J. Math. Anal. Appl. 403 (2013) 155–166

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Homoclinic orbits in degenerate reversible-equivariant systems in $\mathbb{R}^6$

Maurício Firmino Silva Lima [a,*], Marco Antonio Teixeira [b]

[a] *Centro de Matematica Computação e Cognição. Universidade Federal do ABC, 09210-170. Santo Andre. S.P., Brazil*

[b] *Departamento de Matemática, Universidade Estadual de Campinas, Caixa Postal 6065, 13083-970, Campinas, S.P., Brazil*

ARTICLE INFO

Article history:
Received 1 July 2010
Available online 24 February 2013
Submitted by D. O'Regan

Keywords:
Equilibrium point
Periodic orbit
Homoclinic orbit
Reversibility
Normal form
Resonance

ABSTRACT

We study the dynamics near an equilibrium point p_0 of a $\mathbb{Z}_2 x \mathbb{Z}_2$-reversible vector field in $\mathbb{R}^6$ with the reversing symmetry or symmetry φ satisfying $\varphi^2 = I$ and $dimFix(\varphi) = 3$. We deal with systems such that X presents at p_0 a degenerate resonance of type $0 : p : q$ or 0-*non-resonant*. We are assuming that the linearized system of X (at p_0) has as eigenvalues: $\lambda_1 = 0$ $\lambda_j = \pm i\alpha_j, j = 2, 3$. Our main concern is to find conditions for the existence of families of homoclinic orbits associated to periodic orbits near the equilibrium.

1. Introduction

It is known that the dynamics of a vector field around an equilibrium point of elliptical type can be somewhat complex in higher dimensions. In addition there are many open questions regarding the existence and robustness of homoclinic orbits around a saddle-center equilibrium point. Here we consider a reversible system X near an equilibrium $p_0 \in \mathbb{R}^6$ that presents a degenerate resonance of type $0 : p : q$ or 0-*non-resonant*. For a class of quadratic reversible vector fields, to be described later, we show that there exists a two-dimensional invariant manifold, filled with periodic orbits surrounding the equilibrium. Moreover, at each periodic orbit there are two homoclinic orbits. Our main aim is to study the persistence of such one-parameter families of homoclinic orbits when the quadratic model is perturbed. In a comparison paper [6] the persistence of such families in the 4- and 6-dimensional cases for analytic perturbations of a saddle-center equilibrium were studied. In such an approach a family of analytic reversible vector fields is considered, and the persistence of symmetric homoclinic orbits associated to periodic orbits of exponentially small amplitude is stated. In [8] families of reversible-equivariant vector fields in dimension 4 are also studied and, in this context, the persistence of homoclinic orbits also associated to a saddle-center equilibrium is considered.

In this paper we analyze the case of reversible and reversible-equivariant systems. References on reversibility and connections with other problems can be found in [4].

In Section 2, basic concepts and the statement of the main result are presented. In Section 3, homoclinic orbits in 4*D*-systems are discussed. We apply the analysis and results of this section to prove the main theorem of the paper in Section 4.

2. Preliminaries and statement of the main result

In this section we recall some general concepts, establish the terminology and state the main result of the paper.

* Corresponding author.
E-mail addresses: mauricio.lima@ufabc.edu.br, limamauric@gmail.com (M.F.S. Lima), teixeira@ime.unicamp.br (M.A. Teixeira).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2013.02.024

Let X be a vector field in $\mathbb{R}^6$ with $X(0) = 0$.

We say that X is reversible (resp. equivariant) if there is an involutive diffeomorphism $\varphi : \mathbb{R}^6, 0 \to \mathbb{R}^6, 0$ satisfying $D\varphi(x)X(x) = -X(\varphi(x))$, $(resp.\, D\varphi(x)X(x) = X(\varphi(x)))$. An orbit solution γ of X is called φ-symmetric if $\varphi(\gamma) = \gamma$.

Definition 2.1. Let X be a vector field in $\mathbb{R}^6$ and $\varphi_1, \varphi_2 : \mathbb{R}^6, 0 \to \mathbb{R}^6, 0$ be involutive diffeomorphisms. We say that X is (φ_1, φ_2)-reversible-equivariant if X is φ_i-reversible with $i = 1, 2$.

Let $G = [\varphi_1, \varphi_2]$ be the group generated by the involutions (φ_1, φ_2).

We assume:

1- X is (φ_1, φ_2)-reversible-equivariant.
2- $\dim(Fix(\varphi_i)) = 3,\ i = 1, 2$.
3- G is a finite group isomorphic to $\mathbb{Z}_2 x \mathbb{Z}_2$.
4- $spec(DX(0)) = \{0,\ \pm i\alpha_1,\ \pm i\alpha_2\}$. Moreover, some resonant conditions on $\{\alpha_1,\ \alpha_2\}$ will be further assumed.

We start by fixing, throughout the paper, the linear part of the vector field X.

$$A = DX(0) = \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & & & & \\ & & 0 & -\alpha_1 & & \\ & & \alpha_1 & 0 & & \\ & & & & 0 & -\alpha_2 \\ & & & & \alpha_2 & 0 \end{pmatrix}.$$

Definition 2.2. We say that the set of eigenvalues $\{\pm i\alpha_j,\ \ j = 2, 3\}$ of A satisfy the *non-resonance condition* if they are rationally independent. That is:

$$\sum_{j=2}^{3} k_j \alpha_j = 0, \quad k_j \in \mathbb{Z} \Rightarrow k_j = 0,\ j = 2, 3.$$

Definition 2.3. The vector field X, with $X(0) = 0$ is 0-*non-resonant* if the set $\{\pm\alpha_j,\ \ j = 2, 3\}$ satisfies the non-resonance condition.

Definition 2.4. We say that X is $0 : p : q$-*resonant* at 0 if $\pm i\alpha_1$ and $\pm i\alpha_2$ are in $p : q$-*resonance*. That means that $q\alpha_1 - p\alpha_2 = 0$, with $p, q \in \mathbb{Z}_+$.

Definition 2.5. Let $X^{(k)}$ be the k-jet of X at $x = 0$. We say that X is in the Belitskii Normal Form (**BNF**) up to order k if $X^{(k)}$ satisfies $A^*X^{(k)} = DX^{(k)}A^*x$ where $A = DX(0)$ and A^* is the formal adjoint of A.

Since G is a compact group we may apply the Montgomery–Bochner Theorem [1,2] and take both mappings φ_1 and φ_2 as linear. Moreover, we can fix $\varphi_1 = R_1$, with $R_1(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, -x_2, x_3, -x_4, x_5, -x_6)$ and let φ_2 vary in different ways leading the system to have different scenarios. Denote $G_i = [R_i, R_1]$.

The next lemma classifies all the possible choices of $\varphi_i = R_i$ with $i \in \{1, 2, \ldots, 8\}$. Its proof is straightforward and it will be omitted. Note that we have included, for sake of completeness, the case $G_1 = [R_1, R_1]$ which is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Lemma 2.6. *Up to a (A, R_1)-equivariant change of coordinates there are eight possible choices for R_i for which X is $[R_i, R_1]$-reversible. These involutions are given by the following expressions:*

$$\begin{aligned} &R_1(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, -x_2, x_3, -x_4, x_5, -x_6), \\ &R_2(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, -x_2, x_3, -x_4, -x_5, x_6), \\ &R_3(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, -x_2, -x_3, x_4, x_5, -x_6), \\ &R_4(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1, -x_2, -x_3, x_4, -x_5, x_6) \\ &R_5(x_1, x_2, x_3, x_4, x_5, x_6) = (-x_1, x_2, x_3, -x_4, x_5, -x_6), \\ &R_6(x_1, x_2, x_3, x_4, x_5, x_6) = (-x_1, x_2, -x_3, x_4, x_5, -x_6), \\ &R_7(x_1, x_2, x_3, x_4, x_5, x_6) = (-x_1, x_2, x_3, -x_4, -x_5, x_6), \\ &R_8(x_1, x_2, x_3, x_4, x_5, x_6) = (-x_1, x_2, -x_3, x_4, -x_5, x_6). \end{aligned}$$

Denote by $\chi_i^6,\ i \in \{1, \ldots, 8\}$ the space of all jets of G_i-reversible vector fields X at 0 such that $DX(0) = A$ endowed with the C^∞ topology. Moreover assume that the elements in $\chi_i^6,\ i \in \{1, \ldots, 8\}$ are at 0 either 0-non-resonant or $0 : p : q$-resonant with $p + q > 3$. As proved in [5], in both cases the 2-jet of the Belitskii normal forms **BNF** coincide and our results will be stated under generic conditions on the 2-jet of X. In addition, it is possible to find normal forms that preserve the reversible-equivariant structure (see [3]).

Remark 2.7. The 2-jet of $X \in \chi_1^6$, at 0 expressed by **BNF** is given by

$$X^{(2)}(x) = \begin{pmatrix} x_2 \\ a_2x_1^2 + b_1(x_3^2 + x_4^2) + b_2(x_5^2 + x_6^2) \\ -\alpha_1 x_4 - c_1 x_1 x_4 \\ \alpha_1 x_3 + c_1 x_1 x_3 \\ -\alpha_2 x_6 - c_2 x_1 x_6 \\ \alpha_2 x_5 + c_2 x_1 x_5 \end{pmatrix},$$

where $x = (x_1, x_2, x_3, x_4, x_5, x_6)$. We are assuming that $a_2b_1 < 0$, $a_2b_2 < 0$ and $c_1, c_2 \neq 0$.

Remark 2.8. (i) Saying that X is $G_1 = [R_1, R_1]$-reversible is equivalent to saying that it is R_1-reversible.
(ii) If X is $G_i = [R_1, R_i]$-reversible then it is R_1R_i-equivariant.
(iii) $\chi_i^6 \subset \chi_1^6$ for all $i \in \{1, \dots, 8\}$.

We observe that it is no longer true that each G_i-reversible vector field X with $i \in \{5, \dots, 8\}$, satisfies the generic conditions on $X^{(2)}$ as required in Theorem B of [5] concerning the persistence of the families of symmetric periodic orbits via Lyapunov–Schmidt reduction. This result is stated provided that $X^{(2)}$ is written in **BNF** plus some generic conditions (see Remark 2.7). However, examples can be exhibited where families of homoclinic orbits are destroyed when additional high order terms are considered even when the periodic solutions are preserved (see Section 4).

Let $X \in \chi_1^6$. We fix coordinates such that X is in the **BNF** until order 2 and we will denote by $X^{(2)}$ the 2-jet of X written in normal form (see [5]).

The next result describes the dynamics of the polynomial vector field $X^{(2)}$.

Proposition A. *There exists an open set $\mathcal{U}$ in χ_1^6, such that any $X \in \mathcal{U}$, $X^{(2)}$ satisfies:*

(i) *$X^{(2)}$ is completely integrable.*
(ii) *There exist two two-parameter families of flow invariant 2-tori, T_μ^2 and S_μ^2, both terminating at the origin where $\mu = (\mu_1, \mu_2)$.*
(iii) *There is a two-parameter family of flow invariant topological 3-tori, T_μ^3 containing T_μ^2 and terminating at the origin.*
(iv) *There is a three-parameter family of flow invariant 3-tori, $T_{\mu,\nu}^3$, terminating at the origin when $\mu \to (0, 0)$, and for each μ^0, the family originates at $T_{\mu^0}^3$ and terminates at $S_{\mu^0}^2$, when ν goes to $\pm\infty$.*
(v) *There are four one-parameter families of periodic orbits $T_{\mu_1}^1, T_{\mu_2}^1, S_{\mu_1}^1$ and $S_{\mu_2}^1$ contained in T_μ^2 and S_μ^2 for $\mu = (\mu_1, 0)$ and $\mu = (0, \mu_2)$ (with bounded periods) $\gamma_{\mu_1}^i$ and $\delta_{\mu_2}^i$;* (b) *two one-parameter families of homoclinic orbits associated to $T_{\mu_1}^1$ and $T_{\mu_2}^1$ and terminating at origin.*

The main result of this paper is the following:

Theorem A. *Let $X \in \chi_i^6$. Then:*

(a) *If $i = 2, 3$ then there is a non-empty open subset $\mathcal{V}$ in χ_i^6 such that any $X \in \mathcal{V}$ admits two one-parameter families of symmetric homoclinic orbits associated to symmetric periodic orbits around the equilibrium.*
(b) *If $i = 4$ then the homoclinic orbits given in Proposition A*(v-(b)) *are not persistent when high order terms are considered.*
(c) *If $i \in \{5, \dots, 8\}$ then the 2-truncated normal form of the vector field admits families of periodic and homoclinic orbits. Moreover the families of homoclinic solutions can be destroyed when high order terms are considered.*

Remark 2.9. There are a number of ways in which our work differs from that of Lombardi in [6]. First, he deals just with analytic systems, while we are interested in the C^∞ case. More importantly, the topological types of the systems at the equilibrium differ from each other. In Lombardi's approach the stable (or unstable) manifold associated to the saddle-center plays an important role while in our setting the important point is the existence of a 4D-invariant manifold associated to the eigenvalues $\pm i\alpha_1$ and $\pm i\alpha_2$ and the stable/unstable manifolds associated to each symmetric periodic orbit.

Remark 2.10. We observe that as $\chi_i^6 \subset \chi_1^6$ for all $i \in \{1, \dots, 8\}$. So the non persistence of symmetric homoclinic orbits in the R_1-reversible case is immediate from Theorem A(b). The robustness result presented in part (a) of Theorem A follows from the existence of a 4D-invariant manifold fact that the family of two-dimensional stable manifolds (associated to periodic orbits) is transverse to the fixed points set of the associated involution. Due to the fact that a 4D-invariant manifold cannot exists for the parts (b) and (c) one deduces immediately that the intersection between the two-dimensional stable manifold and the set of fixed points of the symmetries can be destroyed by small perturbations.

The proof of Theorem A follows the following strategy: first we prove a version of Theorem A in a 4D-situation for a R-reversible vector field where R is an involution. Such a proof requires many technical steps. We consider the vector field written in the **BNF** until order 2 and we see that the truncated normal form satisfies Proposition A. So we use the Banach Fixed Point Theorem for contraction in a convenient Banach space to show that most of the homoclinic orbits are persistent when the original system is considered. The 6D-case is proved by reducing the system to a 4D-flow-invariant manifold.

3. Homoclinic solutions in 4D

Let $X \in \chi_1^4$ be the space of R-reversible vector field with $X(0) = 0$,

$$B = DX(0) = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}$$

and $R(x_1, x_2, x_3, x_4) = (x_1, -x_2, x_3, -x_4)$.

In [5] is proved that under a C^∞-conjugacy we can write the system in the following way where the 2-jet is given in the **BNF**.

$$X : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = a_2 x_1^2 + b_2(x_3^2 + x_4^2) + f_1(x_1, x_2, x_3, x_4) \\ \dot{x}_3 = -x_4 - c_1 x_1 x_4 + f_2(x_1, x_2, x_3, x_4) \\ \dot{x}_4 = x_3 + c_1 x_1 x_3 + f_3(x_1, x_2, x_3, x_4), \end{cases} \tag{1}$$

where $f_i(x_1, x_2, x_3, x_4) = \mathcal{O}(|x|^3)$ for $i = 1, 2, 3$ and $x = (x_1, x_2, x_3, x_4)$.

Assume on X the following extra assumption:

$$C_1 : \{a_2 > 0,\ b_2 < 0 \text{ and } c_1 \neq 0\}.$$

Such a condition allows us to claim that the truncated normal form of the system possesses R-symmetric periodic and homoclinic solutions.

Note that under the assumption C_1 and under the linear change of coordinates $x_1 = \frac{1}{a_2}X_1$, $x_2 = \frac{1}{a_2}X_2$, $x_3 = \sqrt{-\frac{1}{a_2 b_2}}X_3$ and $x_4 = \sqrt{-\frac{1}{a_2 b_2}}X_4$ we obtain a system as system (1) but with $a_2 = 1$ and $b_2 = -1$. So, without loss of generality we may assume that $a_2 = 1$ and $b_2 = -1$.

Lemma 3.1. *Changing coordinates to (x_1, x_2, Z) with $Z = x_3 + ix_4$ and rescaling by $x_1 = \varepsilon y_1$, $x_2 = \varepsilon^{3/2} y_2$, $Z = \varepsilon z$, and $\tau = \varepsilon^{1/2} t$ for $\varepsilon > 0$, system (1) is written as:*

$$Y_\varepsilon : \begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = y_1^2 - |z|^2 + \varsigma_1(y_1, y_2, z, \bar{z}, \varepsilon) \\ \dot{z} = iz\left(\dfrac{1}{\varepsilon^{1/2}} + c_1 \varepsilon^{1/2} y_1\right) + \varsigma_2(y_1, y_2, z, \bar{z}, \varepsilon), \end{cases}$$

where $\varsigma_1 = \mathcal{O}(\varepsilon\,(|y_1| + |y_2| + |z|)^3)$ and $\varsigma_2 = \mathcal{O}(\varepsilon^{3/2}\,(|y_1| + |y_2| + |z|)^3)$.

Moreover $\varsigma_1(Y', \varepsilon) - \varsigma_1(Y, \varepsilon) = \mathcal{O}\left(\varepsilon|Y' - Y|\right)$ and $\varsigma_2(Y', \varepsilon) - \varsigma_2(Y, \varepsilon) = \mathcal{O}\left(\varepsilon^{3/2}|Y' - Y|\right)$, where $Y(\tau) = (y_1(\tau), y_2(\tau), z(\tau), \bar{z}(\tau))$.

Proof. In the new coordinates and time τ it is easy to see that we can write

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = y_1^2 - |z|^2 + \dfrac{1}{\varepsilon^2} f_1\left(\varepsilon y_1, \varepsilon^{3/2} y_2, \dfrac{\varepsilon(z + \bar{z})}{2}, \dfrac{\varepsilon(z - \bar{z})}{2i}\right) \\ \dot{z} = iz\left(\dfrac{1}{\varepsilon^{1/2}} + c_1 \varepsilon^{1/2} y_1\right) + \dfrac{1}{\varepsilon^{3/2}} \begin{bmatrix} f_2\left(\varepsilon y_1, \varepsilon^{3/2} y_2, \dfrac{\varepsilon(z + \bar{z})}{2}, \dfrac{\varepsilon(z - \bar{z})}{2i}\right) \\ +if_3\left(\varepsilon y_1, \varepsilon^{3/2} y_2, \dfrac{\varepsilon(z + \bar{z})}{2}, \dfrac{\varepsilon(z - \bar{z})}{2i}\right) \end{bmatrix}. \end{cases}$$

So, from the fact that $h_i(x) = \mathcal{O}(|x|^3)$ the proof follows immediately. $\square$

Considering now the coordinates $y_1,\ y_2,\ z = re^{i\theta}$, we have:

$$Y_\varepsilon : \begin{cases} \dfrac{dY}{d\tau} = f_1(Y, \theta) = \begin{pmatrix} y_2 \\ y_1^2 - r^2 + \tilde{\varsigma}_1(y_1, y_2, r, \theta, \varepsilon) \\ \tilde{\varsigma}_r(y_1, y_2, r, \theta, \varepsilon) \end{pmatrix} \\ \dfrac{d\theta}{d\tau} = f_2(Y, \theta) = \dfrac{1}{\varepsilon^{1/2}} + \tilde{\varsigma}_\theta(y_1, y_2, r, \theta, \varepsilon). \end{cases} \tag{2}$$

Observe that $Y(\tau) = (y_1(\tau), y_2(\tau), r(\tau))$.

In these new coordinates the reversibility R is given by: $R' f_1(Y, \theta) = -f_1(R'Y, -\theta)$ with $R'(y_1, y_2, r) = (y_1, -y_2, r)$ and θ being an odd function.

We still have

$$\begin{aligned}
&\tilde{\varsigma}_1 = \mathcal{O}\left(\varepsilon|Y|\right),\\
&\tilde{\varsigma}_1(Y',\theta,\varepsilon) - \tilde{\varsigma}_1(Y,\theta,\varepsilon) = \mathcal{O}\left(\varepsilon|Y'-Y|\right),\\
&\tilde{\varsigma}_r = \mathcal{O}\left(\varepsilon^{3/2}|Y|\right),\\
&\tilde{\varsigma}_r(Y',\theta,\varepsilon) - \tilde{\varsigma}_r(Y,\theta,\varepsilon) = \mathcal{O}\left(\varepsilon^{3/2}|Y'-Y|\right),\\
&\tilde{\varsigma}_\theta = \mathcal{O}\left(\varepsilon^{1/2} + \frac{\varepsilon^{3/2}}{r}|Y|\right),\\
&\tilde{\varsigma}_\theta(Y',\theta,\varepsilon) - \tilde{\varsigma}_\theta(Y,\theta,\varepsilon) = \mathcal{O}\left(\varepsilon^{1/2} + \varepsilon^{3/2}\left[\frac{1}{r'} + \frac{|Y|}{rr'}\right]|Y'-Y|\right).
\end{aligned} \tag{3}$$

Now we state the main result of this section.

Theorem A^***.** *There exists an open set $\mathcal{U} \subset \chi_1(\mathbb{R}^4)$ such that associated to each $X \in \mathcal{U}$ there is a ε-rescaling in the variables $(y_1, y_2, r, \theta, \varepsilon)$, with $\varepsilon > 0$ such that:*

(i) *in these new coordinates the vector field is expressed by Y_ε;*
(ii) *one can find a small number $k_\varepsilon > 0$ such that if $k \geq k_\varepsilon$ then there are two symmetric homoclinic solutions associated to one symmetric periodic solution $\hat{Y}_k(t,\varepsilon)$.*

The proof of part (i) of Theorem A* follows from Lemma 3.1. The rest of this section will be used to prove part (ii).

Given (2), we want to associate to it another auxiliary system having θ as the time. For this purpose the following result is necessary:

Lemma 3.2. *Restricted to*

$$\mathcal{E}_\eta = \left\{(Y,\theta)/\sup_{\tau\in\mathbb{R}}|Y| \leq M_1,\ r \geq \frac{\varepsilon^\eta}{2}\right\} \quad \textit{with } 0 < \eta < \frac{1}{2},$$

the function θ is a local diffeomorphism around 0*. Moreover, if $(Y,\theta) \in \mathcal{E}_\eta$ then*

$$\tilde{\varsigma}_\theta = \mathcal{O}(\varepsilon^{1/2}), \qquad \tilde{\varsigma}_\theta(Y',\theta,\varepsilon) - \tilde{\varsigma}_\theta(Y,\theta,\varepsilon) = \mathcal{O}(\varepsilon^{1/2}|Y'-Y|). \tag{4}$$

Proof. As in $\mathcal{E}_\eta$ we have $r \geq \frac{\varepsilon^\eta}{2}$ and $|Y| \leq M_1$ and taking into account that $0 < \varepsilon < 1$ and $0 < \eta < \frac{1}{2}$ it follows that

$$\frac{\varepsilon^{3/2}}{r}|Y| \leq 2M_1\varepsilon^{3/2-\eta} < 2M_1\varepsilon^{1/2},$$

and

$$\varepsilon^{3/2}\left[\frac{1}{r'} + \frac{|Y|}{rr'}\right] \leq \varepsilon^{3/2}\left[\frac{2}{\varepsilon^\eta} + \frac{4M_1}{\varepsilon^{2\eta}}\right] < c\varepsilon^{1/2}.$$

So from the estimates (3) we obtain (4). Moreover from the Eq. (2) we have $\frac{d\theta}{d\tau} = \frac{1}{\varepsilon^{1/2}} + \mathcal{O}(\varepsilon^{1/2}) > 0$. This implies that θ is a local diffeomorphism around 0. □

Now we focus the analysis on f_1. We have:

$$\begin{cases}
\dfrac{dY}{d\theta} = \mathcal{N}_\varepsilon(Y,\varepsilon) + \bar{\mathcal{R}}(Y,\theta,\varepsilon)\\
\dfrac{d\theta}{d\tau} = \dfrac{1}{\varepsilon^{1/2}} + \mathcal{R}_\theta(Y,\theta,\varepsilon)
\end{cases}$$

where

$$\mathcal{N}_\varepsilon = \sqrt{\varepsilon}\,\mathcal{N}, \quad \mathcal{N} = \begin{pmatrix} y_2 \\ y_1^2 - r^2 \\ 0 \end{pmatrix}$$

is the truncated normal form up to second order.

We proceed to the analysis of:

$$\tilde{Y}_{\theta,\varepsilon} : \frac{dY}{d\theta} = \mathcal{N}_\varepsilon(Y) + \bar{\mathcal{R}}(Y,\theta,\varepsilon). \tag{5}$$

By means of a θ-re-parametrization we get

$$Y_{\theta,\varepsilon} : \frac{dY}{d\theta} = \mathcal{N}(Y) + \mathcal{R}(Y,\theta,\varepsilon), \tag{6}$$

with

$$\mathcal{R} = \begin{pmatrix} s_1(Y,\theta,\varepsilon) = \mathcal{O}\left(\varepsilon|Y|\right) \\ s_2(Y,\theta,\varepsilon) = \mathcal{O}\left(\varepsilon|Y|\right) \\ \mathcal{R}_r(Y,\theta,\varepsilon) = \mathcal{O}\left(\varepsilon^{3/2}|Y|\right) \end{pmatrix}. \tag{7}$$

In this way

$$\mathcal{R}(Y,\theta,\varepsilon) = \varepsilon\hat{\mathcal{R}}(Y,\theta,\varepsilon),$$

and the system (6) is written as:

$$\frac{dY}{d\theta} = \mathcal{N}(Y) + \varepsilon\hat{\mathcal{R}}(Y,\theta,\varepsilon). \tag{8}$$

Lemma 3.3. *The truncated system*

$$\frac{dY}{d\theta} = \mathcal{N}(Y). \tag{9}$$

has

$$H_1 = r^2, \quad \text{and} \quad H_2 = \frac{y_2^2}{2} - \int_0^{y_1} (s^2 - H_1)\, ds$$

as first integrals. Moreover, associated to each $r = k > 0$ there is a R'-symmetric homoclinic solution $\mathcal{H}_k(\theta)$ at the R'-symmetric equilibrium $Y_k = (k, 0, k)$, where

$$\mathcal{H}_k(\theta) = (y_1(\theta), y_2(\theta), k)$$

and

$$y_1(\theta) = k - \frac{3k}{\cosh^2\left(\frac{\sqrt{2k}}{2}\theta\right)}, \qquad y_2(\theta) = \frac{3k\sqrt{2k}\tanh\left(\frac{\sqrt{2k}}{2}\theta\right)}{\cosh^2\left(\frac{\sqrt{2k}}{2}\theta\right)}.$$

Proof. It is easy to see that $H_1 = r^2$ is a first integral of (9). Moreover, for each $r = k > 0$ the vector field

$$\begin{cases} \dot{y}_1 = y_2, \\ \dot{y}_2 = y_1^2 - k^2, \end{cases} \tag{10}$$

is Hamiltonian with Hamiltonian function H_2. Also we can see that system (10) is $\bar{R}$-reversible where $\bar{R}(y_1, y_2) = (y_1, -y_2)$. Moreover $(k, 0)$ is an equilibrium point of saddle type of (10) that admits a symmetric homoclinic orbit given by $H_2(y_1, y_2) = H_2(k, 0)$.

Considering the change $x_1 = y_1 - k$ and $x_2 = y_2$ we obtain from (10):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 2kx_1 + x_2^2 \end{cases} \tag{11}$$

with saddle point in $(0, 0)$ and $H_2(x_1, x_2) = \frac{x_2^2}{2} - kx_1^2 - \frac{x_1^3}{3}$.

In order to obtain a parametric expression of the homoclinic orbit given by

$$x_2^2 = 2x_1^2\left(k + \frac{x_1}{3}\right) \tag{12}$$

we take $x_1(\theta) = \frac{-3k}{\cosh^2(at)}$. Note that $|x_1(\theta)| \to 0$ when $|\theta| \to \infty$. From (12) we obtain $x_2(\theta) = \frac{3k\sqrt{2k}\tanh(a\theta)}{\cosh^2(a\theta)}$. Observe that $(x_1(\theta), x_2(\theta))$ satisfies system (11) provided that $a = \frac{\sqrt{2k}}{2}$. Returning to the (y_1, y_2) variable we obtain the desired result. $\square$

In comparison with the truncated original system we note that the equilibrium Y_k corresponds to a symmetric periodic orbit and $\mathcal{H}_k(\theta)$ corresponds to a symmetric homoclinic orbit.

It is worthwhile to note that the truncated normal form in $\mathbb{R}^4$ is invariant with respect to $S_\alpha(y_1, y_2, z) = (y_1, y_2, ze^{i\alpha})$. This implies in the existence of another symmetric homoclinic solution $S_\pi\mathcal{H}_k(t)$ at $Y_k(t)$. Note that S_π performs a rotation of angle π in the (x_3, x_4)-plane. So if $\mathcal{H}_k(t)$ intersects $Fix(R)$ at the point $(x_1^0, 0, x_3^0, 0)$ with $x_3^0 \neq 0$ then $S_\pi\mathcal{H}_k(t)$ intersects $Fix(R)$ at the point $(x_1^0, 0, x_3^0, 0) \neq (x_1^0, 0, -x_3^0, 0)$. This implies that $S_\pi\mathcal{H}_k(t) \neq \mathcal{H}_k(t)$. So associated to each periodic orbit there are two symmetric homoclinic orbits.

As discussed in [5], such periodic solutions are persistent when the original system is considered. Our concern now is to decide how persistent are the homoclinic orbits detected above when the system is perturbed.

Let us concentrate our attention on the system $\frac{dY}{d\theta}$. Let $Y(\theta,\varepsilon)$ be a solution of (8) written in the following form:

$$Y(\theta,\varepsilon) = \mathcal{H}_k(\theta) - Y_k + U(\theta,\varepsilon) + \hat{Y}_k(\theta,\varepsilon) \tag{13}$$

where $\hat{Y}_k(\theta,\varepsilon)$ is the continuation of Y_k.

Let us assume that the domain of variation of U is concentrated in the set E_γ of all $U = (y_1, y_2, r)$ that satisfy: (i) $U \in C^o(\mathbb{R})$; (ii) U is R'-*reversible* and (iii) $\|U\|_\gamma < \infty$ where

$$\|U\|_\gamma = \sup_{\theta\in\mathbb{R}} \left(|U(\theta,\varepsilon)|e^{\gamma|\theta|}\right) \quad \text{and} \quad \gamma = \sqrt{2k}\,\delta, \quad 0 < \delta < 1.$$

Note that E_γ is a Banach space endowed with the norm $\|.\|_\gamma$.

Remark 3.4. It follows from the fact that $\|U\|_\gamma < \infty$ that solution (13) belongs to the perturbed stable manifold of the symmetric periodic orbit $\hat{Y}_k(\theta,\varepsilon)$. Moreover, this perturbed manifold is $\mathcal{O}(\varepsilon)$-close to the unperturbed one.

Now we state a fundamental result. It will be proved at the end of this section.

Lemma 3.5. *Given a small number $\varepsilon > 0$ there exists $k_\varepsilon > 0$ such that $\hat{Y}_k(\theta,\varepsilon)$ admits two symmetric homoclinic orbits provided that $k \geq k_\varepsilon > 0$.*

Note that the proof of part (ii) of Theorem A* is a direct consequence of Lemma 3.5.
We discuss now the solution (13).

Lemma 3.6. *If U is taken in the convex subset of E_γ given by*

$$E_{\gamma,d} = \left\{U \in E_\gamma /\ \|U\|_\gamma \leq d,\ |\Pi_r(U)| \leq \frac{\varepsilon^\eta}{2}\right\} \tag{14}$$

with $\Pi_r(.)$ being the canonical projection to r-axis then the solution (13) is in $\mathcal{E}_\eta$.

Proof. In fact, to guarantee that Y is in $\mathcal{E}_\eta$ it is sufficient to show that $|\Pi_r(Y)| \geq \frac{\varepsilon^\eta}{2}$.
We have that

$$|\Pi_r(Y)| = |\Pi_r(\mathcal{H}_k(\theta) - Y_k + U(\theta,\varepsilon) + \hat{Y}(\theta,\varepsilon))| \geq \varepsilon^\eta - \frac{\varepsilon^\eta}{2} = \frac{\varepsilon^\eta}{2}. \quad \square$$

Lemma 3.7. *The function $Y(\theta,\varepsilon)$ given by (13) is a solution of (8), provided that $U(\theta,\varepsilon)$ is a solution of*

$$\frac{dU}{d\theta} - D\mathcal{N}\left(\mathcal{H}_k(\theta,\varepsilon)\right)U = \hat{N}\left(U(\theta,\varepsilon),\theta,\varepsilon\right) + \varepsilon\hat{R}\left(U(\theta,\varepsilon),\theta,\varepsilon\right) \tag{15}$$

where

$$\hat{N}(U(\theta,\varepsilon),\theta,\varepsilon) = \mathcal{N}\left(\mathcal{H}_k(\theta) - Y_k + U(\theta,\varepsilon) + \hat{Y}_k(\theta,\varepsilon)\right) - \mathcal{N}\left(\mathcal{H}_k(\theta)\right) - \mathcal{N}\left(\hat{Y}_k(\theta,\varepsilon)\right) - D\mathcal{N}\left(H_k(\theta)\right)U,$$
$$\hat{R}(U(\theta),\theta,\varepsilon) = \hat{\mathcal{R}}\left(\mathcal{H}_k(\theta) - Y_k + U(\theta,\varepsilon) + \hat{Y}_k(\theta,\varepsilon),\theta,\varepsilon\right) - \hat{\mathcal{R}}\left(\hat{Y}_k(\theta,\varepsilon),\theta,\varepsilon\right).$$

Moreover, under these assumptions there exists $M > 0$ such that:

$$\begin{aligned}
&\|\hat{N}(U(\theta),\theta,\varepsilon)\| \leq M\left(\varepsilon e^{-\sqrt{2k}\theta} + \varepsilon\,|U(\theta,\varepsilon)| + |U(\theta,\varepsilon)|^2\right) && \text{(I)}\\
&\|\hat{R}(U(\theta,\varepsilon),\theta,\varepsilon)\| \leq M\left(e^{-\sqrt{2k}\theta} + |U(\theta,\varepsilon)|\right) && \text{(II)}\\
&\|\hat{N}(U',\theta,\varepsilon) - \hat{N}(U,\theta,\varepsilon)\| \leq M\left(\varepsilon + |U(\theta,\varepsilon)| + |U'(\theta,\varepsilon)|\right)|(U'-U)(\theta,\varepsilon)| && \text{(III)}\\
&\|\hat{R}(U',\theta,\varepsilon) - \hat{R}(U,\theta,\varepsilon)\| \leq M|(U-U')(\theta,\varepsilon)|. && \text{(IV)}
\end{aligned} \tag{16}$$

Proof. The proof of this lemma is straightforward, since such inequalities are reached directly from the expansion of the mappings in Taylor series. $\square$

The following example illustrates the inequalities (I)–(IV) of Lemma 3.7.

Example 3.8. Consider the perturbed vector field given by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1^2 - r^2 + \varepsilon x_1^4 \\ \dot{r} = 0. \end{cases}$$

For this system we have: $Y_k = (k, 0, k)$, $\hat{Y}_k(\theta, \varepsilon) = \left(\sqrt{\frac{-1+\sqrt{1+4k^2\varepsilon}}{2\varepsilon}}, 0, k\right)$ and $U(\theta, \varepsilon) = (u_1, u_2, 0)$. So $\hat{N}(U(\theta), \theta, \varepsilon) = \left(0, u_1^2 + \varepsilon k^3 \left[-u_1 + 3k \operatorname{sech}\left(\frac{\sqrt{2k}}{2}\theta\right)\right], k\right)$. And we have part (I) of inequality (16). The other inequalities can be obtained in a similar way. $\square$

Note that inequalities (I) and (II) imply that the sets $\hat{N}(E_\gamma)$ and $\hat{R}(E_\gamma)$ are in E_γ since $\gamma < \sqrt{2k}$. Our objective now is to give extra assumptions in such a way that on the set $E_{\gamma,d}$ the last condition is also satisfied.

Now it remains to show that the system (15) admits a solution in $E_{\gamma,d}$.

Proposition 3.9. *The system (15) admits a solution in $E_{\gamma,d}$.*

Proof. First of all consider the linearization of (15) given by:

$$\frac{dU}{d\theta} = D\mathcal{N}\left(\mathcal{H}_k(\theta)\right) U. \tag{17}$$

It follows that

$$D\mathcal{N}(\mathcal{H}_k(\theta)) = \begin{pmatrix} 0 & 1 & 0 \\ 2y_1(\theta) & 0 & -2k \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $p(\theta) = \frac{\partial \mathcal{H}_k}{\partial \theta}(\theta) = \left(y_2(\theta), \frac{\partial y_2}{\partial \theta}(\theta), 0\right)$ is always a solution of (17).

Observe that

$$R'p(\theta) = -p(-\theta) \quad \text{and} \quad p(\theta) = \mathcal{O}\left(e^{-\sqrt{2k}\theta}\right) \quad \text{when } |\theta| \to \infty.$$

Moreover

$$r(\theta) = \frac{\partial \mathcal{H}_k}{\partial k}(\theta) = \left(\frac{\partial y_1}{\partial k}(\theta), \frac{\partial y_2}{\partial k}(\theta), 1\right)$$

is another solution of (17) in such a way that $R'r(\theta) = r(-\theta)$ and $r(\theta)$ is bounded.

From the R'-reversibility of (17) it follows that associated to the solution $p(\theta)$, there exists another solution $q(\theta)$ satisfying

$$R'q(\theta) = q(-\theta) \quad \text{and} \quad q(\theta) = \mathcal{O}\left(e^{\sqrt{2k}\theta}\right) \quad \text{for } |\theta| \to \infty.$$

Now it is straightforward to check that the set

$$\{p(\theta),\ r(\theta),\ q(\theta)\}$$

is a basis of solution of (17).

Thus

$$\Phi(\theta) = \left(p(\theta) \quad \vdots \quad r(\theta) \quad \vdots \quad q(\theta)\right)$$

is a fundamental matrix of (17).

Considering $T(\theta, \phi) = \Phi(\theta)\Phi^{-1}(\phi)$, the solution of (15) is given by

$$U(\theta, \varepsilon) = T(\theta, 0)U(0) + \int_0^\theta T(\theta, \phi)\left[\hat{N}(U(\phi, \varepsilon), \phi, \varepsilon) + \varepsilon\hat{R}(U(\phi, \varepsilon), \phi, \varepsilon)\right] d\phi. \tag{18}$$

Let $P(\theta)$ be the projection to the $p(\theta)$-axis along the direction $(r(\theta), q(\theta))$. Denote $Q(\theta) = Id - P(\theta)$.

Decomposing Eq. (18) with respect to $\theta > 0$ we get:

$$\begin{aligned}
&\int_0^\theta T(\theta, \phi)\left[\hat{N}(U(\phi, \varepsilon), \phi, \varepsilon) + \varepsilon\hat{R}(U(\phi, \varepsilon), \phi, \varepsilon)\right] d\phi \\
&\quad = \int_0^\theta T(\theta, \phi)P(\phi)\left[\hat{N}(U(\phi, \varepsilon), \phi, \varepsilon) + \varepsilon\hat{R}(U(\phi, \varepsilon), \phi, \varepsilon)\right] d\phi \\
&\quad\quad + \int_0^\theta T(\theta, \phi)Q(\phi)\left[\hat{N}(U(\phi, \varepsilon), \phi, \varepsilon) + \varepsilon\hat{R}(U(\phi, \varepsilon), \phi, \varepsilon)\right] d\phi \\
&\quad = \int_0^\theta T(\theta, \phi)P(\phi)\left[\hat{N}(U(\phi, \varepsilon), \phi, \varepsilon) + \varepsilon\hat{R}(U(\phi, \varepsilon), \phi, \varepsilon)\right] d\phi
\end{aligned}$$

$$+ \int_{\infty}^{\theta} T(\theta,\phi)Q(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi$$
$$+ \int_{0}^{\infty} T(\theta,\phi)Q(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi.$$

The solution (18) can be written as:

$$\begin{aligned} U(\theta) = {} & T(\theta,0)\left(U(0)+\int_0^{\infty} T(0,\phi)Q(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi\right)\\ & + \int_0^{\theta} T(\theta,\phi)P(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi\\ & + \int_{\infty}^{\theta} T(\theta,\phi)Q(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi. \end{aligned}$$

Now it follows from Theorem 4.6 of [7] that such a solution is bounded in $\mathbb{R}_+$, if and only if,

$$Q(0)\left(U(0)+\int_0^{\infty} T(0,\phi)Q(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi\right)=0.$$

Observing that $Q(0)T(0,\phi)=T(0,\phi)Q(\phi)$, $Q=Id-P$ and $Q^2=Q$ we get

$$\begin{aligned} 0 &= Q(0)U(0)+\int_0^{\infty} Q(0)T(0,\phi)Q(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi\\ &= U(0)-P(0)U(0)+\int_0^{\infty} T(0,\phi)Q(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi. \end{aligned}$$

Hence:

$$U(0)+\int_0^{\infty} T(0,\phi)Q(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi = P(0)U(0).$$

This implies that the solution (18) can be written as:

$$\begin{aligned} U(\theta) = {} & T(\theta,0)P(0)U(0)+\int_0^{\theta} T(\theta,\phi)P(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi\\ & - \int_{\theta}^{\infty} T(\theta,\phi)Q(\phi)\left[\hat{N}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\hat{R}(U(\phi,\varepsilon),\phi,\varepsilon)\right]d\phi. \end{aligned} \tag{19}$$

Consider now $\{p^*(\theta),\ r^*(\theta),\ q^*(\theta)\}$ the adjoint basis associated to $\{p(\theta),\ r(\theta),\ q(\theta)\}$, which is a basis of solutions of

$$\frac{dU}{d\theta} = -D\mathcal{N}\left(\mathcal{H}_k(\theta)\right)^T U. \tag{20}$$

The mapping

$$\Psi(\theta) = \left(p^*(\theta) \ \ \vdots \ \ r^*(\theta) \ \ \vdots \ \ q^*(\theta)\right)$$

is a fundamental matrix of (20) where: $p^*(\theta) = \mathcal{O}\left(e^{\sqrt{2k}\theta}\right)$ when $|\theta| \to \infty$, $R'p^*(\theta) = -p^*(-\theta)$, $r^*(\theta)$ is bounded, $R'r^*(\theta) = r^*(-\theta)$, $q^*(\theta) = \mathcal{O}\left(e^{-\sqrt{2k}\theta}\right)$ when $|\theta| \to \infty$, $R'q^*(\theta) = q^*(-\theta)$.

It follows that $\Psi^T(\phi) = \Phi^{-1}(\phi)$. So $T(\theta,\phi) = \Phi(\theta)\Psi^T(\phi)$ and the solution of (15) takes the form:

$$\begin{aligned} U(\theta,\varepsilon) = {} & \omega p(\theta) + \int_0^{\theta}\left\langle \hat{\mathcal{N}}(U(\phi),\phi,\varepsilon)+\varepsilon\hat{\mathcal{R}}(U(\phi),\phi,\varepsilon), p^*(\phi)\right\rangle d\phi\ p(\theta)\\ & - \int_{\theta}^{\infty}\left\langle \hat{\mathcal{N}}(U(\phi),\phi,\varepsilon)+\varepsilon\hat{\mathcal{R}}(U(\phi),\phi,\varepsilon), r^*(\phi)\right\rangle d\phi\ r(\theta)\\ & - \int_{\theta}^{\infty}\left\langle \hat{\mathcal{N}}(U(\phi),\phi,\varepsilon)+\varepsilon\hat{\mathcal{R}}(U(\phi),\phi,\varepsilon), q^*(\phi)\right\rangle d\phi\ q(\theta) \end{aligned}$$

where $\omega = \langle U(0), p^*(0)\rangle \in \mathbb{R}$ and $\langle\cdot,\cdot\rangle$ is the canonical inner product.

Moreover, the R'-reversibility of U implies $\omega = 0$.

Now, define the following mapping

$$G\left(U(\theta,\varepsilon)\right)=\int_0^\theta \left\langle \hat{\mathcal{N}}(U(\phi),\phi,\varepsilon)+\varepsilon\hat{\mathcal{R}}(U(\phi),\phi,\varepsilon),p^*(\phi)\right\rangle d\phi\; p(\theta)$$
$$-\int_\theta^\infty \left\langle \hat{\mathcal{N}}(U(\phi),\phi,\varepsilon)+\varepsilon\hat{\mathcal{R}}(U(\phi),\phi,\varepsilon),r^*(\phi)\right\rangle d\phi\; r(\theta)$$
$$-\int_\theta^\infty \left\langle \hat{\mathcal{N}}(U(\phi),\phi,\varepsilon)+\varepsilon\hat{\mathcal{R}}(U(\phi),\phi,\varepsilon),q^*(\phi)\right\rangle d\phi\; q(\theta).$$

We claim that: *G is a mapping from $E_{\gamma,d}$ to $E_{\gamma,d}$ and it possesses a fixed point in this set.*
Firstly we show that $|\Pi_r\left(G(U)\right)|\le\frac{\varepsilon^\eta}{2}$.
As

$$\Pi_r\left(p(\theta)\right)=0=\Pi_r\left(q(\theta)\right),$$

the problem is reduced to the analysis of the second integral of the expression of $G(U)$.

Observe that from the expression of $D\mathcal{N}\left(\mathcal{H}_k(\theta)\right)$ we may choose in the adjoint basis the following vector $r^*(\theta)=(0,0,1)$.

As $\Pi_r\left(r(\theta)\right)=1$ it follows that

$$\Pi_r\left(G(U(\theta,\varepsilon))\right)=-\int_\theta^\infty \Pi_r\left[\hat{\mathrm{N}}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\,\hat{\mathrm{R}}(U(\phi\varepsilon),\phi,\varepsilon)\right]d\phi.$$

On the other hand $\Pi_r(\mathcal{N})=0$.
So $\Pi_r\left[\hat{\mathrm{N}}(U(\phi,\varepsilon),\phi,\varepsilon)\right]=0$ and

$$\Pi_r\left[\hat{\mathrm{N}}(U(\phi,\varepsilon),\phi,\varepsilon)+\varepsilon\,\hat{\mathrm{R}}(U(\phi,\varepsilon),\phi,\varepsilon)\right]=\Pi_r\left[\varepsilon\,\hat{\mathrm{R}}(U(\phi,\varepsilon),\phi,\varepsilon)\right]$$
$$=\mathcal{R}_r\left(\mathcal{H}_k(\phi)-Y_k+U(\phi,\varepsilon)+\hat{Y}_k(\phi,\varepsilon),\phi,\varepsilon\right)-\mathcal{R}_r\left(\hat{Y}_k(\phi,\varepsilon),\phi,\varepsilon\right).$$

From (7) we know that

$$R_r(Y,\theta,\varepsilon)-R_r(Y',\theta,\varepsilon)=\mathcal{O}\left(\varepsilon^{3/2}|Y-Y'|\right).$$

Thus

$$|\Pi_r\left(G(U)\right)|=\left|\int_\theta^\infty \mathcal{R}_r\left(Y\left(\phi,\varepsilon\right),\phi,\varepsilon\right)-\mathcal{R}_r\left(\hat{Y}_k(\phi,\varepsilon),\phi,\varepsilon\right)d\phi\right|$$
$$\le M_2\frac{\varepsilon^{3/2}}{\gamma}\left(\|U\|_\gamma+e^{-\sqrt{2k}\theta}\right)$$

where $Y(\phi,\varepsilon)=\mathcal{H}_k(\phi)-Y_k+U(\phi,\varepsilon)+\hat{Y}_k(\phi,\varepsilon)$.

As $k\ge\frac{\varepsilon^\eta}{2}$ taking $\gamma\ge\frac{\sqrt{2k}}{2}$ we have:

$$|\Pi_r\left(G(U)\right)|\le\frac{\varepsilon^\eta}{2}.$$

Because of the definition of $G\left(U(\theta)\right)$ we derive that G is R'-reversible.
Next we show that $\|G(U)\|_\gamma\le d$. From (16) we have :

$$\|G\left(U\right)\|_\gamma\le M'\left[\left(\varepsilon+\|U\|_\gamma^2\right)\frac{\sqrt{2k}}{\left(\sqrt{2k}-\gamma\right)\gamma}\right].\tag{21}$$

Furthermore

$$\|G\left(U\right)-G\left(U'\right)\|_\gamma\le M'\left[\left(\varepsilon+\|U\|_\gamma^2+\|U'\|_\gamma^2\right)\frac{\sqrt{2k}}{\left(\sqrt{2k}-\gamma\right)\gamma}\right]\|U-U'\|_\gamma.\tag{22}$$

It is straightforward to show that for each $k>0$, there exists small $\varepsilon_k>0$ and d such that for $0<\varepsilon<\varepsilon_k$, G maps $E_{\gamma,d}$ into $E_{\gamma,d}$ (see (21)). Moreover one deduces immediately from (22) that it is a contraction. Hence G has a unique fixed point in $E_{\gamma,d}$, and the claim is achieved. □

Interchanging (13) by

$$Y(\theta,\varepsilon)=\mathcal{H}_k(\theta)-Y_k+U(\theta,\varepsilon)+\hat{Y}_k(\theta+\pi,\varepsilon) \tag{23}$$

and using the same scheme of proof as above we get similar results for such a solution.

Recall that $\hat{Y}_k(\theta,\varepsilon)$ is a symmetric orbit and meets $Fix(R')$ at $\theta=0$ and $\theta=\pi$.

In conclusion, we have that given $\varepsilon>0$ small enough, there exists $k_\varepsilon>0$ such that: if $k\geq k_\varepsilon$ then at $\hat{Y}_k(\theta,\varepsilon)$ there are two symmetric homoclinic solutions.

It is fairly easy to see that from the above considerations Lemma 3.5 is proved.

4. Proof of Theorem A

In this section we perform the proof of Theorem A. The proof of part (a) follows from Proposition 4.1 and parts (b) and (c) follow from Examples 4.2 and 4.4 (in the sequel).

Let $X\in\chi_i(\mathbb{R}^6)$ where $i\in\{2,\ldots,4\}$. Take a coordinate system around the origin such that the $2-jet$ of X is in **BNF** as given by [5]. So

$$X:\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=a_2x_1^2+b_1(x_3^2+x_4^2)+b_2(x_5^2+x_6^2)+f_1(x_2,x_2,x_3,x_4,x_5,x_6)\\ \dot{x}_3=-\alpha x_4-c_1x_1x_4+f_2(x_2,x_2,x_3,x_4,x_5,x_6)\\ \dot{x}_4=\alpha x_3+c_1x_1x_3+f_3(x_2,x_2,x_3,x_4,x_5,x_6)\\ \dot{x}_5=-\beta x_6+c_2x_1x_6+f_4(x_2,x_2,x_3,x_4,x_5,x_6)\\ \dot{x}_6=\beta x_5+c_2x_1x_5+f_5(x_2,x_2,x_3,x_4,x_5,x_6).\end{cases} \tag{24}$$

Assume the following extra assumption:

$$C_1':\{a_2>0,\ b_1,b_2<0 \text{ and } c_1,c_2\neq 0\}. \tag{25}$$

Such a condition allows us to claim that the 2-truncated normal form of the system possesses R_1-symmetric periodic orbits and homoclinic orbits as stated at Proposition A.

Our problem is to verify under which conditions the homoclinic connections are persistent when high order terms are considered.

Proposition 4.1. *Let X be a vector field in $\chi_2(\mathbb{R}^6)$ (resp. $\chi_3(\mathbb{R}^6)$) satisfying (25). Then there exist families of R_1-symmetric homoclinic orbits to R_1-symmetric periodic orbits near the equilibrium.*

Proof. First of all observe that the vector field is invariant with respect to $S_{12}(x_1,x_2,x_3,x_4,x_5,x_6)=(x_1,x_2,x_3,x_4,-x_5,-x_6)$ where $Fix(S_{12})=\{x_5=x_6=0\}$. As $Fix(S_{12})$ is flow-invariant it follows that it is possible to reduce X to a vector field in dimension four as described in the previous section.

So we can apply the Theorem A* (Section 2) to the subsystem in the variables (x_1,x_2,x_3,x_4) and the result follows.

The proof for $\chi_3(\mathbb{R}^6)$ is obtained in the same way. For this family X is invariant with respect to $S_{13}=(x_1,x_2,x_3,x_4,x_5,x_6)=(x_1,x_2,-x_3,-x_4,x_5,x_6)$. □

Now in what follows we exhibit a model where one can see that how a families of homoclinic orbits can be destroyed when reversible perturbations are considered.

Example 4.2. Consider the following equation:

$$X:\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=x_1^2-r^2\\ \dot{r}=0\\ \dot{z}_2=i\beta z_2+ic\,rx_2^3\end{cases} \tag{26}$$

where $r^2=x_3^2+x_4^4$, $z_2=x_5+ix_6$ and where we are considering $\theta=arctg\left(\frac{x_4}{x_3}\right)$ as the time.

Note that for $c=0$ the system is in the normal form and $\mathcal{H}_k(\theta)=(x_1(\theta),x_2(\theta),k,0)$ is a symmetric homoclinic orbit associated to the equilibrium $Y_k=(k,0,k,0)$, where

$$x_1(\theta)=k-\frac{3k}{\cosh^2\left(\frac{\sqrt{2k}}{2}\theta\right)} \quad\text{and}\quad x_2(\theta)=3k\sqrt{2k}\frac{tgh\left(\frac{\sqrt{2k}}{2}\theta\right)}{\cosh^2\left(\frac{\sqrt{2k}}{2}\theta\right)}.$$

The equilibrium Y_k is associated to a periodic orbit of the original system (as t is the time).

A straightforward calculation shows that the stable manifold of Y_k is given by

$$X_k(\theta)=\left(x_1(\theta),x_2(\theta),k,ikce^{i\beta\theta}\int_\theta^\infty e^{-i\beta s}x_2^3(s)ds\right). \tag{27}$$

To verify that this manifold does not meet Y_k when $\theta \to -\infty$ we have to show that the integral

$$h(k) = \int_{-\infty}^{\infty} e^{-i\beta s} x_2(s)^3 ds$$

is nonzero.

But, using residues we can calculate this integral and obtain

$$h(k) \sim \frac{g(\beta, k)}{\left(e^{\frac{\beta\pi}{\sqrt{2k}}} + e^{-\frac{\beta\pi}{\sqrt{2k}}}\right)}, \quad k \to 0, \ \beta \neq 1$$

where $g(\beta, k)$ is a polynomial of degree five in β satisfying $g(\beta, 0) \equiv 0$ and $g(\beta, k) \not\equiv$ for $k \neq 0$.

So, (27) does not connect Y_k to itself. In this case, $X_k(\theta)$ represents a heteroclinic orbit from Y_k to a periodic orbit of small amplitude. □

The idea developed on this model can be applied to more general systems.

Remark 4.3 (*Degenerate Situation in 6D*)**.** It is easy to see that when $X \in \chi_i(\mathbb{R}^6)$ with $i \in \{5, \dots, 8\}$ then it does not satisfy the generic condition imposed on [5] (condition (25)). In these cases the 2-jet of the vector field in the **BNF** is degenerate. However, although the results in [5] cannot be applied, it is possible to show that, in general, if the truncated normal form allows the existence of families of periodic orbits and homoclinic then it is possible to produce examples in the $\chi_i(\mathbb{R}^6)$ families where the previous families of homoclinic orbits are destroyed.

The next example shows how to obtain such special class of equations. We will just consider the case $i = 5$ and 6. The other situations are similar.

Example 4.4. Consider

$$X : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 + x_1 r^2 \\ \dot{r} = 0 \\ \dot{z}_2 = i\beta z_2 + icx_2^4 \end{cases} \tag{28}$$

where $r^2 = x_3^2 + x_4^4$, $z_2 = x_5 + ix_6$ and where we are considering $\theta = arctg\left(\frac{x_4}{x_3}\right)$ as the time.

Note that for $c = 0$ the system is in the normal form and admits a symmetric homoclinic orbit associated to the equilibrium $Y_k = (0, 0, k, 0)$.

A straightforward calculation using the same ideas of the one used in Example 4.2 can show that for $c \neq 0$ families of periodic orbits are preserved (in fact Y_k is preserved) but the stable manifold of Y_k does not connect Y_k to itself in this example. So, the homoclinic orbit is destroyed. □

As before, the idea developed on this example can be applied to more general systems.

Acknowledgments

The first author was partially supported by CNPq grant number 200293/2010-9. Both authors were supported by the joint project CAPES-MECD grant PHB-2009-0025-PC and by Fapesp grant number 2007/06896-5.

References

[1] S. Bochner, Compact groups of differentiable transformation, Ann. of Math. (2) 46 (3) (1945) 372–381.
[2] S. Bochner, D. Montgomery, Locally compact groups of differenciable transformations, Ann. of Math. (2) 47 (4) (1946) 639–653.
[3] C.A. Buzzi, J.S.W. Lamb, Reversible equivariant Hopf bifurcation, Arch. Ration. Mech. Anal. 175 (2005) 39–84.
[4] J.S.W. Lamb, J.A.G. Roberts, Time-reversal symmetry in dynamical systems: a survey, Physica D 112 (1998) 1–39.
[5] M.F.S. Lima, M.A. Teixeira, Families of periodic orbits in resonant reversible systems, Bull. Braz. Math. Soc. 40 (4) (2009) 511–537.
[6] E. Lombardi, Oscillatory Integral and Phenomena Beyond all Algebraic Orders, in: Lec. Notes in Math., vol. 1741, 2000.
[7] H.R. Rodrigues, Introdução ao Metodo Alternativo, Notes of ICMC-USP, Brazil, Jul./1997.
[8] Ling-Jun Wang, Homoclinic and heteroclinic orbits for the 0^2 or $0^2 i\omega$ singularity in the presence of two reversibility symmetries, Quart. Appl. Math. 67 (1) (2009) 1–38.