 2. If $p > 2$, the assumption $\kappa > 0$ is not necessary. Indeed, if $\kappa \leqslant 0$, then $f(u) + \kappa u = \bar{f}(u) + u$, where $\bar{f}(s) := f(s) + (\kappa - 1)s$, satisfies

$$\big(\bar{f}(s) - \bar{f}(r)\big)(s - r) \geqslant -(l - \kappa + 1)(s - r)^2, \quad \text{for all } s, r \in \mathbb{R},$$

and taking into account Young's inequality, if $p > 2$,

$$\frac{\alpha_1}{2}|s|^p - \beta - \frac{p-2}{p}\left(\frac{4}{p\alpha_1}\right)^{2/(p-2)} (1 - \kappa)^{p/(p-2)} \leqslant s\bar{f}(s) \leqslant \alpha_2 |s|^p + \beta,$$

for all $s \in \mathbb{R}$.

In this paper we study the existence of pullback attractors for the process associated to (1). As we mentioned before, we only have references in the literature of this approach in the stochastic context, with the help of random dynamical systems. In that sense, a particularly interesting situation is treated in [5]. There, the authors obtain the existence of a random attractor for a general class of stochastic parabolic equations with dynamical boundary conditions, under the restrictive assumptions $p = q$ and $|f(s) - g(s)| \leqslant c(1 + |s|)$. We will obtain the existence of pullback attractors for (1) without these assumptions, using a continuous dependence result which is proved using an energy method.

The structure of the paper is as follows. In Section 2 we give a weak formulation of the problem, the concept of weak solution, and establish the existence and uniqueness of solution using the monotonicity method. A continuous dependence result with respect to initial data, which is the main key for the asymptotic compactness we will require later, is addressed in Section 3. There we use an energy method that strengthens the energy equality satisfied by the solutions. A brief recall on abstract results about the existence of minimal pullback attractors is given in Section 4. In Section 5, the main goals of proving the existence of different families of pullback attractors for different universes, and the relation among them under certain suitable assumption, are finally established.

2. Existence and uniqueness of solution

We denote by $(\cdot,\cdot)_\Omega$ (respectively, $(\cdot,\cdot)_{\partial\Omega}$) the inner product in $L^2(\Omega)$ (respectively, in $L^2(\partial\Omega)$), and by $|\cdot|_\Omega$ (respectively, $|\cdot|_{\partial\Omega}$) the associated norm. We will also denote $(\cdot,\cdot)_\Omega$ (respectively, $(\cdot,\cdot)_{\partial\Omega}$) the inner product in $(L^2(\Omega))^N$, and the duality product between $L^{p'}(\Omega)$ and $L^p(\Omega)$ (respectively, the duality product between $L^{q'}(\partial\Omega)$ and $L^q(\partial\Omega)$). If $r\neq 2$, we will denote $|\cdot|_{r,\Omega}$ (respectively $|\cdot|_{r,\partial\Omega}$) the norm in $L^r(\Omega)$ (respectively in $L^r(\partial\Omega)$). By $\|\cdot\|_\Omega$ we denote the norm in $H^1(\Omega)$, which is associated to the inner product $((\cdot,\cdot))_\Omega := (\nabla\cdot,\nabla\cdot)_\Omega + (\cdot,\cdot)_\Omega$.

We use the notation γ_0 for the trace operator $u\mapsto u|_{\partial\Omega}$. The trace operator belongs to $\mathcal{L}(H^1(\Omega),H^{1/2}(\partial\Omega))$, and we will use $\|\gamma_0\|$ to denote the norm of γ_0 in this space.

Finally, we will use $\|\cdot\|_{\partial\Omega}$ to denote the norm in $H^{1/2}(\partial\Omega)$, which is given by $\|\phi\|_{\partial\Omega}=\inf\{\|v\|_\Omega\colon\ \gamma_0(v)=\phi\}$. We remember that with this norm, $H^{1/2}(\partial\Omega)$ is a Hilbert space.

Definition 3. A weak solution of (1) is a pair of functions (u,ψ), satisfying

$$u\in C\big([\tau,\infty);L^2(\Omega)\big),\qquad \psi\in C\big([\tau,\infty);L^2(\partial\Omega)\big), \tag{9}$$

$$u\in L^2\big(\tau,T;H^1(\Omega)\big)\cap L^p\big(\tau,T;L^p(\Omega)\big),\quad \text{for all } T>\tau, \tag{10}$$

$$\psi\in L^2\big(\tau,T;H^{1/2}(\partial\Omega)\big)\cap L^q\big(\tau,T;L^q(\partial\Omega)\big),\quad \text{for all } T>\tau, \tag{11}$$

$$\gamma_0\big(u(t)\big)=\psi(t),\quad \text{a.e. } t\in(\tau,\infty), \tag{12}$$

$$\begin{cases} \dfrac{d}{dt}(u(t),v)_\Omega+\dfrac{d}{dt}\big(\psi(t),\gamma_0(v)\big)_{\partial\Omega}+\big(\nabla u(t),\nabla v\big)_\Omega+\kappa\big(u(t),v\big)_\Omega \\ \quad +\big(f\big(u(t)\big),v\big)_\Omega+\big(g\big(\gamma_0(u(t))\big),\gamma_0(v)\big)_{\partial\Omega}=\big(h(t),v\big)_\Omega+\big(\rho(t),\gamma_0(v)\big)_{\partial\Omega} \\ \quad \text{in } \mathcal{D}'(\tau,\infty), \text{ for all } v\in H^1(\Omega)\cap L^p(\Omega) \text{ such that } \gamma_0(v)\in L^q(\partial\Omega), \end{cases} \tag{13}$$

$$u(\tau)=u_\tau,\quad \text{and}\quad \psi(\tau)=\psi_\tau. \tag{14}$$

Remark 4. If a pair of functions (u,ψ) satisfies (10)–(13), then there exists a version of these functions satisfying (9). The function ψ is the $L^2(\partial\Omega)$-continuous version of $\gamma_0(u)$ (see (17)–(19) below).

We have the following result.

Theorem 5. *Under the assumptions* (2)–(7)*, there exists a unique solution* $(u,\psi)=(u(\cdot;\tau,u_\tau,\psi_\tau),\psi(\cdot;\tau,u_\tau,\psi_\tau))$ *of the problem* (1)*. Moreover, this solution satisfies the energy equality*

$$\begin{aligned}&\frac12\frac{d}{dt}\big(|u(t)|_\Omega^2+|\psi(t)|_{\partial\Omega}^2\big)+|\nabla u(t)|_\Omega^2+\kappa|u(t)|_\Omega^2+\big(f\big(u(t)\big),u(t)\big)_\Omega+\big(g\big(\psi(t)\big),\psi(t)\big)_{\partial\Omega}\\ &\quad=\big(h(t),u(t)\big)_\Omega+\big(\rho(t),\psi(t)\big)_{\partial\Omega},\quad \textit{a.e. } t>\tau.\end{aligned} \tag{15}$$

Proof. The proof of this result is standard (see for example [9]). For the sake of completeness, we give a sketch of a proof.

Let us consider the Hilbert space

$$H:=L^2(\Omega)\times L^2(\partial\Omega),$$

with the natural inner product $((v,\phi),(w,\varphi))_H=(v,w)_\Omega+(\phi,\varphi)_{\partial\Omega}$, which in particular induces the norm $|(\cdot,\cdot)|_H$ given by

$$\big|(v,\phi)\big|_H^2=|v|_\Omega^2+|\phi|_{\partial\Omega}^2,\quad (v,\phi)\in H.$$

Let us also consider the space

$$V_1:=\big\{\big(v,\gamma_0(v)\big)\colon\ v\in H^1(\Omega)\big\}.$$

We note that V_1 is a closed vector subspace of $H^1(\Omega)\times H^{1/2}(\partial\Omega)$, and therefore, with the norm $\|(\cdot,\cdot)\|_{V_1}$ given by

$$\big\|\big(v,\gamma_0(v)\big)\big\|_{V_1}^2=\|v\|_\Omega^2+\big\|\gamma_0(v)\big\|_{\partial\Omega}^2,\quad \big(v,\gamma_0(v)\big)\in V_1,$$

V_1 is a Hilbert space.

On the other hand, V_1 is densely embedded in H. In fact, if we consider $(w,\phi)\in H$ such that

$$(v,w)_\Omega+\big(\gamma_0(v),\phi\big)_{\partial\Omega}=0,\quad \text{for all } v\in H^1(\Omega),$$

in particular, we have

$$(v, w)_\Omega = 0, \quad \text{for all } v \in H_0^1(\Omega),$$

and therefore $w = 0$. Consequently,

$$\big(\gamma_0(v), \phi\big)_{\partial\Omega} = 0, \quad \text{for all } v \in H^1(\Omega),$$

and then, as $H^{1/2}(\partial\Omega) = \gamma_0(H^1(\Omega))$ is dense in $L^2(\partial\Omega)$, we have that $\phi = 0$.

Now, on the space V_1 we define a continuous symmetric linear operator $A_1 : V_1 \to V_1'$, given by

$$\big\langle A_1\big(\big(v, \gamma_0(v)\big)\big), \big(w, \gamma_0(w)\big)\big\rangle = (\nabla v, \nabla w)_\Omega + \kappa (v, w)_\Omega, \quad \forall v, w \in H^1(\Omega).$$

We observe that A_1 is coercive. In fact, we have

$$\begin{aligned}\big\langle A_1\big(\big(v, \gamma_0(v)\big), \big(v, \gamma_0(v)\big)\big)\big\rangle &\geqslant \min\{1, \kappa\}\|v\|_\Omega^2 \\ &= \frac{1}{1 + \|\gamma_0\|^2}\min\{1, \kappa\}\|v\|_\Omega^2 + \frac{\|\gamma_0\|^2}{1 + \|\gamma_0\|^2}\min\{1, \kappa\}\|v\|_\Omega^2 \\ &\geqslant \frac{1}{1 + \|\gamma_0\|^2}\min\{1, \kappa\}\big\|\big(v, \gamma_0(v)\big)\big\|_{V_1}^2, \end{aligned} \tag{16}$$

for all $v \in H^1(\Omega)$.

Let us denote

$$\begin{aligned}&V_2 = L^p(\Omega) \times L^2(\partial\Omega), \qquad V_3 = L^2(\Omega) \times L^q(\partial\Omega), \\ &A_2(v, \phi) = \big(f(v), 0\big), \qquad A_3(v, \phi) = \big(0, g(\phi)\big), \qquad \vec{h}(t) = \big(h(t), \rho(t)\big).\end{aligned}$$

From (8) one deduces that $A_i : V_i \to V_i'$, for $i = 2, 3$.

Observe also that by (3),

$$\vec{h} \in L^2_{loc}(\mathbb{R}; H) \subset L^2_{loc}\big(\mathbb{R}; V_1'\big).$$

With this notation, and denoting $V = \bigcap_{i=1}^3 V_i$, $p_1 = 2$, $p_2 = p$, $p_3 = q$, $\vec{u} = (u, \psi)$, one has that (9)–(14) is equivalent to

$$\vec{u} \in C\big([\tau, \infty); H\big), \qquad \vec{u} \in \bigcap_{i=1}^3 L^{p_i}(\tau, T; V_i), \quad \text{for all } T > \tau, \tag{17}$$

$$(\vec{u})'(t) + \sum_{i=1}^3 A_i\big(\vec{u}(t)\big) = \vec{h}(t) \quad \text{in } \mathcal{D}'\big(\tau, \infty; V'\big), \tag{18}$$

$$\vec{u}(\tau) = (u_\tau, \psi_\tau). \tag{19}$$

Applying a slight modification of [13, Ch. 2, Th. 1.4], it is not difficult to see that problem (17)–(19) has a unique solution. Moreover, $\vec{u}$ satisfies the energy equality

$$\frac{1}{2}\frac{d}{dt}\big\|\vec{u}(t)\big\|_H^2 + \sum_{i=1}^3 \big\langle A_i\big(\vec{u}(t)\big), \vec{u}(t)\big\rangle_i = \big(\vec{h}(t), \vec{u}(t)\big)_H \quad \text{a.e. } t > \tau,$$

where $\langle\cdot,\cdot\rangle_i$ denotes the duality product between V_i' and V_i.

This last equality turns out to be just (15). □

Remark 6. The assumption $\kappa > 0$ is not necessary for the existence and uniqueness of weak solution to (1).

3. A continuous dependence result

In this section, we prove a result on continuous dependence of the solutions of (1) with respect to the initial datum (u_τ, ϕ_τ). This result will be crucial in the proof of the existence of pullback attractors for (1).

Theorem 7. *Under the assumptions* (2)–(7)*, let* $\{(u_\tau^{(n)}, \psi_\tau^{(n)})\}_{n\geqslant 1} \subset L^2(\Omega) \times L^2(\partial\Omega)$ *be a sequence such that*

$$\big(u_\tau^{(n)}, \psi_\tau^{(n)}\big) \rightharpoonup (u_\tau, \psi_\tau) \quad \text{weakly in } L^2(\Omega) \times L^2(\partial\Omega). \tag{20}$$

Let us denote $\vec{u}^{(n)} = (u^{(n)}, \psi^{(n)}) = (u(\cdot; \tau, u_\tau^{(n)}, \psi_\tau^{(n)}), \psi(\cdot; \tau, u_\tau^{(n)}, \psi_\tau^{(n)}))$ *and* $\vec{u} = (u, \psi) = (u(\cdot; \tau, u_\tau, \psi_\tau), \psi(\cdot; \tau, u_\tau, \psi_\tau))$, *the corresponding weak solutions of* (1). *Then, for all* $T > \tau$,

$$\begin{aligned}
&\vec{u}^{(n)} \rightharpoonup \vec{u} && \text{weakly in } L^2(\tau, T; H^1(\Omega)) \times L^2(\tau, T; H^{1/2}(\partial\Omega)),\\
&\vec{u}^{(n)} \overset{*}{\rightharpoonup} \vec{u} && \text{weakly-star in } L^\infty(\tau, T; L^2(\Omega)) \times L^\infty(\tau, T; L^2(\partial\Omega)),\\
&\vec{u}^{(n)} \rightharpoonup \vec{u} && \text{weakly in } L^p(\tau, T; L^p(\Omega)) \times L^q(\tau, T; L^q(\partial\Omega)),\\
&f(u^{(n)}) \rightharpoonup f(u) && \text{weakly in } L^{p'}(\tau, T; L^{p'}(\Omega)),\\
&g(\psi^{(n)}) \rightharpoonup g(\psi) && \text{weakly in } L^{q'}(\tau, T; L^{q'}(\partial\Omega)),\\
&\vec{u}^{(n)} \to \vec{u} && \text{strongly in } L^2(\tau, T; L^2(\Omega)) \times L^2(\tau, T; L^2(\partial\Omega)), && (21)\\
&\vec{u}^{(n)}(t) \to \vec{u}(t) && \text{strongly in } L^2(\Omega) \times L^2(\partial\Omega), \text{ for all } t > \tau. && (22)
\end{aligned}$$

Proof. For the sake of clarity, we split the proof in two parts. Firstly, for all but last of the above convergences we only require to obtain suitable a priori estimates and well-known compactness results; secondly, for the last convergence, we use an energy method that strength the energy equality satisfied by the solutions.

Step 1. All but last of the convergences in the above statement hold.

By (15) applied to $\vec{u}^{(n)}$, and taking into account (4), (5) and (16), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\left|u^{(n)}(t)\right|_\Omega^2 + \left|\psi^{(n)}(t)\right|_{\partial\Omega}^2\right) + \frac{2\min\{1, \kappa\}}{1 + \|\gamma_0\|^2}\left(\left\|u^{(n)}(t)\right\|_\Omega^2 + \left\|\psi^{(n)}(t)\right\|_{\partial\Omega}^2\right) + 2\alpha_1\left(\left|u^{(n)}(t)\right|_{p,\Omega}^p + \left|\psi^{(n)}(t)\right|_{q,\partial\Omega}^q\right)\\
&\quad \leqslant 2\beta\left(|\Omega| + |\partial\Omega|\right) + \left|h(t)\right|_\Omega^2 + \left|\rho(t)\right|_{\partial\Omega}^2 + \left|u^{(n)}(t)\right|_\Omega^2 + \left|\psi^{(n)}(t)\right|_{\partial\Omega}^2, \qquad (23)
\end{aligned}$$

a.e. $t > \tau$.

By (20) in particular we know that there exists a constant $C > 0$ such that

$$\left|u_\tau^{(n)}\right|_\Omega^2 + \left|\psi_\tau^{(n)}\right|_{\partial\Omega}^2 \leqslant C \quad \text{for all } n \geqslant 1.$$

Thus, integrating (23) between τ and t, and applying Gronwall lemma, we see that the sequence $\{u^{(n)}\}$ is bounded in $L^2(\tau, T; H^1(\Omega)) \cap C([\tau, T]; L^2(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$, and the sequence $\{\psi^{(n)}\}$ is bounded in $L^2(\tau, T; H^{1/2}(\partial\Omega)) \cap C([\tau, T]; L^2(\partial\Omega)) \cap L^q(\tau, T; L^q(\partial\Omega))$, for all $T > \tau$.

Then, taking into account (8) and (13) for $(u^{(n)}, \psi^{(n)})$, we deduce that the sequence $\{f(u^{(n)})\}$ is bounded in $L^{p'}(\tau, T; L^{p'}(\Omega))$ and the sequence $\{g(\psi^{(n)})\}$ is bounded in $L^{q'}(\tau, T; L^{q'}(\partial\Omega))$. Moreover, the sequence of time derivatives $\{(u^{(n)})'\}$ is bounded in $L^2(\tau, T; (H^1(\Omega))') + L^{p'}(\tau, T; L^{p'}(\Omega)) \subset L^{p'}(\tau, T; (H^1(\Omega) \cap L^p(\Omega))')$, and finally, the sequence of time derivatives $\{(\psi^{(n)})'\}$ is bounded in $L^2(\tau, T; (H^{1/2}(\partial\Omega))') + L^{q'}(\tau, T; L^{q'}(\partial\Omega)) \subset L^{q'}(\tau, T; (H^{1/2}(\partial\Omega) \cap L^q(\partial\Omega))')$, for all $T > \tau$.

Let us fix $T > \tau$. Taking into account the compactness of the injection of $H^1(\Omega)$ into $L^2(\Omega)$, and the compactness of the injection of $H^{1/2}(\partial\Omega)$ into $L^2(\partial\Omega)$, from the boundedness results above and the Aubin–Lions compactness lemma (e.g., cf. [13]), we deduce that there exist a subsequence $\{(u^{(n')}, \psi^{(n')})\}_{n' \geqslant 1} \subset \{(u^{(n)}, \psi^{(n)})\}_{n \geqslant 1}$ and functions $\hat{u} \in L^2(\tau, T; H^1(\Omega)) \cap L^\infty(\tau, T; L^2(\Omega)) \cap L^p(\tau, T; L^p(\Omega))$, $\hat{\psi} \in L^2(\tau, T; H^{1/2}(\partial\Omega)) \cap L^\infty(\tau, T; L^2(\partial\Omega)) \cap L^q(\tau, T; L^q(\partial\Omega))$, $\hat{f} \in L^{p'}(\tau, T; L^{p'}(\Omega))$, $\hat{g} \in L^{q'}(\tau, T; L^{q'}(\partial\Omega))$, $\xi_T \in L^2(\Omega)$, and $\eta_T \in L^2(\partial\Omega)$, such that

$$\begin{aligned}
&\vec{u}^{(n')} \rightharpoonup (\hat{u}, \hat{\psi}) && \text{weakly in } L^2(\tau, T; H^1(\Omega)) \times L^2(\tau, T; H^{1/2}(\partial\Omega)),\\
&\vec{u}^{(n')} \overset{*}{\rightharpoonup} (\hat{u}, \hat{\psi}) && \text{weakly-star in } L^\infty(\tau, T; L^2(\Omega)) \times L^\infty(\tau, T; L^2(\partial\Omega)),\\
&\vec{u}^{(n')} \rightharpoonup (\hat{u}, \hat{\psi}) && \text{weakly in } L^p(\tau, T; L^p(\Omega)) \times L^q(\tau, T; L^q(\partial\Omega)),\\
&f(u^{(n')}) \rightharpoonup \hat{f} && \text{weakly in } L^{p'}(\tau, T; L^{p'}(\Omega)), && (24)\\
&g(\psi^{(n')}) \rightharpoonup \hat{g} && \text{weakly in } L^{q'}(\tau, T; L^{q'}(\partial\Omega)), && (25)\\
&\vec{u}^{(n')} \to (\hat{u}, \hat{\psi}) && \text{strongly in } L^2(\tau, T; L^2(\Omega)) \times L^2(\tau, T; L^2(\partial\Omega)),\\
&u^{(n')} \to \hat{u} && \text{a.e. in } \Omega \times (\tau, T), && (26)\\
&\psi^{(n')} \to \hat{\psi} && \text{a.e. in } \partial\Omega \times (\tau, T), && (27)\\
&\vec{u}^{(n')}(T) \rightharpoonup (\xi_T, \eta_T) && \text{weakly in } L^2(\Omega) \times L^2(\partial\Omega).
\end{aligned}$$

By the continuity of f and g, from (24), (25), (26), and (27), one deduces (see [13, Ch. 1, Lem. 1.3]) that $\hat{f} = f(\hat{u})$ and $\hat{g} = g(\hat{\psi})$. Now, it is a standard matter to deduce from (20) and the above convergences, that

$$\gamma_0(\hat{u}(t)) = \hat{\psi}(t), \quad \text{a.e. } t \in (\tau, T), \tag{28}$$

$$\begin{cases} \dfrac{d}{dt}(\hat{u}(t), v)_\Omega + \dfrac{d}{dt}(\hat{\psi}(t), \gamma_0(v))_{\partial\Omega} + (\nabla\hat{u}(t), \nabla v)_\Omega + \kappa(\hat{u}(t), v)_\Omega + (f(\hat{u}(t)), v)_\Omega \\ \quad + (g(\gamma_0(\hat{u}(t))), \gamma_0(v))_{\partial\Omega} = (h(t), v)_\Omega + (\rho(t), \gamma_0(v))_{\partial\Omega} \\ \text{in } \mathcal{D}'(\tau, T), \text{ for all } v \in H^1(\Omega) \cap L^p(\Omega), \text{ such that } \gamma_0(v) \in L^q(\partial\Omega), \end{cases} \tag{29}$$

$$\hat{u}(\tau) = u_\tau, \qquad \hat{\psi}(\tau) = \psi_\tau, \tag{30}$$

and

$$(\hat{u}(T), \hat{\psi}(T)) = (\xi_T, \eta_T). \tag{31}$$

Consequently, by uniqueness of solution to (28)–(30), we deduce that $(\hat{u}, \hat{\psi})$ coincides with the restriction to $[\tau, T]$ of $\vec{u} = (u, \psi)$, the above convergences hold for the whole sequence $\{(u^{(n)}, \psi^{(n)})\}_{n\geqslant 1}$, and therefore, by the arbitrariness of $T > \tau$, all but last convergences in the statement are satisfied, as we wanted to prove.

Step 2. We prove now that (22) holds.

From above, and by (31), we also deduce that

$$\left(u^{(n)}(t), \psi^{(n)}(t)\right) \rightharpoonup \left(u(t), \psi(t)\right) \quad \text{weakly in } L^2(\Omega) \times L^2(\partial\Omega), \text{ for all } t > \tau. \tag{32}$$

Now, we will prove that

$$\left|u^{(n)}(t)\right|_\Omega^2 + \left|\psi^{(n)}(t)\right|_{\partial\Omega}^2 \to \left|u(t)\right|_\Omega^2 + \left|\psi(t)\right|_{\partial\Omega}^2, \quad \text{for all } t > \tau, \tag{33}$$

which jointly with (32) will imply (22).

In order to prove (33), observe that from (21) we deduce in particular that for any subsequence $\{(u^{(n')}, \psi^{(n')})\}_{n'\geqslant 1} \subset \{(u^{(n)}, \psi^{(n)})\}_{n\geqslant 1}$ there exists another subsequence $\{(u^{(n'')}, \psi^{(n'')})\}_{n''\geqslant 1} \subset \{(u^{(n')}, \psi^{(n')})\}_{n'\geqslant 1}$ such that

$$\left|u^{(n'')}(t)\right|_\Omega^2 + \left|\psi^{(n'')}(t)\right|_{\partial\Omega}^2 \to \left|u(t)\right|_\Omega^2 + \left|\psi(t)\right|_{\partial\Omega}^2, \quad \text{a.e. } t > \tau. \tag{34}$$

Let us define

$$J(t) := \frac{1}{2}\left(\left|u(t)\right|_\Omega^2 + \left|\psi(t)\right|_{\partial\Omega}^2\right) - \beta\left(|\Omega| + |\partial\Omega|\right)t - \int_\tau^t \left[\left(h(s), u(s)\right)_\Omega + \left(\rho(s), \psi(s)\right)_{\partial\Omega}\right] ds,$$

and

$$J_{n''}(t) := \frac{1}{2}\left(\left|u^{(n'')}(t)\right|_\Omega^2 + \left|\psi^{(n'')}(t)\right|_{\partial\Omega}^2\right) - \beta\left(|\Omega| + |\partial\Omega|\right)t - \int_\tau^t \left[\left(h(s), u^{(n'')}(s)\right)_\Omega + \left(\rho(s), \psi^{(n'')}(s)\right)_{\partial\Omega}\right] ds,$$

for all $t \geqslant \tau$.

It is clear that J and $J_{n''}$ are well defined continuous functions on $[\tau, \infty)$, and by (21), if we prove that

$$J_{n''}(t) \to J(t) \quad \text{for all } t > \tau, \tag{35}$$

then (33) will hold.

From (21) and (34), we have that

$$J_{n''}(t) \to J(t) \quad \text{a.e. } t \in (\tau, \infty). \tag{36}$$

On the other hand, from the energy equality, (4), and (5), we obtain that J and $J_{n''}$ are non-increasing functions of t.

Let us fix $t \in (\tau, \infty)$, and $\varepsilon > 0$. From (36) and the continuity of J, we can take $t_2 < t < t_1$ such that

$$J_{n''}(t_i) \to J(t_i), \quad \text{as } n'' \to \infty, i = 1, 2, \tag{37}$$

and

$$J(t_2) - J(t_1) = \left|J(t_2) - J(t)\right| + \left|J(t) - J(t_1)\right| \leqslant \varepsilon.$$

From this inequality and the non-increasing character of $J_{n''}$, we have

$$\begin{aligned} J_{n''}(t)-J(t) &= J_{n''}(t)-J_{n''}(t_2)+J_{n''}(t_2)-J(t_2)+J(t_2)-J(t) \\ &\leqslant \left|J_{n''}(t_2)-J(t_2)\right|+\left|J(t_2)-J(t)\right| \\ &\leqslant \left|J_{n''}(t_2)-J(t_2)\right|+\varepsilon. \end{aligned} \tag{38}$$

Analogously, we have

$$\begin{aligned} J(t)-J_{n''}(t) &= J(t)-J(t_1)+J(t_1)-J_{n''}(t_1)+J_{n''}(t_1)-J_{n''}(t) \\ &\leqslant \left|J(t)-J(t_1)\right|+\left|J(t_1)-J_{n''}(t_1)\right| \\ &\leqslant \varepsilon+\left|J(t_1)-J_{n''}(t_1)\right|. \end{aligned} \tag{39}$$

From (37)–(39), we deduce that

$$\limsup_{n''\to\infty}\left|J(t)-J_{n''}(t)\right|\leqslant\varepsilon,$$

and therefore, as $\varepsilon>0$ is arbitrary, we obtain (35). $\square$

4. Abstract results on minimal pullback attractors

In this section we remember some abstract results on pullback attractors theory. We present a resume of some results on the existence of minimal pullback attractors obtained in [11] (see also [14,3,4]). In particular, we consider the process U being closed (see below Definition 8).

Consider given a metric space (X,d_X), and let us denote $\mathbb{R}^2_d=\{(t,\tau)\in\mathbb{R}^2\colon\ \tau\leqslant t\}$.

A process on X is a mapping U such that $\mathbb{R}^2_d\times X\ni(t,\tau,x)\mapsto U(t,\tau)x\in X$ with $U(\tau,\tau)x=x$ for any $(\tau,x)\in\mathbb{R}\times X$, and $U(t,r)(U(r,\tau)x)=U(t,\tau)x$ for any $\tau\leqslant r\leqslant t$ and all $x\in X$.

Definition 8. Let U be a process on X.

(a) U is said to be continuous if for any pair $\tau\leqslant t$, the mapping $U(t,\tau):X\to X$ is continuous.
(b) U is said to be closed if for any $\tau\leqslant t$, and any sequence $\{x_n\}\subset X$, if $x_n\to x\in X$ and $U(t,\tau)x_n\to y\in X$, then $U(t,\tau)x=y$.

Remark 9. It is clear that every continuous process is closed. More generally, every strong-weak continuous process (see [14] for the definition) is a closed process.

Let us denote $\mathcal{P}(X)$ the family of all nonempty subsets of X, and consider a family of nonempty sets $\widehat{D}_0=\{D_0(t)\colon t\in\mathbb{R}\}\subset\mathcal{P}(X)$ [observe that we do not require any additional condition on these sets as compactness or boundedness].

Definition 10. We say that a process U on X is pullback $\widehat{D}_0$-asymptotically compact if for any $t\in\mathbb{R}$ and any sequences $\{\tau_n\}\subset(-\infty,t]$ and $\{x_n\}\subset X$ satisfying $\tau_n\to-\infty$ and $x_n\in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Let be given $\mathcal{D}$ a nonempty class of families parameterized in time $\widehat{D}=\{D(t)\colon\ t\in\mathbb{R}\}\subset\mathcal{P}(X)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(X)$.

Definition 11. It is said that $\widehat{D}_0=\{D_0(t)\colon\ t\in\mathbb{R}\}\subset\mathcal{P}(X)$ is pullback $\mathcal{D}$-absorbing for the process U on X if for any $t\in\mathbb{R}$ and any $\widehat{D}\in\mathcal{D}$, there exists a $\tau_0(t,\widehat{D})\leqslant t$ such that

$$U(t,\tau)D(\tau)\subset D_0(t)\quad\text{for all }\tau\leqslant\tau_0(t,\widehat{D}).$$

Observe that in the definition above $\widehat{D}_0$ does not belong necessarily to the class $\mathcal{D}$.

Definition 12. A process U on X is said to be pullback $\mathcal{D}$-asymptotically compact if it is $\widehat{D}$-asymptotically compact for any $\widehat{D}\in\mathcal{D}$, i.e. if for any $t\in\mathbb{R}$, any $\widehat{D}\in\mathcal{D}$, and any sequences $\{\tau_n\}\subset(-\infty,t]$ and $\{x_n\}\subset X$ satisfying $\tau_n\to-\infty$ and $x_n\in D(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in X.

Denote

$$\Lambda(\widehat{D}_0,t) := \bigcap_{s\leqslant t}\overline{\bigcup_{\tau\leqslant s} U(t,\tau)D_0(\tau)}^X \quad \text{for all } t\in\mathbb{R},$$

where $\overline{\{\cdots\}}^X$ is the closure in X.

We denote by $\mathrm{dist}_X(\mathcal{O}_1,\mathcal{O}_2)$ the Hausdorff semi-distance in X between two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, defined as

$$\mathrm{dist}_X(\mathcal{O}_1,\mathcal{O}_2) = \sup_{x\in\mathcal{O}_1}\inf_{y\in\mathcal{O}_2} d_X(x,y) \quad \text{for } \mathcal{O}_1,\mathcal{O}_2\subset X.$$

We have the following result (cf. [11]) on existence of minimal pullback attractors.

Theorem 13. *Consider a closed process $U:\mathbb{R}^2_d\times X\to X$, a universe $\mathcal{D}$ in $\mathcal{P}(X)$, and a family $\widehat{D}_0=\{D_0(t)\colon t\in\mathbb{R}\}\subset\mathcal{P}(X)$ which is pullback $\mathcal{D}$-absorbing for U, and assume also that U is pullback $\widehat{D}_0$-asymptotically compact.*

Then, the family $\mathcal{A}_\mathcal{D}=\{\mathcal{A}_\mathcal{D}(t)\colon t\in\mathbb{R}\}$ defined by

$$\mathcal{A}_\mathcal{D}(t) = \overline{\bigcup_{\widehat{D}\in\mathcal{D}}\Lambda(\widehat{D},t)}^X, \quad t\in\mathbb{R},$$

has the following properties:

(a) *for any $t\in\mathbb{R}$, the set $\mathcal{A}_\mathcal{D}(t)$ is a nonempty compact subset of X, and*

$$\mathcal{A}_\mathcal{D}(t)\subset\Lambda(\widehat{D}_0,t),$$

(b) *$\mathcal{A}_\mathcal{D}$ is pullback $\mathcal{D}$-attracting, i.e.*

$$\lim_{\tau\to-\infty}\mathrm{dist}_X\big(U(t,\tau)D(\tau),\mathcal{A}_\mathcal{D}(t)\big)=0 \quad \textit{for all } \widehat{D}\in\mathcal{D},\ t\in\mathbb{R},$$

(c) *$\mathcal{A}_\mathcal{D}$ is invariant, i.e. $U(t,\tau)\mathcal{A}_\mathcal{D}(\tau)=\mathcal{A}_\mathcal{D}(t)$ for all $\tau\leqslant t$,*

(d) *if $\widehat{D}_0\in\mathcal{D}$, then $\mathcal{A}_\mathcal{D}(t)=\Lambda(\widehat{D}_0,t)\subset\overline{D_0(t)}^X$ for all $t\in\mathbb{R}$.*

The family $\mathcal{A}_\mathcal{D}$ is minimal in the sense that if $\widehat{C}=\{C(t)\colon t\in\mathbb{R}\}\subset\mathcal{P}(X)$ is a family of closed sets such that for any $\widehat{D}=\{D(t)\colon t\in\mathbb{R}\}\in\mathcal{D}$,

$$\lim_{\tau\to-\infty}\mathrm{dist}_X\big(U(t,\tau)D(\tau),C(t)\big)=0,$$

then $\mathcal{A}_\mathcal{D}(t)\subset C(t)$.

Remark 14. Under the assumptions of Theorem 13, the family $\mathcal{A}_\mathcal{D}$ is called the minimal pullback $\mathcal{D}$-attractor for the process U.

If $\mathcal{A}_\mathcal{D}\in\mathcal{D}$, then it is the unique family of closed subsets in $\mathcal{D}$ that satisfies (b)–(c).

A sufficient condition for $\mathcal{A}_\mathcal{D}\in\mathcal{D}$ is to have that $\widehat{D}_0\in\mathcal{D}$, the set $D_0(t)$ is closed for all $t\in\mathbb{R}$, and the family $\mathcal{D}$ is inclusion-closed (i.e. if $\widehat{D}\in\mathcal{D}$, and $\widehat{D}'=\{D'(t)\colon t\in\mathbb{R}\}\subset\mathcal{P}(X)$ with $D'(t)\subset D(t)$ for all t, then $\widehat{D}'\in\mathcal{D}$).

We will denote $\mathcal{D}^X_F$ the universe of fixed nonempty bounded subsets of X, i.e. the class of all families $\widehat{D}$ of the form $\widehat{D}=\{D(t)=D\colon t\in\mathbb{R}\}$ with D a fixed nonempty bounded subset of X. In the particular case of the universe $\mathcal{D}^X_F$, the corresponding minimal pullback $\mathcal{D}^X_F$-attractor for the process U is the pullback attractor defined by Crauel, Debussche, and Flandoli, [6, Th. 1.1, p. 311], and will be denoted $\mathcal{A}_{\mathcal{D}^X_F}$.

Now, it is easy to conclude the following result.

Corollary 15. *Under the assumptions of Theorem* 13, *if the universe $\mathcal{D}$ contains the universe $\mathcal{D}_F(X)$, then both attractors, $\mathcal{A}_{\mathcal{D}_F(X)}$ and $\mathcal{A}_\mathcal{D}$, exist, and the following relation holds*:

$$\mathcal{A}_{\mathcal{D}_F(X)}(t)\subset\mathcal{A}_\mathcal{D}(t) \quad \textit{for all } t\in\mathbb{R}.$$

Remark 16. It can be proved (see [14]) that, under the assumptions of the preceding corollary, if for some $T\in\mathbb{R}$, the set $\bigcup_{t\leqslant T}D_0(t)$ is a bounded subset of X, then

$$\mathcal{A}_{\mathcal{D}_F(X)}(t)=\mathcal{A}_\mathcal{D}(t) \quad \text{for all } t\leqslant T.$$

5. Existence of pullback attractors

Now, by the previous results, we are able to define correctly a process U on $H = L^2(\Omega) \times L^2(\partial\Omega)$ associated to (1), and to obtain the existence of minimal pullback attractors.

Proposition 17. *Assume that $\kappa > 0$, and the assumptions* (3)–(7), *are satisfied. Then, the bi-parametric family of maps $U(t,\tau): H \to H$, with $\tau \leqslant t$, given by*

$$U(t,\tau)(u_\tau, \psi_\tau) = \big(u(t), \psi(t)\big), \tag{40}$$

where $(u,\psi) = (u(\cdot;\tau,u_\tau,\psi_\tau), \psi(\cdot;\tau,u_\tau,\psi_\tau))$ is the unique weak solution of (1)*, defines a continuous process on H.*

Proof. It is a consequence of Theorem 5 and (22) in Theorem 7. □

For the obtention of a pullback absorbing family for the process U, let us observe that the space $H^1(\Omega) \times H^{1/2}(\partial\Omega)$ is compactly embedded in H, and therefore, for the symmetric and coercive linear continuous operator $A_1 : V_1 \to V_1'$, defined in the proof of Theorem 5, there exists a non-decreasing sequence $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ of eigenvalues associated to the operator A_1. In particular, one has for the first eigenvalue

$$\lambda_1 = \min_{v \in H^1(\Omega),\, v \neq 0} \frac{|\nabla v|_\Omega^2 + \kappa |v|_\Omega^2}{|v|_\Omega^2 + |\gamma_0(v)|_{\partial\Omega}^2} > 0. \tag{41}$$

We have the following result.

Lemma 18. *Under the assumptions of Theorem 5, for any $\mu \in (0, 2\lambda_1)$ the solution (u,ψ) of* (1) *satisfies*

$$\begin{aligned} \big|u(t)\big|_\Omega^2 + \big|\psi(t)\big|_{\partial\Omega}^2 \leqslant{}& e^{-\mu(t-\tau)}\big(|u_\tau|_\Omega^2 + |\psi_\tau|_{\partial\Omega}^2\big) + \frac{2\beta}{\mu}\big(|\Omega| + |\partial\Omega|\big) \\ &+ \frac{e^{-\mu t}}{2\lambda_1 - \mu} \int_\tau^t e^{\mu s}\big(\big|h(s)\big|_\Omega^2 + \big|\rho(s)\big|_{\partial\Omega}^2\big)\, ds, \end{aligned} \tag{42}$$

for all $t \geqslant \tau$.

Proof. From (15), and taking into account (4), (5) and (41), we obtain

$$\begin{aligned} &\frac{d}{dt}\big[e^{\mu t}\big(\big|u(t)\big|_\Omega^2 + \big|\psi(t)\big|_{\partial\Omega}^2\big)\big] + (2\lambda_1 - \mu)e^{\mu t}\big(\big|u(t)\big|_\Omega^2 + \big|\psi(t)\big|_{\partial\Omega}^2\big) + 2\alpha_1 e^{\mu t}\big(\big|u(t)\big|_{p,\Omega}^p + \big|\psi(t)\big|_{q,\partial\Omega}^q\big) \\ &\quad \leqslant 2\beta e^{\mu t}\big(|\Omega| + |\partial\Omega|\big) + 2e^{\mu t}\big[\big(h(t), u(t)\big)_\Omega + \big(\rho(t), \psi(t)\big)_{\partial\Omega}\big], \end{aligned}$$

a.e. $t > \tau$, and then, observing that

$$2e^{\mu t}\big[\big(h(t), u(t)\big)_\Omega + \big(\rho(t), \psi(t)\big)_{\partial\Omega}\big] \leqslant (2\lambda_1 - \mu)e^{\mu t}\big(\big|u(t)\big|_\Omega^2 + \big|\psi(t)\big|_{\partial\Omega}^2\big) + \frac{e^{\mu t}}{2\lambda_1 - \mu}\big(\big|h(t)\big|_\Omega^2 + \big|\rho(t)\big|_{\partial\Omega}^2\big),$$

we have in particular

$$\frac{d}{dt}\big[e^{\mu t}\big(\big|u(t)\big|_\Omega^2 + \big|\psi(t)\big|_{\partial\Omega}^2\big)\big] \leqslant 2\beta e^{\mu t}\big(|\Omega| + |\partial\Omega|\big) + \frac{e^{\mu t}}{2\lambda_1 - \mu}\big(\big|h(t)\big|_\Omega^2 + \big|\rho(t)\big|_{\partial\Omega}^2\big),$$

a.e. $t > \tau$.

Integrating in this last inequality, we obtain (42). □

Taking into account the estimate (42), we define the following universe.

Definition 19. For any $\mu \in (0, 2\lambda_1)$, we will denote by $\mathcal{D}_\mu^H$ the class of all families of nonempty subsets $\widehat{D} = \{D(t): t \in \mathbb{R}\} \subset \mathcal{P}(H)$ such that

$$\lim_{\tau \to -\infty}\Big(e^{\mu\tau} \sup_{(v,\phi) \in D(\tau)}\big(|v|_\Omega^2 + |\phi|_{\partial\Omega}^2\big)\Big) = 0.$$

Accordingly to the notation introduced in the previous section, $\mathcal{D}_F^H$ will denote the class of families $\widehat{D} = \{D(t) = D: t \in \mathbb{R}\}$ with D a fixed nonempty bounded subset of H.

Remark 20. Observe that $\mathcal{D}_F^H \subset \mathcal{D}_\mu^H$ and that both are inclusion-closed.

As an evident consequence of Lemma 18, we have the following result.

Corollary 21. *Assume that $\kappa > 0$, and the assumptions* (3)–(7)*, are satisfied. Suppose moreover that there exists some $\mu \in (0, 2\lambda_1)$ such that*

$$\int_{-\infty}^{0} e^{\mu s}\big[\big|h(s)\big|_\Omega^2 + \big|\rho(s)\big|_{\partial\Omega}^2\big]\,ds < +\infty. \tag{43}$$

Then, the family $\widehat{D}_0 = \{D_0(t)\colon t \in \mathbb{R}\}$ defined by $D_0(t) = \overline{B}_H(0, R_H^{1/2}(t))$, the closed ball in H of center zero and radius $R_H^{1/2}(t)$, where

$$R_H(t) = 1 + \frac{2\beta}{\mu}\big(|\Omega| + |\partial\Omega|\big) + \frac{e^{-\mu t}}{2\lambda_1 - \mu}\int_{-\infty}^{t} e^{\mu s}\big[\big|h(s)\big|_\Omega^2 + \big|\rho(s)\big|_{\partial\Omega}^2\big]\,ds,$$

is pullback $\mathcal{D}_\mu^H$-absorbing for the process $U : \mathbb{R}_d^2 \times H \to H$ given by (40) (*and therefore $\mathcal{D}_F^H$-absorbing too*)*, and $\widehat{D}_0 \in \mathcal{D}_\mu^H$.*

We also have the character $\mathcal{D}_\mu^H$-pullback asymptotically compact of the process U.

Lemma 22. *Under the assumptions of Corollary* 21*, the process U defined by* (40) *is pullback $\mathcal{D}_\mu^H$-asymptotically compact.*

Proof. Let us consider $\widehat{D} = \{D(t)\colon t \in \mathbb{R}\} \in \mathcal{D}_\mu^H$, $t \in \mathbb{R}$, and sequences $\{\tau_n\} \subset (-\infty, t]$ and $\{(u_{\tau_n}, \psi_{\tau_n})\} \subset H$ satisfying $\tau_n \to -\infty$ and $(u_{\tau_n}, \psi_{\tau_n}) \in D(\tau_n)$ for all n. We must prove that the sequence $\{U(t, \tau_n)(u_{\tau_n}, \psi_{\tau_n})\}$ is relatively compact in H.

As $\tau_n \to -\infty$ and $(u_{\tau_n}, \psi_{\tau_n}) \in D(\tau_n)$ for all n, by Corollary 21, there exists n_0 such that $\tau_n < t - 1$, and

$$U(t-1, \tau_n)(u_{\tau_n}, \psi_{\tau_n}) \in D_0(t-1) = \overline{B}_H\big(0, R_H^{1/2}(t-1)\big),$$

for all $n \geqslant n_0$.

Thus, the sequence $\{U(t-1, \tau_n)(u_{\tau_n}, \psi_{\tau_n})\colon n \geqslant n_0\}$ is bounded in H, and therefore, there exist $(u_{t-1}, \psi_{t-1}) \in H$, and a subsequence $\{U(t-1, \tau_\nu)(u_{\tau_\nu}, \psi_{\tau_\nu})\} \subset \{U(t-1, \tau_n)(u_{\tau_n}, \psi_{\tau_n})\colon n \geqslant n_0\}$, such that

$$U(t-1, \tau_\nu)(u_{\tau_\nu}, \psi_{\tau_\nu}) \rightharpoonup (u_{t-1}, \psi_{t-1}) \quad \text{weakly in } H, \text{ as } \nu \to \infty.$$

But then, from (22) in Theorem 7, we deduce that

$$U(t, \tau_\nu)(u_{\tau_\nu}, \psi_{\tau_\nu}) = U(t, t-1)\big(U(t-1, \tau_\nu)(u_{\tau_\nu}, \psi_{\tau_\nu})\big) \to U(t, t-1)(u_{t-1}, \psi_{t-1})$$

strongly in H, as $\nu \to \infty$. $\square$

As a consequence of the above results, we obtain the existence of minimal pullback attractors for the process $U : \mathbb{R}_d^2 \times H \to H$ defined by (40).

Theorem 23. *Assume that $\kappa > 0$ and the assumptions* (3)–(7) *are satisfied. Suppose moreover that there exists some $\mu \in (0, 2\lambda_1)$ such that the condition* (43) *holds. Then, there exist the minimal pullback $\mathcal{D}_F^H$-attractor*

$$\mathcal{A}_{\mathcal{D}_F^H} = \big\{\mathcal{A}_{\mathcal{D}_F^H}(t)\colon t \in \mathbb{R}\big\}$$

and the minimal pullback $\mathcal{D}_\mu^H$-attractor

$$\mathcal{A}_{\mathcal{D}_\mu^H} = \big\{\mathcal{A}_{\mathcal{D}_\mu^H}(t)\colon t \in \mathbb{R}\big\},$$

for the process U defined by (40)*. The family $\mathcal{A}_{\mathcal{D}_\mu^H}$ belongs to $\mathcal{D}_\mu^H$, and the following relation holds*:

$$\mathcal{A}_{\mathcal{D}_F^H}(t) \subset \mathcal{A}_{\mathcal{D}_\mu^H}(t) \subset \overline{B}_H\big(0, R_H^{1/2}(t)\big) \quad \forall t \in \mathbb{R}.$$

If moreover the pair (h, ρ) satisfies

$$\sup_{s \leqslant 0}\left(e^{-\mu s}\int_{-\infty}^{s} e^{\mu\theta}\big[\big|h(\theta)\big|_\Omega^2 + \big|\rho(\theta)\big|_{\partial\Omega}^2\big]\,d\theta\right) < +\infty, \tag{44}$$

then

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^H}(t) \quad \textit{for all } t \in \mathbb{R}. \tag{45}$$

Proof. All but last results are consequence of Theorem 13 and Corollary 15. Finally, (45) follows from (44) and Remark 16, taking into account the expression $R_H(t)$ given in Corollary 21. $\square$

Remark 24. Observe that if the pair (h, ρ) satisfies (3) and (43) for some $\mu \in (0, 2\lambda_1)$, then it also satisfies

$$\int_{-\infty}^{0} e^{\sigma s}\left[\left|h(s)\right|_{\Omega}^2 + \left|\rho(s)\right|_{\partial\Omega}^2\right] ds < \infty, \quad \text{for all } \sigma \in (\mu, 2\lambda_1).$$

Thus, for any $\sigma \in (\mu, 2\lambda_1)$ there exists the corresponding minimal $\mathcal{D}_\sigma^H$-pullback attractor, $\mathcal{A}_{\mathcal{D}_\sigma^H}$.

Since $\mathcal{D}_\mu^H \subset \mathcal{D}_\sigma^H$, it is evident that, for any $t \in \mathbb{R}$,

$$\mathcal{A}_{\mathcal{D}_\mu^H}(t) \subset \mathcal{A}_{\mathcal{D}_\sigma^H}(t) \quad \text{for all } \sigma \in (\mu, 2\lambda_1).$$

Moreover, if the pair (h, ρ) satisfies (44), then, by (45),

$$\mathcal{A}_{\mathcal{D}_F^H}(t) = \mathcal{A}_{\mathcal{D}_\mu^H}(t) = \mathcal{A}_{\mathcal{D}_\sigma^H}(t) \quad \text{for all } t \in \mathbb{R}, \text{ and any } \sigma \in (\mu, 2\lambda_1).$$

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