



Fractional stochastic differential equations with applications to finance

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ARTICLE INFO

Article history:

Received 30 July 2011

Available online 4 August 2012

Submitted by U. Stadtmueller

Keywords:

Stochastic differential equations

Fractional Brownian motion

Malliavin calculus

Filtering

Optimal portfolio

ABSTRACT

In this paper we use a definition of the fractional stochastic integral given by Carmona et al. (2003) in [19] and develop a simple approximation method to study quasi-linear stochastic differential equations by fractional Brownian motion. We also propose a stochastic process, namely fractional semimartingale, to model for the noise driving in some financial models.

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1. Introduction

A fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$ is a centered Gaussian process defined by

$$B_t^H = \int_0^t \bar{K}(t, s) dW_s, \quad (1.1)$$

where W is a standard Brownian motion and the kernel $\bar{K}(t, s)$, $t \geq s$, is given by

$$\bar{K}(t, s) = C_H \left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}} (t-s)^{H-\frac{1}{2}} - \left(H - \frac{1}{2} \right) \int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}} (u-s)^{H-\frac{1}{2}} du \right],$$

where $C_H = \sqrt{\frac{\pi H(2H-1)}{\Gamma(2-2H)\Gamma(H+\frac{1}{2})^2 \sin(\pi(H-\frac{1}{2}))}}$.

In [1] Mandelbrot and Van Ness have given a representation of B_t^H of the form:

$$B_t^H = \frac{1}{\Gamma(1+\alpha)} \left(Z_t + \int_0^t (t-s)^\alpha dW_s \right),$$

where $\alpha = H - \frac{1}{2}$, Z_t is a stochastic process of absolutely continuous trajectories, and $W_t^H := \int_0^t (t-s)^\alpha dW_s$ is called a Liouville fBm. A Liouville fBm shares many properties of a fBm except that it has non-stationary increments (for example, see [2]). Moreover, Comte and Renault in [3] have given an excellent application of Liouville fBm to finance. Because of these reasons and for simplicity we use W_t^H throughout this paper.

The main difficulty in studying fractional stochastic calculus is that we cannot apply stochastic calculus developed by Itô since fBm is neither a Markov process nor a semimartingale, except for $H = \frac{1}{2}$. Recently, there have been numerous attempts to define a stochastic integral with respect to fBm. The main approaches are the following ones (refer [4,5] for a detailed survey).

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- The pathwise approach was introduced by Lin [6]. Since the trajectories of the fBm are β -Hölder continuous, $\beta < H$ and by the work of Young in [7], the pathwise Riemann–Stieltjes integral exists for any integrand with sample paths γ -Hölder continuous with $\gamma + H > 1$. In [8] Nualart and Răşcanu have studied stochastic differential equations with respect to fBm with Hurst index $H > 1/2$.
- The regularization approach was introduced by Russo and Vallois in [9,10] and further developed by Cheridito and Nualart in [11]. This approach has been used by Nourdin in [12] to prove the existence and uniqueness of stochastic differential equations and an approximation scheme for all $H \in (0, 1)$.
- The Malliavin calculus approach was introduced by Decreusefond and Üstünel in [13] for fBm and extended to more general Gaussian processes by Alòs et al. in [14], Decreusefond in [15], etc. This approach leads to some different definitions for the fractional stochastic integrals such as the divergence integral, the Skorohod integral and the Stratonovich integral. By using the Skorohod integral some stochastic differential equations have been studied in [16,17] for the case of linear equations and in [18] for the case of quasi-linear equations with Hurst index $H < 1/2$.

It is known that the study of the stochastic differential equations (SDEs) depends on the definitions of the stochastic integrals involved. One of the definitions of the fractional stochastic integrals is given by Carmona et al. in [19]. This kind of fractional stochastic integral belongs to the third approach mentioned above and turns out to be equal to the divergence integral plus a complementary term (see Remark 18 in [4]). Thus it can be considered as a new definition of fractional stochastic integrals and naturally, the theory of SDEs needs studying independently.

In this paper we use Carmona et al.'s definition to study the SDEs driven by fractional Brownian motion. When the integrand is deterministic, our fractional stochastic integral coincides with the Wiener integral and SDEs of the form

$$dX_t = b(t, X_t)dt + \sigma(t)dW_t^H, \quad X_0 = x_0, \quad (1.2)$$

have been studied by Mishura (Section 3.5 in [5]). As a new contribution to (1.2), we will point out a way to find explicitly its solution.

More generally, our work deals with the following form of SDEs:

$$dX_t = b(t, X_t)dt + \sigma(t)X_t dW_t^H, \quad X_0 = x_0. \quad (1.3)$$

In order to prove the existence and uniqueness of the solution of Eq. (1.3) we make the following standard assumptions on coefficients. The volatility $\sigma : [0, T] \rightarrow \mathbb{R}$ is a deterministic function on $[0, T]$, bounded by a constant M and the drift coefficient $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function in all their arguments and satisfies the following conditions, for a positive constant L_0 :

(C₁) $b(t, x)$ is a continuously differentiable function in x and

$$|b(t, x) - b(t, y)| \leq L_0|x - y| \quad (1.4)$$

for all $x, y \in \mathbb{R}$, $t \in [0, T]$;

(C₂) linear growth

$$|b(t, x)| \leq L_0(1 + |x|), \quad \forall x \in \mathbb{R}, \forall t \in [0, T]. \quad (1.5)$$

Since Eq. (1.3) is an anticipate SDE, similar to the Brownian case, the traditional methods cannot be applied. A method of approximation equations has been introduced recently by Thao (see [20] and the references therein) to solve simple linear SDEs and then used by N.T. Dung to solve more complicated SDEs such as the fractional SDEs with polynomial drift [21] and the fractional geometric mean-reversion equations [22]. We continue to develop this method in current work with the main idea being that the existence and uniqueness of the solution for Eq. (1.3) can be proved via an “approximation” equation, which is driven by semimartingales, and that the limit in $L^2(\Omega)$ of approximation solution will be the solution of (1.3). Thus, advantages of this method are that we can still use classical Itô calculus and do not need any other fractional stochastic calculus. Based on our obtained approximation results, we propose a stochastic process, namely fractional semimartingale, to model for noise driving in some financial models.

This paper is organized as follows. In Section 2, we recall the definition of fractional stochastic integral given in [19] and some moment inequalities for fractional stochastic integral of deterministic integrands. Section 3 contains the main result of this paper which proved the existence and uniqueness of the solution of Eq. (1.3). In Section 4, the European option pricing formula in the fractional semimartingale Black–Scholes model is found and the optimal portfolio in a stochastic drift model is investigated.

2. Preliminaries

Let us recall some elements of stochastic calculus of variations.

For $h \in L^2([0, T], \mathbb{R})$, we denote by $W(h)$ the Wiener integral

$$W(h) = \int_0^T h(t)dW_t.$$

Let \mathcal{H} denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of those classes of random variables of the form

$$F = f(W(h_1), \dots, W(h_n)), \quad (2.1)$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n, L^2([0, T], \mathbb{R}))$, $h_1, \dots, h_n \in L^2([0, T], \mathbb{R})$. If F has the form (2.1), we define its derivative as the process $D^W F := \{D_t^W F, t \in [0, T]\}$ given by

$$D_t^W F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(W(h_1), \dots, W(h_n)) h_k(t).$$

For any $1 \leq p < \infty$, we shall denote by $\mathbb{D}_W^{1,p}$ the closure of \mathcal{H} with respect to the norm

$$\|F\|_{1,p} := [E|F|^p]^{\frac{1}{p}} + E \left[\int_0^T |D_u^W F|^p du \right]^{\frac{1}{p}},$$

and then $\mathbb{D}_W^{1,\infty} = \bigcap_{p \geq 1} \mathbb{D}_W^{1,p}$.

For every $\varepsilon > 0$ we define

$$W_t^{H,\varepsilon} = \int_0^t K(t + \varepsilon, s) dW_s,$$

where $K(t, s) = (t - s)^\alpha$.

It is well known from [20,23] that $W_t^{H,\varepsilon}$ is a semimartingale with the following decomposition

$$W_t^{H,\varepsilon} = \varepsilon^\alpha W_t + \int_0^t \varphi_s^\varepsilon ds, \quad (2.2)$$

where $\varphi_s^\varepsilon = \int_0^s \alpha(s + \varepsilon - u)^{\alpha-1} dW_u$. Moreover, $W_t^{H,\varepsilon}$ converges in $L^p(\Omega)$, $p > 1$ uniformly in $t \in [0, T]$ to W_t^H as $\varepsilon \rightarrow 0$:

$$E|W_t^{H,\varepsilon} - W_t^H|^p \leq c_p \varepsilon^{pH}.$$

It is well known from [19,4] that for an adapted process f belonging to the space $\mathbb{D}_W^{1,2}$ we have

$$\begin{aligned} \int_0^t f_s dW_s^{H,\varepsilon} &= \int_0^t f_s K(s + \varepsilon, s) dW_s + \int_0^t f_s \varphi_s^\varepsilon ds \\ &= \int_0^t f_s K(t + \varepsilon, s) dW_s + \int_0^t \int_s^t (f_u - f_s) \partial_1 K(u + \varepsilon, s) du \delta W_s \\ &\quad + \int_0^t \int_0^u D_s^W f_u \partial_1 K(u + \varepsilon, s) ds du, \end{aligned} \quad (2.3)$$

where $\partial_1 K(u, s) = \frac{\partial}{\partial u} K(u, s)$ and the second integral on the right-hand side is a Skorokhod integral (we refer the reader to [24] for more details about the Skorokhod integral).

Hypothesis (H). Assume that f is an adapted process belonging to the space $\mathbb{D}_W^{1,2}$ and that there exists β fulfilling $\beta + H > 1/2$ and $p > 1/H$ such that

- (i) $\|f\|_{L_{\beta}^{1,2}}^2 := \sup_{0 < s < u < T} \frac{E[(f_u - f_s)^2 + \int_0^T (D_r^W f_u - D_r^W f_s)^2 dr]}{|u - s|^{2\beta}}$ is finite,
- (ii) $\sup_{0 < s < T} |f_s|$ belongs to $L^p(\Omega)$.

For $f \in (H)$, $\int_0^t f_s dW_s^{H,\varepsilon}$ converges in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$. Each term on the right-hand side of (2.3) converges to the same term where $\varepsilon = 0$. Then, it is “natural” to define the following.

Definition 2.1. Let $f \in (H)$. The fractional stochastic integral of f with respect to W^H is defined by

$$\int_0^t f_s dW_s^H = \int_0^t f_s K(t, s) dW_s + \int_0^t \int_s^t (f_u - f_s) \partial_1 K(u, s) du \delta W_s + \int_0^t du \int_0^u D_s^W f_u \partial_1 K(u, s) ds, \quad (2.4)$$

where $K(t, s) = (t - s)^\alpha$, $\partial_1 K(t, s) = \alpha(t - s)^{\alpha-1}$.

If f is a deterministic function in $L^{\frac{1}{H}}[0, T]$ then our fractional stochastic integral coincides with the Wiener integral with respect to fBm. Indeed, we have

$$I_t := \int_0^t f_s dW_s^H = \alpha \int_0^t \int_s^t f_u (u - s)^{\alpha-1} du dW_s. \quad (2.5)$$

Proposition 2.1. Suppose that $H > \frac{1}{2}$. Let $f \in L^{\frac{1}{H}}[0, T]$; then for any $p > 0$, there exists a constant $C(p, H)$ such that

$$E|I_T^*|^p \leq C(p, H) \|f\|_{L^{\frac{1}{H}}[0, T]}^p, \quad (2.6)$$

where $I_T^* = \sup_{0 \leq t \leq T} |I_t|$. Consequently,

$$E \exp(I_T^*) \leq 2 \exp \left(4\sqrt{2} \bar{C}_H \|f\|_{L^{\frac{1}{H}}[0, T]} + \frac{1}{2} C(2, H) \|f\|_{L^{\frac{1}{H}}[0, T]}^2 \right), \quad (2.7)$$

where $\bar{C}_H = \frac{1}{2} (C(2, H))^{\frac{1}{2}} H \int_{\ln 2}^{\infty} z^{\frac{1}{2}} \frac{e^z dz}{(e^z - 1)^{H+1}}$.

Proof. The inequality (2.6) was proved by Mishura in Theorem 1.10.3 [5]. Denote $\gamma^2 = \sup_{0 \leq t \leq T} E[I_t^2]$, then for any $r > 4\sqrt{2}D(T, \frac{\gamma}{2})$ we have

$$P(I_T^* > r) \leq 2 \left(1 - \Phi \left(\frac{r - 4\sqrt{2}D(T, \frac{\gamma}{2})}{\gamma} \right) \right), \quad (2.8)$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$ and $D(T, \frac{\gamma}{2})$ is the Dudley integral (see, [25]).

$$\begin{aligned} E \exp(I_T^*) &= 1 + \int_0^\infty e^x P(I_T^* > x) dx \leq 1 + \int_0^{4\sqrt{2}D(T, \frac{\gamma}{2})} e^x P(I_T^* > x) dx \\ &\quad + \int_{4\sqrt{2}D(T, \frac{\gamma}{2})}^\infty 2e^x \left(1 - \Phi \left(\frac{x - 4\sqrt{2}D(T, \frac{\gamma}{2})}{\gamma} \right) \right) dx \\ &\leq e^{4\sqrt{2}D(T, \frac{\gamma}{2})} + 2 \int_0^\infty e^{x+4\sqrt{2}D(T, \frac{\gamma}{2})} \left(1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/\gamma} e^{-\frac{y^2}{2}} dy \right) dx \\ &\leq e^{4\sqrt{2}D(T, \frac{\gamma}{2})} + 2e^{4\sqrt{2}D(T, \frac{\gamma}{2})} \int_0^\infty e^x \left(\frac{1}{\sqrt{2\pi}} \int_{x/\gamma}^\infty e^{-\frac{y^2}{2}} dy \right) dx \\ &\leq e^{4\sqrt{2}D(T, \frac{\gamma}{2})} + 2e^{4\sqrt{2}D(T, \frac{\gamma}{2})} \left(e^{\frac{\gamma^2}{2}} - \frac{1}{2} \right) = 2e^{4\sqrt{2}D(T, \frac{\gamma}{2}) + \frac{\gamma^2}{2}}. \end{aligned}$$

Evidently,

$$\gamma^2 \leq C(2, H) \|f\|_{L^{\frac{1}{H}}[0, T]}^2.$$

Moreover, from the proof of Theorem 1.10.3 in [5] we know that

$$D\left(T, \frac{\gamma}{2}\right) \leq \bar{C}_H \|f\|_{L^{\frac{1}{H}}[0, T]}.$$

Thus the proposition is proved. \square

Proposition 2.2. Suppose that $H \in (0, 1)$. Let f be a bounded deterministic function on $[0, T] : |f(t)| \leq M \forall t \in [0, T]$. Then the integral process $I_t \in C^{H-}[0, T] = \bigcap_{0 < h < H} C^h[0, T]$ a.s.

Proof. Since I_t is a Gaussian process, $I_t \in C^{H-}[0, T]$ follows from the following inequality

$$\begin{aligned} E|I_t - I_r|^p &\leq C_p \left(\int_0^r \left(\int_r^t \frac{\alpha f_u}{(u-s)^{1-\alpha}} du \right)^2 ds \right)^{\frac{p}{2}} + C_p \left(\int_r^t \left(\int_s^t \frac{\alpha f_u}{(u-s)^{1-\alpha}} du \right)^2 ds \right)^{\frac{p}{2}} \\ &\leq C_p \left(\int_0^r \left(\int_r^t \frac{\alpha M}{(u-s)^{1-\alpha}} du \right)^2 ds \right)^{\frac{p}{2}} + C_p \left(\int_r^t \left(\int_s^t \frac{\alpha M}{(u-s)^{1-\alpha}} du \right)^2 ds \right)^{\frac{p}{2}} \\ &\leq C_p M^p \left(\int_0^r [(t-s)^\alpha - (r-s)^\alpha]^2 ds \right)^{\frac{p}{2}} + C_p M^p \left(\int_r^t (t-s)^{2\alpha} ds \right)^{\frac{p}{2}} \\ &= C_p M^p (E|W_t^H - W_r^H|^2)^{\frac{p}{2}} + C_p M^p \left(\frac{(t-r)^{2H}}{2H} \right)^{\frac{p}{2}} \\ &\leq C_p M^p \left(1 + \frac{1}{(2H)^{p/2}} \right) |t-r|^{pH}, \end{aligned}$$

where C_p is a finite constant. In the above inequalities we used the Burkholder–Davis–Gundy inequality and the fundamental inequality $(a + b)^p \leq c_p(a^p + b^p)$, where $c_p = 1$ if $0 < p \leq 1$ and $c_p = 2^{p-1}$ if $p > 1$. \square

3. Fractional stochastic differential equations

In this whole section we consider only $H > \frac{1}{2}$. Since the Malliavin derivative $D_u^W f_s = 0$ for any deterministic function f_s , we have the following definition.

Definition 3.1. The solution of (1.3) is a stochastic process X_t belonging to the space (H) and has a form for all $t \in [0, T]$

$$\begin{aligned} X_t = X_0 &+ \int_0^t b(s, X_s) ds + \int_0^t \sigma(s) X_s K(t, s) dW_s + \int_0^t \int_s^t (\sigma(u) X_u - \sigma(s) X_s) \partial_1 K(u, s) du dW_s \\ &+ \int_0^t \int_0^s \sigma(s) D_u^W X_s \partial_1 K(s, u) duds. \end{aligned} \quad (3.1)$$

We can see that (3.1) contains the Skorokhod integral and the Malliavin derivative, so we cannot apply standard methods (for instance, Picard iteration procedure) to prove the existence and uniqueness of the solution. However, Definition 2.1 for the fractional stochastic integral leads us to consider the “approximation” equation corresponding to (3.1)

$$\begin{aligned} X_t^\varepsilon = X_0 &+ \int_0^t b(s, X_s^\varepsilon) ds + \int_0^t \sigma(s) X_s^\varepsilon K(t + \varepsilon, s) dW_s \\ &+ \int_0^t \int_s^t (\sigma(u) X_u^\varepsilon - \sigma(s) X_s^\varepsilon) \partial_1 K(u + \varepsilon, s) du dW_s + \int_0^t \int_0^s \sigma(s) D_u^W X_s^\varepsilon \partial_1 K(s + \varepsilon, u) duds, \end{aligned} \quad (3.2)$$

which can be reduced to

$$dX_t^\varepsilon = b(t, X_t^\varepsilon) dt + \sigma(t) X_t^\varepsilon dW_t^{H, \varepsilon}, \quad X_0^\varepsilon = x_0. \quad (3.3)$$

Note that the Hypothesis (H) ensures the convergence of approximation integrals to the fractional stochastic integrals. Hence, if both X_t^ε and its $L^2 - \lim_{\varepsilon \rightarrow 0} X_t^\varepsilon$ belong to (H) then we can take the limit of (3.2) as $\varepsilon \rightarrow 0$ to get (3.1). Thus $L^2 - \lim_{\varepsilon \rightarrow 0} X_t^\varepsilon$ will be the solution of (3.1).

Theorem 3.1. Fix $\varepsilon > 0$. Eq. (3.3) has a unique solution X_t^ε satisfying the Hypothesis (H) and converging in $L^2(\Omega)$ when $\varepsilon \rightarrow 0$.

Proof (Existence and Uniqueness). By (2.2) we can rewrite (3.3) as a classical Itô equation

$$dX_t^\varepsilon = [b(t, X_t^\varepsilon) + \varphi_t^\varepsilon \sigma(t) X_t^\varepsilon] dt + \varepsilon^\alpha \sigma(t) X_t^\varepsilon dW_t, \quad 0 \leq t \leq T. \quad (3.4)$$

As a consequence, the explicit solution of (3.3) is given by

$$X_t^\varepsilon = \frac{Z_t^\varepsilon}{Y_t^\varepsilon},$$

where $Y_t^\varepsilon = \exp\left(\frac{1}{2} \varepsilon^{2\alpha} \int_0^t \sigma^2(s) ds - \int_0^t \sigma(s) dW_s^{H, \varepsilon}\right) \in \bigcap_{0 < \gamma < \frac{1}{2}} C^\gamma[0, T]$ a.s. and Z_t^ε is the solution of the ordinary differential equation

$$dZ_t^\varepsilon = Y_t^\varepsilon b\left(t, \frac{Z_t^\varepsilon}{Y_t^\varepsilon}\right) dt, \quad Z_0^\varepsilon = X_0^\varepsilon = x_0. \quad (3.5)$$

For P -almost all $\omega \in \Omega$, the function $f^\varepsilon(t, x, \omega) = Y_t^\varepsilon b\left(t, \frac{x}{Y_t^\varepsilon}\right)$ satisfies global Lipschitzian condition in $x \in \mathbb{R}$. Hence, Eq. (3.5) has a unique solution $Z_t^\varepsilon \in C^1[0, T]$ a.s., so (3.3) does.

Hypothesis (H). First, we show that $X_t^\varepsilon \in \mathbb{D}_W^{1,2}$ by using Theorem 2.2.1 in [24]. Since the stochastic process φ_t^ε is not bounded we need to introduce the increasing sequence of stopping times

$$\tau_M = \inf \left\{ t \in [0, T] : \int_0^t (\varphi_s^\varepsilon)^2 ds > M \right\} \wedge T,$$

and consider the sequence of stopped equations corresponding to (3.4)

$$dX_{t \wedge \tau_M}^\varepsilon = [b(t \wedge \tau_M, X_{t \wedge \tau_M}^\varepsilon) + \varphi_{t \wedge \tau_M}^\varepsilon \sigma(t \wedge \tau_M) X_{t \wedge \tau_M}^\varepsilon] dt + \varepsilon^\alpha \sigma(t \wedge \tau_M) X_{t \wedge \tau_M}^\varepsilon dW_t, \quad 0 \leq t \leq T. \quad (3.6)$$

It is easy to see that the coefficients of (3.6) are globally Lipschitz functions with linear growth. Hence, $X_{t \wedge \tau_M}^\varepsilon \in \mathbb{D}_W^{1,\infty} \subset \mathbb{D}_W^{1,2}$, and then by taking limit $M \rightarrow \infty$ and by closability of the Malliavin derivative we obtain $X_t^\varepsilon \in \mathbb{D}_W^{1,\infty}$. Moreover, $b(t, x)$ is continuously differentiable in x ; from Theorem 2.2.1 in [24] we know that $D_s^W X_t^\varepsilon$ solves the following linear equation

$$D_s^W X_t^\varepsilon = \varepsilon^\alpha \sigma(s) X_s^\varepsilon + \int_s^t [(b'_x(r, X_r^\varepsilon) + \sigma(r) \varphi_r^\varepsilon) D_s^W X_r^\varepsilon + \alpha \sigma(r) X_r^\varepsilon (r - s + \varepsilon)^{\alpha-1}] dr + \varepsilon^\alpha \int_s^t \sigma(r) D_s^W X_r dW_r \quad (3.7)$$

for $s \leq t$.

The explicit solution of the above equation is given by

$$D_s^W X_t^\varepsilon = \Theta_{s,t}^\varepsilon \left(\varepsilon^\alpha \sigma(s) X_s^\varepsilon + \int_s^t \alpha \sigma(r) X_r^\varepsilon (r - s + \varepsilon)^{\alpha-1} (\Theta_{s,r}^\varepsilon)^{-1} dr \right) \mathbf{1}_{\{s \leq t\}},$$

or equivalently

$$D_s^W X_t^\varepsilon = \left(\varepsilon^\alpha \sigma(s) X_s^\varepsilon \Theta_{s,t}^\varepsilon + \int_s^t \alpha \sigma(r) X_r^\varepsilon (r - s + \varepsilon)^{\alpha-1} \Theta_{r,t}^\varepsilon dr \right) \mathbf{1}_{\{s \leq t\}},$$

where

$$\Theta_{s,t}^\varepsilon = \exp \left(\int_s^t \left[b'_x(r, X_r^\varepsilon) - \frac{1}{2} \varepsilon^{2\alpha} \sigma^2(r) \right] dr + \int_s^t \sigma(r) dW_r^{H,\varepsilon} \right).$$

Next, we prove that X_t^ε satisfies the condition (ii): put

$$I_t^{(\varepsilon)} := \int_0^t \sigma(s) dW_s^{H,\varepsilon} = \int_0^t \left(\varepsilon^\alpha \sigma(s) + \int_s^t \sigma(u) \partial_1 K(u + \varepsilon, s) du \right) dW_s,$$

$$I_t := \int_0^t \sigma(s) dW_s^H = \int_0^t \int_s^t \sigma(u) \partial_1 K(u, s) du dW_s.$$

Then $I_t^{(\varepsilon)} \rightarrow I_t$ uniformly in $t \in [0, T]$ as $\varepsilon \rightarrow 0$ since $I_t^{(\varepsilon)}$, I_t are Gaussian processes and

$$\begin{aligned} E |I_t^{(\varepsilon)} - I_t|^p &\leq C_p \left(\int_0^t \varepsilon^{2\alpha} \sigma^2(s) ds \right)^{p/2} + C_p \left\{ \int_0^t \left(\int_s^t \alpha \sigma(u) [(u - s + \varepsilon)^{\alpha-1} - (u - s)^{\alpha-1}] du \right)^2 ds \right\}^{p/2} \\ &\leq C_p M^p \varepsilon^{p\alpha} + C_p M^p \left\{ \int_0^t ((t - s)^\alpha - (t - s + \varepsilon)^\alpha + \varepsilon^\alpha)^2 ds \right\}^{p/2} \\ &\leq C_p M^p (1 + 2T^{p/2}) \varepsilon^{p\alpha}. \end{aligned} \quad (3.8)$$

Hence, from (2.7), $E \exp \left(\sup_{0 \leq t \leq T} |p I_t^{(\varepsilon)}| \right)$ is bounded for small enough values of ε . We have for all $p > 1$

$$\begin{aligned} E \left| \sup_{0 \leq t \leq T} Y_t^\varepsilon \right|^p &\leq E \exp \left\{ \sup_{0 \leq t \leq T} \left(\frac{p}{2} \varepsilon^{2\alpha} \int_0^t \sigma^2(s) ds - p \int_0^t \sigma(s) dW_s^{H,\varepsilon} \right) \right\} \\ &\leq \exp \left(\frac{p}{2} \varepsilon^{2\alpha} \int_0^T \sigma^2(s) ds \right) E \exp \left(\sup_{0 \leq t \leq T} |p I_t^{(\varepsilon)}| \right) \leq C_{p,H,T}^{(1)} < \infty, \end{aligned}$$

where $C_{p,H,T}^{(1)}$ is a constant depending only on p, H, T . Similarly,

$$E \left| \sup_{0 \leq t \leq T} \frac{1}{Y_t^\varepsilon} \right|^p \leq C_{p,H,T}^{(1)}. \quad (3.9)$$

From (3.5) and by assumption (1.5) we get

$$\begin{aligned} Z_u^\varepsilon &= x_0 + \int_0^u Y_s^\varepsilon b \left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon} \right) ds, \\ \sup_{0 \leq u \leq t} |Z_u^\varepsilon| &\leq |x_0| + \int_0^t \left| Y_s^\varepsilon b \left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon} \right) \right| ds \leq |x_0| + \int_0^t \sup_{0 \leq u \leq s} \left| Y_u^\varepsilon b \left(s, \frac{Z_u^\varepsilon}{Y_u^\varepsilon} \right) \right| ds, \\ E \left(\sup_{0 \leq u \leq t} |Z_u^\varepsilon| \right)^p &\leq 2^{p-1} |x_0|^p + 2^{p-1} I_0^p \int_0^t E \left(\sup_{0 \leq u \leq s} |Y_u^\varepsilon + Z_u^\varepsilon| \right)^p ds \\ &\leq 2^{p-1} |x_0|^p + 4^{p-1} I_0^p \int_0^t \left[E \left(\sup_{0 \leq u \leq s} |Y_u^\varepsilon| \right)^p + E \left(\sup_{0 \leq u \leq s} |Z_u^\varepsilon| \right)^p \right] ds. \end{aligned}$$

An application of Gronwall's Lemma to the latest inequality yields

$$E \left(\sup_{0 \leq u \leq t} |Z_u^\varepsilon| \right)^p \leq (2^{p-1} |x_0|^p + 4^{p-1} L_0^p T C_{p,H,T}) e^{4^{p-1} L_0^p T} := C_{p,H,T}^{(2)}. \quad (3.10)$$

We now combine (3.9) and (3.10) to get

$$\begin{aligned} E \left(\sup_{0 \leq u \leq t} |X_t^\varepsilon| \right)^p &\leq \left\{ E \left(\sup_{0 \leq u \leq t} |Z_t^\varepsilon| \right)^{2p} E \left(\sup_{0 \leq u \leq t} \left| \frac{1}{Y_t^\varepsilon} \right| \right)^{2p} \right\}^{\frac{1}{2}} \\ &\leq \left\{ C_{2p,H,T}^{(1)} C_{2p,H,T}^{(2)} \right\}^{\frac{1}{2}} := C_{p,H,T}^{(3)}. \end{aligned} \quad (3.11)$$

Finally, we prove that X_t^ε satisfies the condition (i).

Since $b'_x(t, x)$ is bounded and $\Theta_{s,t}^\varepsilon = \exp \left(\int_s^t b'_x(r, X_r^\varepsilon) \right) \frac{Y_s^\varepsilon}{Y_t^\varepsilon}$ we get

$$E \left(\sup_{0 \leq s \leq t \leq T} \Theta_{s,t}^\varepsilon \right)^p < \infty.$$

Then using the inequality (3.12) below we can see that there exists a constant $C_{p,H,T}^{(4)}$ depending only on p, H, T such that

$$\begin{aligned} E |\Theta_{s,t_2}^\varepsilon - \Theta_{s,t_1}^\varepsilon|^p &\leq C_{p,H,T}^{(4)} |t_1 - t_2|^{p/2}, \quad \forall t_1, t_2 \in [0, T]. \\ E |D_s^W X_{t_2}^\varepsilon - D_s^W X_{t_1}^\varepsilon|^2 &\leq E |\varepsilon^\alpha \sigma(s) X_s^\varepsilon (\Theta_{s,t_2}^\varepsilon - \Theta_{s,t_1}^\varepsilon)|^2 + E \left| \int_s^{t_1} \alpha \sigma(r) X_r^\varepsilon (r - s + \varepsilon)^{\alpha-1} (\Theta_{s,t_2}^\varepsilon - \Theta_{s,t_1}^\varepsilon) dr \right|^2 \\ &\quad + E \left| \int_{t_1}^{t_2} \alpha \sigma(r) X_r^\varepsilon (r - s + \varepsilon)^{\alpha-1} \Theta_{r,t_2}^\varepsilon dr \right|^2 := A_1 + A_2 + A_3. \end{aligned}$$

It is easy to see that $A_1 \leq C_{H,T}^{(5)} |t_1 - t_2|$, $\forall t_1, t_2 \in [0, T]$. Applying Hölder's inequality to $p_1 = \frac{2-\alpha}{2-2\alpha} \in (1, \frac{1}{1-\alpha})$, $q_1 = \frac{p_1}{p_1-1}$ we have

$$\begin{aligned} A_2 &\leq \left(\int_s^{t_1} (r - s + \varepsilon)^{p_1 \alpha - p_1} dr \right)^{\frac{2}{p_1}} E \left| \int_s^{t_1} (\alpha \sigma(r) X_r^\varepsilon (\Theta_{s,t_2}^\varepsilon - \Theta_{s,t_1}^\varepsilon))^{q_1} dr \right|^{\frac{2}{q_1}} \\ &\leq \left(\int_s^{t_1} (r - s + \varepsilon)^{p_1 \alpha - p_1} dr \right)^{\frac{2}{p_1}} \int_s^{t_1} E |\alpha \sigma(r) X_r^\varepsilon (\Theta_{s,t_2}^\varepsilon - \Theta_{s,t_1}^\varepsilon)|^2 dr \\ &\leq C_{H,T}^{(6)} |t_1 - t_2|. \end{aligned}$$

Similarly, we have also $A_3 \leq C_{H,T}^{(7)} |t_1 - t_2|$. Thus,

$$E |D_s^W X_{t_2}^\varepsilon - D_s^W X_{t_1}^\varepsilon|^2 \leq C_{H,T}^{(8)} |t_1 - t_2|.$$

Since $X_t^\varepsilon = \frac{Z_t^\varepsilon}{Y_t^\varepsilon}$, it is easy to show that

$$E |X_{t_2}^\varepsilon - X_{t_1}^\varepsilon|^2 \leq C_{H,T}^{(9)} |t_1 - t_2|.$$

Consequently, if we choose β such that $\frac{1}{2} - H < \beta < \frac{1}{2}$ then

$$\|X^\varepsilon\|_{L_{\beta}^{1,2}}^2 \leq C_{H,T}^{(10)} \sup_{0 < t_1 < t_2 < T} |t_2 - t_1|^{1-2\beta} < \infty.$$

Convergence. Let X_1, X_2 be two random variables. By Lagrange's theorem and Hölder's inequality we have

$$\begin{aligned} E |e^{X_1} - e^{X_2}|^p &\leq E \left| (X_1 - X_2) \sup_{\min(X_1, X_2) \leq x \leq \max(X_1, X_2)} e^x \right|^p \\ &\leq E |X_1 - X_2| e^{|X_1| + |X_2|} \leq (E[e^{2p|X_1|} + e^{2p|X_2|}] E |X_1 - X_2|^{2p})^{\frac{1}{2}}. \end{aligned} \quad (3.12)$$

We now put $Y_t = \exp \left(- \int_0^t \sigma(s) dW_s^H \right)$ and apply (3.12) to $X_1 = \frac{1}{2} \varepsilon^{2\alpha} \int_0^t \sigma^2(s) ds - \int_0^t \sigma(s) dW_s^{H,\varepsilon}$, $X_2 = - \int_0^t \sigma(s) dW_s^H$ to get

$$E |Y_t^\varepsilon - Y_t|^p \leq (E[e^{2p|X_1|} + e^{2p|X_2|}] E |X_1 - X_2|^{2p})^{\frac{1}{2}}. \quad (3.13)$$

It is obvious that $E[e^{2p|X_1|} + e^{2p|X_2|}]$ is finite because $I_t^{(\varepsilon)}$ and I_t are centered Gaussian processes with finite variances in $[0, T]$. Moreover, by the fundamental inequality $(a + b)^p \leq c_p(a^p + b^p)$, where $c_p = 1$ if $0 < p \leq 1$ and $c_p = 2^{p-1}$ if $p > 1$

$$E|X_1 - X_2|^{2p} \leq c_{2p} \left[E|I_t^{(\varepsilon)} - I_t|^{2p} + \frac{1}{2^{2p}} \varepsilon^{4p\alpha} \left(\int_0^t \sigma^2(s) ds \right)^{2p} \right] \leq C \varepsilon^{2p\alpha}, \quad (3.14)$$

where C is a finite constant. Consequently, $Y_t^\varepsilon \rightarrow Y_t$ in $L^p(\Omega)$ as $\varepsilon \rightarrow 0$.

We consider

$$dZ_t = Y_t b\left(t, \frac{Z_t}{Y_t}\right) dt, \quad Z_0 = x_0. \quad (3.15)$$

For P -almost all $\omega \in \Omega$, the function $f(t, x, \omega) = Y_t b\left(t, \frac{x}{Y_t}\right)$ satisfies the global Lipschitzian condition in $x \in \mathbb{R}$. Hence,

Eq. (3.15) has a unique solution $Z_t \in C^1[0, T]$. Moreover, $Z_t^\varepsilon \xrightarrow{L^2(\Omega)} Z_t$ uniformly in $t \in [0, T]$. Indeed, we have

$$\begin{aligned} E|Z_t^\varepsilon - Z_t|^2 &\leq \int_0^t E \left[Y_s^\varepsilon b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) - Y_s b\left(s, \frac{Z_s}{Y_s}\right) \right]^2 ds \\ &\leq 2 \int_0^t E \left[Y_s^\varepsilon b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) - Y_s b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) \right]^2 ds + 2 \int_0^t E \left[Y_s b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) - Y_s b\left(s, \frac{Z_s}{Y_s}\right) \right]^2 ds \\ &\leq 2 \int_0^t E \left[Y_s^\varepsilon b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) - Y_s b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) \right]^2 ds + 2L_0^2 \int_0^t E|Z_s^\varepsilon - Z_s|^2 ds \\ &:= A_4 + 2L_0^2 \int_0^t E|Z_s^\varepsilon - Z_s|^2 ds. \end{aligned} \quad (3.16)$$

By the assumptions (1.4), (1.5) we have

$$\begin{aligned} \left| Y_s^\varepsilon b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) - Y_s b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) \right| &\leq \left| (Y_s^\varepsilon - Y_s) b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) \right| + \left| Y_s^\varepsilon \left(b\left(s, \frac{Z_s^\varepsilon}{Y_s^\varepsilon}\right) - b\left(s, \frac{Z_s}{Y_s}\right) \right) \right| \\ &\leq L_0 \left| (Y_s^\varepsilon - Y_s) \left(\frac{Z_s^\varepsilon}{Y_s^\varepsilon} + 1 \right) \right| + L_0 \left| Y_s^\varepsilon \left(\frac{Z_s^\varepsilon}{Y_s^\varepsilon} - \frac{Z_s}{Y_s} \right) \right|, \end{aligned}$$

which means that $A_4 \xrightarrow{L^2(\Omega)} 0$ uniformly in $t \in [0, T]$. As a consequence, the fact that $Z_t^\varepsilon \xrightarrow{L^2(\Omega)} Z_t$ uniformly in $t \in [0, T]$ follows from (3.16) and by Gronwall's Lemma.

Now we put $X_t = \frac{Z_t}{Y_t}$ then $X_t^\varepsilon \xrightarrow{L^2(\Omega)} X_t$ as $\varepsilon \rightarrow 0$. We note also $X_t \in C^{H^-}[0, T]$ because $Y_t \in C^{H^-}[0, T]$ (this fact follows from Proposition 2.2) and $Z_t \in C^1[0, T]$ a.s.

Thus the theorem is proved. \square

By taking limit when $\varepsilon \rightarrow 0$, the main result of this section is formulated in the theorem given below.

Theorem 3.2. *The fractional stochastic differential equation (1.3) has a solution which is given by*

$$X_t = \frac{Z_t}{Y_t}, \quad (3.17)$$

where $Y_t = \exp\left(-\int_0^t \sigma(s) dW_s^H\right)$ and Z_t is the unique solution of the ordinary differential equation

$$dZ_t = Y_t b\left(t, \frac{Z_t}{Y_t}\right) dt, \quad Z_0 = x_0. \quad (3.18)$$

Moreover, $X_t \in C^{H^-}[0, T] \cap \mathbb{D}_W^{1,\infty}$ and

$$D_s^W X_t = \left(\int_s^t \alpha \sigma(r) X_r (r-s)^{\alpha-1} \Theta_{r,t} dr \right) \mathbf{1}_{\{s \leq t\}},$$

where $\Theta_{s,t} = \exp\left(\int_s^t b'_x(r, X_r) dr + \int_s^t \sigma(r) dW_r^H\right)$.

Proof. By using similar estimates as above we can see that X_t defined by (3.17) satisfies the Hypothesis (H) and then we can take the limit on both sides of (3.2) when $\varepsilon \rightarrow 0$, each term on the right-hand side converges to the same term where $\varepsilon = 0$. So X_t solves (1.3). \square

Example. The solution of the fractional Black–Scholes equation of the form

$$dX_t = b(t)X_t dt + \sigma(t)X_t dW_t^H, \quad X_0 = x_0,$$

is given by

$$X_t = x_0 \exp \left(\int_0^t b(s) ds + \int_0^t \sigma(s) dW_s^H \right).$$

Remark 3.1. We now turn our attention to Eq. (1.2) mentioned in Section 1:

$$dX_t = b(t, X_t)dt + \sigma(t)dW_t^H, \quad X_0 = x_0, \quad (3.19)$$

where $\sigma(t)$ is a deterministic function and b is Lipschitz continuous and satisfies the linear growth condition. The existence and the uniqueness of the solution of (3.19) were discussed by Mishura (see Section 3.5 in [5]).

Consider the approximation equations corresponding to (3.19):

$$dX_t^\varepsilon = b(t, X_t^\varepsilon)dt + \sigma(t)dW_t^{H,\varepsilon}, \quad X_0^\varepsilon = x_0. \quad (3.20)$$

By Gronwall's lemma it is obvious that $X_t^\varepsilon \rightarrow X_t$ in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$ since the definition of our fractional stochastic integral and

$$E|X_t^\varepsilon - X_t|^2 \leq 2L_0^2 \int_0^t E|X_s^\varepsilon - X_s|^2 ds + 2E \left| \int_0^t \sigma(s) d(W_s^{H,\varepsilon} - W_s^H) \right|^2.$$

Thus our approximate method works for all $H \in (0, 1)$. This gives us the following scheme to solve (3.19).

Step 1: Solving (3.20) by using the classical Itô differential formula.

Step 2: Taking limit of the solution found in Step 1 in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$ to get the solution of (3.19).

We illustrate this scheme by finding the solution for the equation with nonlinear drift of the form

$$dX_t = (e^{cX_t} + b)dt + \sigma dW_t^H, \quad X_0 = x_0, \quad (3.21)$$

where b, c, σ, x_0 are real constants and $c \neq 0$. The approximation equation corresponding to (3.21) is

$$dX_t^\varepsilon = (e^{cX_t^\varepsilon} + b)dt + \sigma dW_t^{H,\varepsilon}, \quad X_0^\varepsilon = x_0,$$

or equivalently,

$$dX_t^\varepsilon = (e^{cX_t^\varepsilon} + b + \sigma \varphi_t^\varepsilon)dt + \sigma \varepsilon^\alpha dW_t.$$

An application of the classical Itô differential formula to $Y_t^\varepsilon := e^{-cX_t^\varepsilon}$ yields

$$Y_t^\varepsilon = \left(\left(\frac{1}{2} c^2 \sigma^2 \varepsilon^{2\alpha} - bc - c \varphi_t^\varepsilon \right) Y_t^\varepsilon - c \right) dt - c \sigma \varepsilon^\alpha Y_t^\varepsilon dW_t. \quad (3.22)$$

Eq. (3.22) is a linear SDE and its solution is given by

$$Y_t^\varepsilon = \exp(-bct - \sigma c W_t^{H,\varepsilon} - cx_0) \left(1 - \int_0^t c \exp(cx_0 + bcs + \sigma c W_s^{H,\varepsilon}) ds \right).$$

As a consequence,

$$X_t^\varepsilon = bt + \sigma W_t^{H,\varepsilon} + x_0 - \frac{1}{c} \ln \left(1 - c \int_0^t \exp(cx_0 + bcs + \sigma c W_s^{H,\varepsilon}) ds \right),$$

and then the solution of (3.21) is

$$X_t = bt + \sigma W_t^H + x_0 - \frac{1}{c} \ln \left(1 - c \int_0^t \exp(cx_0 + bcs + \sigma c W_s^H) ds \right).$$

4. Applications to finance

It is known that fBm with Hurst index $H \neq \frac{1}{2}$ is a Gaussian process that has a memory. More precisely, let $\rho_H(n) := E(W_1^H(W_{n+1}^H - W_n^H))$; then (see, [1])

$$\rho_H(n) \approx H(2H - 1)n^{2H-2} \quad \text{as } n \rightarrow \infty.$$

Thus, if $H > \frac{1}{2}$, then $\sum_n \rho_H(n) = \infty$ and according to Beran's definition [26], fBm is called a long-memory process. If $H < \frac{1}{2}$, then $\sum_n \rho_H(n) < \infty$ and fBm is called a short-memory process.

The long-memory property makes fBm a potential candidate to model for noise in a variety of models (for a survey on theory and applications of long memory processes, see [27]). However, one has found that models driven by fBm are difficult to study because of the non-semimartingale property of fBm, as well as the complexity of the fractional stochastic calculus. In order to avoid this difficult, it would be desirable to find a long-memory process that has semimartingale property. Proposition 4.1 below implies that $W_t^{H,\varepsilon}$ is such a process.

Proposition 4.1. $W^{H,\varepsilon}$ has long-memory if $H > \frac{1}{2}$ and has short-memory if $H < \frac{1}{2}$.

Proof. Consider the auto-variance functions $\rho_{H,\varepsilon}(n) := E(W_1^{H,\varepsilon}(W_{n+1}^{H,\varepsilon} - W_n^{H,\varepsilon}))$, $n \geq 1$. By Itô isometry formula we have

$$\begin{aligned}\rho_{H,\varepsilon}(n) &= \int_0^1 (1-s+\varepsilon)^\alpha [(n+1-s+\varepsilon)^\alpha - (n-s+\varepsilon)^\alpha] ds \\ &= \int_{\varepsilon-1}^\varepsilon (1+s)^\alpha [(n+1+s)^\alpha - (n+s)^\alpha] ds.\end{aligned}$$

We now apply the Mean Value Theorem and then Lagrange's theorem to obtain

$$\begin{aligned}\rho_{H,\varepsilon}(n) &= (1+s_0)^\alpha [(n+1+s_0)^\alpha - (n+s_0)^\alpha] \\ &= \alpha(1+s_0)^\alpha (s_0+\theta)^{\alpha-1},\end{aligned}$$

where $s_0 \in (\varepsilon-1, \varepsilon)$ and $\theta \in (n, n+1)$.

Consequently,

$$\rho_{H,\varepsilon}(n) \approx \alpha(1+s_0)^\alpha n^{\alpha-1} \quad \text{as } n \rightarrow \infty,$$

which implies that $\sum_n \rho_{H,\varepsilon}(n) = \infty$ if $\alpha = H - \frac{1}{2} > 0$ and $\sum_n \rho_{H,\varepsilon}(n) < \infty$ if $\alpha < 0$.

Thus the proposition is proved. \square

Since $W_t^{H,\varepsilon}$ is a semimartingale and, similar to fBm, has long memory, we suggest the name “fractional semimartingale” for $W_t^{H,\varepsilon}$.

Definition 4.1. The stochastic process $W_t^{H,\varepsilon}$ defined by

$$W_t^{H,\varepsilon} = \int_0^t (t-s+\varepsilon)^\alpha dW_s, \quad \alpha = H - \frac{1}{2}$$

is called a fractional semimartingale with two parameters $(H, \varepsilon) \in (0, 1) \times (0, \infty)$.

A fractional semimartingale $W_t^{H,\varepsilon}$, contrarily to fBm, is not a self-similar process and has non-stationary increments. However, a simulation example in Plienpanich et al. [28] has showed a significant reduction of error in a stock price model driven by $W_t^{H,\varepsilon}$ as compared to the classical stock price model. This result means that $W_t^{H,\varepsilon}$, from empirical point of view, seems to be the potential candidate to model for noise in mathematical finance. In the remaining of this paper, let us study two models driven by fractional semimartingale.

4.1. The Black–Scholes model

We consider the Black–Scholes model with fractional noise that contains a stock S_t and a bond B_t .

Bond price:

$$dB_t = rB_t dt; \quad B_0 = 1. \quad (4.1)$$

Stock price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^H, \quad (4.2)$$

where S_0 is a positive real number, the coefficients r, μ, σ are assumed to be constants symbolizing the riskless interest rate, the drift of the stock and its volatility, respectively. Then, from Theorem 3.2 we have

$$S_t = S_0 \exp(\mu t + \sigma W_t^H), \quad B_t = e^{rt}.$$

It is obvious that $S_t \in C^{H-}[0, T]$ and this implies that the fractional stochastic integral can be understood as the Riemann–Stieltjes integral. As a consequence, our fractional Black–Scholes model admits an arbitrage opportunity (see, for instance, [29]) and we cannot use the traditional method to find the price of a European call option. However, for the Black–Scholes model with fractional semimartingale noise we have a surprising result in the following.

Theorem 4.1. Consider the fractional semimartingale Black–Scholes model containing a bond (4.1) and a stock S_t^ε

$$dS_t^\varepsilon = \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon dW_t^{H,\varepsilon}, \quad S_0^\varepsilon = S_0. \quad (4.3)$$

Then this model has no arbitrage and is complete. The European call option price at time $t = 0$ is given by

$$C_0(\varepsilon) = S_0 N(d_1) - e^{-rT} KN(d_2), \quad (4.4)$$

where K is strike price at maturity time T , $d_1 = \frac{\ln \frac{S_0}{K} + \left(r + \frac{\sigma^2 \varepsilon^{2\alpha}}{2}\right)T}{\sigma \varepsilon^\alpha \sqrt{T}}$, $d_2 = \frac{\ln \frac{S_0}{K} + \left(r - \frac{\sigma^2 \varepsilon^{2\alpha}}{2}\right)T}{\sigma \varepsilon^\alpha \sqrt{T}}$ and $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt$.

Proof. Refer [23, Theorem 4.1 and Remark 4.1]. \square

4.2. Optimal portfolio

Let $(\Omega, \mathcal{F}, P, \mathcal{F} = \{\mathcal{F}_t, 0 \leq t \leq T\})$ be a complete filtered probability space. Suppose that we observe continuously a single security with the price process $\{S_t; 0 \leq t \leq T\}$ which follows the equation

$$dS_t = S_t(\mu_t dt + \sigma dW_t^{H_1, \varepsilon_1}), \quad (4.5)$$

where σ and initial price S_0 are known positive constants, but the drift coefficient is an unobserved mean-reverting process with the dynamics

$$d\mu_t = \beta \mu_t dt + \nu dW_t^{H_2, \varepsilon_2}, \quad (4.6)$$

where β, ν are unknown real constants, the initial condition μ_0 is a Gaussian random variable and independent of $(W^{H_1, \varepsilon_1}, W^{H_2, \varepsilon_2})$.

We assume the Brownian motions $W^{(i)}, i = 1, 2$ are independent, so do $W^{H_i, \varepsilon_i}, i = 1, 2$. We shall denote by $\mathcal{F}^S = \{\mathcal{F}_t^S, 0 \leq t \leq T\}$ the P -augmentation of the filtration generated by the price process S . A portfolio is an \mathbb{R} -valued \mathcal{F}^S -adapted process $\pi = \{\pi_t, 0 \leq t \leq T\}$ such that

$$\int_0^T \pi_t^2 dt < \infty \quad \text{a.s.}$$

We regard π_t as the number of shares invested in the stock at time t . Given an initial wealth $x \geq 0$, the wealth process corresponding to a self-financing portfolio π is defined by $X_0^{x, \pi} = x$ and

$$dX_t^{x, \pi} = \pi_t dS_t.$$

We denote by $\mathcal{A}(x)$ the set of admissible portfolios π such that

$$X_t^{x, \pi} \geq 0 \quad \text{a.s., } 0 \leq t \leq T$$

and consider the following expected logarithmic utility maximization problem from terminal wealth over the class $\mathcal{A}(x)$:

$$\max_{\pi \in \mathcal{A}(x)} E[\log(X_T^{x, \pi})]. \quad (4.7)$$

Theorem 4.2. There exists an optimal portfolio $\pi^* = \{\pi_t^*, 0 \leq t \leq T\}$ for the utility maximization problem (4.7):

$$\pi_t^* = \frac{x(\hat{\mu}_t + \sigma \hat{\varphi}_t^{(1)})}{S_t \hat{L}_t}, \quad 0 \leq t \leq T, \quad (4.8)$$

where

$$\hat{L}_t = \exp\left(-\frac{1}{\sigma^2} \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)}) dL_s - \frac{1}{2\sigma^2} \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)})^2 ds\right),$$

where $\hat{\mu}_s = E[\mu_s | \mathcal{F}_s^S]$, $\hat{\varphi}_s^{(1)} = E[\varphi_s^{(1)} | \mathcal{F}_s^S]$.

The optimal wealth process corresponding to π^* is given by

$$X_T^{x, \pi^*} = \frac{x}{\hat{L}_T}, \quad X_t^{x, \pi^*} = \frac{\pi_t^* S_t}{\hat{\mu}_t + \sigma \hat{\varphi}_t^{(1)}}, \quad 0 \leq t \leq T. \quad (4.9)$$

Proof. From the decomposition (2.2), Eq. (4.5) can be rewritten as follows

$$dS_t = S_t(\mu_t + \sigma \varphi_t^{(1)})dt + \sigma S_t \varepsilon_1^{H_1 - \frac{1}{2}} dW_t^{(1)},$$

where $\varphi_t^{(1)} = \int_0^t (H_1 - \frac{1}{2})(t + \varepsilon_1 - u)^{H_1 - \frac{3}{2}} dW_u$ and we can use usual Itô calculus to find its solution

$$S_t = S_0 \exp\left(\int_0^t \left(\mu_s - \frac{1}{2}\sigma^2 \varepsilon_1^{2H_1 - 1}\right) ds + \sigma W_t^{H_1, \varepsilon_1}\right). \quad (4.10)$$

Denote by $U = \{U_t, 0 \leq t \leq T\}$ the stochastic process defined by

$$U_t = \int_0^t \mu_s ds + \sigma W_t^{H_1, \varepsilon_1} = \sigma W_t^{(1)} + \int_0^t (\mu_s + \sigma \varphi_s^{(1)}) ds \quad (4.11)$$

then $\mathcal{F}^S = \mathcal{F}^U$ and we can consider U as the observation process standing for S .

As in [30, Theorem 7.16] we define the innovation process $I = \{I_t, 0 \leq t \leq T\}$

$$I_t = \frac{1}{\sigma} \left[U_t - \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)}) ds \right].$$

I is a \mathcal{F}^U -standard Brownian motion satisfying $\mathcal{F}^I = \mathcal{F}^U$.

We can see that the following stochastic process is a \mathcal{F} -martingale

$$\begin{aligned} L_t &:= \exp \left(-\frac{1}{\sigma} \int_0^t (\mu_s + \sigma \varphi_s^{(1)}) dW_s^{(1)} - \frac{1}{2\sigma^2} \int_0^t (\mu_s + \sigma \varphi_s^{(1)})^2 ds \right) \\ &= \exp \left(-\frac{1}{\sigma^2} \int_0^t (\mu_s + \sigma \varphi_s^{(1)}) dU_s + \frac{1}{2\sigma^2} \int_0^t (\mu_s + \sigma \varphi_s^{(1)})^2 ds \right). \end{aligned} \quad (4.12)$$

We take the conditional expectation on both sides of (4.12) with respect to \mathcal{F}^U and obtain

$$\begin{aligned} \hat{L}_t &= \exp \left(-\frac{1}{\sigma^2} \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)}) dU_s + \frac{1}{2\sigma^2} \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)})^2 ds \right) \\ &= \exp \left(-\frac{1}{\sigma^2} \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)}) dI_s - \frac{1}{2\sigma^2} \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)})^2 ds \right). \end{aligned} \quad (4.13)$$

Applying the Itô formula we can see that $\hat{L}_t X_t^{x, \pi}$ is a local martingale. Moreover, for $x > 0$ and $\pi \in \mathcal{A}(x)$ we have $X_t^{x, \pi} \geq 0$, whence $\hat{L}_t X_t^{x, \pi}$ is an \mathcal{F}^U -supermartingale.

Now using a simple inequality that $\log(u) - uv \leq \log \frac{1}{v} - 1, \forall u, v > 0$ we obtain the inequality below for all $y > 0$

$$\begin{aligned} E[\log(X_T^{x, \pi})] &\leq E \left[\log(X_T^{x, \pi}) - y \hat{L}_T X_T^{x, \pi} \right] + yx \\ &\leq E \left[\log \left(\frac{1}{y \hat{L}_T} \right) - 1 \right] + yx. \end{aligned} \quad (4.14)$$

The equalities in (4.14) hold if and only if

$$X_T^{x, \pi} = \frac{x}{\hat{L}_T} \quad \text{a.s.} \quad (4.15)$$

Thus the strategy π satisfying (4.15) is optimal.

From (4.13) we have

$$d \frac{x}{\hat{L}_t} = \frac{x(\hat{\mu}_t + \sigma \hat{\varphi}_t^{(1)})}{\hat{L}_t} dY_t = \frac{x(\hat{\mu}_t + \sigma \hat{\varphi}_t^{(1)})}{S_t \hat{L}_t} dS_t;$$

hence, if we put

$$\pi_t^* = \frac{x(\hat{\mu}_t + \sigma \hat{\varphi}_t^{(1)})}{S_t \hat{L}_t}, \quad 0 \leq t \leq T,$$

then $dX_t^{x, \pi^*} = \pi_t^* dS_t = d \frac{x}{\hat{L}_t}$. This yields

$$X_t^{x, \pi^*} = \frac{x}{\hat{L}_t}, \quad 0 \leq t \leq T$$

because $X_0^{x, \pi^*} = x = \frac{x}{\hat{L}_0}$.

In particular, π^* satisfies (4.15) and it is the desired optimal strategy.

Thus the proof is complete. \square

At the end of this paper, we put $Z_t = (\mu_t, \varphi_s^{(1)})$ and our aim is to compute the optimal filter $\hat{Z}_t = (\hat{\mu}_t, \hat{\varphi}_s^{(1)})$ appeared in Theorem 4.2. First, in our filtering problem the dynamics of observation U and state process μ are given by, respectively

$$\begin{aligned} dU_t &= \mu_t dt + \sigma dW_t^{H_1, \varepsilon_1}, \\ d\mu_t &= \beta \mu_t dt + \nu dW_t^{H_2, \varepsilon_2}. \end{aligned}$$

The error matrix $P(t, s)$ is defined by

$$P(t, s) = E[Z_t(Z_s - \hat{Z}_s)^\tau], \quad 0 \leq s \leq t \leq T.$$

Theorem 4.3. The optimal filter $\hat{Z}_t = (\hat{\mu}_t, \hat{\varphi}_t^{(1)})$ satisfies the stochastic integral equation

$$\hat{Z}_t = \frac{1}{\sigma^2} \int_0^t [P(t, s) + D(t, s)] a [dU_s - a^\tau \hat{Z}_s ds], \quad 0 \leq t \leq T, \quad (4.16)$$

where $a = (1, \sigma)^\tau$, τ denotes the transposition and

$$D(t, s) = \begin{pmatrix} 0 & 0 \\ \sigma \left(H_1 - \frac{1}{2} \right) (t - s + \varepsilon_1)^{H_1 - \frac{3}{2}} & 0 \end{pmatrix}.$$

The error matrix $P(t, s)$ is the solution of the following Riccati-type equation

$$P(t, s) = \frac{-1}{\sigma^2} \int_0^s [P(t, u) + D(t, u)] a a^\tau [P(s, u) + D(s, u)]^\tau du + \Gamma_{ZZ}(t, s), \quad 0 \leq s \leq t \leq T, \quad (4.17)$$

where $\Gamma_{ZZ}(t, s) = E[Z_t Z_s^\tau]$.

Proof. We have

$$U_t = \int_0^t (\mu_s + \sigma \varphi_s^{(1)}) ds + \sigma W_t^{(1)}$$

and the associated innovation process $\{I_t, 0 \leq t \leq T\}$ is given by

$$I_t = \frac{1}{\sigma} \left[U_t - \int_0^t (\hat{\mu}_s + \sigma \hat{\varphi}_s^{(1)}) ds \right], \quad (4.18)$$

and is a \mathcal{F}^U -standard Brownian motion satisfying $\mathcal{F}^I = \mathcal{F}^U$. For convenience, we rewrite I_t in the matrix form:

$$I_t = \frac{1}{\sigma} \int_0^t (Z_s - \hat{Z}_s)^\tau a ds + W_t^{(1)}.$$

Since the Brownian motions $W^{(1)}, W^{(2)}$ are independent, the system $(Z, U) = (\mu, \varphi^{(1)}, U)$ is Gaussian. Hence, the optimal filter \hat{Z} is a linear function of the observation process $\{U_s, 0 \leq s \leq t\}$ and it is also a Gaussian system. By Liptser and Shiryaev [30, Theorem 5.6], there exists a deterministic Volterra function $F(t, s) = (F_1(t, s), F_2(t, s))$ on $0 \leq s \leq t \leq T$ such that

$$\begin{aligned} \int_0^t |F(t, s)|^2 ds &< \infty, \quad 0 \leq t \leq T, \\ \hat{Z}_t &= \int_0^t F(t, s) dI_s, \quad 0 \leq t \leq T. \end{aligned} \quad (4.19)$$

Now from (4.19) we can find the function $F(t, s)$ as follows

$$\begin{aligned} F(t, s) &= \frac{d}{ds} E(\hat{Z}_t I_s) = \frac{d}{ds} E(Z_t I_s) \\ &= \frac{d}{ds} \left(\frac{1}{\sigma} \int_0^s E[Z_t (Z_u - \hat{Z}_u)^\tau] a du + E[Z_t W_s^{(1)}] \right) \\ &= \frac{1}{\sigma} P(t, s) a + \left(0, \left(H_1 - \frac{1}{2} \right) (t - s + \varepsilon_1)^{H_1 - \frac{3}{2}} \right)^\tau \\ &= \frac{1}{\sigma} [P(t, s) + D(t, s)] a. \end{aligned}$$

Thus we have

$$\hat{Z}_t = \frac{1}{\sigma} \int_0^t [P(t, s) + D(t, s)] a \, dI_s, \quad 0 \leq t \leq T, \quad (4.20)$$

and the filtering equation (4.16) follows from (4.18).

From the definition of the error matrix we have

$$P(t, s) = E[Z_t Z_s^T] - E[\hat{Z}_t \hat{Z}_s^T];$$

hence Eq. (4.17) follows from (4.20) and Itô isometry formula.

We give an explicit computation of $\Gamma_{ZZ}(t, s)$, $0 \leq s \leq t \leq T$. We have

$$\Gamma_{ZZ}(t, s) = \begin{pmatrix} E[\mu_t \mu_s] & E[\mu_t \varphi_s^{(1)}] \\ E[\varphi_t^{(1)} \mu_s] & E[\varphi_t^{(1)} \varphi_s^{(1)}] \end{pmatrix}.$$

We recall from [31, Proposition 4.2] that

$$\mu_t = \mu_0 e^{\beta t} + v \varepsilon_2^{H_2 - \frac{1}{2}} \int_0^t e^{\beta(t-s)} dW_s^{(2)} + v \int_0^t e^{\beta(t-s)} \varphi_s^{(2)} ds.$$

Applying stochastic Fubini's theorem we have

$$\begin{aligned} \int_0^t e^{\beta(t-s)} \varphi_s^{(2)} ds &= \int_0^t \int_0^s \left(H_2 - \frac{1}{2} \right) e^{\beta(t-s)} (s - u + \varepsilon_2)^{H_2 - \frac{3}{2}} dW_u^{(2)} ds \\ &= \int_0^t \int_u^t \left(H_2 - \frac{1}{2} \right) e^{\beta(t-s)} (s - u + \varepsilon_2)^{H_2 - \frac{3}{2}} ds dW_u^{(2)}. \end{aligned}$$

Thus,

$$\mu_t = \mu_0 e^{\beta t} + \int_0^t b(t - u) dW_u^{(2)},$$

where

$$\begin{aligned} b(t - u) &= v \varepsilon_2^{H_2 - \frac{1}{2}} e^{\beta(t-u)} + v \int_u^t \left(H_2 - \frac{1}{2} \right) e^{\beta(t-s)} (s - u + \varepsilon_2)^{H_2 - \frac{3}{2}} ds \\ &= e^{\beta(t-u)} \left(v \varepsilon_2^{H_2 - \frac{1}{2}} + v \int_0^{t-u} \left(H_2 - \frac{1}{2} \right) e^{-\beta s} (s + \varepsilon_2)^{H_2 - \frac{3}{2}} ds \right). \end{aligned}$$

As a consequence, we have

$$\begin{aligned} E[\mu_t \mu_s] &= e^{\beta(t+s)} E[\mu_0^2] + \int_0^s b(t - u) b(s - u) du, \\ E[\mu_t \varphi_s^{(1)}] &= E[\varphi_t^{(1)} \mu_s] = 0, \\ E[\varphi_t^{(1)} \varphi_s^{(1)}] &= \left(H_1 - \frac{1}{2} \right)^2 \int_0^s (t - u + \varepsilon_1)^{H_1 - \frac{3}{2}} (s - u + \varepsilon_1)^{H_1 - \frac{3}{2}} du. \quad \square \end{aligned}$$

When $H_1 = H_2 = \frac{1}{2}$ and $\mu_0 = 0$ we obtain the following corollary for the classical Kalman–Bucy linear filtering problem.

Corollary 4.1. Consider the filtering problem with observation and state process given by, respectively

$$\begin{aligned} dU_t &= \mu_t dt + \sigma dW_t^{(1)}, & U_0 &= 0, \\ d\mu_t &= \beta \mu_t dt + v dW_t^{(2)}, & \mu_0 &= 0. \end{aligned}$$

Then the optimal filter $\hat{\mu}_t$ satisfies

$$\hat{\mu}_t = \frac{\beta - \gamma(t)}{\sigma^2} \hat{\mu}_t dt + \frac{\gamma(t)}{\sigma^2} dU_t,$$

where the error $\gamma(t) = E[\mu_t(\mu_t - \hat{\mu}_t)]$ is the solution of the Riccati equation

$$\frac{d\gamma(t)}{dt} = \frac{v^2}{\sigma^2} + \frac{2\beta}{\sigma^2} \gamma(t) - \frac{1}{\sigma^2} \gamma^2(t).$$

Proof. It is obvious since $D(t, s) \equiv 0 \equiv \varphi_t^{(1)}$,

$$\Gamma_{ZZ}(t, s) = \begin{pmatrix} \frac{v^2}{2\beta}(e^{\beta(t+s)} - e^{\beta(t-s)}) & 0 \\ 0 & 0 \end{pmatrix},$$

$$P(t, s) = \begin{pmatrix} E[\mu_t(\mu_s - \hat{\mu}_s)] & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$E[\mu_t(\mu_s - \hat{\mu}_s)] = E[E[\mu_t | \mathcal{F}_s](\mu_s - \hat{\mu}_s)] = e^{\beta(t-s)}\gamma(s). \quad \square$$

Acknowledgments

The author would like to thank the anonymous referees for their valuable comments for improving the paper.

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2011.12 (16–101).

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