J. Math. Anal. Appl. 389 (2012) 367–378

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Limit cycles of some polynomial Liénard systems

Weijiao Xu *, Cuiping Li

*LMIB-School of Mathematics and Systems Science, Beihang University, Beijing 100191, China*

ARTICLE INFO

*Article history:*
Received 30 May 2011
Available online 28 November 2011
Submitted by W. Sarlet

*Keywords:*
Liénard system
Homoclinic loop
Heteroclinic loop
Melnikov function

ABSTRACT

In this paper, we study the number of limit cycles of some polynomial Liénard systems. By the Melnikov functions and the methods of Hopf, homoclinic and heteroclinic bifurcation theory, we prove that $H(2,5) \geqslant 3$, $H(4,5) \geqslant 5$, $H(6,5) \geqslant 10$, $H(8,5) \geqslant 10$.



## 1. Introduction

In general, it is difficult to get the number of limit cycles for differential equations. The well-known Hilbert's 16th problem is just about the number of limit cycles and their relative locations for polynomial vector fields. This problem attracts many famous mathematicians in world to do a lot of important works. In this paper, we consider the following Liénard system [1]

$$\dot{x} = y, \qquad \dot{y} = -g(x) - \varepsilon f(x) y, \tag{1.1}$$

where $\varepsilon$ is a small parameter, $f(x)$ and $g(x)$ are polynomials of $x$ with $\deg f = m$ and $\deg g = n$. The maximal number of limit cycles of (1.1) is denoted by $H(m,n)$. There are many results on the problem. For example, Han [2] proved $H(m,2) \geqslant [\frac{2m+1}{3}]$ for $m \geqslant 2$. Christopher and Lynch [3] proved $H(m,2) \geqslant [\frac{2m+1}{3}]$ for $m \geqslant 3$ and $H(m,3) \geqslant [\frac{3(m+2)}{8}]$ for $3 \leqslant m \leqslant 50$. Yang, Han and Romanovski [4] proved $H(3,3) \geqslant 5$, $H(4,3) \geqslant 6$, $H(5,3) \geqslant 6$, $H(6,3) \geqslant 8$, $H(7,3) \geqslant 8$, $H(8,3) \geqslant 9$. Han, Yan et al. [5] proved $H(m,4) \geqslant m+3$, for $m = 2, \dots, 8$, $m \neq 4$ and $H(4,4) \geqslant 6$.

The above results are on $n = 2$, $n = 3$ and $n = 4$. Now we study the case on $n = 5$. Take $g(x) = x(x^2-1)(x^2-\alpha^2)$ in (1.1) with $0 < \alpha < \frac{\sqrt{3}}{3}$ that will ensure the local maximum of the Hamiltonian function of $(1.1)|_{\varepsilon=0}$ at the value $x = 1$ will be positive and two heteroclinic orbits exist for $\varepsilon = 0$ as described in assumption (A4) in the next section.

For the convenience of computation, we will take $\alpha = \frac{1}{2}$ from now on. Then (1.1) becomes

$$\dot{x} = y, \qquad \dot{y} = \frac{1}{4}x - \frac{5}{4}x^3 + x^5 - \varepsilon f(x,\delta) y, \tag{1.2}$$

where

$$f(x,\delta) = a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6 + a_4 x^8 \tag{1.3}$$

is a polynomial of degree 8, $\delta = (a_0, a_1, a_2, a_3, a_4) \in R^5$.

* Corresponding author.
*E-mail addresses:* xuweijiao@sohu.com (W. Xu), cuipingli@buaa.edu.cn (C. Li).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2011.11.070

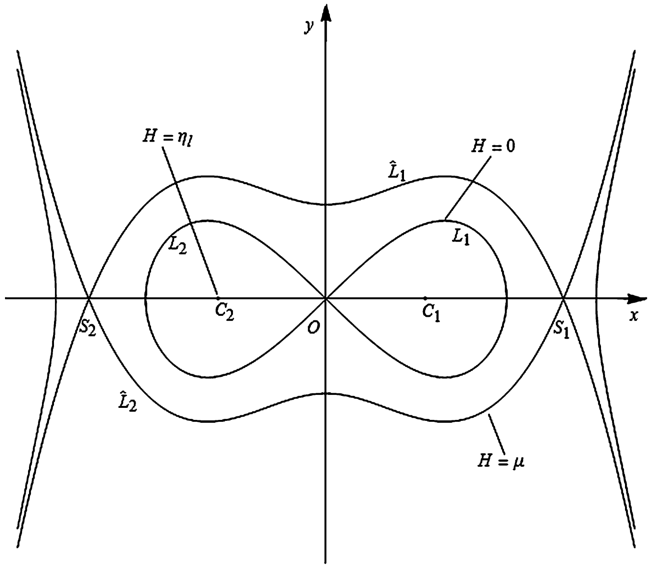


**Fig. 1.** The phrase portrait of (2.2) with assumptions (A1)–(A4).

For the system (1.2), we get the following result.

**Theorem 1.1.** *Let* $a_4 \neq 0$ *in* (1.3), *the system* (1.2) *has at least* 10 *limit cycles.*

## 2. Preliminaries

In this section we introduce some known results used by this paper.
Consider the system

$$\dot{x} = H_y(x, y) + \varepsilon p(x, y, \varepsilon, \delta), \qquad \dot{y} = -H_x(x, y) + \varepsilon q(x, y, \varepsilon, \delta), \tag{2.1}$$

where $H(x, y)$, $p(x, y, \varepsilon, \delta)$ and $q(x, y, \varepsilon, \delta)$ are analytic functions, $\varepsilon \geqslant 0$ is small enough and $\delta \in D \subset R^m$ is a vector parameter with $D$ compact. When $\varepsilon = 0$, (2.1) becomes a Hamiltonian system

$$\dot{x} = H_y, \qquad \dot{y} = -H_x. \tag{2.2}$$

So, we will call Eq. (2.1) a near-Hamiltonian system.
We make the following assumptions on system (2.2):

(A1) The system (2.2) is $y$-axis symmetric.
(A2) The system (2.2) has a double homoclinic loop $L = L_1 \cup L_2$ consisting of two homoclinic loops $L_1 = L|_{x\geqslant 0}$ and $L_2 = L|_{x\leqslant 0}$ with hyperbolic saddle $O(0, 0)$, where $L$ is defined by the equation $H(x, y) = 0$.
(A3) The center $C_l(x_{c_l}, y_{c_l})$, $l = 1, 2$ is surrounded by the homoclinic loop $L_l$, where $H(C_l) = \eta_l < 0$.
(A4) A 2-polycycle $\Gamma^2$ composed of 2 hyperbolic saddles $S_1$ and $S_2$ and 2 heteroclinic orbits $\widehat{L}_1$ and $\widehat{L}_2$ connecting them

$$\Gamma^2 = (\widehat{L}_1 \cup S_1) \cup (\widehat{L}_2 \cup S_2),$$

where $\Gamma^2$ is defined by the equation $H(x, y) = \mu > 0$. (See Fig. 1.)

Then there exist three families of periodic orbits given by

$$\begin{aligned} &L_l(h): \quad H(x, y) = h, \quad h \in (\eta_l, 0),\ l = 1, 2, \\ &L(h): \quad H(x, y) = h, \quad h \in (0, \mu). \end{aligned} \tag{2.3}$$

Let

$$M_l(h, \delta) = \oint\limits_{L_l(h)} (q\,dx - p\,dy)|_{\varepsilon=0}, \quad h \in (\eta_l, 0),\ l = 1, 2, \tag{2.4}$$

and

$$M(h,\delta) = \oint_{L(h)} (q\,dx - p\,dy)|_{\varepsilon=0}, \quad h \in (0,\mu), \tag{2.5}$$

which are called Melnikov functions. By the symmetry, we have $M_1(h,\delta) = M_2(h,\delta)$.

From [6], for $(x,y)$ near the saddle $O(0,0)$, we have

**Lemma 2.1.** *(See [6].) Suppose*

$$H(x,y) = \frac{\lambda}{2}\left(y^2 - x^2\right) + \sum_{i+j\geqslant 3} h_{ij}x^i y^j, \quad \lambda \neq 0$$

*and*

$$p(x,y,0,\delta) = \sum_{i+j\geqslant 0} a_{ij}x^i y^j, \qquad q(x,y,0,\delta) = \sum_{i+j\geqslant 0} b_{ij}x^i y^j,$$

*then*

$$M_1(h,\delta) = c_{01} + c_1 h\ln|h| + c_{21}h + c_3h^2\ln|h| + O\left(h^2\right), \quad 0 < -h \ll 1, \tag{2.6}$$

$$M(h,\delta) = c_0 + 2c_1 h\ln|h| + c_2 h + 2c_3 h^2\ln|h| + O\left(h^2\right), \quad 0 < h \ll 1, \tag{2.7}$$

*where*

$$c_{01} = M_1(0,\delta) = \oint_{L_1} q\,dx - p\,dy|_{\varepsilon=0}, \qquad c_0 = 2c_{01},$$

$$c_1 = -\frac{1}{|\lambda|}(a_{10}+b_{01}) = -\frac{1}{|\lambda|}(p_x+q_y)(0,0,0,\delta),$$

$$c_{21} = \oint_{L_1} (p_x + q_y - a_{10} - b_{01})|_{\varepsilon=0}\,dt|_{c_1=0}, \qquad c_2 = 2c_{21},$$

$$c_3 = \frac{-1}{2|\lambda|\lambda}\left\{(-3a_{30} - b_{21} + a_{12} + 3b_{03}) - \frac{1}{\lambda}\left[(2b_{02}+a_{11})(3h_{03}-h_{21}) + (2a_{20}+b_{11})(3h_{30}-h_{12})\right]\right\}.$$

**Lemma 2.2.** *(See [7].) In Lemma 2.1, denote*

$$\bar{c}_0 = c_{01}, \qquad \bar{c}_1 = c_1, \qquad \bar{c}_2 = c_{21}, \qquad \bar{c}_3 = c_3|_{c_1=0}. \tag{2.8}$$

*If there exist $\delta_0 \in D$ and $1 \leqslant k \leqslant 3$ such that*

$$\bar{c}_j(\delta_0) = 0, \quad j = 0,\dots,k-1, \qquad \bar{c}_k \neq 0,$$

$$\mathit{rank}\,\frac{\partial(\bar{c}_0,\dots,\bar{c}_{k-1})}{\partial(\delta_1,\dots,\delta_m)}(\delta_0) = k,$$

*then system (2.1) can have $[\frac{5k}{2}]$ limit cycles near $L$ for some $(\varepsilon,\delta)$ near $(0,\delta_0)$.*

From [8], for $(x,y)$ near $(x_{c_1}, y_{c_1})$, we have

**Lemma 2.3.** *(See [8].) Suppose*

$$H(x,y) = \eta_1 + \frac{1}{2}\left((x-x_{c_1})^2 + (y-y_{c_1})^2\right) + \sum_{i+j\geqslant 3} h_{ij}(x-x_{c_1})^i(y-y_{c_1})^j$$

*and*

$$(p_x+q_y)|_{\varepsilon=0} = \sum_{i+j\geqslant 0} c_{ij}(x-x_{c_1})^i(y-y_{c_1})^j,$$

*then for $0 < h - \eta_1 \ll 1$, we have*

$$M_1(h,\delta) = b_{01}(h-\eta_1) + b_{11}(h-\eta_1)^2 + \cdots + b_{k1}(h-\eta_1)^{k+1} + O\left(|h-\eta_1|^{k+2}\right), \tag{2.9}$$

*where*

$$b_{01} = 2\pi c_{00}, \qquad b_{11} = -c_{10}\pi(h_{12}+3h_{30}) - c_{01}\pi(h_{21}+3h_{03}) + c_{20}\pi + c_{02}\pi.$$

From [6], for $(x, y)$ near $(x_{s_l}, y_{s_l})$, $l = 1, 2$, we have

**Lemma 2.4.** *(See [6].) Suppose*

$$H(x, y) = \mu + \frac{\lambda}{2}\big((y - y_{s_l})^2 - (x - x_{s_l})^2\big) + \sum_{i+j\geqslant 3} h_{ij}(x - x_{s_l})^i (y - y_{s_l})^j, \quad \lambda \neq 0,$$

*then for* $0 < \mu - h \ll 1$, *we have*

$$M(h, \delta) = \widehat{c}_0 + \widehat{c}_1(h - \mu)\ln|h - \mu| + \widehat{c}_2(h - \mu) + \widehat{c}_3(h - \mu)^2 \ln|h - \mu| + O\big((h - \mu)^2\big), \tag{2.10}$$

*where*

$$\begin{aligned}
&\widehat{c}_0(\delta) = \int_{\widehat{L}_1} q\,dx - p\,dy|_{\varepsilon=0} + \int_{\widehat{L}_2} q\,dx - p\,dy|_{\varepsilon=0},\\
&\widehat{c}_1(\delta) = \widehat{c}_1(S_1, \delta) + \widehat{c}_1(S_2, \delta), \qquad \widehat{c}_3(\delta) = \widehat{c}_3(S_1, \delta) + \widehat{c}_3(S_2, \delta),\\
&\widehat{c}_2(\delta) = \oint_{\Gamma^2} (p_x + q_y)|_{\varepsilon=0}\,dt|_{\widehat{c}_1(\delta)=0} = \int_{\widehat{L}_1} (p_x + q_y)|_{\varepsilon=0}\,dt|_{\widehat{c}_1(S_1,\delta)=0} + \int_{\widehat{L}_2} (p_x + q_y)|_{\varepsilon=0}\,dt|_{\widehat{c}_1(S_2,\delta)=0}.
\end{aligned}$$

Summarizing the above lemmas, we can get the following theorem.

**Theorem 2.1.** *Let the system* (2.1) *satisfy the assumptions* (A1)–(A4) *and* (2.6), (2.7), (2.8), (2.9) *and* (2.10) *hold. Suppose that there exist* $\delta_0 \in D$ *and* $k_1 \geqslant 0$, $k_2 \geqslant 0$, $1 \leqslant k \leqslant 3$ *such that*

$$\begin{aligned}
&b_{s1}(\delta_0) = 0, \quad s = 0, \ldots, k_1 - 1, \qquad b_{k_1,1}(\delta_0) \neq 0,\\
&\widehat{c}_s(\delta_0) = 0, \quad s = 0, \ldots, k_2 - 1, \qquad \widehat{c}_{k_2}(\delta_0) \neq 0,\\
&\bar{c}_s = 0, \quad s = 0, \ldots, k - 1, \qquad \bar{c}_k \neq 0,
\end{aligned}$$

*and*

$$\operatorname{rank} \frac{\partial(b_{01}, \ldots, b_{k_1-1,1}, \bar{c}_0, \ldots, \bar{c}_{k-1}, \widehat{c}_0, \ldots, \widehat{c}_{k_2-1})}{\partial(\delta_1, \ldots, \delta_m)}(\delta_0) = k_1 + k_2 + k.$$

*Let* $\rho_1 = (-1)^{[\frac{k}{2}]+1} b_{k_1,1}(\delta_0)\bar{c}_k(\delta_0)$, $\rho_2 = (-1)^{k+[\frac{k_2}{2}]-1}\bar{c}_k(\delta_0)\widehat{c}_{k_2}(\delta_0)$. *Then system* (2.1) *can have* $2k_1 + k_2 + [\frac{5k}{2}] + 3$ *limit cycles for some* $(\varepsilon, \delta)$ *near* $(0, \delta_0)$ *if* $\rho_1$ *and* $\rho_2$ *are both positive.*

**Proof.** We only prove the theorem for $k_2 = 1, k = 3$. The proof for the other cases is similar. When (2.6), (2.7), (2.8), (2.9) and (2.10) hold, we have

$$\begin{aligned}
&M_1(h, \delta_0) = b_{k_1,1}(\delta_0)(h - \eta_1)^{k_1+1} + O\big(|h - \eta_1|^{k_1+2}\big), \quad 0 < h - \eta_1 \ll 1,\\
&M_1(h, \delta_0) = \bar{c}_3(\delta_0)h^2 \ln|h| + O\big(h^2\big), \quad 0 < -h \ll 1,\\
&M(h, \delta_0) = 2\bar{c}_3(\delta_0)h^2 \ln|h| + O\big(h^2\big), \quad 0 < h \ll 1,
\end{aligned}$$

and

$$M(h, \delta_0) = \widehat{c}_1(\delta_0)(h - \mu)\ln|h - \mu|, \quad 0 < \mu - h \ll 1.$$

If $b_{k_1,1}(\delta_0)\bar{c}_3(\delta_0) > 0$ and $\bar{c}_3(\delta_0)\widehat{c}_1(\delta_0) > 0$, then there exist $h_1 \in (\eta_1, 0)$ and $h \in (0, \mu)$ such that

$$\begin{aligned}
&M_1(h_1, \delta_0) = 0, \qquad M_1(h_1 - \varepsilon_0, \delta_0)M_1(h_1 + \varepsilon_0, \delta_0) < 0,\\
&M(h, \delta_0) = 0, \qquad M(h - \varepsilon_0, \delta_0)M(h + \varepsilon_0, \delta_0) < 0,
\end{aligned} \tag{2.11}$$

when $\varepsilon_0$ is sufficiently small.

For definiteness, we take $\bar{c}_3(\delta_0) > 0$. By the assumptions, we can choose $b_{01}, \ldots, b_{k_1-1,1}$, $\bar{c}_0$, $\bar{c}_1$, $\bar{c}_2$, $\widehat{c}_0$ as free parameters. Then (2.6), (2.7), (2.9) and (2.10) under (2.8) can be rewritten as

$$\begin{aligned}
&M_1(h, \delta_0) = b_{01}(h - \eta_1) + \cdots + b_{k_1-1,1}(h - \eta_1)^{k_1} + \widetilde{b}_{k_1,1}(h - \eta_1)^{k_1+1} + O\big((h - \eta_1)^{k_1+2}\big), \quad 0 < h - \eta_1 \ll 1,\\
&M_1(h, \delta_0) = \bar{c}_0 + \bar{c}_1 h \ln|h| + \bar{c}_2 h + \widetilde{\bar{c}}_3 h^2 \ln|h| + O\big(h^2\big), \quad 0 < -h \ll 1,\\
&M(h, \delta_0) = 2\bar{c}_0 + 2\bar{c}_1 h \ln|h| + 2\bar{c}_2 h + 2\widetilde{\bar{c}}_3 h^2 \ln|h| + O\big(h^2\big), \quad 0 < h \ll 1,
\end{aligned}$$

and

$$M(h,\delta)=\widehat{c}_0+\widetilde{\widehat{c}}_1(h-\mu)\ln|h-\mu|+O\big((h-\mu)\big),\quad 0<\mu-h\ll 1,$$

where

$$\widetilde{b}_{k_1,1}=b_{k_1,1}(\delta_0)+O\big(|b_{01},\dots,b_{k_1-1,1},\bar{c}_0,\bar{c}_1,\bar{c}_2,\widehat{c}_0|\big),$$
$$\widetilde{\bar{c}}_3=\bar{c}_3(\delta_0)+O\big(|b_{01},\dots,b_{k_1-1,1},\bar{c}_0,\bar{c}_1,\bar{c}_2,\widehat{c}_0|\big),$$
$$\widetilde{\widehat{c}}_1=\widehat{c}_1(\delta_0)+O\big(|b_{01},\dots,b_{k_1-1,1},\bar{c}_0,\bar{c}_1,\bar{c}_2,\widehat{c}_0|\big).$$

Then it is easy to see that if

$$b_{k_1-1,1}\widetilde{b}_{k_1,1}<0,\qquad b_{k_1-s-1,1}\widetilde{b}_{k_1-s,1}<0,\quad s=1,2,\dots,k_1-1,$$
$$0<|b_{01}|\ll|b_{11}|\ll\cdots\ll|b_{k_1-1,1}|\ll 1,\tag{2.12}$$

then $M_1(h,\delta)$ has $k_1$ zeros near $h=\eta_1$. If

$$0<\bar{c}_0\ll-\bar{c}_1\ll-\bar{c}_2\ll 1,\tag{2.13}$$

then $M_1(h,\delta)$ has 3 negative zeros and $M(h,\delta)$ has 1 positive zero near $h=0$. And if

$$0<\widehat{c}_0\ll 1,\tag{2.14}$$

then $M(h,\delta)$ has 1 zero near $h=\mu$. Moreover, it follows from (2.11) that under (2.12), (2.13) and (2.14) there exist $h_1^\star$ near $h_1$ and $h^\star$ near $h$ such that

$$M_1\big(h_1^\star,\delta_0\big)=0,\qquad M_1\big(h_1^\star-\varepsilon_0,\delta_0\big)M_1\big(h_1^\star+\varepsilon_0,\delta_0\big)<0,$$
$$M\big(h^\star,\delta_0\big)=0,\qquad M\big(h^\star-\varepsilon_0,\delta_0\big)M\big(h^\star+\varepsilon_0,\delta_0\big)<0.$$

Then for $0<|\varepsilon_0|\ll 1$, system (2.1) can have at least $2k_1+1+7+3=2k_1+k_2+\frac{[5k]}{2}+3$ limit cycles. □

## 3. Main results

In this section we prove our main result Theorem 1.1.

We consider the system (1.2). For $\varepsilon=0$, the system (1.2) is a $y$-axis symmetric Hamiltonian system with

$$H(x,y)=\frac{1}{2}y^2-\frac{1}{8}x^2+\frac{5}{16}x^4-\frac{1}{6}x^6\tag{3.1}$$

and has 5 equilibria: centers $C_1(\frac{1}{2},0)$, $C_2(-\frac{1}{2},0)$ and saddles $O(0,0)$, $S_1(1,0)$, $S_2(-1,0)$. Note that $H(C_l)=-\frac{11}{768}$, $H(O)=0$, $H(S_l)=\frac{1}{48}$, $l=1,2$. The equation $H(x,y)=0$ defines a double homoclinic loop $L=L_1\cup L_2$ with $L_1=L|_{x\geqslant 0}$, $L_2=L|_{x\leqslant 0}$ and the equation $H(x,y)=\frac{1}{48}$ defines a 2-polycycle $\Gamma^2$ consisting of 2 hyperbolic saddles $S_1,S_2$ and 2 heteroclinic orbits $\widehat{L}_1$ and $\widehat{L}_2$ satisfying $\omega(\widehat{L}_1)=S_1$, $\alpha(\widehat{L}_1)=S_2$ and $\omega(\widehat{L}_2)=S_2$, $\alpha(\widehat{L}_2)=S_1$, respectively. Then the equation $H(x,y)=h$ defines three families of periodic orbits $L_l(h)$, $l=1,2$ for $-\frac{11}{768}<h<0$ and $L(h)$ for $0<h<\frac{1}{48}$. (See Fig. 2.)

By (2.4) and (2.5), we have the Melnikov functions as follows

$$M_l(h,\delta)=-\oint\limits_{L_l(h)} f(x,\delta)y\,dx,\quad -\frac{11}{768}<h<0,\ l=1,2,$$
$$M(h,\delta)=-\oint\limits_{L(h)} f(x,\delta)y\,dx,\quad 0<h<\frac{1}{48},\tag{3.2}$$

where $M_1(h,\delta)=M_2(h,\delta)$.

**Proof of Theorem 1.1.** By Lemma 2.1, for $0<-h\ll 1$, we have

$$M_1(h,\delta)=c_{01}+c_1h\ln|h|+c_{21}h+c_3(O,\delta)h^2\ln|h|+O\big(h^2\big),$$

where $\lambda=\frac{1}{2}$ denotes an eigenvalue of $O$. By (3.1), the homoclinic orbit $L_1$ has the expression:

$$L_1:\quad y_\pm=\pm\frac{1}{12}x\sqrt{36-90x^2+48x^4},\quad 0\leqslant x\leqslant T_0,\quad T_0=\frac{1}{4}\sqrt{15-\sqrt{33}}.$$

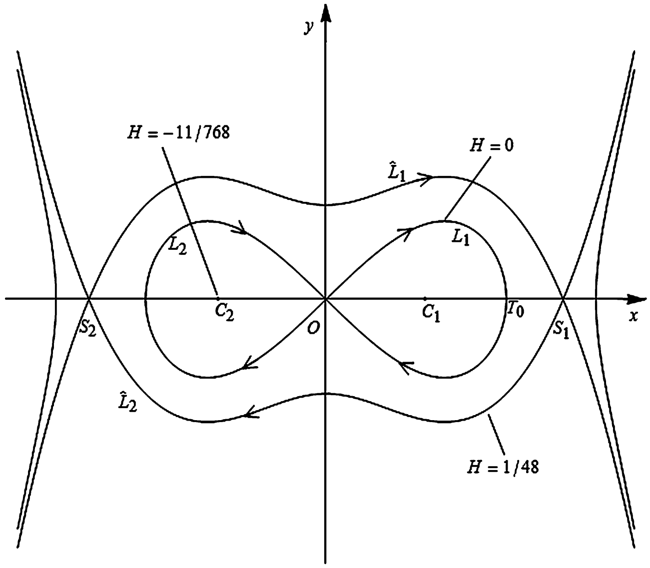


**Fig. 2.** The phrase portrait of (1.2) for $\varepsilon = 0$.

Then by Lemma 2.1 and (1.2) we have

$$c_{01} = -\oint_{L_1} f(x,\delta) y\, dx = -2\int_0^{T_0} \left(a_0 + a_1x^2 + a_2x^4 + a_3x^6 + a_4x^8\right) y_+\, dx = 2\sum_{j=0}^{4} a_j I_j, \tag{3.3}$$

with

$$\begin{aligned}
I_0 &= -\int_0^{T_0} y_+\, dx = -\frac{15}{128} + \frac{11}{2048}\sqrt{3}\Phi_1 - \frac{11}{2048}\sqrt{3}\Phi_2,\\
I_1 &= -\int_0^{T_0} x^2 y_+\, dx = -\frac{97}{2048} + \frac{165}{32\,768}\sqrt{3}\Phi_1 - \frac{165}{32\,768}\sqrt{3}\Phi_2,\\
I_2 &= -\int_0^{T_0} x^4 y_+\, dx = -\frac{4395}{131\,072} + \frac{10\,263}{2\,097\,152}\sqrt{3}\Phi_1 - \frac{10\,263}{2\,097\,152}\sqrt{3}\Phi_2,\\
I_3 &= -\int_0^{T_0} x^6 y_+\, dx = -\frac{312\,483}{10\,485\,760} + \frac{164\,835}{33\,554\,432}\sqrt{3}\Phi_1 - \frac{164\,835}{33\,554\,432}\sqrt{3}\Phi_2,\\
I_4 &= -\int_0^{T_0} x^8 y_+\, dx = -\frac{1\,968\,507}{67\,108\,864} + \frac{5\,447\,079}{1\,073\,741\,824}\sqrt{3}\Phi_1 - \frac{5\,447\,079}{1\,073\,741\,824}\sqrt{3}\Phi_2,
\end{aligned}$$

$$c_1 = 2a_0, \tag{3.4}$$

$$\begin{aligned}
c_{21} &= -\oint_{L_1} \left(f(x,\delta) - a_0\right) dt\Big|_{c_1=0} = -2\int_0^{T_0} \left(a_1x^2 + a_2x^4 + a_3x^6 + a_4x^8\right) dt\\
&= -2\int_0^{T_0} \left(a_1x^2 + a_2x^4 + a_3x^6 + a_4x^8\right)\frac{1}{y_+}\, dx = 2\sum_{j=1}^{4} a_j J_j,
\end{aligned} \tag{3.5}$$

with

$$J_1 = -\int_0^{T_0} \frac{x^2}{y_+}\,dx = -\frac{1}{4}\sqrt{3}\Phi_1 + \frac{1}{4}\sqrt{3}\Phi_2,$$

$$J_2 = -\int_0^{T_0} \frac{x^4}{y_+}\,dx = \frac{3}{4} - \frac{15}{64}\sqrt{3}\Phi_1 + \frac{15}{64}\sqrt{3}\Phi_2,$$

$$J_3 = -\int_0^{T_0} \frac{x^6}{y_+}\,dx = \frac{135}{128} - \frac{483}{2048}\sqrt{3}\Phi_1 + \frac{483}{2048}\sqrt{3}\Phi_2,$$

$$J_4 = -\int_0^{T_0} \frac{x^8}{y_+}\,dx = \frac{2607}{2048} - \frac{8235}{32\,768}\sqrt{3}\Phi_1 + \frac{8235}{32\,768}\sqrt{3}\Phi_2,$$

where $\Phi_1 = \ln(5\sqrt{3}+8)$, $\Phi_2 = \ln(5\sqrt{3}-8)$.

In order to find $c_3(\delta)$ we make a change of variables of the form $x = 2u$, $y = v$. Then the system (1.2) becomes

$$\dot{u} = \frac{v}{2}, \qquad \dot{v} = \frac{u}{2} - 10u^3 + 32u^5 - \varepsilon f(2u,\delta)v. \tag{3.6}$$

For $\varepsilon = 0$, the Hamiltonian function of (3.6) is

$$\widetilde{H}(u,v) = \frac{1}{4}\left(v^2 - u^2\right) + \frac{5}{2}u^4 - \frac{16}{3}u^6 = \frac{1}{2}H(2u,v).$$

Let

$$\widetilde{M}_1(h,\delta) = -\oint\limits_{\widetilde{H}(u,v)=h} f(2u,\delta)v\,du$$

denote the Melnikov function of the new system (3.6). Then by Lemma 2.1, we have

$$\widetilde{M}_1 = \widetilde{c}_{01} + \widetilde{c}_1 h\ln|h| + \widetilde{c}_{21}h + \widetilde{c}_3 h^2\ln|h| + O\left(|h|^2\right),$$

where

$$\widetilde{c}_3(O,\delta) = -8a_1.$$

Note that

$$M_1(h,\delta) = 2\widetilde{M}_1\left(\frac{1}{2}h,\delta\right),$$

we have

$$c_3(O,\delta) = \frac{1}{2}\widetilde{c}_3(O,\delta) = -4a_1. \tag{3.7}$$

Next, we discuss the property of $M_1(h,\delta)$ for $0 < h + \frac{11}{768} \ll 1$. We move the center $C_1(\frac{1}{2},0)$ into the origin by letting $x = x_1 + \frac{1}{2}$, $y = \frac{\sqrt{6}}{4}y_1$ and $t = \frac{4}{\sqrt{6}}\tau$, system (1.2) becomes

$$\frac{dx_1}{d\tau} = y_1, \qquad \frac{dy_1}{d\tau} = -x_1 - \frac{5}{3}x_1^2 + \frac{10}{3}x_1^3 + \frac{20}{3}x_1^4 + \frac{8}{3}x_1^5 - \varepsilon\frac{4}{\sqrt{6}}f\left(x_1 + \frac{1}{2},\delta\right)y_1. \tag{3.8}$$

For $\varepsilon = 0$, the Hamiltonian function of (3.8) is

$$\overline{H}(x_1,y_1) = -\frac{11}{288} + \frac{1}{2}\left(x_1^2 + y_1^2\right) + \frac{5}{9}x_1^3 - \frac{5}{6}x_1^4 - \frac{4}{3}x_1^5 - \frac{4}{9}x_1^6 = \frac{8}{3}H\left(x_1 + \frac{1}{2}, \frac{\sqrt{6}}{4}y_1\right).$$

From Lemma 2.3, we have

$$\bar{b}_{01} = 2\pi c_{00} = 2\pi\left[\left(-\frac{4}{\sqrt{6}}\right)\left(a_0 + \frac{1}{4}a_1 + \frac{1}{16}a_2 + \frac{1}{64}a_3 + \frac{1}{256}a_4\right)\right],$$

$$\bar{b}_{11} = \pi(c_{20} - 3h_{30}c_{10}) = \pi\left[\left(-\frac{4}{\sqrt{6}}\right)\left(-\frac{2}{3}a_1 + \frac{2}{3}a_2 + \frac{5}{8}a_3 + \frac{1}{3}a_4\right)\right].$$

Let

$$\overline{M}_1(h,\delta) = -\oint_{\widetilde{H}(x_1,y_1)=h} \frac{4}{\sqrt{6}} f\left(x_1+\frac{1}{2},\delta\right) y_1\,dx_1$$

denote the Melnikov function of the new system (3.8). Then by Lemma 2.3, we have

$$\overline{M}_1 = \bar{b}_{01}\left(h+\frac{11}{288}\right) + \bar{b}_{11}\left(h+\frac{11}{288}\right)^2 + O\left(\left(h+\frac{11}{288}\right)^3\right).$$

Note that

$$M_1(h,\delta) = \frac{3}{8}\overline{M}_1\left(\frac{8}{3}h,\delta\right),$$

we have

$$b_{01}(\delta) = \bar{b}_{01}(\delta) = 2\pi\left[\left(-\frac{4}{\sqrt{6}}\right)\left(a_0+\frac{1}{4}a_1+\frac{1}{16}a_2+\frac{1}{64}a_3+\frac{1}{256}a_4\right)\right], \tag{3.9}$$

$$b_{11}(\delta) = \frac{8}{3}\bar{b}_{11}(\delta) = \frac{8}{3}\pi\left[\left(-\frac{4}{\sqrt{6}}\right)\left(-\frac{2}{3}a_1+\frac{2}{3}a_2+\frac{5}{8}a_3+\frac{1}{3}a_4\right)\right]. \tag{3.10}$$

Then by Lemma 2.4, for $0 < \frac{1}{48} - h \ll 1$, we have

$$M(h,\delta) = \widehat{c}_0 + \widehat{c}_1\left(h-\frac{1}{48}\right)\ln\left|h-\frac{1}{48}\right| + \widehat{c}_2\left(h-\frac{1}{48}\right) + \widehat{c}_3\left(h-\frac{1}{48}\right)^2\ln\left|h-\frac{1}{48}\right| + O\left(\left(h-\frac{1}{48}\right)^2\right),$$

where $\widehat{\lambda} = \frac{3}{\sqrt{6}}$ denotes an eigenvalue of $S_1$. By (3.1), the homoclinic orbits $\widehat{L}_1$ and $\widehat{L}_2$ have the expressions:

$$\widehat{L}_1:\quad y_+ = -\frac{1}{12}\left(x^2-1\right)\sqrt{48x^2+6},\quad -1 \leqslant x \leqslant 1,$$

$$\widehat{L}_2:\quad y_- = \frac{1}{12}\left(x^2-1\right)\sqrt{48x^2+6},\quad -1 \leqslant x \leqslant 1.$$

By (1.2), we have

$$\widehat{c}_0 = -\oint_{\widehat{L}_1+\widehat{L}_2} f(x,\delta)y\,dx = -2\int_{-1}^{1}\left(a_0+a_1x^2+a_2x^4+a_3x^6+a_4x^8\right)y_+\,dx = 2\sum_{j=0}^{4} a_jK_j, \tag{3.11}$$

with

$$K_0 = -\int_{-1}^{1} y_+\,dx = -\frac{15}{128}\sqrt{6} - \frac{11}{512}\sqrt{3}\Psi_1 + \frac{11}{512}\sqrt{3}\Psi_2,$$

$$K_1 = -\int_{-1}^{1} x^2y_+\,dx = -\frac{97}{2048}\sqrt{6} + \frac{17}{24\,576}\sqrt{3}\Psi_1 - \frac{17}{24\,576}\sqrt{3}\Psi_2,$$

$$K_2 = -\int_{-1}^{1} x^4y_+\,dx = -\frac{2859}{131\,072}\sqrt{6} - \frac{23}{524\,288}\sqrt{3}\Psi_1 + \frac{23}{524\,288}\sqrt{3}\Psi_2,$$

$$K_3 = -\int_{-1}^{1} x^6y_+\,dx = -\frac{135\,843}{10\,485\,760}\sqrt{6} + \frac{29}{8\,388\,608}\sqrt{3}\Psi_1 - \frac{29}{8\,388\,608}\sqrt{3}\Psi_2,$$

$$K_4 = -\int_{-1}^{1} x^8y_+\,dx = -\frac{574\,331}{67\,108\,864}\sqrt{6} - \frac{245}{805\,306\,368}\sqrt{3}\Psi_1 + \frac{245}{805\,306\,368}\sqrt{3}\Psi_2,$$

where $\Psi_1 = \ln(3+2\sqrt{2})$, $\Psi_2 = \ln(3-2\sqrt{2})$.

By (3.3), (3.4), (3.5) and (3.9), letting $c_{01} = c_1 = c_{21} = b_{01} = 0$, we can obtain

$$a_0 = 0, \qquad a_1 = -\frac{3a_4}{2048} \times \frac{N_1}{N_0}, \qquad a_2 = \frac{a_4}{512} \times \frac{N_2}{N_0}, \qquad a_3 = -20a_4 \times \frac{N_3}{N_0} \tag{3.12}$$

with

$$\begin{aligned}
N_0 &= 3\,927\,968\sqrt{3}(\Phi_1 - \Phi_2) - 439\,230(\Phi_1 + \Phi_2 - \Phi_1\Phi_2) - 19\,657\,984,\\
N_1 &= 769\,843\,360\sqrt{3}(\Phi_1 - \Phi_2) - 205\,998\,870(\Phi_1 + \Phi_2 - \Phi_1\Phi_2) - 3\,225\,305\,344,\\
N_2 &= 3\,405\,444\,064\sqrt{3}(\Phi_1 - \Phi_2) - 828\,827\,010(\Phi_1 + \Phi_2 - \Phi_1\Phi_2) - 14\,696\,514\,304,\\
N_3 &= 477\,191\sqrt{3}(\Phi_1 - \Phi_2) - 87\,846(\Phi_1 + \Phi_2 - \Phi_1\Phi_2) - 2\,206\,896,
\end{aligned}$$

where $\Phi_1 = \ln(5\sqrt{3} + 8)$, $\Phi_2 = \ln(5\sqrt{3} - 8)$.

Then substituting (3.12) into (3.7), (3.10) and (3.11), we have

$$\begin{aligned}
c_3 &= \frac{3a_4}{512} \times \frac{N_1}{N_0} \approx 2.122151961a_4,\\
b_{11} &= -\frac{\pi(3N_1 + 4N_2 - 38\,400N_3 + 1024N_0)a_4}{768\sqrt{6}N_0} \approx -0.5930873636a_4,\\
\widehat{c}_0 &= -\frac{(3K_1N_1 - 4K_2N_2 + 40\,960K_3N_3 - 2048K_4N_0)a_4}{1024N_0} \approx 0.0164348984a_4.
\end{aligned} \tag{3.13}$$

Define $\bar{c}_0 = c_{01}$, $\bar{c}_1 = c_1$, $\bar{c}_2 = c_{21}$ and $\bar{c}_3 = c_3$. By (3.3), (3.4), (3.5) and (3.9), it is obvious that

$$\begin{aligned}
\det\frac{\partial(\bar{c}_0, \bar{c}_1, \bar{c}_2, b_{01})}{\partial(a_0, a_1, a_2, a_3)} &= \begin{vmatrix} 2I_0 & 2I_1 & 2I_2 & 2I_3\\ 2 & 0 & 0 & 0\\ 0 & 2J_1 & 2J_2 & 2J_3\\ -\frac{4\sqrt{6}\pi}{3} & -\frac{\sqrt{6}\pi}{3} & -\frac{\sqrt{6}\pi}{12} & -\frac{\sqrt{6}\pi}{48}\end{vmatrix}\\
&= \frac{\sqrt{6}\pi}{6}(I_1J_2 - J_1I_2) - \frac{2\sqrt{6}\pi}{3}(I_1J_3 - J_1I_3) + \frac{8\sqrt{6}\pi}{3}(I_2J_3 - J_2I_3)\\
&\approx 0.00004580684306 \neq 0.
\end{aligned}$$

Then by (3.12), we have $a_0 \equiv a_0^\star$, $a_1 \equiv a_1^\star$, $a_2 \equiv a_2^\star$ and $a_3 \equiv a_3^\star$. Let us take $\delta_0 = (a_0^\star, a_1^\star, a_2^\star, a_3^\star)$. Thus, with $a_4 \neq 0$, we have $\bar{c}_3(\delta_0)b_{11}(\delta_0) < 0$, which indicates that there exist no roots $h \in (-\frac{11}{768}, 0)$. And we also have $\bar{c}_3(\delta_0)\widehat{c}_0(\delta_0) > 0$, which indicates that there exists a root $h^\star \in (0, \frac{1}{48})$ such that $M(h^\star, \delta_0) = 0$ under (3.12). $\square$

From the above theorem, we get the following corollaries.

**Corollary 3.1.** *Let $a_4 = 0$ and $a_3 \neq 0$ in* (1.3), *the system* (1.2) *has at least* 10 *limit cycles.*

**Proof.** Define $\bar{c}_0 = c_{01}$, $\bar{c}_1 = c_1$, $\bar{c}_2 = c_{21}$ and $\bar{c}_3 = c_3$. According to (3.3), (3.4), (3.5), (3.9) and (3.11), we get

$$\begin{aligned}
\bar{c}_0 &= 2\Bigg[\left(-\frac{15}{128} + \frac{11\sqrt{3}}{2048}\Phi_1 - \frac{11\sqrt{3}}{2048}\Phi_2\right)a_0 + \left(-\frac{97}{2048} + \frac{165\sqrt{3}}{32\,768}\Phi_1 - \frac{165\sqrt{3}}{32\,768}\Phi_2\right)a_1\\
&\quad + \left(-\frac{4395}{131\,072} + \frac{10\,263\sqrt{3}}{2\,097\,152}\Phi_1 - \frac{10\,263\sqrt{3}}{2\,097\,152}\Phi_2\right)a_2\\
&\quad + \left(-\frac{312\,483}{10\,485\,760} + \frac{164\,835\sqrt{3}}{33\,554\,432}\Phi_1 - \frac{164\,835\sqrt{3}}{33\,554\,432}\Phi_2\right)a_3\Bigg],\\
\bar{c}_1 &= 2a_0,\\
\bar{c}_2 &= 2\Bigg[\left(-\frac{\sqrt{3}}{4}\Phi_1 + \frac{\sqrt{3}}{4}\Phi_2\right)a_1 + \left(\frac{3}{4} - \frac{15\sqrt{3}}{64}\Phi_1 + \frac{15\sqrt{3}}{64}\Phi_2\right)a_2 + \left(\frac{135}{128} - \frac{483\sqrt{3}}{2048}\Phi_1 + \frac{483\sqrt{3}}{2048}\Phi_2\right)a_3\Bigg],\\
\bar{c}_3 &= -4a_1,\\
b_{01} &= 2\pi\left[\left(-\frac{4}{\sqrt{6}}\right)\left(a_0 + \frac{1}{4}a_1 + \frac{1}{16}a_2 + \frac{1}{64}a_3\right)\right],\\
\widehat{c}_0 &= 2\Bigg[\left(-\frac{15\sqrt{6}}{128} - \frac{11\sqrt{3}}{512}\Psi_1 + \frac{11\sqrt{3}}{512}\Psi_2\right)a_0 + \left(-\frac{97\sqrt{6}}{2048} + \frac{17\sqrt{3}}{24\,576}\Psi_1 - \frac{17\sqrt{3}}{24\,576}\Psi_2\right)a_1
\end{aligned}$$

$$+\left(-\frac{2859\sqrt{6}}{131\,072}-\frac{23\sqrt{3}}{524\,288}\Psi_1+\frac{23\sqrt{3}}{524\,288}\Psi_2\right)a_2$$
$$+\left(-\frac{135\,843\sqrt{6}}{10\,485\,760}+\frac{29\sqrt{3}}{8\,388\,608}\Psi_1-\frac{29\sqrt{3}}{8\,388\,608}\Psi_2\right)a_3\bigg], \tag{3.14}$$

where $\Phi_1=\ln(5\sqrt{3}+8)$, $\Phi_2=\ln(5\sqrt{3}-8)$, $\Psi_1=\ln(3+2\sqrt{2})$ and $\Psi_2=\ln(3-2\sqrt{2})$.

In (3.14), letting $\bar{c}_0=\bar{c}_1=\bar{c}_2=0$, we can obtain

$$a_0=0,\qquad a_1=\frac{3a_3N_5}{2560N_4},\qquad a_2=-\frac{a_3N_6}{80N_4} \tag{3.15}$$

with

$$N_4=-99\,328+726(\Phi_1+\Phi_2-\Phi_1\Phi_2)+18\,160\sqrt{3}(\Phi_1-\Phi_2),$$
$$N_5=-31\,053\,568-39\,930(\Phi_1+\Phi_2-\Phi_1\Phi_2)+5\,727\,840\sqrt{3}(\Phi_1-\Phi_2),$$
$$N_6=-11\,174\,400+54\,450(\Phi_1+\Phi_2-\Phi_1\Phi_2)+2\,020\,144\sqrt{3}(\Phi_1-\Phi_2),$$

where $\Phi_1=\ln(5\sqrt{3}+8)$, $\Phi_2=\ln(5\sqrt{3}-8)$.

Then substituting (3.15) into (3.14), we have

$$\bar{c}_3=-\frac{3a_3N_5}{640N_4}\approx-0.5980230234a_3,$$
$$b_{01}=-\frac{\pi(1280N_4+3N_5-4N_6)a_3}{1280\sqrt{6}N_4}\approx-0.002671375946a_3,$$
$$\widehat{c}_0=\frac{(3N_5K_1-32N_6K_2+5120N_4K_3)a_3}{1280N_4}\approx-0.00622553280a_3. \tag{3.16}$$

By (3.14), it is obvious that

$$\det\frac{\partial(\bar{c}_0,\bar{c}_1,\bar{c}_2)}{\partial(a_0,a_1,a_2)}=\begin{vmatrix}2I_0 & 2I_1 & 2I_2\\ 2 & 0 & 0\\ 0 & 2J_1 & 2J_2\end{vmatrix}=8(I_2J_1-I_1J_2)\approx-0.01714723524\neq0.$$

Then by (3.15), we have $a_0\equiv a_0^\star$, $a_1\equiv a_1^\star$, $a_2\equiv a_2^\star$. Let us take $\delta_0=(a_0^\star,a_1^\star,a_2^\star)$. Thus, with $a_3\neq0$, we have $\bar{c}_3(\delta_0)b_{01}(\delta_0)>0$, which indicates that there exists a root $h_1^\star\in(-\frac{11}{768},0)$ such that $M_1(h_1^\star,\delta_0)=0$ under (3.15). And we also have $\bar{c}_3(\delta_0)\widehat{c}_0(\delta_0)>0$, which indicates that there exist no roots $h\in(0,\frac{1}{48})$. $\square$

**Corollary 3.2.** *Let $a_3=a_4=0$ and $a_2\neq0$ in (1.3), system (1.2) has at least 5 limit cycles.*

**Proof.** Define $\bar{c}_0=c_{01}$ and $\bar{c}_1=c_1$. According to (3.3), (3.4), (3.9), (3.10) and (3.11), we get

$$\bar{c}_0=2\bigg[\left(-\frac{15}{128}+\frac{11\sqrt{3}}{2048}\Phi_1-\frac{11\sqrt{3}}{2048}\Phi_2\right)a_0+\left(-\frac{97}{2048}+\frac{165\sqrt{3}}{32\,768}\Phi_1-\frac{165\sqrt{3}}{32\,768}\Phi_2\right)a_1$$
$$+\left(-\frac{4395}{131\,072}+\frac{10\,263\sqrt{3}}{2\,097\,152}\Phi_1-\frac{10\,263\sqrt{3}}{2\,097\,152}\Phi_2\right)a_2\bigg],$$
$$\bar{c}_1=2a_0,$$
$$b_{01}=2\pi\left[\left(-\frac{4}{\sqrt{6}}\right)\left(a_0+\frac{1}{4}a_1+\frac{1}{16}a_2\right)\right],$$
$$b_{11}=\frac{8}{3}\pi\left[\left(-\frac{4}{\sqrt{6}}\right)\left(-\frac{2}{3}a_1+\frac{2}{3}a_2\right)\right],$$
$$\widehat{c}_0=2\bigg[\left(-\frac{15\sqrt{6}}{128}-\frac{11\sqrt{3}}{512}\Psi_1+\frac{11\sqrt{3}}{512}\Psi_2\right)a_0+\left(-\frac{97\sqrt{6}}{2048}+\frac{17\sqrt{3}}{24\,576}\Psi_1-\frac{17\sqrt{3}}{24\,576}\Psi_2\right)a_1$$
$$+\left(-\frac{2859\sqrt{6}}{131\,072}-\frac{23\sqrt{3}}{524\,288}\Psi_1+\frac{23\sqrt{3}}{524\,288}\Psi_2\right)a_2\bigg], \tag{3.17}$$

where $\Phi_1=\ln(5\sqrt{3}+8)$, $\Phi_2=\ln(5\sqrt{3}-8)$, $\Psi_1=\ln(3+2\sqrt{2})$ and $\Psi_2=\ln(3-2\sqrt{2})$.

In (3.17), letting $\bar{c}_0 = b_{01} = 0$, we can obtain

$$a_0 = \frac{a_2}{256} \times \frac{45\,488\sqrt{3} + 22\,869(\Phi_2 - \Phi_1)}{592\sqrt{3} + 363(\Phi_2 - \Phi_1)},$$
$$a_1 = -\frac{3a_2}{64} \times \frac{18\,320\sqrt{3} + 9559(\Phi_2 - \Phi_1)}{592\sqrt{3} + 363(\Phi_2 - \Phi_1)}. \tag{3.18}$$

Then substituting (3.18) into (3.17), we have

$$\bar{c}_1 \approx -0.2647515758a_2,$$
$$b_{11} \approx -6.571186554a_2,$$
$$\widehat{c}_0 \approx -0.05915704115a_2. \tag{3.19}$$

By (3.17), it is obvious that

$$\det \frac{\partial(\bar{c}_0, b_{01})}{\partial(a_0, a_1)} = \begin{vmatrix} 2I_0 & 2I_1 \\ -\frac{4\sqrt{6}\pi}{3} & -\frac{\sqrt{6}\pi}{3} \end{vmatrix} = -\frac{2\sqrt{6}\pi}{3} I_0 + \frac{8\sqrt{6}\pi}{3} I_1 \approx 0.0529496805 \neq 0.$$

Then by (3.18), we have $a_0 \equiv a_0^\star$, $a_1 \equiv a_1^\star$. Let us take $\delta_0 = (a_0^\star, a_1^\star)$. Thus, with $a_2 \neq 0$, we have $\bar{c}_1(\delta_0)b_{11}(\delta_0) > 0$, which indicates that there exist no roots $h \in (-\frac{11}{768}, 0)$. And we also have $\bar{c}_1(\delta_0)\widehat{c}_0(\delta_0) > 0$, which indicates that there exists a root $h^\star \in (0, \frac{1}{48})$ such that $M(h^\star, \delta_0) = 0$ under (3.18). □

**Corollary 3.3.** *Let $a_2 = a_3 = a_4 = 0$ and $a_1 \neq 0$ in* (1.3), *system* (1.2) *has at least* 3 *limit cycles.*

**Proof.** Define $\bar{c}_0 = c_{01}$ and $\bar{c}_1 = c_1$. According to (3.3), (3.4), (3.9) and (3.11), we get

$$\bar{c}_0 = 2\left[\left(-\frac{15}{128} + \frac{11\sqrt{3}}{2048}\Phi_1 - \frac{11\sqrt{3}}{2048}\Phi_2\right)a_0 + \left(-\frac{97}{2048} + \frac{165\sqrt{3}}{32\,768}\Phi_1 - \frac{165\sqrt{3}}{32\,768}\Phi_2\right)a_1\right],$$
$$\bar{c}_1 = 2a_0,$$
$$b_{01} = 2\pi\left[\left(-\frac{4}{\sqrt{6}}\right)\left(a_0 + \frac{1}{4}a_1\right)\right],$$
$$\widehat{c}_0 = 2\left[\left(-\frac{15\sqrt{6}}{128} - \frac{11\sqrt{3}}{512}\Psi_1 + \frac{11\sqrt{3}}{512}\Psi_2\right)a_0 + \left(-\frac{97\sqrt{6}}{2048} + \frac{17\sqrt{3}}{24\,576}\Psi_1 - \frac{17\sqrt{3}}{24\,576}\Psi_2\right)a_1\right]. \tag{3.20}$$

In (3.20), letting $\bar{c}_0 = 0$, we can obtain

$$a_0 = -\frac{a_1}{16} \times \frac{-1552 + 165\sqrt{3}(\Phi_1 - \Phi_2)}{-240 + 11\sqrt{3}(\Phi_1 - \Phi_2)}. \tag{3.21}$$

Then substituting (3.21) into (3.20), we have

$$\bar{c}_1 \approx -0.4407890354a_1,$$
$$b_{01} \approx -0.3037640503a_1,$$
$$\widehat{c}_0 \approx -0.0392280783a_1. \tag{3.22}$$

Then by (3.21), we have $a_0 \equiv a_0^\star$. Let us take $\delta_0 = a_0^\star$. Thus, with $a_1 \neq 0$, we have $\bar{c}_1(\delta_0)b_{01}(\delta_0) > 0$, which indicates that there exist no roots $h \in (-\frac{11}{768}, 0)$. And we also have $\bar{c}_1(\delta_0)\widehat{c}_0(\delta_0) > 0$, which indicates that there exists a root $h^\star \in (0, \frac{1}{48})$ such that $M(h^\star, \delta_0) = 0$ under (3.21). □

## References


[1] A. Liénard, Étude des oscillations entretenues, Rev. Gen. Electr. 23 (1928) 946–954.
[2] M. Han, Liapunov constants and Hopf cyclicity of Liénard systems, Ann. Differential Equations 15 (1999) 113–126.
[3] C.J. Christopher, S. Lynch, Small-amplitude limit cycle bifurcations for Liénard systems with quadratic or cubic damping or restoring forces, Nonlinearity 12 (1999) 1099–1112.
[4] J. Yang, M. Han, V.G. Romanovski, Limit cycle bifurcations of some Liénard systems, J. Math. Anal. Appl. 366 (2010) 242–255.

[5] M. Han, H. Yan, J. Yang, C. Lhotka, On the number of limit cycles of some Liénard systems, Can. Appl. Math. Q. 17 (2009) 61–83.
[6] M. Han, J. Yang, A. Tarta, Y. Gao, Limit cycles near homoclinic and heteroclinic loops, J. Dynam. Differential Equations 20 (2008) 923–944.
[7] M. Han, On the number of limit cycles bifurcating from a homoclinic or heteroclinic loop, Sci. China Ser. A 36 (1993) 113–132.
[8] Y. Hou, M. Han, Melnikov functions for planar near-Hamiltonian systems and Hopf bifurcations, J. Shanghai Normal Univ. (Nat. Sci.) 35 (2006) 1–10.