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# On the solutions of operator equation $CAX = C = XAC$

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ABSTRACT

In this paper, we find the explicit solutions of the equation $CAX = C = XAC$ for linear bounded operators on Hilbert spaces, where the unknown operator $X$ is called the inverse of $A$ along $C$. This solution is expressed in terms of the inner inverses of the operators $AC$ and $CA$. From these formulas we derive necessary and sufficient conditions for the solution to be unique.

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## 1. Introduction

Let $\mathcal{H}$, $\mathcal{K}$ denote complex Hilbert spaces, and $\mathcal{B}(\mathcal{H},\mathcal{K})$ the set of all bounded linear operators between $\mathcal{H}$ and $\mathcal{K}$. We write $\mathcal{R}(T)$ and $\mathcal{N}(T)$ for the range and the null space of $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Recall that the adjoint $T^*$ of a linear operator $T$ on a Hilbert space $\mathcal{H}$ is defined as a linear operator satisfying the condition $(Tx, y) = (x, T^*y)$, $x, y \in \mathcal{H}$. The orthogonal projection onto closed subspace $U \subset \mathcal{H}$ is denoted by $P_U$. Let $P_{U,V}$ denote the idempotent with $\mathcal{R}(P_{U,V}) = U$ and $\mathcal{N}(P_{U,V}) = V$. An operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is regular if there is an operator $S$ such that (I) $TST = T$. The operator $S$ is not unique in general. In order to force its uniqueness, further conditions have to be imposed:

$$\text{(II) } STS = S, \qquad \text{(III) } (TS)^* = TS, \qquad \text{(IV) } (ST)^* = ST, \qquad \text{(V) } TS = ST, \qquad (\mathrm{I}_k)\ T^kST = T^k$$

with some $k \in \mathbb{Z}^+$. Clearly, (I) $= (\mathrm{I}_1)$. Elements $S \in \mathcal{B}(\mathcal{H},\mathcal{K})$ satisfying (I) are called inner inverse (or (I)-inverses) of $T$, denoted by $S = T^-$. We observe that both $T^-T$ and $TT^-$ are idempotents. In this paper, $\mathcal{B}(\mathcal{H},\mathcal{K})^-$ will denote the set of all regular elements of $\mathcal{B}(\mathcal{H},\mathcal{K})$. A bounded linear $T \in \mathcal{B}(\mathcal{H})$ has a bounded inner inverse if and only if $T$ has closed range. A proof of this fact can be found in [1]. We call that (I, II, III, IV)-inverses, Moore–Penrose inverses (MP-inverse), denoted by $S = T^+$ (see [2]). The MP-inverse of $T$ is unique and $TT^+ = P_{\mathcal{R}(T)}$ and $T^+T = P_{\mathcal{R}(T^*)}$. In fact, if $S_1$ and $S_2$ are two (I, II, III, IV)-inverses of $T$, then

$$\begin{aligned} S_1 &= S_1TS_1 = S_1(TS_2T)S_1 = S_1(TS_2)^*(TS_1)^* = S_1(TS_1TS_2)^* = S_1(TS_2)^* = S_1TS_2 \\ &= S_1(TS_2T)S_2 = (S_1T)^*(S_2T)^*S_2 = (S_2TS_1T)^*S_2 = (S_2T)^*S_2 = S_2TS_2 = S_2. \end{aligned}$$

And $(\mathrm{I}_k$, II, V)-inverses are called Drazin inverses, denoted by $S = T^D$, where $k$ is the Drazin index of $T$ [2,3]. The uniqueness of the Drazin inverse can be seen as follows. If $S_1'$ and $S_2'$ are two $(\mathrm{I}_k$, II, V)-inverses of $T$, then $P := TS_1' = S_1'T$ and $Q := TS_2' = S_2'T$ are idempotents. Moreover

$$P = TS_1' = T^k(S_1')^k = T^kS_2'T(S_1')^k = TS_2'T^k(S_1')^k = QTS_1' = QP$$

and

$$Q = S_2'T = (S_2')^kT^k = (S_2')^kT^kS_1'T = (S_2')^kT^kP = S_2'TP = QP.$$

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Hence $P = Q$ and

$$S_1' = T(S_1')^2 = PS_1' = QS_1' = S_2'TS_1' = S_2'P = S_2'Q = S_2'TS_2' = S_2'.$$

Similarly, (I, II, V)-inverses are called group inverses, denoted by $S = T^{\#}$. For $A, B \in B(\mathcal{H}, \mathcal{K})$, the $\mathcal{N}$-related, the $\mathcal{R}$-related and $\mathcal{H}$-related are defined, respectively, by

$$\begin{aligned} &(1)\ A_{\mathcal{N}}B =: \mathcal{N}(A) = \mathcal{N}(B),\\ &(2)\ A_{\mathcal{R}}B =: \mathcal{R}(A) = \mathcal{R}(B),\\ &(3)\ A_{\mathcal{H}}B =: A_{\mathcal{N}}B \quad \text{and} \quad A_{\mathcal{R}}B. \end{aligned} \tag{1.1}$$

Parallel with these equivalent relations we have the preorder relations:

$$\begin{aligned} &(1')\ A \le_{\mathcal{N}} B =: \mathcal{N}(A) \supset \mathcal{N}(B),\\ &(2')\ A \le_{\mathcal{R}} B =: \mathcal{R}(A) \subset \mathcal{R}(B),\\ &(3')\ A \le_{\mathcal{H}} B =: A \le_{\mathcal{N}} B \quad \text{and} \quad A \le_{\mathcal{R}} B. \end{aligned} \tag{1.2}$$

The solutions of operator equations involving generalized inverses are fundamental in the theory of operators. They have attracted considerable attention. Much progress has been made on the study of matrix equations for finite matrices [4], Hilbert space operators [5], elements of $C^*$-algebras [6] and elements of $C^*$-modules [7,8]. This paper studies the equation $CAX = C = XAC$ for bounded linear operators between Hilbert spaces. We derive general existence criteria and properties of solutions, give necessary and sufficient conditions for the uniqueness of solutions, and obtain the formula for the general form of these solutions. It appears that these solutions, which are called the inverse of $A$ along $C$, encompass the classical generalized inverses but are of richer type. That is to say, for some given $C$, the solution $X \in \mathcal{B}(\mathcal{H})$ with additive condition $X \le_{\mathcal{H}} C$ reduces as the classical generalized inverses: group inverse, Drazin inverse and MP-inverse.

## 2. Some lemmas and relations

In this section some auxiliary lemmas are given. Suppose that every bounded linear operator is defined on convenient Hilbert spaces. We start with a known elementary result which was given in [7] for Hilbert $C^*$-modules.

**Lemma 1** ([7, Proposition 2.1]). *Suppose that $A \in \mathcal{B}(\mathcal{H})^-$. $AX = C$ has a solution $X \in \mathcal{B}(\mathcal{H})$ if and only if $\mathcal{R}(C) \subset \mathcal{R}(A)$. In this case, the general solution has the form*

$$X = A^-C + (I - A^-A)T, \tag{2.1}$$

*where $T \in \mathcal{B}(\mathcal{H})$ is arbitrary.*

Let $A \in \mathcal{B}(\mathcal{H})$. By Remark 1.1 in [8] we know that

$$\mathcal{R}(A) \text{ is closed } \Longleftrightarrow R(A^*) \text{ is closed } \Longleftrightarrow R(AA^*) \text{ is closed } \Longleftrightarrow R(A^*A) \text{ is closed.} \tag{2.2}$$

If $\mathcal{R}(A)$ is closed, we have the following result.

**Lemma 2** ([2],[5, Lemma 2.1] and [9]).

(i) *Let $A \in \mathcal{B}(\mathcal{H})^-$. Then $AA^-$ and $A^-A$ are idempotents with*

$$\mathcal{R}(AA^-) = \mathcal{R}(A) = \mathcal{N}(A^*)^{\perp}, \quad \mathcal{N}(A^-A) = \mathcal{N}(A) = \mathcal{R}(A^*)^{\perp}.$$

(ii) *If $AC \in \mathcal{B}(\mathcal{H})^-$, then*

$$C(AC)^-AC = C \Longleftrightarrow \mathcal{N}(AC) = \mathcal{N}(C), \qquad AC(AC)^-A = A \Longleftrightarrow \mathcal{R}(AC) = \mathcal{R}(A).$$

(iii) *If $\mathcal{L}$ and $\mathcal{M}$ are closed subspaces of $\mathcal{H}$ and $P_{\mathcal{L},\mathcal{M}}$ is an idempotent on $\mathcal{L}$ along $\mathcal{M}$, then*

$$P_{\mathcal{L},\mathcal{M}}T = T \Longleftrightarrow \mathcal{R}(T) \subset \mathcal{L}, \qquad TP_{\mathcal{L},\mathcal{M}} = T \Longleftrightarrow \mathcal{N}(T) \supset \mathcal{M}.$$

Throughout this work the next well-known criterion due to Douglas [10] (see also [11]) about range inclusions and factorization of operators will be crucial.

**Lemma 3** (Douglas). *If $A, B \in \mathcal{B}(\mathcal{H})$, then the following are equivalent:*

(i) *$A = BC$ for some operator $C \in \mathcal{B}(\mathcal{H})$;*
(ii) *$\|A^*x\| \le k\|B^*x\|$ for some $k > 0$ and all $x \in \mathcal{H}$;*
(iii) *$\mathcal{R}(A) \subset \mathcal{R}(B)$.*

If one of these conditions holds then there exists a unique solution $C_0 \in \mathcal{B}(\mathcal{H})$ of the equation $BX = A$ such that $R(C_0) \subset \overline{\mathcal{R}(B^*)}$ and $\mathcal{N}(C_0) = \mathcal{N}(A)$. This solution is called the Douglas reduced solution.

If $A$ and $C$ are $n \times n$ complex matrices, the Cline's formula is $(AC)^D = A[(CA)^D]^2 C$ (see [12]). For $A, C \in \mathcal{B}(\mathcal{H})$, if $CA$ is group invertible, then it is easy to get the following results.

**Lemma 4.** *Let $A, C \in \mathcal{B}(\mathcal{H})$. If $CA$ is group invertible, then $AC$ is Drazin invertible with $(AC)^D = A\left[(CA)^\#\right]^2 C$ and $\operatorname{ind}(AC) \le 2$. If $CA$ and $CA$ are group invertible, then*

$$(AC)^\# A = A(CA)^\#, \qquad C(AC)^\# = (CA)^\# C. \tag{2.3}$$

Next, we give some equivalent forms of definition (1.1) that we will frequently use in the sequel.

**Theorem 5.** *Let $A, B \in \mathcal{B}(\mathcal{H})$.*

(i) $A_{\mathcal{R}}B \Longleftrightarrow$ *There exist $X, Y \in \mathcal{B}(\mathcal{H})$ such that $AX = B$ and $A = BY$.*
(ii) $A_{\mathcal{R}}B \Longrightarrow A^*_{\mathcal{N}}B^*$. *Moreover, if $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, then $A_{\mathcal{R}}B \Longleftrightarrow A^*_{\mathcal{N}}B^*$.*

**Proof.** (i) By Lemma 3.

(ii) By item (i), if $A_{\mathcal{R}}B$, then there exist $X, Y \in \mathcal{B}(\mathcal{H})$ such that $AX = B$ and $A = BY$. It follows that $X^*A^* = B^*$ and $A^* = Y^*B^*$. We get $\mathcal{N}(A^*) \subset \mathcal{N}(B^*)$ and $\mathcal{N}(A^*) \supset \mathcal{N}(B^*)$. Thus $A^*_{\mathcal{N}}B^*$ holds. If $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are closed, then

$$A^*_{\mathcal{N}}B^* \Longrightarrow \mathcal{N}(A^*) = \mathcal{N}(B^*) \Longrightarrow \mathcal{R}(A)^\perp = \mathcal{R}(B)^\perp \Longrightarrow \mathcal{R}(A) = \mathcal{R}(B) \Longrightarrow A_{\mathcal{R}}B. \quad \square$$

We consider two special cases: (i) $CAX = 0$; (ii) $CAX$ invertible. As for case (i), it is easy to get the general solution $X$. If $CA \in \mathcal{B}(\mathcal{H})^-$, then

$$CAX = 0 \Longleftrightarrow X = T - (CA)^- CAT, \quad T \in \mathcal{B}(\mathcal{H}).$$

In fact, if $X = T$ satisfying $CAT = 0$, then also $(CA)^-CAT = 0$, and $X = T - (CA)^-CAT$. The converse is clear.

Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ with orthocomplement $\mathcal{M}^\perp$. According to the orthogonal decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, every operator $M \in \mathcal{B}(\mathcal{H})$ can be written in a block-form $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. It is well-known that, if $A \in \mathcal{B}(\mathcal{M})$ is invertible, then $M$ is invertible if and only if the Schur complement $S = D - CA^{-1}B$ of $A$ in $\mathcal{M}$ is invertible. The inverse of $M$ is $M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}$. This expression is called the Banachiewicz–Schur form of the operator $M$ and can be found in standard textbooks on linear algebra. In the following we study the inverse of $CAX$. We give the equivalent conditions which ensure that $CAX$ is invertible. The explicit expression for the inverse of $CAX$ is obtained.

**Theorem 6.** *Let $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be invertible, $X, C \in \mathcal{B}(\mathcal{H}, \mathcal{K})^-$. Denote $S = C^+CAXX^+$ and $T = (I - XX^+)A^{-1}(I - C^+C)$.*

(i) *$CAX$ is invertible if and only if $X$ is injective, $C$ is surjective, $S$ (or $T$) is MP-invertible and one of the following conditions holds:*

$$(1)\ SS^+ = C^+C; \qquad (2)\ S^+S = XX^+; \qquad (3)\ TT^+ + XX^+ = I_{\mathcal{K}}; \qquad (4)\ T^+T + C^+C = I_{\mathcal{H}}.$$

(ii) *If $CAX$ is invertible, then $(CAX)^{-1} = X^+S^+C^+$. An interesting result is that*

$$(CAX)^{-1} = X^+A^{-1}(I - T^+A^{-1})C^+.$$

*This second relation shows that there always exist some $S, T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $SA^{-1}T$ is an inverse of $CAX$.*

**Proof.** (i) The invertibility of $CAX$ implies that $X$ is injective and $\mathcal{K} = \mathcal{R}(CAX) \subset \mathcal{R}(C)$. In the following, we suppose that $X$ is injective and $C$ is surjective.

It is clear that $\mathcal{H} = \mathcal{R}(C^*) \oplus \mathcal{N}(C)$ and $\mathcal{K} = \mathcal{R}(X) \oplus \mathcal{N}(X^*)$. There exist invertible operators $X_1, C_1$ such that $C, X, A, C^+, X^+, CC^+, X^+X, C^+C$ and $XX^+$ have the operator matrix representations as

$$C = \begin{pmatrix} C_1 & 0 \end{pmatrix}, \qquad X = \begin{pmatrix} X_1 \\ 0 \end{pmatrix}, \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad C^+ = \begin{pmatrix} C_1^{-1} \\ 0 \end{pmatrix}, \qquad X^+ = \begin{pmatrix} X_1^{-1} & 0 \end{pmatrix},$$

$CC^+ = I_{\mathcal{K}}, X^+X = I_{\mathcal{H}}, C^+C = I_{\mathcal{R}(C^*)} \oplus 0$ and $XX^+ = I_{\mathcal{R}(X)} \oplus 0$, respectively. Using these matrix representations, $CAX = C_1A_{11}X_1$ is invertible if and only if $A_{11}$ is invertible. Note that $S = C^+CAXX^+ = A_{11} \oplus 0$,

$$A^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}S_0^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S_0^{-1} \\ -S_0^{-1}A_{21}A_{11}^{-1} & S_0^{-1} \end{pmatrix} \tag{2.4}$$

and $T = (I - XX^+)A^{-1}(I - C^+C) = 0 \oplus S_0^{-1}$, where the Schur complement $S_0 = A_{22} - A_{21}A_{11}^{-1}A_{12}$. So $A_{11}$ is invertible if and only if $S$ (or $T$) is MP-invertible and $SS^+ = C^+C$, or $S^+S = XX^+$, or $TT^+ + XX^+ = I_{\mathcal{K}}$, or $T^+T + C^+C = I_{\mathcal{H}}$.

On the other hand, if one of (1)–(4) holds, then $A_{11}$ is invertible, which is equivalent to, $S_0$ is invertible, because $A$ is invertible. Note that $X_1$ and $C_1$ are invertible. Hence, $CAX = C_1A_{11}X_1$ is invertible.

(ii) By item (i), we get

$$(CAX)^{-1} = \left[\begin{pmatrix} C_1 & 0 \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} X_1 \\ 0 \end{pmatrix}\right]^{-1} = (C_1A_{11}X_1)^{-1} = X^+S^+C^+$$

and

$$\begin{aligned} X^+A^{-1}(I-T^+A^{-1})C^+ &= X^+A^{-1}\left[\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & S_0 \end{pmatrix}\begin{pmatrix} A_{11}^{-1}+A_{11}^{-1}A_{12}S_0^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S_0^{-1} \\ -S_0^{-1}A_{21}A_{11}^{-1} & S_0^{-1} \end{pmatrix}\right]\begin{pmatrix} C_1^{-1} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} X_1^{-1} & 0 \end{pmatrix}\begin{pmatrix} A_{11}^{-1}+A_{11}^{-1}A_{12}S_0^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S_0^{-1} \\ -S_0^{-1}A_{21}A_{11}^{-1} & S_0^{-1} \end{pmatrix}\begin{pmatrix} C_1^{-1} \\ A_{21}A_{11}^{-1}C_1^{-1} \end{pmatrix} \\ &= X_1^{-1}A_{11}^{-1}C_1^{-1} = (CAX)^{-1}. \quad \square \end{aligned}$$

Now, let $CAX$ neither be zero nor invertible. The equation $CAX = C = XAC$ is solvable if there exists a bounded linear operator $X_0$ such that $CAX_0 = C = X_0AC$. We get the following result.

**Theorem 7.** *Let $A \in \mathcal{B}(\mathcal{H})^-$. The equation $A^-AX = A^- = XAA^-$ is solvable if and only if $A^-$ is a $\{\mathrm{I}, \mathrm{II}\}$-inverse of $A$. In this case, the general solution is*

$$X = A^- + (I - A^-A)S(I - AA^-),$$

*where $S \in \mathcal{B}(\mathcal{H})$ is arbitrary.*

**Proof.** It is clear that $X = A^-$ is one solution if $A^-$ is a $\{\mathrm{I}, \mathrm{II}\}$-inverse of $A$.

On the other hand, $A \in \mathcal{B}(\mathcal{H})^-$ implies that $\mathcal{R}(A)$ is closed. Then $A$ can be written as $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}$ where $A_1 \in \mathcal{B}(\mathcal{R}(A^*), \mathcal{R}(A))$ is invertible. Since $AA^-A = A$, we get that $A^- = \begin{pmatrix} A_1^{-1} & A_3 \\ A_4 & A_2 \end{pmatrix}$, where $A_i$, $i = 2, 3, 4$ are corresponding arbitrary bounded operators. We consider the partition $X$ conforming with $A$ as $X = \begin{pmatrix} X_1 & X_3 \\ X_4 & X_2 \end{pmatrix}$, where $X_i$, $i = 1, 2, 3, 4$ are corresponding bounded operators. $A^-AX = A^- = XAA^-$ implies that

$$\begin{pmatrix} A_1^{-1} & A_3 \\ A_4 & A_2 \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X_1 & X_3 \\ X_4 & X_2 \end{pmatrix} = \begin{pmatrix} A_1^{-1} & A_3 \\ A_4 & A_2 \end{pmatrix} = \begin{pmatrix} X_1 & X_3 \\ X_4 & X_2 \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1^{-1} & A_3 \\ A_4 & A_2 \end{pmatrix}.$$

So

$$\begin{pmatrix} X_1 & X_3 \\ A_4A_1X_1 & A_4A_1X_3 \end{pmatrix} = \begin{pmatrix} A_1^{-1} & A_3 \\ A_4 & A_2 \end{pmatrix} = \begin{pmatrix} X_1 & X_1A_1A_3 \\ X_4 & X_4A_1A_3 \end{pmatrix}.$$

Comparing two sides of above equations, it follows that $A^- = \begin{pmatrix} A_1^{-1} & A_3 \\ A_4 & A_4A_1A_3 \end{pmatrix}$ is the $\{\mathrm{I}, \mathrm{II}\}$-inverse of $A$ and

$$X = \begin{pmatrix} A_1^{-1} & A_3 \\ A_4 & X_2 \end{pmatrix} = A^- + (I - A^-A)S(I - AA^-)$$

for arbitrary $S \in \mathcal{B}(\mathcal{H})$. $\square$

**Theorem 8.** *Let $A, C, X \in \mathcal{B}(\mathcal{H})$. Then*

$$CAX = C = XAC \quad \text{and} \quad X \leq_{\mathcal{H}} C \Longleftrightarrow X = XAX \text{ and } X_{\mathcal{H}}C. \tag{2.5}$$

*Moreover, for given $A, C$ in (2.5), if $X$ exists, then $X$ is unique.*

**Proof.** Suppose that $CAX = C = XAC$ and $X \leq_{\mathcal{H}} C$. By definition (1′) in (1.2) and $CAX = C$, we deduce that $\mathcal{N}(C) \subset \mathcal{N}(X) \subset \mathcal{N}(C)$. By definition (2′) in (1.2) and $C = XAC$, we deduce that $\mathcal{R}(X) \subset \mathcal{R}(C) \subset \mathcal{R}(X)$. Hence, $\mathcal{N}(C) = \mathcal{N}(X)$ and $\mathcal{R}(X) = \mathcal{R}(C)$, i.e., $X_{\mathcal{H}}C$ holds by definition (1.1). Moreover, $(I - XA)C = 0$ and $\mathcal{R}(X) \subset \mathcal{R}(C)$ imply that $(I - XA)X = 0$.

Conversely, let $X = XAX$ and $X_{\mathcal{H}}C$. We have $\mathcal{N}(C) = \mathcal{N}(X)$ and $\mathcal{R}(X) = \mathcal{R}(C)$. Hence, $X \leq_{\mathcal{H}} C$ and

$$(I - XA)X = 0 \Longrightarrow \mathcal{R}(X) \subset \mathcal{N}(I - XA) \Longrightarrow \mathcal{R}(C) \subset \mathcal{N}(I - XA) \Longrightarrow C = XAC,$$
$$X(I - AX) = 0 \Longrightarrow \mathcal{R}(I - AX) \subset \mathcal{N}(X) \Longrightarrow \mathcal{R}(I - AX) \subset \mathcal{N}(C) \Longrightarrow C = CAX.$$

For given $A, C$ in (2.5), if $X, X'$ satisfy the right hand side of (2.5), then $\mathcal{N}(X) = \mathcal{N}(C) = \mathcal{N}(X')$ and $\mathcal{R}(X) = \mathcal{R}(C) = \mathcal{R}(X')$, which imply

$$(I - XA)X = 0 \Longrightarrow (I - XA)X' = 0 \Longrightarrow X' = XAX',$$
$$X'(I - AX') = 0 \Longrightarrow X(I - AX') = 0 \Longrightarrow X = XAX'.$$

Hence, $X = X'$. $\square$

The equivalency (2.5) was originally obtained by Mary [13, Lemma 3] on semigroups by means of Greens relations. In [13], Mary defined $X$ as an inverse of $A$ along $C$ and studied its properties. Theorem 8 implies that, under the condition $X \leq_{\mathcal{H}} C$, the solution of $CAX = C = XAC$ is an out inverse of an operator $A$ over complex field with prescribed range space $\mathcal{R}(C)$ and null space $\mathcal{N}(C)$ (see [1,2]).

## 3. The solutions of operator equation $CAX = C = XAC$

In this section, we will study the general solution to the equation

$$CAX = C = XAC \tag{3.1}$$

and the unique solution $X$ of Eq. (3.1). We say that the solution $X$ is an inverse of $A$ along $C$. The technique results are Lemmas 1–4, which are crucial in our study of the solutions to Eq. (3.1).

First, we consider some particular cases. If $AC \in \mathcal{B}(\mathcal{H})^-$, by Lemmas 2 and 3,

$$C^*A^*X^* = C^* \Longrightarrow \mathcal{R}(C^*) = \mathcal{R}(C^*A^*) \Longrightarrow \mathcal{N}(C) = \mathcal{N}(AC).$$

So $X = C(AC)^-$ is one solution of the equation $C = XAC$; if $AC$ and $CA$ are idempotents, then $CAX = XAC$ implies that $(I - CA)XAC = 0$. Hence, $X$ is the solution of $CAX = XAC$ if and only if there exists $S \in \mathcal{B}(\mathcal{H})$ such that $X = SAC - (I - CA)S$. Moreover, we have the following result.

**Theorem 9.** *Let $A, C \in \mathcal{B}(\mathcal{H})$. If there exist orthogonal projections $P$ and $Q$ such that $C \leq_{\mathcal{H}} QA^*P$ and $PAQ \in \mathcal{B}(\mathcal{H})^-$. Then $X = (PAQ)^+$ is one solution of Eq. (3.1).*

**Proof.** let $T = (PAQ)^+$. We have $\mathcal{R}(T) \subset \mathcal{R}(Q)$ and $\mathcal{R}(T^*) \subset \mathcal{R}(P)$. Hence $QT = T$ and $TP = T$. From $C \leq_{\mathcal{H}} QA^*P$ we obtain

$$\mathcal{N}(P) \subset \mathcal{N}(QA^*P) \subset \mathcal{N}(C), \qquad \mathcal{R}(C) \subset \mathcal{R}(QA^*P) \subset \mathcal{R}(Q).$$

By Lemma 2, we get

$$CP = C, \qquad QC = C, \qquad CPAQ(PAQ)^+ = C, \qquad (PAQ)^+PAQC = C.$$

Hence,

$$CAT = CAQ(PAQ)^+ = CPAQ(PAQ)^+ = C$$

and

$$TAC = (PAQ)^+PAC = (PAQ)^+PAQC = C,$$

i.e., $T$ is one solution of Eq. (3.1). □

**Theorem 10.** *Let $A, C \in \mathcal{B}(\mathcal{H})$, $AC$ and $CA \in \mathcal{B}(\mathcal{H})^-$. The following conditions are equivalent:*

(i) *Eq. (3.1) is solvable;*
(ii) *$CA(CA)^-C(AC)^-AC = C$;*
(iii) *$\mathcal{R}(C) = \mathcal{R}(CA)$ and $\mathcal{R}(C^*) = \mathcal{R}(C^*A^*)$.*

**Proof.** (i) $\Longrightarrow$ (2) If the equation $CAX = C = XAC$ is solvable then, by Lemma 3, we know $\mathcal{R}(C) \subset \mathcal{R}(CA) \subset \mathcal{R}(C)$ and $\mathcal{N}(C) \supset \mathcal{N}(AC) \supset \mathcal{N}(C)$, which implies $\mathcal{R}(C) = \mathcal{R}(CA)$ and $\mathcal{N}(C) = \mathcal{N}(AC)$. By Lemma 2, it is straightforward that $CA(CA)^-C(AC)^-AC = C$ for every inner inverse, $(CA)^-$, $(AC)^-$, of $CA$ and $AC$, respectively.

(2) $\Longrightarrow$ (3) Since $\mathcal{R}(AC)$ and $\mathcal{R}(CA)$ are closed, $R(C^*A^*)$ and $\mathcal{R}(C^*)$ are closed by (2.2). $\mathcal{N}(C) = \mathcal{N}(AC)$ if and only if $\mathcal{R}(C^*) = \mathcal{R}(C^*A^*)$. It is trivial that (iii) holds.

(3) $\Longrightarrow$ (1) By Lemma 1, $\mathcal{R}(C) = \mathcal{R}(CA)$ implies $CAX = C$ is solvable with one solution

$$X_0 = (CA)^+C + (I - (CA)^+CA)C(AC)^+. \tag{3.2}$$

$\mathcal{R}(C^*) = \mathcal{R}(C^*A^*)$ implies that $\mathcal{N}(C) = \mathcal{N}(AC)$ and, hence

$$\begin{aligned} X_0AC &= (CA)^+CAC + (I - (CA)^+CA)C(AC)^+AC \\ &= (CA)^+CAC + C(AC)^+AC - (CA)^+CAC \\ &= C. \end{aligned}$$

It follows that Eq. (3.1) is solvable and $X_0 = (CA)^+C + (I - (CA)^+CA)C(AC)^+$ is one solution. □

**Remark.** In Theorem 10, if $AC$ and $CA \notin \mathcal{B}(\mathcal{H})^-$, then (i) $\nRightarrow$ (iii) but (iii) $\Rightarrow$ (i) holds. Even though $(C^*A^*)^+ \notin \mathcal{B}(\mathcal{H})$, we get $(C^*A^*)^+C^*A^* = P_{\overline{\mathcal{R}(AC)}}$ and $(C^*A^*)^+C^* = (C^*A^*)^+C^*A^*X^0 = P_{\overline{\mathcal{R}(AC)}}X^0 \in \mathcal{B}(\mathcal{H})$, i.e., $C(AC)^+ \in \mathcal{B}(\mathcal{H})$. Similarly, we have $(CA)^+C \in \mathcal{B}(\mathcal{H})$. $X_0$ in (3.2) is bounded and it is straightforward that $CAX_0 = C = X_0AC$. In this paper, we only consider the case in which $AC$ and $CA$ have closed range.

**Theorem 11.** *Let $A, C \in \mathcal{B}(\mathcal{H})$, $AC$ and $CA \in \mathcal{B}(\mathcal{H})^-$ such that Eq. (3.1) is solvable. Then the general solution is*

$$X = (CA)^- C + [I - (CA)^- CA]C(AC)^- + [I - (CA)^- CA]S[I - AC(AC)^-],$$

*where $S \in \mathcal{B}(\mathcal{H})$ is arbitrary.*

**Proof.** By Lemma 1, the equation $CAX = C$ has the general solution $X = (CA)^- C + [I - (CA)^- CA]U$, where $U \in \mathcal{B}(\mathcal{H})$ is arbitrary. If $X$ is also the general solution of the equation $C = XAC$, then

$$(CA)^- CAC + [I - (CA)^- CA]UAC = C.$$

We get $[I - (CA)^- CA][C - UAC] = 0$. Hence, $C - UAC = (CA)^- CAW$, where $W \in \mathcal{B}(\mathcal{H})$ is arbitrary. Now, we get $UAC = C - (CA)^- CAW$. Taking $*$-operation, we get

$$C^*A^*U^* = C^* - W^*A^*C^*(A^*C^*)^-.$$

Since $C^*A^*(C^*A^*)^-$ is idempotent and $\mathcal{R}(C^*A^*(C^*A^*)^-) = \mathcal{R}(C^*A^*) = \mathcal{R}(C^*)$, by Lemma 1, $U^*$ exists if and only if

$$C^*A^*(C^*A^*)^-[C^* - W^*A^*C^*(A^*C^*)^-] = C^* - W^*A^*C^*(A^*C^*)^-.$$

We get $C^*A^*(C^*A^*)^- W^*A^*C^*(A^*C^*)^- = W^*A^*C^*(A^*C^*)^-$ and

$$U^* = (C^*A^*)^-[C^* - C^*A^*(C^*A^*)^- W^*A^*C^*(A^*C^*)^-] + [I - (C^*A^*)^- C^*A^*]S^*,$$

where $S \in \mathcal{B}(\mathcal{H})$ is arbitrary. Hence, $U = C(AC)^- - (CA)^- CAW(AC)^- AC(AC)^- + S[I - AC(AC)^-]$ and

$$\begin{aligned} X &= (CA)^- C + [I - (CA)^- CA]U \\ &= (CA)^- C + [I - (CA)^- CA]C(AC)^- + [I - (CA)^- CA]S[I - AC(AC)^-]. \quad \square \end{aligned}$$

Since $\mathcal{R}(CA)$ is closed, $CA$ and $(CA)^+$ as operators from $\mathcal{H} = \mathcal{R}((CA)^*) \oplus \mathcal{N}(CA)$ into $\mathcal{H} = \mathcal{R}(CA) \oplus \mathcal{N}((CA)^*)$ can be written as $CA = T_1 \oplus 0$ and $(CA)^+ = T_1^{-1} \oplus 0$, respectively. By the definition of (I)-inverses,

$$(CA)^- = \begin{pmatrix} T_1^{-1} & W_3 \\ W_4 & W_2 \end{pmatrix} = (CA)^+ + W - (CA)^+ CAWCA(CA)^+,$$

where $W = \begin{pmatrix} W_1 & W_3 \\ W_4 & W_2 \end{pmatrix}$ is arbitrary.

If $AC$ and $CA$ are group invertible, by Lemma 4, $C(AC)^\# = (CA)^\# C$ and $A(CA)^\# = (AC)^\# A$. Theorem 11 implies that

$$\begin{aligned} X &= (CA)^\# C + [I - (CA)^\# CA]C(AC)^\# + [I - (CA)^\# CA]S[I - AC(AC)^\#] \\ &= (CA)^\# C + [I - (CA)^\# CA]S[I - AC(AC)^\#] \end{aligned}$$

is one solution of $CAX = C = XAC$. Let $A, C \in \mathcal{B}(\mathcal{H})$ satisfy Eq. (3.1). If $C$ is group invertible and $AC = CA$, then we can deduce that $AC$ is group invertible.

**Theorem 12.** *Let $A, C \in \mathcal{B}(\mathcal{H})$.*

(i) *If $C$ is group invertible and $AC = CA$ in Eq. (3.1), then $AC$ is group invertible and any solution of Eq. (3.1) has the form*

$$X = C(AC)^\# + [I - (CA)^\# CA]S[I - AC(AC)^\#] \quad \textit{for arbitrary } S \in \mathcal{B}(\mathcal{H}).$$

(ii) *If $AC$ and $CA$ are group invertible, then the unique solution $X$ of Eq. (3.1) which satisfies $X \leq_{\mathcal{H}} C$ is $X = C(AC)^\#$.*

**Proof.** (i) Since $C$ is group invertible, $C$ can be written as $C = C_1 \oplus 0$, where $C_1 \in \mathcal{B}(\mathcal{R}(C))$ is invertible. We consider the partitions $A$ and $X$ conforming with $C$. Since $AC = CA$, $A$ can be written as $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Let $X = \begin{pmatrix} X_1 & X_3 \\ X_4 & X_2 \end{pmatrix}$. From $CAX = C = XAC$ we get

$$\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} X_1 & X_3 \\ X_4 & X_2 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_1 & X_3 \\ X_4 & X_2 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{pmatrix} C_1A_1X_1 & C_1A_1X_3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X_1A_1C_1 & 0 \\ X_4A_1C_1 & 0 \end{pmatrix}.$$

Comparing two sides of the above equations and, using the invertibility of $C_1$, we have $A_1X_1 = I = X_1A_1$, $X_4A_1 = 0$ and $A_1X_3 = 0$. These imply that $A_1$ is invertible and $X_1 = A_1^{-1}$, $X_3 = 0$, $X_4 = 0$ and $X_2$ is arbitrary. Hence, $(AC)^\# = (CA)^\# = A_1^{-1}C_1^{-1} \oplus 0$ and

$$X = X_1 \oplus X_2 = A_1^{-1} \oplus X_2 = C(AC)^\# + [I - (CA)^\# CA]S[I - AC(AC)^\#]$$

for arbitrary $S \in \mathcal{B}(\mathcal{H})$.

(ii) Let $X = C(AC)^{\#}$. If $AC$ and $CA$ are group invertible, then $\mathcal{R}(X) \subset \mathcal{R}(C)$ and $\mathcal{N}(X) = \mathcal{N}(C(AC)^{\#}) = \mathcal{N}((CA)^{\#}C) \supset \mathcal{N}(C)$. So, $X \leq_{\mathcal{H}} C$. Moreover, we have $CAX = CAC(AC)^{\#} = C(AC)^{\#}AC = CA(CA)^{\#}C = C$ by Lemmas 2 and 4. Hence $XAC = C(AC)^{\#}AC = CA(CA)^{\#}C = C = CAX$. By Theorem 8 we know $X = C(AC)^{\#}$ is the unique solution. $\square$

**Example.** We will show that, for some special $C$ such that $X \leq_{\mathcal{H}} C$, the unique $X \in \mathcal{B}(\mathcal{H})$ reduces as the classical generalized inverses: group inverse, Drazin inverse and MP-inverse. This can be seen from the following table.

| $C$ | $X \leq_{\mathcal{H}} C$ $(\mathcal{R}(X) \subset \mathcal{R}(C)),\ \mathcal{N}(X) \supset \mathcal{N}(C)$ | The unique solution X of $CAX = C = XAC$ |
|---|---|---|
| $C = I$ | $\mathcal{R}(X) = \mathcal{H},\ \mathcal{N}(X) = \{0\}$ | $X = A^{-1}$ |
| $C = A$ ($A$ group invertible) | $\mathcal{R}(X) = \mathcal{R}(A),\ \mathcal{N}(X) = \mathcal{N}(A)$ | $X = A^{\#}$ |
| $C = A^m,\ \mathrm{ind}(A) = m$ ($A$ Drazin invertible) | $\mathcal{R}(X) = \mathcal{R}(A^m),\ \mathcal{N}(X) = \mathcal{N}(A^m)$ | $X = A^D$ |
| $C = A^*$ ($A$ MP-invertible) | $\mathcal{R}(X) = \mathcal{R}(A^*),\ \mathcal{N}(X) = \mathcal{N}(A^*)$ | $X = A^+$ |

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