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On the existence of embedded eigenvalues

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ABSTRACT

I relate the existence of eigenfunctions of decaying perturbations of the free Laplacian to the semi-classical behavior of the spectrum of quasi-periodic Schrödinger operators. In particular, I construct embedded eigenvalues in the spectrum of the discrete Laplacian perturbed by a potential of generalized bounded variation.

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1. Introduction

I consider discrete Schrödinger operators $H = \Delta + V$ with decaying potential V. My goal in this paper is to exhibit the existence of eigenvalues embedded in the absolutely continuous spectrum. For this, I exploit the mechanisms used in [1–5] to prove Anderson localization. My main motivation for this is that the construction is relatively simple and deserves to be better known. The proof of this paper is different from the constructions described in [6–9]. Instead of studying the solution of the Schrödinger equation directly, one studies local restrictions of the operator and then infer decay properties of the solution from these.

Consider the Schrödinger operator

$$\begin{aligned} &H_\beta : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+), \\ &H_\beta u(n) = \begin{cases} u(2) + (V(1) + \beta)u(1), & n = 1; \\ u(n+1) + V(n)u(n) + u(n-1), & n \geq 2, \end{cases} \end{aligned} \tag{1.1}$$

where $V(n)$ is a bounded real valued sequence called the *potential*, and $\beta \in \mathbb{R}$ parametrizes the boundary condition. We also define H_∞ as the restriction of H_0 to $\ell^2(\{2, 3, \ldots\})$.

We will be interested in potentials of the form

$$V(n) = \kappa_n W(n), \quad W(n) = \left(\sum_{j=1}^{d} b_j \cos(2\pi(\alpha_j n + x_j)) \right), \tag{1.2}$$

where κ_n is a sequence monotonically decaying to 0 and $b_j \neq 0$. Since $V(n) \to 0$ as $n \to \infty$, we have $\sigma_{\mathrm{ess}}(H_\beta) = [-2, 2]$. Since $\sum_{n=1}^{\infty} (\kappa_n - \kappa_{n+1}) < \infty$, V is of *generalized bounded variation* see [10,11] for a definition. In particular, it is shown in [10].

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Theorem 1.1 ([10]). *Let p be an integer and $\alpha_1, \dots, \alpha_d \in \mathbb{R}$ arbitrary. If $\kappa \in \ell^p(\mathbb{Z}_+)$, then $\sigma_{\mathrm{ac}}(H_\beta) = [-2, 2]$.*
Furthermore, the pure point spectrum of H_β in $(-2, 2)$ satisfies

$$\sigma_{\mathrm{pp}}(H_\beta) \cap (-2, 2) \subseteq \{\pm 2\cos(\pi \underline{\alpha} \cdot \underline{\ell}) \, \underline{\ell} \in \mathbb{Z}^d, |\underline{\ell}|_1 \leq p - 1\}. \tag{1.3}$$

Here we used the notation $|\underline{\ell}|_1 = \sum_{j=1}^d |\ell_j|$ and $\underline{x} \cdot \underline{y} = \sum_{j=1}^d x_j y_j$. The results of this paper show that the exceptional set given in this theorem is optimal. We will first state a simple ($d = 1$, specific choice of κ_n) version of our main result.

Theorem 1.2. *Let $\kappa_n = n^{-\tau}$ for $\tau > 0$ and $W(n) = 2\cos(2\pi\alpha n)$ for α irrational. Let $1 \leq \ell < \frac{1}{\tau}$ and $\sigma \in \{-1, 1\}$. Then there exists $\beta \in \mathbb{R} \cup \{\infty\}$ and $c_2, \dots, c_\ell \in \mathbb{R}$ such that $E_0 = \sigma 2\cos(\pi \ell \alpha)$ is an eigenvalue of*

$$H_\beta + \widetilde{V}, \qquad \widetilde{V}(n) = \sum_{j=2}^{\ell} c_j \cdot (\kappa_n)^j. \tag{1.4}$$

One can easily check that $E_0 \in (-2, 2) \subseteq \sigma_{\mathrm{ess}}(H)$. Hence, E_0 is an embedded eigenvalue. Adding the correction $\widetilde{V}$ is necessary to deal with higher order corrections on the location of gaps of quasi-periodic Schrödinger operators. In fact, let $Q_\lambda = \Delta + \lambda W$, $W(n) = 2\cos(2\pi\alpha n)$ acting on $\ell^2(\mathbb{Z})$, which is just the *Almost–Mathieu operator*. Then the spectrum of Q_λ has a gap of λ^ℓ with center $E_0 + \sum_{j=2}^{\ell} c_j \lambda^j$ for $\lambda > 0$ small enough as we show in Theorem 2.2.

We have that $\kappa_n = n^{-\tau}$ is in $\ell^p(\mathbb{Z}_+)$ if $\tau p > 1$. Hence, we see that the range of allowed ℓ in Theorem 1.2 is in this case $1 \leq \ell < p$, which is exactly the exceptional set of Theorem 1.1.

We now begin with stating the more sophisticated version of Theorem 1.2. For $\underline{\ell} \in \mathbb{Z}^d$, denote by $\widetilde{\Gamma}_{\underline{\ell}}$ the set of all paths of length $|\ell|_1$ in $\mathbb{Z}^d$ connecting 0 to $\underline{\ell}$. Formally, we can define $\widetilde{\Gamma}_{\underline{\ell}}$ as the set of all sequences

$$(\underline{\gamma}_1, \dots, \underline{\gamma}_{|\underline{\ell}|_1 - 1}) \in (\mathbb{Z}^d)^{|\underline{\ell}|_1 - 1} \tag{1.5}$$

such that $|\underline{\gamma}_q - \underline{\gamma}_{q-1}| = 1$ for $q = 1, \dots, |\underline{\ell}|_1$ with $\underline{\gamma}_0 = 0$ and $\underline{\gamma}_{|\underline{\ell}|_1} = \underline{\ell}$. Now, define the quantity

$$\widetilde{W}_{\pm,\underline{\ell}} = -\sum_{\gamma \in \widetilde{\Gamma}_{\underline{\ell}}} \prod_{q=1}^{|\underline{\ell}|_1 - 1} \frac{|\underline{b} \cdot (\underline{\gamma}_q - \underline{\gamma}_{q-1})|}{\pm(2\cos(2\pi\underline{\alpha} \cdot \underline{\ell}) - 2\cos(\pi(2\underline{\gamma}_q - \underline{\ell}) \cdot \underline{\alpha}))}. \tag{1.6}$$

It is not hard to see that $\widetilde{W}_{\pm,\underline{\ell}}$ is finite as long as for all $\underline{m} \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\underline{m} \cdot \underline{\alpha} = \sum_{j=1}^{d} m_j \alpha_j \notin \mathbb{Z}. \tag{1.7}$$

We are now ready to state the more sophisticated version of our result.

Theorem 1.3. *Let $\tau > 0$ and assume (1.7) and that κ_n satisfies for $n \geq 2$*

$$\kappa_n \geq \max\left(\frac{1}{n^\tau}, \left(\kappa_{\lceil \frac{n}{2} \rceil} - \kappa_{\lceil \frac{n}{2} \rceil + 1}\right)^{\frac{\tau}{2}}\right) \tag{1.8}$$

and $\kappa_n - \kappa_{n+1}$ is a decreasing sequence.
Let $\sigma \in \{-1, +1\}$, $\underline{\ell} \in \mathbb{Z}^d \setminus \{0\}$ satisfying $\tau|\underline{\ell}|_1 < 1$, and define

$$E_0 = \sigma 2\cos(\pi \underline{\ell} \cdot \underline{\alpha}). \tag{1.9}$$

Assume that $\widetilde{W}_{\sigma,\underline{\ell}} \neq 0$. Then there exists $\beta = \beta(\underline{\ell}, \sigma) \in \mathbb{R} \cup \{\infty\}$ and $c_2, \dots, c_{|\underline{\ell}|_1}$ such that E_0 is an eigenvalue of

$$H_\beta + \widetilde{V}, \qquad \widetilde{V}(n) = \sum_{j=2}^{|\underline{\ell}|_1} c_j \cdot (\kappa_n)^j. \tag{1.10}$$

Theorem 1.3 clearly implies Theorem 1.2. As another example of a sequence κ_n, I want to mention $\kappa_n = \frac{1}{\log(n+4)}$. Then (1.8) holds for any $\tau > 1$, so there are infinitely many possible eigenvalues. Of course due to the necessary correction, Theorem 1.3 only asserts the existence of a single eigenvalue.

The assumption that α is irrational in Theorem 1.2 respectively that (1.7) holds in Theorem 1.3 is not strictly necessary. However, dropping these assumptions entails additional bookkeeping, so I have decided to work under them.

If $\widetilde{W}_{\pm,\ell} = 0$, then the result of Theorem 1.3 no longer holds. In fact an inspection of the proof of Theorem 2.2 shows that then the gaps of the underlying quasi-periodic operator are smaller. It should still be possible to prove a version of Theorem 1.3 when $\widetilde{W}_{\pm,\ell} = 0$, but the range of allowed $|\ell|_1$ would be smaller.

The proof of Theorems 1.2 and 1.3 is explained in the next section. For the readers orientation, I will point out some highlights at this point. The condition $\widetilde{W}_{+,\underline{\ell}} \neq 0$ implies that the Schrödinger operator $H_N = \Delta + \kappa_N W$ has a gap in its spectrum given by

$$\sigma(H_N) \cap [E_1 - c(\kappa_N)^{|\underline{\ell}|_1}, +c(\kappa_N)^{|\underline{\ell}|_1}] = \emptyset, \quad E_1 = 2\cos(\pi \underline{\alpha} \cdot \underline{\ell}) + \widetilde{V}(N) \tag{1.11}$$

for a constant $c > 0$. This fact is proven in Theorem 2.2 and the discussion following it. Theorem 2.2 is an assertion about the spectrum of quasi-periodic Schrödinger operators and should be considered of interest for its own sake. Its proof is given in Section 4 and takes up about half of this paper.

The rest of the proof is somewhat technical. One sees the usual ingredients of proving point spectrum like the Combes–Thomas estimate (Proposition 3.2). Maybe the key new aspect is to use compactness to assert that an eigenfunction exists.

Finally, let me point out that the methods of this paper are quite general. For example, it should be possible to analyze the potential $V(n) = \kappa_n \chi_{[1-\alpha,1)}(n\alpha \pmod 1)$, where $\alpha = \frac{1}{2}(\sqrt{5}-1)$ is the golden mean, combining my results with the ones of [12].

2. The skeleton of the proof

The goal of this section is to give the proof of Theorem 1.3. The main technical ingredient involves the analysis of the gaps of the following multi-frequency quasi-periodic operator. Define for $\lambda > 0$ and $\underline{x} \in \mathbb{T}^d$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the quasi-periodic Schrödinger operator

$$\begin{aligned} &Q_{\lambda,\underline{x}} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \\ &Q_{\lambda,\underline{x}} u(n) = u(n+1) + u(n-1) + 2\lambda \left(\sum_{\ell=1}^{d} b_j \cos(2\pi(n\alpha_j + x_j)) \right) u(n). \end{aligned} \tag{2.1}$$

By (1.7), $\sigma(Q_{\lambda,\underline{x}})$ is independent of $\underline{x}$. In order to state the existence of gaps, we first need to define some things. We denote by Γ_p the set of all length p paths connecting 0 to itself in $\mathbb{Z}^d$. Similarly to $\widetilde{\Gamma}_{\underline{\ell}}$, Γ_p can be defined as the set of all sequences

$$(\underline{\gamma}_1, \dots, \underline{\gamma}_{p-1}) \in (\mathbb{Z}^d \setminus \{0\})^{p-1} \tag{2.2}$$

such that $|\underline{\gamma}_q - \underline{\gamma}_{q-1}|_1 = 1$ for $q = 1, \dots, p$ and $\underline{\gamma}_0 = \underline{\gamma}_p = 0$.

Furthermore given $\underline{\ell} \in \mathbb{Z}^d \setminus \{0\}$, we define $E_{\underline{\ell},\pm} = \pm 2\cos(\pi \underline{\alpha} \cdot \underline{\ell})$. If $|\underline{\ell}|_1 = 1$, there is a gap of size λ around $E_{\underline{\ell},\pm}$. However for $|\underline{\ell}|_1 \geq 2$, there are higher order corrections, which are defined in the next theorem.

Theorem 2.1. *Let $\underline{\ell} \in \mathbb{Z}^d \setminus \{0\}$, $\underline{\alpha}$ satisfying (1.7), $\underline{b} \in (\mathbb{R} \setminus \{0\})^d$, $\sigma \in \{-1, +1\}$. Then for $\lambda > 0$ small enough, there exists an analytic function E defined in a small neighborhood of 0 such that*

$$E(\lambda) = E_{\underline{\ell},\sigma} + \sum_{p=2}^{|\underline{\ell}|_1} \lambda^p \mathcal{W}_{\sigma,\underline{\ell},p}(E(\lambda)) \tag{2.3}$$

where the functions

$$\mathcal{W}_{\sigma,\underline{\ell},p}(E) = -\sum_{\underline{\gamma} \in \Gamma_p} \prod_{q=1}^{p-2} \frac{|\underline{b} \cdot (\underline{\gamma}_q - \underline{\gamma}_{q-1})|}{E - \sigma 2\cos(\pi(2\underline{\gamma}_q - \underline{\ell}) \cdot \underline{\alpha})} \tag{2.4}$$

are uniformly bounded in a neighborhood of $E_{\underline{\ell},\sigma}$.

We give the proof of this theorem at the end of this section. We clearly have that $E(\lambda) = E_{\ell,\sigma} + O(\lambda^2)$. We will prove it in Section 4.

Theorem 2.2. *Let $x \in \mathbb{T}^d$, $\underline{\ell} \in \mathbb{Z}^d \setminus \{0\}$, $\sigma \in \{\pm\}$, $\underline{b} \in (\mathbb{R} \setminus \{0\})^d$, and $\underline{\alpha}$ satisfying (1.7). If $|\underline{\ell}|_1 \geq 2$ assume that $\widetilde{W}_{\sigma,\underline{\ell}} \neq 0$. Then for $c = c(\underline{\ell}, \underline{\alpha}, \underline{b}) > 0$ and $\lambda > 0$ small enough*

$$\sigma(Q_{\lambda,\underline{x}}) \cap [E(\lambda) - c\lambda^{|\underline{\ell}|_1}, E(\lambda) + c\lambda^{|\underline{\ell}|_1}] = \emptyset. \tag{2.5}$$

We now return to the proof of Theorem 1.3. In particular, we have that $E_0 = E_{\underline{\ell},\sigma}$ for the appropriate choice of $\underline{\ell}$ and σ. We first explain the meaning of the constants $c_2, \dots, c_{|\underline{\ell}|_1}$.

Remark 2.3. Since $E(\lambda)$ is an analytic function, it can be expanded as a power series. Combining this with (2.3), we obtain

$$E(\lambda) = E_0 - \sum_{j=2}^{|\underline{\ell}|_1} c_j \cdot \lambda^j + O(\lambda^{|\underline{\ell}|_1+1}). \tag{2.6}$$

These c_j are the ones in Theorem 1.3.

In particular, this implies that

$$\widetilde{V}(n) = E_0 - E(\kappa_n) + O((\kappa_n)^{|\underline{\ell}|_1+1}). \tag{2.7}$$

For $N \geq 3$, define the operator

$$H_{\beta,N} : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+), \tag{2.8}$$
$$H_{\beta,N}u(n) = \begin{cases} u(2) + (\kappa_1 W(1) + \widetilde{V}(1) + \beta)u(1), & n = 1; \\ u(n+1) + (\kappa_n W(n) + \widetilde{V}(n))u(n) + u(n-1), & 2 \leq n < N, \\ u(n+1) + (\kappa_N W(n) + \widetilde{V}(N))u(n) + u(n-1), & n \geq N \end{cases}$$

and also the operator

$$\begin{aligned} &\widetilde{H}_N : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \\ &\widetilde{H}_N u(n) = u(n+1) + (\kappa_N W(n) + \widetilde{V}(N))u(n) + u(n-1). \end{aligned} \tag{2.9}$$

We note that the action of these two operators agree on $[N+1, \infty)$.

Theorem 2.4. *There exists $\delta = \delta(N) > 0$ such that*

$$\sigma(\widetilde{H}_N) \cap [E_0 - \delta, E_0 + \delta] = \emptyset. \tag{2.10}$$

Proof. By (2.7), we have for some $\underline{x}$ that

$$\widetilde{H}_N = Q_{\kappa_N,\underline{x}} - (E(\kappa_N) - E_0) + O((\kappa_N)^{|\underline{\ell}|_1+1}).$$

By Theorem 2.2, we have that

$$\sigma(Q_{\kappa_N,\underline{x}} - (E(\kappa_N) - E_0)) \cap [E_0 - c(\kappa_N)^{|\underline{\ell}|_1}, E_0 - c(\kappa_N)^{|\underline{\ell}|_1}] = \emptyset$$

for some $c > 0$. This implies the claim with $\delta = \frac{c}{2}(\kappa_N)^{|\underline{\ell}|_1}$. □

From this theorem, we obtain that

Corollary 2.5. *There exists a non-zero $\psi_N \in \ell^2(\mathbb{Z}_+)$ solving $H_{\beta,N}\psi_N = E_0\psi_N$ for some $\beta \in \widehat{\mathbb{R}}$.*

Proof. Lemma 2.5. in [13] implies that there exists $u : \mathbb{Z} \to \mathbb{C}$ decaying exponentially at infinity solving $\widetilde{H}_N u = E_0 u$. Set $\psi_N u(n) = u(n)$ for $n > N$ and continues to be the solution of $H_{\beta,N}\psi_N = E_0\psi_N$ for $2 \leq n \leq N$. Choose β such that the equation at 1 is satisfied. □

The next step again exploits the mechanism behind Theorem 2.4, but is independent of the considerations so far.

Proposition 2.6. *There exists $n_0 \geq 1$, $\varepsilon > 0$, and $C > 0$ such that if $\psi \in \ell^2(\mathbb{Z}_+)$ with $\|\psi\| = 1$ solves $H_{\beta,N}\psi = E_0\psi$ for some $\beta \in \mathbb{R} \cup \{\infty\}$ and $N \geq 1$, then for $n \geq n_0$ we have*

$$|\psi(n)| \leq C \exp(-\varepsilon n^{\frac{1}{2}(1-\tau|\underline{\ell}|_1)}). \tag{2.11}$$

We will give the proof of this proposition in the next section.

Denote by $\psi_N \neq 0$ the solutions of $H_{\beta_N,N}\psi_N = E_0\psi_N$ constructed in Corollary 2.5. By replacing ψ_N by $(\|\psi_N\|)^{-1}\psi_N$, we can assume $\|\psi_N\| = 1$. Proposition 2.6 implies that there exists $X \geq 1$ such that for every $N \geq 1$

$$\sum_{x=1}^{X} |\psi_N(x)|^2 \geq \frac{1}{2}. \tag{2.12}$$

Proof of Theorem 1.3. Since the set $\{\psi_N\}_{N=1}^{\infty}$ is compact in the topology of pointwise convergence, there exists a sequence $N_j \to \infty$ and ψ_∞ with $\|\psi_\infty\| \leq 1$ such that

$$\lim_{j\to\infty} \psi_{N_j}(x) = \psi_\infty(x)$$

for $x \geq 1$. (2.12) implies that $\sum_{x=1}^{X} |\psi_\infty(x)|^2 \geq \frac{1}{2}$ in particular $\psi_\infty \neq 0$. Since for $2 \leq x \leq N$

$$\psi_N(x+1) + \psi_N(x-1) + (\kappa_x W(x) + \widetilde{V}(x))\psi_N(x) = E_0 \psi_N(x), \tag{2.13}$$

we obtain that $H_\beta \psi_\infty(x) = E_0 \psi_\infty(x)$ for $x \geq 2$. Hence, we can choose β such that $H_\beta \psi_\infty = E_0 \psi_\infty$. Hence, E_0 is an eigenvalue of H_β and we are done. □

We now give the proof of Theorem 2.1. First, we need

Lemma 2.7. *There exists a neighborhood U of $E_{\underline{\ell},\sigma}$ such that for $1 \leq p \leq |\underline{\ell}|_1$,*

$$\mathcal{W}_{\sigma,\underline{\ell},p} : U \to \mathbb{C} \tag{2.14}$$

is an analytic function.

Proof. By (1.7), we have for $1 \leq |\underline{n}|_1 \leq |\underline{\ell}|_1 - 1$, that

$$E_{\underline{\ell},\pm} \mp 2\cos(\pi(2\underline{n} - \underline{\ell}) \cdot \underline{\alpha}) \neq 0.$$

Since these are finitely many conditions, there exists some $c > 0$ such that $|E_{\underline{\ell},\pm} \mp 2\cos(\pi(2\underline{n} - \underline{\ell}) \cdot \underline{\alpha})| \geq c$. This implies that $|\mathcal{W}_{\sigma,\underline{\ell},p}(E)| \leq C^p$ for some other constant C if $|E - E_{\underline{\ell},\sigma}| \leq \frac{c}{2}$. □

Proof of Theorem 2.1. Define the function of two variable

$$f(E,\lambda) = E_{\underline{\ell},\sigma} - E + \sum_{p=2}^{|\underline{\ell}|_1} \lambda^p \mathcal{W}_{\underline{\ell},p}(E).$$

Clearly $f(E_{\underline{\ell},\sigma}, 0) = 0$ and for $(E - E_{\underline{\ell},\sigma}, \lambda)$ small enough $|\partial_E f(E,\lambda)| \geq \frac{1}{2}$. Hence, by the implicit function theorem there exists a solution $E(\lambda)$ such that $f(E(\lambda),\lambda) = 0$ and this solution is analytic. □

3. Proof of Proposition 2.6

For the proof of Proposition 2.6, we will first need the following two facts. Here we denote by Δ the discrete Laplacian $\Delta u(n) = u(n+1) + u(n-1)$ acting on $\ell^2(\mathbb{Z})$. For $\Lambda \subseteq \Xi \subseteq \mathbb{Z}$ and $H : \ell^2(\Xi) \to \ell^2(\Xi)$, we denote by H^Λ the restriction of H to $\ell^2(\Lambda)$. Furthermore, we will denote by $\{e_x\}_{x\in\mathbb{Z}}$ the standard basis of $\ell^2(\mathbb{Z})$:

$$e_x(n) = \begin{cases} 1, & n = x; \\ 0, & n \neq x. \end{cases} \tag{3.1}$$

Lemma 3.1. *Let $[a,b] \subseteq \mathbb{Z}$ be a finite interval and $V, \widetilde{V}$ be potentials. Let $\Lambda \supseteq [a-1, b+1]$ and assume that $\psi \in \ell^2(\Lambda)$ solves $(\Delta^\Lambda + V - E)\psi = 0$ on Λ and that*

$$V(n) = \widetilde{V}(n), \quad n \in [a-1, b+1]. \tag{3.2}$$

Let $\widetilde{G}(E,x,y) = \langle e_x, (\Delta + \widetilde{V} - E)^{-1} e_y \rangle$ be the Green's function of the whole line operator $\Delta + \widetilde{V} - E$. Then we have for $x \in [a,b]$ that

$$u(x) = \widetilde{G}(E,x,a)u(a-1) - \widetilde{G}(E,x,a-1)u(a) - \widetilde{G}(E,x,b+1)u(b) + \widetilde{G}(E,x,b)u(b+1). \tag{3.3}$$

Proof. Denote by $\chi_{[a,b]}$ the characteristic function of $[a,b]$. Then $u(x) = \langle e_x, \chi_{[a,b]} u \rangle$. Furthermore, by assumption

$$\chi_{[a,b]}(\widetilde{H} - E)u = 0.$$

The claim now follows by noting that

$$u(x) = \langle (\widetilde{H} - E)^{-1} e_x, (\widetilde{H} - E)\chi_{[a,b]} u \rangle = \langle (\widetilde{H} - E)^{-1} e_x, [\Delta, \chi_{[a,b]}] u \rangle$$

where $[\Delta, \chi_{[a,b]}] = \Delta\chi_{[a,b]} - \chi_{[a,b]}\Delta$ is the commutator. The claim now follows by evaluating the commutator, see (2.5) in [4]. □

We will also need the Combes–Thomas lemma (see [14] and [13, Lemma 2.5] for a proof in the discrete setting).

Proposition 3.2. *Let $H : \ell^2(\Lambda) \to \ell^2(\Lambda)$ be a Schrödinger operator, $\varepsilon > 0$, $\delta > 0$, and $E \in \mathbb{R}$. Assume that*

$$\sigma(H) \cap [E - \delta, E + \delta] = \emptyset. \tag{3.4}$$

Then for $|x - y| \geq \delta^{-1-\varepsilon}$, we have

$$|\langle e_x, (H - E)^{-1} e_y \rangle| \leq \exp(-c\delta|x - y|) \tag{3.5}$$

for a universal constant $c > 0$ and $\delta \in (0, \delta_0)$, where $\delta_0 = \delta_0(\varepsilon) > 0$.

Since $|\underline{\ell}|_1\tau < 1, \sigma = \frac{1}{2}\left(|\underline{\ell}|_1 + \frac{1}{\tau}\right)$ satisfies $|\underline{\ell}|_1 < \sigma < \frac{1}{\tau}$. Define $k_m = \lfloor(\kappa_m)^{-\sigma}\rfloor$ and $\Lambda_m = [m - k_m, m + k_m]$. By (1.8), we have $k_m \le m^{\sigma\tau}$ and thus $k_m \le \frac{1}{2}m$ for m large enough since $\sigma\tau \in (0, 1)$.

Lemma 3.3. *For $m \ge 1$ large enough, we have for some $\underline{x} \in \mathbb{T}^d$*

$$\|H_{\beta,N}^{\Lambda_m} - Q_{\tilde{\kappa}_{m,N},\underline{x}}^{\Lambda_m}\| \le (\kappa_m)^{\frac{2}{\tau}-\sigma} \tag{3.6}$$

where

$$\tilde{\kappa}_{m,N} = \begin{cases} \kappa_m, & 1 \le m \le N; \\ \kappa_N, & m \ge N+1. \end{cases} \tag{3.7}$$

Proof. For $x > y > 0$ and $j \ge 1$, we have that $|x^j - y^j| \le jx^{j-1}|x - y|$. Hence,

$$|\widetilde{V}(n) - \widetilde{V}(n+1)| \le \left(\sum_{j=2}^{|\underline{\ell}|_1} j|c_j|(\kappa_n)^j\right) \cdot (\kappa_n - \kappa_{n+1}).$$

Since $\kappa_n \to 0$, we obtain for n large enough that $|\widetilde{V}(n) - \widetilde{V}(n+1)| \le |\kappa_n - \kappa_{n+1}|$. Hence, we have for some $\underline{x}$

$$\|H_{\beta,N}^{\Lambda_m} - Q_{\tilde{\kappa}_{m,N},\underline{x}}^{\Lambda_m}\| \le 2\max_{n\in\Lambda_m}|\kappa_n - \kappa_m| \le 2k_m \max_{n\in\Lambda_m}|\kappa_{n+1} - \kappa_n|.$$

The claim follows. □

We let

$$\widetilde{H}_m = Q_{\tilde{\kappa}_{m,N},\underline{x}} + \chi_{\Lambda_m}(H_{\beta,N}^{\Lambda_m} - Q_{\tilde{\kappa}_{m,N},\underline{x}}^{\Lambda_m}). \tag{3.8}$$

Lemma 3.4. *We have that*

$$\sigma(\widetilde{H}_m) \cap [E_0 - c(\kappa_m)^{|\underline{\ell}|_1}, E_0 + c(\kappa_m)^{|\underline{\ell}|_1}] = \emptyset. \tag{3.9}$$

Proof. By Theorem 4.3, we have that

$$\sigma(Q_{\tilde{\kappa}_{m,N},\underline{x}}) \cap [E_0 - 2c(\kappa_m)^{|\underline{\ell}|_1}, E_0 + 2c(\kappa_m)^{|\underline{\ell}|_1}] = \emptyset$$

for some constant $c > 0$. The claim follows from the previous lemma, since $\kappa_m^{\frac{2}{\tau}-\sigma-|\underline{\ell}|_1} \to 0$ as $m \to \infty$. □

Proof of Proposition 2.6. By Lemma 3.1 and $\|\psi\| = 1$, we have that

$$\begin{aligned} |\psi(m)| \le\ & |\langle e_m, (\widetilde{H}_m - E_0)^{-1}e_{m+k_m}\rangle| + |\langle e_m, (\widetilde{H}_m - E_0)^{-1}e_{m+k_m-1}\rangle| \\ & + |\langle e_m, (\widetilde{H}_m - E_0)^{-1}e_{m-k_m}\rangle| + |\langle e_m, (\widetilde{H}_m - E_0)^{-1}e_{m-k_m+1}\rangle|. \end{aligned}$$

By the previous lemma combined with Proposition 3.2, we obtain that this is

$$\le 4\exp\left(-\tilde{c}(\kappa_m)^{|\underline{\ell}|_1}k_m\right)$$

for some constant $\tilde{c} > 0$. As $k_m = \lfloor(\kappa_m)^{-\sigma}\rfloor$, we have

$$k_m \cdot (\kappa_m)^{|\underline{\ell}|_1} \ge \frac{1}{2}\left(\frac{1}{\kappa_m}\right)^{\sigma-|\underline{\ell}|_1}.$$

Thus the claim follows by (1.8). □

4. The spectrum of quasi-periodic Schrödinger operators

This section derives quantitative bounds on the size of gaps in the spectrum of one-dimensional Schrödinger operators with multi-frequency potential at small coupling. The analysis goes through analyzing Schrödinger operators with a one-frequency potential on a higher dimensional lattice and then obtaining the desired result through Aubry duality. We recall that we defined for $\lambda > 0$ and $\underline{x} \in \mathbb{T}^d$ the operator

$$\begin{aligned} & Q_{\lambda,\underline{x}} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \\ & Q_{\lambda,\underline{x}}u(n) = u(n+1) + u(n-1) + 2\lambda\left(\sum_{\ell=1}^{d} b_j\cos(2\pi(n\alpha_j + x_j))\right)u(n). \end{aligned} \tag{4.1}$$

We note that the spectrum of $Q_{\lambda,\underline{x}}$ is independent of $\underline{x}$ by our assumption on $\underline{\alpha} = \{\alpha_j\}_{j=1}^d$. We can assume without loss of generality that $b_j \neq 0$ for $j = 1, \ldots, d$. We also define for $y \in \mathbb{T}$ and $\lambda > 0$, the operator

$$\begin{aligned} &\widehat{Q}_{\lambda,y} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d) \\ &\widehat{Q}_{\lambda,y} u(\underline{n}) = \lambda(\Delta_{\underline{b}} u)(\underline{n}) + 2\cos(2\pi(\underline{\alpha}\cdot\underline{n} + y))u(\underline{n}), \end{aligned} \tag{4.2}$$

where

$$\Delta_{\underline{b}} u(\underline{n}) = \sum_{j=1}^{d} b_j (u(\underline{n} + e_j) + u(\underline{n} - e_j)) \tag{4.3}$$

is a weighted version of the Laplacian. Again our assumptions on $\underline{\alpha}$ imply that the spectrum of $\widehat{Q}_{\lambda,y}$ is independent of y. We have that

Proposition 4.1. *We have for any $\underline{x} \in \mathbb{T}^d$, $y \in \mathbb{T}$*

$$\sigma(Q_{\lambda,\underline{x}}) = \sigma(\widehat{Q}_{\lambda,y}). \tag{4.4}$$

Proof. One can shown as done in [15] that these two operators have the same integrated density of states and thus the same spectrum. □

The advantage of working with $\widehat{Q}_{\lambda,y}$ is that semi-classical methods apply. This way we can obtain the desired gaps. We will first prove

Theorem 4.2. *Let $\underline{\ell} \in \mathbb{Z}^d \setminus \{0\}$ with $|\underline{\ell}|_1 = 1$ and define*

$$E_{\underline{\ell},\pm} = \pm 2\cos(\pi\underline{\ell}\cdot\underline{\alpha}). \tag{4.5}$$

Then we have for $c = c(\underline{\ell}, \underline{\alpha}, \underline{b}) > 0$ that for $\lambda > 0$ small enough and any $y \in \mathbb{T}$

$$\sigma(\widehat{Q}_{\lambda,y}) \cap [E_{\underline{\ell},\pm} - c\lambda, E_{\underline{\ell},\pm} + c\lambda] = \emptyset. \tag{4.6}$$

Afterwards, we will prove the following more sophisticated version.

Theorem 4.3. *Let $\underline{\ell} \in \mathbb{Z}^d \setminus \{0\}$ and $\sigma \in \{-1, +1\}$ such that $\widetilde{w}_{\sigma,\underline{\ell}} \neq 0$. Then there exists $c = c(\underline{\ell}, \underline{\alpha}, \underline{b}) > 0$ such that*

$$\sigma(\widehat{Q}_{\lambda,y}) \cap [E(\lambda) - c\lambda^{|\underline{\ell}|_1}, E(\lambda) + c\lambda^{|\underline{\ell}|_1}] = \emptyset. \tag{4.7}$$

Here $E(\lambda)$ is as defined in Theorem 2.1. It is clear that Theorems 4.2 and 4.3 imply Theorem 2.2. The strategy of the proof of these theorems is similar to the proof of Theorem 1 in [16], see also [17]. To simplify the notation, we will write $x \lesssim y$ if there exists $C = C(\underline{\ell}, \underline{b}) > 0$ such that $x \leq Cy$ for all $\lambda > 0$ small enough. We write $x \asymp y$ for $x \lesssim y \lesssim x$.

We restrict ourself to

$$E(\lambda) = E_{\underline{\ell},+} + O(\lambda^2), \quad E_{\underline{\ell},+} = 2\cos(\pi\underline{\alpha}\cdot\underline{\ell}). \tag{4.8}$$

The other case requires some minor modifications of the following argument. Furthermore, we restrict ourself to the case $y = 0$ and write in the following $\widehat{Q}_\lambda = \widehat{Q}_{\lambda,0}$. The first step of the proof is to analyze the potential $V : \mathbb{Z}^d \to \mathbb{R}$

$$V(\underline{n}) = 2\cos(2\pi(\underline{\alpha}\cdot\underline{n})) \tag{4.9}$$

and understand where resonances, i.e. $V(\underline{n}) \approx V(\underline{m}) \approx E_{\underline{\ell},+}$ for $1 \leq |\underline{n} - \underline{m}|_1 \lesssim 1$, can occur. Define the points

$$x_1 = \frac{1}{2}\underline{\ell}\cdot\underline{\alpha} \pmod 1, \qquad x_2 = 1 - x_1 \tag{4.10}$$

which satisfy that $2\cos(2\pi x_j) = E_{\underline{\ell},+}$ for $j = 1, 2$. For $x \in \mathbb{R}$, we define

$$\|x\| = \operatorname{dist}(x, \mathbb{Z}). \tag{4.11}$$

Let $\varepsilon > 0$ and define the set

$$\Lambda_{\underline{\ell},\varepsilon} = \left\{ \underline{n} \in \mathbb{Z}^d : \min_{j=1,2} \|\underline{n}\cdot\underline{\alpha} - x_j\| \leq \varepsilon \right\}. \tag{4.12}$$

We will later choose $\varepsilon = \frac{2|\underline{b}|_1}{L}\cdot\lambda$, where L is the constant from the next lemma. We have

Lemma 4.4. *Let $K \geq 1$, $\varepsilon > 0$ small enough. The set $\Lambda_{\underline{\ell},\varepsilon}$ has the following properties*

(i) $\underline{n} \in \Lambda_{\underline{\ell},\varepsilon}$ *implies*

$$|V(\underline{n}) - E_{\underline{\ell},+}| \leq 4\pi\varepsilon. \tag{4.13}$$

(ii) *There exists $L = L(\underline{\ell}) > 0$ such that*

$$|V(\underline{n}) - E_{\underline{\ell},+}| \leq L\varepsilon \tag{4.14}$$

implies $\underline{n} \in \Lambda_{\underline{\ell},\varepsilon}$.

(iii) *If $\underline{n} \in \Lambda_{\underline{\ell},\varepsilon}$, then either $\underline{n} + \underline{\ell} \in \Lambda_{\underline{\ell},\varepsilon}$ or $\underline{n} - \underline{\ell} \in \Lambda_{\underline{\ell},\varepsilon}$.*

(iv) *There is $\tau > 0$ independent of $\varepsilon > 0$ such that*

$$1 \leq \inf_{\underline{n} \in \Lambda_{\underline{\ell},\varepsilon}} |\underline{m} - \underline{n}|_1 \leq K \tag{4.15}$$

implies $|V(\underline{m}) - E_{\ell,+}| \geq \tau$.

(v) *For $\{\underline{n}, \underline{n} + \underline{\ell}\} \subseteq \Lambda_{\underline{\ell},\varepsilon}$, we have*

$$(E_{\underline{\ell},+} - V(\underline{n}))(E_{\underline{\ell},+} - V(\underline{n} + \underline{\ell})) < 0. \tag{4.16}$$

(vi) *For $\{\underline{n}, \underline{n} + \underline{\ell}\} \subseteq \Lambda_{\underline{\ell},\varepsilon}$, we have*

$$\|\underline{\alpha} \cdot (2\underline{n} + \underline{\ell})\| \lesssim |E_{\underline{\ell},+} - V(\underline{n})|. \tag{4.17}$$

Proof. (i) follows from $2\cos(x)$ being Lipschitz. For (ii), first observe that

$$|V(\underline{n}) - E_{\underline{\ell},+}| = 4|\sin(\pi x_1 - \underline{n} \cdot \underline{\alpha})| \cdot |\sin(\pi x_2 - \underline{n} \cdot \underline{\alpha})|.$$

Hence, we would have that $\operatorname{dist}(\underline{n} \cdot \underline{\alpha}, \{x_1, x_2\}) \lesssim \sqrt{\varepsilon}$. In this range the other sine term is of order 1, using this the claim follows.

For (iii), observe that $x_2 + \underline{\ell} \cdot \underline{\alpha} = x_1$. For (iv), observe that $x_2 + \underline{n} \cdot \underline{\alpha} \notin \{x_1, x_2\}$ for $\underline{n} \in \mathbb{Z}^d \setminus \{0, \underline{\ell}\}$ by our assumption on α. The claim now follows by requiring $\varepsilon > 0$ to be small enough and continuity of the cosine.

For (v), let $\underline{n} \cdot \underline{\alpha} = x_2 + \delta \pmod 1$ for some $\delta \in [-\varepsilon, \varepsilon]$. Then $(\underline{n} + \underline{\ell}) \cdot \underline{\alpha} = x_1 + \delta \pmod 1$. Hence, we can compute that

$$(E_{\underline{\ell},+} - V(\underline{n}))(E_{\underline{\ell},+} - V(\underline{n} + \underline{\ell})) = 16\left(\sin\left(\frac{\pi\delta}{2}\right)\right)^2 \sin(\pi(2x_2 + \delta))\sin(\pi(2x_1 + \delta)).$$

Since $\sin(\pi(2x_2 + \delta)) = \sin(-\pi(2x_1 - \delta))$, the claim follows.

(vi) follows from the computations done to prove (i). □

We have that $E(\lambda) = E_{\underline{\ell},+} + O(\lambda^2)$. Hence, when $\varepsilon \asymp \lambda$, statements (i)–(iv) of the previous lemma remain valid with $E_{\underline{\ell},+}$ replaced by $E(\lambda)$. We will need the Schur complement formula.

Lemma 4.5 (*Schur Complement*)**.** *Assume A is invertible. Then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.18}$$

is invertible, if and only if

$$S = D - CA^{-1}B \tag{4.19}$$

is invertible. Furthermore then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}. \tag{4.20}$$

Define A to be the restriction of $\widehat{Q}_\lambda - E(\lambda)$ to $\ell^2(\mathbb{Z}^d \setminus \Lambda_{\underline{\ell},\varepsilon})$ and by D the restriction to $\ell^2(\Lambda_{\underline{\ell},\varepsilon})$. Define B and $C = B^*$ by

$$\widehat{Q}_\lambda - E(\lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{4.21}$$

We note that $\|B\|, \|C\| \lesssim \lambda$. The first key lemma analyzes A.

Lemma 4.6. *Let $\lambda > 0$. Assume that $4|\underline{b}|_1\lambda < L\varepsilon$, then*

$$\|A^{-1}\| \leq \frac{2}{L\varepsilon}. \tag{4.22}$$

Proof. In this proof, we consider all operators as restricted to $\ell^2(\mathbb{Z}^d \setminus \Lambda_{\underline{\ell},\varepsilon})$. We have that $\widehat{Q}_\lambda - E(\lambda) = (V - E(\lambda))(\mathbb{I} + (V - E(\lambda))^{-1}\lambda\Delta_{\underline{b}})$. By assumptions, we have that $\|(V - E(\lambda))^{-1}\| \leq \frac{1}{L\varepsilon}$ and thus $\|(V - E(\lambda))^{-1}\lambda\Delta_{\underline{b}}\| \leq \frac{1}{2}$, which implies the claim since $\|\Delta\| = 2|\underline{b}|_1$. □

We now set $\varepsilon = \frac{8|\underline{b}|_1}{L} \cdot \lambda$. Then we have from Lemma 4.5 that

$$\|(\widehat{Q}_\lambda - E(\lambda))^{-1}\| \lesssim \|(D - CA^{-1}B)^{-1}\|. \tag{4.23}$$

Denote by $\{\underline{n}_j, \underline{n}_j + \underline{\ell}\}_{j=1}^{\infty}$ an enumeration of the elements of $\Lambda_{\underline{\ell},\varepsilon}$. The next proposition shows which properties of the Schur complement $D - CA^{-1}B$ imply the necessary estimates.

Proposition 4.7. *If for $\lambda > 0$ small enough, there exist α_j, β_j with*

$$\alpha_j\beta_j < 0, \qquad \alpha_j \asymp \beta_j \tag{4.24}$$

such that

$$D - CA^{-1}B = \bigoplus_{j=1}^{\infty} D_j + O(\lambda^{|\underline{\ell}|_1+1}) \tag{4.25}$$

where for some $\kappa > 0$

$$D_j = \begin{pmatrix} \alpha_j & \kappa\lambda^{|\underline{\ell}|_1} \\ \kappa\lambda^{|\underline{\ell}|_1} & \beta_j \end{pmatrix}, \tag{4.26}$$

then for some constant $c = c(\underline{\ell}, \underline{b}) > 0$, we have for $\lambda > 0$ small enough

$$\sigma(\widehat{Q}_\lambda) \cap [E(\lambda) - c\kappa\lambda^{|\underline{\ell}|_1}, E(\lambda) + c\kappa\lambda^{|\underline{\ell}|_1}] = \emptyset. \tag{4.27}$$

Proof. We have that

$$\det(D_j) = \alpha_j\beta_j - \lambda^{2|\underline{\ell}|_1}(\kappa)^2 \leq -\max(|\alpha_j\beta_j|, \lambda^{2|\underline{\ell}|_1}(\kappa)^2).$$

Hence, we obtain by

$$D_j^{-1} = \frac{1}{\det(D_j)} \begin{pmatrix} \beta_j & -\kappa\lambda^{|\underline{\ell}|_1} \\ -\kappa\lambda^{|\underline{\ell}|_1} & \alpha_j \end{pmatrix}$$

that

$$\|(D_j)^{-1}\| \leq \max\left(\frac{1}{|\kappa|\lambda^{|\underline{\ell}|_1}}, \min\left(\frac{1}{\min(|\alpha_j|, |\beta_j|)}, \frac{\max(|\alpha_j|, |\beta_j|)}{\kappa^2\lambda^{2|\underline{\ell}|_1}}\right)\right).$$

We thus see that $\|(D_j)^{-1}\| \lesssim \frac{1}{\kappa}\lambda^{-|\underline{\ell}|_1}$. Hence, there exists some $c_1 > 0$ such that

$$\sigma\left(\bigoplus_{j=1}^{\infty} D_j\right) \cap [-\kappa c_1\lambda^{|\underline{\ell}|_1}, \kappa c_1\lambda^{|\underline{\ell}|_1}] = \emptyset.$$

By (4.25), for $\lambda > 0$ small enough also

$$\sigma\left(D - CA^{-1}B\right) \cap \left[-\frac{c_1\kappa}{2}\lambda^{|\underline{\ell}|_1}, \frac{c_1\kappa}{2}\lambda^{|\underline{\ell}|_1}\right] = \emptyset.$$

The claim now follows by (4.23). □

We are now ready for

Proof of Theorem 4.2. Since $|\underline{\ell}|_1 = 1$, we have that $E(\lambda) = E_{\underline{\ell},\pm}$. A computation shows that D has already the desired form. In particular,

$$D|_{\ell^2(\{\underline{n}_j, \underline{n}_j+\underline{\ell}\})} = \begin{pmatrix} V(\underline{n}_j) - E_{\underline{\ell},+} & \lambda \\ \lambda & V(\underline{n}_j + \underline{\ell}) - E_{\underline{\ell},+} \end{pmatrix}.$$

This means it suffices to show that $CA^{-1}B = O(\lambda^2)$. Recall that

$$A^{-1} = (\mathbb{I} - (E_{\underline{\ell},+} - V)^{-1}\lambda\Delta_{\underline{b}})^{-1}(V - E_{\underline{\ell},+})^{-1},$$

where all operators are acting on $\ell^2(\mathbb{Z}^d \setminus \Lambda_{\underline{\ell},\varepsilon})$. For $j \neq k$, we have that $|\underline{n}_j - \underline{n}_k| \geq 10$. Hence, we have that

$$P_j A^{-1} P_k = P_j((V - E_{\underline{\ell},+})^{-1}\lambda\Delta_{\underline{b}})^2(\mathbb{I} + (V - E_{\underline{\ell},+})^{-1}\lambda\Delta_{\underline{b}})^{-1}(V - E_{\underline{\ell},+})^{-1}P_j$$

where P_j denotes the projection on $\ell^2(\{\underline{n}_j, \underline{n}_j + \underline{\ell}\})$. Here, we used the identity

$$(\mathbb{I} - B)^{-1} = \sum_{n=0}^{N} B^n + B^{N+1}(\mathbb{I} - B)^{-1}$$

and that $\langle e_x, (\Delta_{\underline{b}})^p e_y\rangle = 0$ when $|x - y|_1 \geq p + 1$. By Lemma 4.4(iv), we have that $\|(E_{\underline{\ell},+} - V)^{-1}\| \lesssim \frac{1}{\tau}$, and thus the claim follows. □

We now start with the proof of Theorem 4.3, that is we have $|\underline{\ell}|_1 \geq 2$. We first note that D is now a diagonal operator. Define the operator $B_{\lambda,j} : \ell^2(\Lambda_{\underline{\ell},\varepsilon}) \to \ell^2(\mathbb{Z}^d \setminus \Lambda_{\underline{\ell},\varepsilon})$ by

$$B_{\lambda,j}\psi(\underline{n}) = \begin{cases} \lambda\Delta_{\underline{b}}\psi(\underline{n}), & \underline{n} \in \{\underline{n}_j, \underline{n}_j + \underline{\ell}\}; \\ 0, & \text{otherwise.} \end{cases} \tag{4.28}$$

Then $B = \bigoplus_{j=1}^{\infty} B_j$. Also define $C_j = (B_j)^*$. We first show

Lemma 4.8. *Let $j \neq k$, then*

$$\|C_j A^{-1} B_k\| \lesssim \lambda^{|\underline{\ell}|_1 + 1}. \tag{4.29}$$

Proof. This follows from $\langle e_x, (\Delta_{\underline{b}})^p e_y\rangle = 0$ when $|x - y|_1 \geq p + 1$ and $|\underline{n}_j - \underline{n}_k| \geq 2|\underline{\ell}|_1 + 10$. □

This means that all we have to analyze are the terms of the form $C_j A^{-1} B_j$. For this denote by $\Gamma_p^{\underline{n},\underline{m}}$ the set of all sequences

$$\left(\underline{\gamma}_1, \underline{\gamma}_2, \ldots, \underline{\gamma}_{p-1}\right) \in (\mathbb{Z}^d \setminus \Lambda_{\underline{\ell},\varepsilon})^{p-1}$$

such that $|\underline{\gamma}_q - \underline{\gamma}_{q-1}|_1 = 1$ for $q = 1, \ldots, p$, with $\underline{\gamma}_0 = \underline{n}$, $\underline{\gamma}_p = \underline{m}$. Next, we define

$$\mathcal{V}_p^{\underline{n},\underline{m}}(\lambda) = -\sum_{\gamma \in \Gamma_p^{\underline{n},\underline{m}}} \prod_{q=1}^{p-1} \frac{|\underline{b} \cdot (\underline{\gamma}_q - \underline{\gamma}_{q-1})|_1}{E(\lambda) - V(\underline{\gamma}_q)}. \tag{4.30}$$

The definition of $\mathcal{V}_p^{\underline{n},\underline{m}}(\lambda)$ generalizes the definition of $\mathcal{W}_{\underline{\ell},p}$ and $\widetilde{\mathcal{W}}_{\underline{\ell},\pm}$, in fact we have

$$\mathcal{W}_{\underline{\ell},p} = \mathcal{V}_p^{0,0}(0), \qquad \widetilde{\mathcal{W}}_{\underline{\ell},\sigma}(E(\lambda)) = \mathcal{V}_{|\underline{\ell}|_1}^{0,\underline{\ell}}(\lambda). \tag{4.31}$$

Hence, we define λ dependent versions by

$$\mathcal{W}_{\underline{\ell},p}(\lambda) = \mathcal{V}_p^{0,0}(\lambda), \qquad \widetilde{\mathcal{W}}_{\underline{\ell},\sigma}(\lambda) = \mathcal{V}_{|\underline{\ell}|_1}^{0,\underline{\ell}}(\lambda). \tag{4.32}$$

We have

Lemma 4.9. *$\mathcal{V}_p^{\underline{n},\underline{m}} \neq 0$ implies $|\underline{n} - \underline{m}| \geq p$. Furthermore, we have that*

$$\begin{aligned} &|\mathcal{V}_p^{\underline{n}_j,\underline{n}_j}(\lambda) - \mathcal{W}_{\underline{\ell},p}(\lambda)|, |\mathcal{V}_{|\underline{\ell}|_1}^{\underline{n}_j,\underline{n}_j+\underline{\ell}}(\lambda) - \widetilde{\mathcal{W}}_{\underline{\ell}}(\lambda)|, \ |\mathcal{V}_p^{\underline{n}_j+\underline{\ell},\underline{n}_j+\underline{\ell}}(\lambda) - \mathcal{W}_{\underline{\ell},p}(\lambda)| \\ &\quad \lesssim |V(\underline{n}_j) - E(\lambda)|. \end{aligned} \tag{4.33}$$

Proof. We have $\underline{m} - \underline{n} = \sum_{q=1}^{p} \underline{\gamma}_q - \underline{\gamma}_{q-1}$ for $\gamma \in \Gamma_p^{\underline{n},\underline{m}}$. Hence, that $|\underline{m} - \underline{n}|_1 \geq p$ if $\Gamma_p^{\underline{n},\underline{m}} \neq \emptyset$. This implies the claim.

The second term follows by

$$\begin{aligned} V(\underline{\gamma}_q) &= 2\cos(\pi\underline{\alpha} \cdot (2(\gamma_q - \underline{n}_j) - \underline{\ell}) + \pi\underline{\alpha} \cdot (2\underline{n}_j + \underline{\ell})) \\ &= 2\cos(\pi\underline{\alpha} \cdot (2(\gamma_q - \underline{n}_j) - \underline{\ell}) + O(\|\underline{\alpha} \cdot (2\underline{n}_j + \underline{\ell})\|)) \end{aligned}$$

and (vi) of Lemma 4.4. □

We now expand the operator $C_j A^{-1} B_j$ into powers of λ. Define

$$T_{j,p}(\lambda) = -\frac{1}{\lambda^p} C_{\lambda,j} \left(\lambda(E(\lambda) - V)^{-1}\Delta_{\underline{b}}\right)^{p-2} (E(\lambda) - V)^{-1} B_{\lambda,j}. \tag{4.34}$$

This is just the coefficient of λ^p in the expansion of $C_j A^{-1} B_j$, if we do not expand $E(\lambda)$. The next lemma describes the structure of $T_{j,p}(\lambda)$.

Lemma 4.10. *For $p = 0, 1$ we have $T_{j,p} = 0$. For $2 \leq p \leq |\underline{\ell}|_1 - 1$, we have*

$$T_{j,p}(\lambda) = \begin{pmatrix} \mathcal{W}_{\underline{\ell},p}(\lambda) & 0 \\ 0 & \mathcal{W}_{\underline{\ell},p}(\lambda) \end{pmatrix} + O(|V(\underline{n}_j) - E(\lambda)|). \tag{4.35}$$

Finally, for $p = |\underline{\ell}|_1$ we have

$$T_{j,p}(\lambda) = \begin{pmatrix} \mathcal{W}_{\underline{\ell},|\underline{\ell}|_1}(\lambda) & \widetilde{\mathcal{W}}_{\underline{\ell}}(\lambda) \\ \widetilde{\mathcal{W}}_{\underline{\ell}}(\lambda) & \mathcal{W}_{\underline{\ell},|\underline{\ell}|_1}(\lambda) \end{pmatrix} + O(|V(\underline{n}_j) - E(\lambda)|). \tag{4.36}$$

Proof. We just compute that $\langle e_{\underline{n}_j}, T_{j,p}(\lambda) e_{\underline{n}_j} \rangle$ matrix element, i.e. the top left corner. First a counting argument, shows that the powers of λ, exactly cancel. We compute

$$\begin{aligned}
\langle e_{\underline{n}_j}, T_{j,p}(\lambda) e_{\underline{n}_j} \rangle &= \sum_{\substack{|\underline{\gamma}_q - \underline{\gamma}_{q-1}|_1 = 1 \\ q=1,p}} \frac{B_1}{E(\lambda) - V(\underline{\gamma}_1)} B_p \langle e_{\underline{\gamma}_{p-2}}, ((E(\lambda) - V)^{-1} \Delta_{\underline{b}})^{p-2} e_{\underline{\gamma}_1} \rangle \\
&= \sum_{\substack{|\underline{\gamma}_q - \underline{\gamma}_{q-1}|_1 = 1 \\ q=1,\dots,t+1,p}} \prod_{s=1}^{t+1} \frac{B_s}{E(\lambda) - V(\underline{\gamma}_s)} B_p \langle e_{\underline{\gamma}_{p-2}}, ((E(\lambda) - V)^{-1} \Delta_{\underline{b}})^{p-2-t} e_{\underline{\gamma}_{t+1}} \rangle \\
&= \sum_{\substack{|\underline{\gamma}_q - \underline{\gamma}_{q-1}|_1 = 1 \\ q=1,\dots,p}} \prod_{s=1}^{p-1} \frac{B_s}{E(\lambda) - V(\underline{\gamma}_s)} B_p \\
&= \mathcal{V}_p^{\underline{n}_j, \underline{n}_j}(\lambda)
\end{aligned}$$

where $B_s = |\underline{b} \cdot (\underline{\gamma}_s - \underline{\gamma}_{s-1})|_1$ and $\underline{\gamma}_0 = \underline{n}_j$, $\underline{\gamma}_p = \underline{n}_j$. This finishes the proof. □

Proof of Theorem 4.3. Let P_j be the projection onto $\ell^2(\{\underline{n}_j, \underline{n}_j + \underline{\ell}\})$. Using the previous results, we obtain that

$$P_j(D - CA^{-1}B)P_j = \begin{pmatrix} \alpha_j & \lambda^{|\underline{\ell}|_1} \widetilde{\mathcal{W}}_{\underline{\ell}}(\lambda) \\ \lambda^{|\underline{\ell}|_1} \widetilde{\mathcal{W}}_{\underline{\ell}}(\lambda) & \beta_j \end{pmatrix} + O(\lambda^{|\underline{\ell}|_1 + 1})$$

where

$$\alpha_j = V(\underline{n}_j) - E(\lambda) + \sum_{p=2}^{\underline{\ell}_1} \lambda^p \mathcal{W}_{\underline{\ell},p}(\lambda) + O(\lambda^2 |V(\underline{n}_j) - E(\lambda)|)$$

$$\beta_j = V(\underline{n}_j + \underline{\ell}) - E(\lambda) + \sum_{p=2}^{\underline{\ell}_1} \lambda^p \mathcal{W}_{\underline{\ell},p}(\lambda) + O(\lambda^2 |V(\underline{n}_j) - E(\lambda)|).$$

By definition of $E(\lambda)$, we obtain that

$$\alpha_j = (V(\underline{n}_j) - E_{\underline{\ell},\pm})(1 + O(\lambda^2)), \qquad \beta_j = (V(\underline{n}_j + \underline{\ell}) - E_{\underline{\ell},\pm})(1 + O(\lambda^2)).$$

The claim now follows by Proposition 4.7. □

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