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Følner sequences and finite operators

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ABSTRACT

This article analyzes Følner sequences of projections for bounded linear operators and their relationship to the class of finite operators introduced by Williams in the 70s. We prove that each essentially hyponormal operator has a proper Følner sequence (i.e., an increasing Følner sequence of projections strongly converging to $\mathbb{1}$). In particular, any quasinormal, any subnormal, any hyponormal and any essentially normal operator has a proper Følner sequence. Moreover, we show that an operator is finite if and only if it has a proper Følner sequence or if it has a non-trivial finite dimensional reducing subspace. We also analyze the structure of operators which have no Følner sequence and give examples of them. For this analysis we introduce the notion of strongly non-Følner operators, which are far from finite block reducible operators, in some uniform sense, and show that this class coincides with the class of non finite operators.

Contents

1. Introduction

The notion of a Følner sequence was introduced in the context of groups to give a new characterization of amenability. A discrete countable group Γ is amenable if it has an invariant mean, i.e., there is a state ψ on the von Neumann algebra $\ell^\infty(\Gamma)$ such that

$$\psi(u_\gamma g) = \psi(g), \quad \gamma \in \Gamma, g \in \ell^\infty(\Gamma),$$

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where u is the left-regular representation of Γ on $\ell^2(\Gamma)$. A Følner sequence for a countable discrete group Γ is an increasing sequence of non-empty finite subsets $\Gamma_n \subset \Gamma$ with $\Gamma = \cup_n \Gamma_n$ and that satisfy

$$\lim_n \frac{|(\gamma\Gamma_n)\Delta\Gamma_n|}{|\Gamma_n|} = 0 \quad \text{for all } \gamma \in \Gamma, \tag{1.1}$$

where Δ denotes the symmetric difference and $|\Gamma_n|$ is the cardinality of Γ_n. Then, Γ has a Følner sequence if and only if Γ is amenable (cf. Chapter 4 in [33]; see also [39] for a review stressing the fundamental fact that amenable groups are those which can be approximated by finite groups).

Følner sequences were introduced in the context of operator algebras by Connes in Section 5 of his seminal paper [14] (see also [15, Section 2]). This notion is an algebraic analogue of Følner's characterization of amenable discrete groups and was used by Connes as an essential tool in the classification of injective type II_1 factors. If $\mathcal{H}$ is a Hilbert space, we will denote by $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. In this article all Hilbert spaces will be complex and separable.

Definition 1.1. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$. A sequence of non-zero finite rank orthogonal projections $\{P_n\}_{n\in\mathbb{N}}$ on $\mathcal{H}$ is called a *Følner sequence for* $\mathcal{T}$ if

$$\lim_n \frac{\|TP_n - P_nT\|_2}{\|P_n\|_2} = 0, \quad T \in \mathcal{T}, \tag{1.2}$$

where $\|\cdot\|_2$ denotes the Hilbert–Schmidt norm. If the Følner sequence $\{P_n\}_{n\in\mathbb{N}}$ satisfies, in addition, that it is increasing and converges strongly to $\mathbb{1}$, then we say it is a proper Følner sequence.

Normal operators, compact operators and Toeplitz operators with L^∞ symbol are examples of operators with a proper Følner sequence (cf. [22, Chapter 7]).

To simplify expressions we will often use the following notation: for $T \in \mathcal{L}(\mathcal{H})$ and P a non-zero finite rank orthogonal projection, put

$$\varphi(T,P) := \frac{\|TP - PT\|_2}{\|P\|_2}. \tag{1.3}$$

There is a canonical relation between the group theoretic and operator algebraic notions of Følner sequences in terms of the group algebra. Let Γ be a discrete, countable and amenable group and $\{\Gamma_n\}_{n\in\mathbb{N}} \subset \Gamma$ a Følner sequence. Denote by P_n the orthogonal finite-rank projections from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_n)$. Then $\{P_n\}_{n\in\mathbb{N}}$ is a proper Følner sequence for the group C*-algebra of Γ. (In Proposition 4 in [5], a stronger result is shown: the sequence $\{P_n\}_{n\in\mathbb{N}}$ mentioned before is a proper Følner sequence even for the group von Neumann algebra, i.e., for the weak operator closure of the algebra generated by the left-regular representation of Γ on $\ell^2(\Gamma)$. In general, if a C*-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ has a Følner sequence, then its weak closure needs not have one.)

In addition to these theoretical developments, Følner sequences have been used in spectral approximation problems: given an operator T on a complex Hilbert space $\mathcal{H}$ and a sequence of matrices (or linear operators) $\{T_n\}_{n\in\mathbb{N}}$ that approximate T in some sense, a natural question is whether the spectral objects of T_n tend to those of T as n grows. There are many references that treat this question from different points of view. Some standard textbooks that contain many examples and an extensive list of references are [1,12,22]. Bédos used Følner sequences for operators in the context of eigenvalue distribution problems (cf. [5]) and refined earlier results by Arveson stated in [3,4]. See also the introduction in [28] and references cited therein. It is worth mentioning that in the last two decades, the relation between spectral approximation problems and Følner sequences for non-selfadjoint and non-normal operators has been also explored, see for instance [41,38,7,35]. Notice that in the context of spectral approximation proper Følner sequences are important.

The second important concept for this article is the class of finite operators. They were introduced and analyzed in a classical article by Williams (cf., [42]). Finite operators $T \in \mathcal{L}(\mathcal{H})$ on an infinite dimensional Hilbert space $\mathcal{H}$ have the property that 0 is in the closure of the numerical range of the commutator $TX - XT$ for all $X \in \mathcal{L}(\mathcal{H})$ (see also Section 2 for a formal definition and additional results). As Williams explains in his article the name 'finite' is admittedly *ad hoc*. It comes from the fact that the class of finite operators contains the closure of all finite block reducible operators (i.e., operators having a non-trivial finite dimensional reducing subspace).

The aim of the present paper is to analyze the role of Følner sequences in the context of a single operator and in relation to the class of finite operators. If $\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for $T \in \mathcal{L}(\mathcal{H})$, then we have with respect to the decomposition $\mathcal{H} = P_n\mathcal{H} \oplus (P_n\mathcal{H})^\perp$

$$T \cong \begin{pmatrix} T_1^{(n)} & T_2^{(n)} \\ T_3^{(n)} & T_4^{(n)} \end{pmatrix} \quad \text{and} \quad TP_n - P_nT \cong \begin{pmatrix} 0 & -T_2^{(n)} \\ T_3^{(n)} & 0 \end{pmatrix}$$

where $T_1^{(n)}$ is a finite dimensional operator on $P_n\mathcal{H}$. Therefore

$$\frac{\|TP_n - P_nT\|_2}{\|P_n\|_2} = \frac{\|T_2^{(n)}\|_2 + \|T_3^{(n)}\|_2}{\|P_n\|_2},$$

so that the condition in (1.2) describes the growth of the off-diagonal blocks of T (in the Hilbert–Schmidt norm) relative to the dimension of the corresponding subspaces as $n \to \infty$.

In Section 2 we will present some consequences of the existence of Følner sequences for operators and give useful characterizations of it. We also mention standard results for the class of finite operators.

In Section 3 we explore the structure of operators without a Følner sequence. We define a strongly non-Følner operator as an operator T such that there is some positive number ε with $\varphi(T, P) \geq \varepsilon$ for all non-zero finite rank projections P on $\mathcal{H}$. In a sense, this condition means that T has to be far from the set of finite block reducible operators. In Theorem 3.2, we show that any operator with no proper Følner sequence is the orthogonal sum of an operator on a finite-dimensional space (which can be possibly zero) and a strongly non-Følner operator.

In Section 4, we show the natural relation between the notion of proper Følner sequences and the class of finite operators. To describe our results with more detail, let us divide the set of all operators in $\mathcal{L}(\mathcal{H})$ into four mutually disjoint classes, according to the following table:

Table 1
Classification of operators in $\mathcal{L}(\mathcal{H})$.

	Operators with a proper Følner sequence	Operators with no proper Følner sequence
Finite block reducible	$\mathcal{W}_{0+}$	$\mathcal{W}_{0-}$
Non finite block reducible	$\mathcal{W}_{1+}$	$\mathcal{S}$

As we will prove in Theorem 4.1,

an operator is finite if and only if it has a proper Følner sequence or it is finite block reducible.

Therefore the class of finite operators is the disjoint union of the operators in the classes $\mathcal{W}_{0+}$, $\mathcal{W}_{0-}$ and $\mathcal{W}_{1+}$. Moreover, an operator is strongly non-Følner if and only if it is not finite, i.e., it belongs to $\mathcal{S}$. This implies that the class of strongly non-Følner operators is open and dense in $\mathcal{L}(\mathcal{H})$. We refer to Section 4 and to the end of Section 6 for more details.

In Section 5 we analyze several classes of non-normal operators and show that each operator from any of these classes has a proper Følner sequence. In Theorem 5.1, we show that any essentially hyponormal operator (i.e., any T such that $T^*T - TT^*$ defines a nonnegative element of the Calkin algebra) has a proper Følner sequence. This implies that important classes of operators, like, e.g., essentially normal or hyponormal operators, have also proper Følner sequences.

In Section 6, we give examples of operators which have no Følner sequence. In the example stated in Proposition 6.1 we use the fact that the Cuntz algebra is singly generated as a C*-algebra. In a subsequent paper [29] we will discuss asymptotic properties of finite square matrices, related to the property of the existence of a proper Følner sequence for an infinite dimensional linear operator.

2. Basic properties of Følner sequences and finite operators

In this section we recall the basic definition and results concerning Følner sequences for operators. We will also discuss some standard properties of finite operators.

In what follows, if T is a linear operator on a Hilbert space $\mathcal{H}$, we denote by $\|T\|_p$ its norm in the Schatten–von Neumann class. We denote by $\mathcal{P}_{\text{fin}}(\mathcal{H})$ the set of all non-zero finite rank orthogonal projections on $\mathcal{H}$.

The existence of a proper Følner sequence for an operator T is a weaker property than quasidiagonality. Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *quasidiagonal* if there exists a sequence of finite rank orthogonal projections $\{P_n\}_{n\in\mathbb{N}}$ converging strongly to $\mathbb{1}$ and such that

$$\lim_n \|TP_n - P_nT\| = 0. \tag{2.1}$$

This notion was introduced by Halmos in [23] (see also [40] for a review that also relates the concept of quasidiagonality to other fields like, e.g., C*-algebras). The existence of a proper Følner sequence can be understood as a kind of quasidiagonality condition relative to the growth of the dimension of the underlying spaces. It can be shown that if the sequence of non-zero finite rank orthogonal projections $\{P_n\}_n$ quasidiagonalizes an operator T, then it is also a proper Følner sequence for T. Examples of quasidiagonal operators are compact operators, block-diagonal operators or normal operators. Abelian C*-algebras or the set of compact operators $\mathcal{K}(\mathcal{H})$ are examples of quasidiagonal C*-algebras (cf. [10]).

The next result collects some easy consequences of the definition of a (proper) Følner sequence for operators.

Proposition 2.1. *Let $\mathcal{T} \subset \mathcal{L}(\mathcal{H})$ be a set of operators and $\{P_n\}_{n\in\mathbb{N}}$ a sequence of non-zero finite rank orthogonal projections. Then we have*

(i) *$\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for $\mathcal{T}$ if and only if it is a Følner sequence for $C^*(\mathcal{T}, \mathbb{1})$, where $C^*(\cdot)$ is the C^*-algebra generated by its argument. Moreover, $\{P_n\}_{n\in\mathbb{N}}$ is a proper Følner sequence for $\mathcal{T}$ if and only if it is a proper Følner sequence for*

$$C^*\left(\mathcal{T}, \mathcal{K}(\mathcal{H}), \mathbb{1}\right).$$

(ii) *Let* $\dim \mathcal{H} = \infty$ *and* $\{P_n\}_{n\in\mathbb{N}}$ *be a proper Følner sequence for* $\mathcal{T}$. *Given a sequence* $\{L_l\}_{l\in\mathbb{N}}$ *of natural numbers with* $L_l \to \infty$, *there exists a sequence* $\{Q_l\}_{l\in\mathbb{N}} \subset \mathcal{K}(\mathcal{H})$ *of finite rank orthogonal projections which is a proper Følner sequence for* $\mathcal{T}$ *and* $\dim Q_l\mathcal{H} \geq L_l$, $l \in \mathbb{N}$.

(iii) $\{P_n\}_{n\in\mathbb{N}}$ *is a Følner sequence for* $\mathcal{T}$ *if and only if the following condition holds:*

$$\lim_n \frac{\|TP_n - P_nT\|_1}{\|P_n\|_1} = 0. \tag{2.2}$$

Proof. (i) It is obvious that $\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for $\mathcal{T}$ if and only if it is a Følner sequence for $C^*(\mathcal{T}, \mathbb{1})$. Moreover, if $\{P_n\}_{n\in\mathbb{N}}$ is proper, then it also satisfies $\|KP_n - K\| \to 0$ for any $K \in \mathcal{K}(\mathcal{H})$. This implies that $\|KP_n - P_nK\| \to 0$, i.e., $\{P_n\}_{n\in\mathbb{N}}$ quasidiagonalizes any compact operator and, therefore, $\{P_n\}_{n\in\mathbb{N}}$ is also a proper Følner sequence for $T + K$ for any $T \in \mathcal{T}$, $K \in \mathcal{K}(\mathcal{H})$.

For (ii), just notice that we can choose an increasing subsequence $Q_l = P_{n_l}$ such that $\dim Q_l\mathcal{H} \geq L_l$ and $\lim_{l\to\infty} Q_l = \mathbb{1}$. Then Eq. (1.2) will be satisfied replacing P_n by Q_n.

(iii) By item (i) we have that $\{P_n\}_{n\in\mathbb{N}}$ is a Følner sequence for $\mathcal{T}$ if and only if it is a Følner sequence for $C^*(\mathcal{T}, \mathbb{1})$ and we can apply Lemma 1 in [5]. □

If P and Q are orthogonal projections, we will denote by $P \vee Q$ the orthogonal projections onto the closure of $P\mathcal{H} + Q\mathcal{H}$. Finally, we will use the common notation for the commutator of two operators: $[A, B] := AB - BA$.

Next we give two useful formulations of the existence of a proper Følner sequence.

Proposition 2.2. *Let* $T \in \mathcal{L}(\mathcal{H})$ *with* $\dim \mathcal{H} = \infty$. *Then the following assertions are equivalent.*

(i) *T has a proper Følner sequence.*

(ii) *For each* $\varepsilon > 0$ *and each* $n \in \mathbb{N}$ *there exists a projection* $P \in \mathcal{P}_{\text{fin}}(\mathcal{H})$ *such that* $\operatorname{rank} P \geq n$ *and* $\varphi(T, P) < \varepsilon$ *(see (1.3)).*

(iii) *For each* $Q \in \mathcal{P}_{\text{fin}}(\mathcal{H})$ *and each* $\varepsilon > 0$ *there exists a projection* $R \in \mathcal{P}_{\text{fin}}(\mathcal{H})$ *such that* $R \geq Q$ *and* $\varphi(T, R) < \varepsilon$.

Proof. (i) $\Longrightarrow$ (ii). This is by the definition of a proper Følner sequence.

(ii) $\Longrightarrow$ (iii). Suppose (ii) holds. Let $Q \in \mathcal{P}_{\text{fin}}(\mathcal{H})$ and $\varepsilon > 0$ be given. By (ii), there exists a projection $P \in \mathcal{P}_{\text{fin}}(\mathcal{H})$ such that $\varphi(T, P) < \varepsilon/2$ and

$$\|T\| \cdot \|Q\|_2 \leq \frac{\varepsilon}{4}\,\|P\|_2.$$

Put $R := P \vee Q$ and note that $\|P\|_2 \leq \|R\|_2$, $R \geq Q$ and $R \geq P$. We assert that R has the desired properties. First notice that $\operatorname{rank} R - \operatorname{rank} P \leq \operatorname{rank} Q$, which implies that $\|R - P\|_2 \leq \|Q\|_2$. So we get

$$\begin{aligned}\|[T, R]\|_2 &\leq \|[T, P]\|_2 + \|[T, R - P]\|_2 \leq \|[T, P]\|_2 + 2\|T\| \cdot \|R - P\|_2 \\ &\leq \varphi(T, P)\|P\|_2 + 2\|T\|\,\|Q\|_2 < \frac{\varepsilon}{2}\,\|R\|_2 + \frac{\varepsilon}{2}\,\|P\|_2 \leq \varepsilon\|R\|_2,\end{aligned}$$

which yields $\varphi(T, R) < \varepsilon$.

(iii) $\Longrightarrow$ (i). Suppose (iii) holds. Choose any sequence $\{L_n\}$ of non-zero finite dimensional orthogonal projections such that $L_1 \leq L_2 \leq \cdots \leq L_n \leq \cdots$ and $s - \lim L_n = \mathbb{1}$. Then a proper Følner sequence $\{P_n\}$ for T can be constructed inductively as follows. Take $P_1 = L_1 \in \mathcal{P}_{\text{fin}}(\mathcal{H})$. If $P_1, \ldots, P_n$ have been defined, use (iii) to choose $P_{n+1} \in \mathcal{P}_{\text{fin}}(\mathcal{H})$ that satisfies $P_{n+1} \geq P_n \vee L_n$ and $\varphi(T, P_{n+1}) \leq \frac{1}{n+1}$. Then $P_{n+1} \geq P_n$, $P_{n+1} \geq L_n$ for any n and $\varphi(T, P_n) \to 0$ as $n \to \infty$, which implies that $\{P_n\}_n$ is a proper Følner sequence. □

Remark 2.3. (i) The preceding proposition also shows that given an operator T and a sequence of finite rank projections $\{Q_n\}_n$ such that $\dim Q_n$ is unbounded and $\varphi(T, Q_n) \to 0$ one can construct a proper Følner sequence for T in the sense of Definition 1.1.

(ii) The equivalent formulations stated above are usual when dealing with asymptotic properties. Adapting from the context of quasidiagonality (cf. [10, Section 7.2]) one can also prove that T has a proper Følner sequence if and only if for each $\varepsilon > 0$ and each finite set $\mathcal{F} \subset \mathcal{H}$ there exists a $P \in \mathcal{P}_{\text{fin}}(\mathcal{H})$ such that $\varphi(T, P) < \varepsilon$ and $\|Px - x\| < \varepsilon$ for all $x \in \mathcal{F}$.

The preceding result immediately implies that the existence of a proper Følner sequence for a direct sum can be localized in one of the direct summands.

Proposition 2.4. *Let* $\mathcal{H}$ *and* $\mathcal{H}'$ *be separable Hilbert spaces with* $\dim \mathcal{H} = \infty$. *If* T *has a proper Følner sequence, then* $T \oplus X \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H}')$ *has a proper Følner sequence for any* $X \in \mathcal{L}(\mathcal{H}')$.

Proof. Assume that T has a proper Følner sequence and X is any operator on $\mathcal{H}'$. Then for each $\varepsilon > 0$ and each $n \in \mathbb{N}$ there exists a $P \in \mathcal{P}_{\text{fin}}(\mathcal{H})$ such that $\operatorname{rank} P \geq n$ and $\varphi(T, P) < \varepsilon$. Then $\varphi(T \oplus X, P \oplus 0) = \varphi(T, P) < \varepsilon$, which shows that $P \oplus 0$ satisfies the properties required in Proposition 2.2(iii) (with respect to $T \oplus X$, instead of T). □

The following proposition concerning tensor products of operators follows from Proposition 2.13 in [6]. For convenience of the reader we give an elementary proof of it.

Proposition 2.5. *If $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces and $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$ are linear operators that have proper Følner sequences, then the tensor product $A \otimes B \in \mathcal{L}(\mathcal{H} \otimes \mathcal{K})$ also has a proper Følner sequence. More precisely, if $\{P_n\}_n$ is a proper Følner sequence for A and $\{Q_n\}_n$ is a proper Følner sequence for B, then $\{P_n \otimes Q_n\}_n$ is a proper Følner sequence for $A \otimes B$.*

Proof. Suppose that $\{P_n\}_n$ and $\{Q_n\}_n$ are as above. Then $\{P_n \otimes Q_n\}_n$ is an increasing sequence strongly converging to the identity. Moreover we have

$$\begin{aligned}(A \otimes B)(P_n \otimes Q_n) - (P_n \otimes Q_n)(A \otimes B) &= (AP_n) \otimes (BQ_n) - (P_nA) \otimes (Q_nB) \\ &= [A, P_n] \otimes (BQ_n) + (P_nA) \otimes [B, Q_n].\end{aligned}$$

This equality and the property $\|C \otimes D\|_2 = \|C\|_2\|D\|_2$ imply that

$$\begin{aligned}\varphi\,(A \otimes B, P_n \otimes Q_n) &= \frac{\|[A \otimes B, P_n \otimes Q_n]\|_2}{\|P_n\|_2\,\|Q_n\|_2} \\ &\le \frac{\|[A, P_n]\|_2}{\|P_n\|_2} \cdot \|B\| + \|A\| \cdot \frac{\|[B, Q_n]\|_2}{\|Q_n\|_2} \to 0 \quad \text{as } n \to \infty. \qquad \square\end{aligned}$$

The existence of a Følner sequence for a unital C*-algebra has important structural consequences. For the next result we need to recall the following notion: a state τ on the unital C*-algebra $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ (i.e., a positive and normalized linear functional on $\mathcal{A}$) is called an *amenable trace* if there exists a state ψ on $\mathcal{L}(\mathcal{H})$ such that $\psi \restriction \mathcal{A} = \tau$ and

$$\psi(XA) = \psi(AX), \quad X \in \mathcal{L}(\mathcal{H}), A \in \mathcal{A}.$$

Note that the previous equation already implies that τ is a trace on $\mathcal{A}$. The state ψ is also referred in the literature as a hypertrace for $\mathcal{A}$. Hypertraces are the algebraic analogue of the invariant mean mentioned at the beginning of the Introduction. Later we will need the following standard result (see [14,15] for the original statement and more results in the context of operator algebras; see also [6,2] for additional results in the context of C*-algebras related to the existence of a hypertrace).

Proposition 2.6. *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a separable unital C*-algebra. Then $\mathcal{A}$ has a Følner sequence if and only if $\mathcal{A}$ has an amenable trace.*

In general, it is not true that if $\mathcal{A}$ has an amenable trace, then it must also have a *proper* Følner sequence.
Finally we recall the following definition from the Introduction.

Definition 2.7. $T \in \mathcal{L}(\mathcal{H})$ is called a *finite operator* if

$$0 \in \overline{W\,([T, X])} \quad \text{for all } X \in \mathcal{L}(\mathcal{H}),$$

where $W(T)$ denotes the numerical range of the operator T, i.e.,

$$W(T) = \{\langle Tx, x\rangle \mid x \in \mathcal{H}, \|x\| = 1\}\,,$$

and where the bar means the closure of the corresponding subset in $\mathbb{C}$.

In this context the following class of operators plays a distinguished role:

Definition 2.8. Let $T \in \mathcal{L}(\mathcal{H})$. We say that T is *finite block reducible* if T has a non-trivial finite-dimensional reducing subspace, i.e., there is an orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, which reduces T, where $\mathcal{H}_0$ is finite dimensional and non-zero.

We collect in the following theorem some standard results due to Williams about the class of finite operators (cf. [42]).

Theorem (*Williams*)**.** *An operator $T \in \mathcal{L}(\mathcal{H})$ is finite if and only if $C^*(T, \mathbb{1})$ has an amenable trace. The class of finite operators is closed in the operator norm and contains all finite block reducible operators.*

It follows that the closure of the set of all finite block reducible operators is contained in the class of finite operators.
Combining Williams' Theorem with Proposition 2.6, we get the following fact.

Corollary 2.9. *For any operator $T \in \mathcal{L}(H)$, the following properties are equivalent:*
(i) *T is finite;*
(ii) *T has a Følner sequence;*
(iii) *$C^*(T, \mathbb{1})$ has an amenable trace.*

Remark 2.10. Note that in the reverse implication of Proposition 2.6 the sequence of projections does not have to be a proper Følner sequence in the sense of Definition 1.1. In fact, one can easily construct the following counterexample: consider a finite block reducible operator $T = T_0 \oplus T_1$ on the Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, with $\dim H_0 < \infty$ and where T_1 has no Følner sequence (examples of these type of operators are given in Section 6). Then, by Williams theorem it follows that $C^*(T, \mathbb{1})$ has a hypertrace and by Proposition 2.6 it has a Følner sequence also. The simplest choice of Følner sequence is the constant sequence $P_n = \mathbb{1}_{\mathcal{H}_0} \oplus 0$ which trivially satisfies (1.2) for T. But T cannot have a proper Følner sequence because T_1 has no Følner sequence (see Proposition 3.6 below).

3. Strongly non-Følner operators

In the present section we begin the analysis of operators with no Følner sequence.

Definition 3.1. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and T an operator on $\mathcal{H}$. We will say that T is *strongly non-Følner* if there exists an $\varepsilon > 0$ such that all projections $P \in \mathcal{P}_{\mathrm{fin}}(\mathcal{H})$ satisfy

$$\varphi(T, P) \geq \varepsilon.$$

Theorem 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ with $\dim \mathcal{H} = \infty$. Then T has no proper Følner sequence if and only if T has an orthogonal sum representation $T = A \oplus \widetilde{T}$ on $\mathcal{H} = \mathcal{H}_0 \oplus \widetilde{\mathcal{H}}$, where $\dim \mathcal{H}_0 < \infty$ (so that A is a finite dimensional operator) and $\widetilde{T}$ is strongly non-Følner.*

To prove the preceding theorem we will need the following lemmas that involve projections.

Lemma 3.3. *Let $\{P_n\}_{n\in\mathbb{N}}$ and $\widetilde{P}$ be orthogonal projections in $\mathcal{H}$. If $\widetilde{P}$ has finite rank and the sequence $\{P_n\}_{n\in\mathbb{N}}$ tends to zero in the strong operator topology, i.e. $P_n \xrightarrow{\mathrm{SOT}} 0$ as $n \to \infty$, then $\|\widetilde{P}P_n\| \to 0$.*

Proof. It suffices to prove the assertion for the case when $\widetilde{P}$ has rank one: let $\widetilde{P} = ff^*$ for a unit vector f in $\mathcal{H}$. Then $\|\widetilde{P}P_n\| \leq \|f\| \|P_n f\| = \|P_n f\| \to 0$ as $n \to \infty$. $\square$

Lemma 3.4. *Let $s \in \mathbb{N}$, $P_1, \ldots, P_s \in \mathcal{P}_{\mathrm{fin}}(\mathcal{H})$, $j = 1, \ldots, s$. If $\|P_j P_k\| \leq \delta := 1/(3s^3)$ for all indices $j \neq k$, then the ranges of projections P_j are linearly independent spaces and*

$$P_1 + \cdots + P_s \geq \frac{1}{2} \left(P_1 \vee \cdots \vee P_s\right).$$

Proof. Let $f \in (P_1 \vee \cdots \vee P_s)\mathcal{H}$. Then $f = \sum_j f_j$, where $f_j \in P_j\mathcal{H}$. If either $j \neq k$ or $j \neq \ell$, then $\left|\langle P_j f_k, f_\ell\rangle\right| \leq \delta \|f_k\| \|f_\ell\|$. Hence

$$\begin{aligned}\sum_{j=1}^{s} \langle P_j f, f\rangle &= \sum_j \sum_{k,\ell} \langle P_j f_k, f_\ell\rangle = \sum_k \|f_k\|^2 + \sum_j \sum_{\substack{j\neq k \text{ or}\\ j\neq \ell}} \mathrm{Re}\, \langle P_j f_k, f_\ell\rangle \\ &\geq \sum_k \|f_k\|^2 - s\delta \left(\sum_k \|f_k\|\right)^2 \geq (1 - s^3\delta) \sum_k \|f_k\|^2 = \frac{2}{3} \sum_k \|f_k\|^2.\end{aligned}$$

In particular, if $f = 0$, then $f_j = 0$ for all j. Hence the ranges of P_j are linearly independent.

Similar arguments show that $\sum_j \|f_j\|^2 \geq \frac{1}{(1+s^2\delta)} \|f\|^2$. Combining the last two inequalities and since $\delta = \frac{1}{3s^3}$ we obtain the estimate $\sum_j \langle P_j f, f\rangle \geq \frac{1}{2}\|f\|^2$ which proves the last statement. $\square$

Lemma 3.5. *If $P, Q \in \mathcal{P}_{\mathrm{fin}}(\mathcal{H})$ and $L \in \mathcal{L}(\mathcal{H})$, then*

$$|\varphi(L, P) - \varphi(L, Q)| \leq 4\, \|L\| \cdot \frac{\|P - Q\|_2}{\max\left(\|P\|_2, \|Q\|_2\right)}.$$

Proof. Without loss of generality, let us assume that $\|P\|_2 \leq \|Q\|_2$. Then we have

$$\begin{aligned}\|Q\|_2\, \|P\|_2\, |\varphi(L, P) - \varphi(L, Q)| &= |\|Q\|_2\, \|[L, P]\|_2 - \|P\|_2\, \|[L, Q]\|_2| \\ &\leq |\|Q\|_2 - \|P\|_2| \cdot \|[L, P]\|_2 + \|P\|_2\, \|[L, Q - P]\|_2 \\ &\leq 4\, \|L\|\, \|Q - P\|_2 \|P\|_2,\end{aligned}$$

which implies the desired estimate. $\square$

Proposition 3.6. *Let $T = A \oplus \widetilde{T}$ on $\mathcal{H} = \mathcal{H}_0 \oplus \widetilde{\mathcal{H}}$, where $\dim \mathcal{H}_0 < \infty$ (hence A is a finite dimensional operator). Then T has a proper Følner sequence if and only if $\widetilde{T}$ has a proper Følner sequence.*

Proof. The implication "⇐" follows from Proposition 2.4. To prove the implication "⇒" suppose that T has a proper Følner sequence and put $d := \dim \mathcal{H}_0 < \infty$. For any ε and any $N > d$ there exists a $P \in \mathcal{P}_{\mathrm{fin}}(\mathcal{H})$ such that $\operatorname{rank} P \geq N$ and

$\varphi(T,P) < \varepsilon$. Now, for each such P there exists also a $\widetilde{P} \in \mathcal{P}_{\text{fin}}(\widetilde{\mathcal{H}})$ such that $0 \oplus \widetilde{P} \le P$ and $\operatorname{rank}\widetilde{P} + d \ge \operatorname{rank} P$. (Take as $\widetilde{P}$, e.g., the orthogonal projection onto $P\mathcal{H} \cap (0 \oplus \widetilde{\mathcal{H}})$.) Using Lemma 3.5 we get

$$\begin{aligned}\varphi\left(\widetilde{T},\widetilde{P}\right) = \varphi\left(T, 0\oplus\widetilde{P}\right) \ &\le\ \varphi(T,P) + \left|\varphi(T,P) - \varphi\left(T,0\oplus\widetilde{P}\right)\right| \\ &\le \varphi(T,P) + \frac{4\|T\|\ \|P-(0\oplus\widetilde{P})\|_2}{\|P\|_2} \\ &\le \varphi(T,P) + \frac{4\|T\|\ d^{\frac{1}{2}}}{\|P\|_2} < 2\varepsilon,\end{aligned}$$

where for the last inequality we have chosen P so that $\|P\|_2 > \frac{4\|T\|d^{\frac{1}{2}}}{\varepsilon}$. By Proposition 2.2(ii) it follows that $\widetilde{T}$ has also a proper Følner sequence. □

Proposition 3.7. *Let $T \in \mathcal{L}(\mathcal{H})$ and suppose that $\varphi(T,P) \neq 0$ for all $P \in \mathcal{P}_{\text{fin}}(\mathcal{H})$. If there is a Følner sequence of projections $\{P_n\}_n \subset \mathcal{P}_{\text{fin}}(\mathcal{H})$ of a constant rank, then T has a proper Følner sequence.*

Proof. Let $\{P_n\}_n$ be a sequence of projections such that the rank $m := \operatorname{rank} P_n$ is constant and non-zero. We can represent them as $P_n = f_n f_n^*$, where $f_n\colon \mathbb{C}^m \to \mathcal{H}$ are isometries. Moreover, by weak compactness of the unit ball in $\mathcal{H}$ there is a contraction $g\colon \mathbb{C}^m \to \mathcal{H}$, $\|g\| \le 1$, such that (passing possibly to a subsequence)

$$f_n \xrightarrow{\text{WOT}} g. \tag{3.1}$$

First we prove that $g = 0$. For this, suppose that $g \neq 0$ and we will show that this leads to a contradiction. If $g \neq 0$ there exists some k, with $1 \le k \le m$, and an isometry $g_0\colon \mathbb{C}^k \to \mathcal{H}$ such that $\operatorname{Ran} g = \operatorname{Ran} g_0$. Notice that

$$(I-P_n)TP_n = \left(Tf_n - f_n(f_n^* T f_n)\right) f_n^*.$$

Put $\alpha_n = f_n^* T f_n\colon \mathbb{C}^m \to \mathbb{C}^m$ and

$$h_n = Tf_n - f_n\alpha_n\colon \mathbb{C}^m \to \mathcal{H}, \tag{3.2}$$

so that

$$(I-P_n)TP_n = h_n f_n^*.$$

Since projections P_n have constant rank and $\varphi(T,P_n) \to 0$ it follows that

$$\|(I-P_n)TP_n\| \to 0 \quad \text{as } n \to \infty.$$

Thus $\|h_n\| \to 0$. Passing possibly to a subsequence, we can assume that there is a limit $\lim_{n\to\infty}\alpha_n = \alpha \in \mathcal{L}(\mathbb{C}^m)$. By (3.2), $f_n\alpha_n = Tf_n - h_n$, so by (3.1), we get

$$f_n\alpha_n \xrightarrow{\text{WOT}} Tg \quad \text{as } n \to \infty.$$

By applying (3.1) once again, it follows that $g\alpha = Tg$. In particular, $\operatorname{Ran}(Tg) \subset \operatorname{Ran} g$. Notice that $P_{g_0} := g_0 g_0^* \neq 0$ is the orthogonal projection onto $\operatorname{Ran} g_0$. Since $\operatorname{Ran} g = \operatorname{Ran} g_0$, we arrive at the equality $(I-P_{g_0})TP_{g_0} = 0$. In the same way, we can prove that $(I-P_{g_0})T^*P_{g_0} = 0$ (for the same isometry g_0). Hence $\varphi\left(T, P_{g_0}\right) = 0$, which contradicts the assumption. Therefore we must have $g = 0$, that is, $f_n \xrightarrow{\text{WOT}} 0$ as $n \to \infty$. Hence $|\langle P_n a, b\rangle| \le \|f_n^* a\|\|f_n^* b\| \to 0$ for any $a, b \in \mathcal{H}$, that is, $P_n \xrightarrow{\text{WOT}} 0$, hence $P_n \xrightarrow{\text{SOT}} 0$.

To show that T has a proper Følner sequence let $\varepsilon > 0$ and $N \in \mathbb{N}$. Consider also a positive $\delta < \min\left(\frac{\varepsilon}{4N}, \frac{1}{3N^3}\right)$. From the assumption $\varphi(T,P_n) \to 0$ and Lemma 3.3 we can choose projections $P_{n_1}, P_{n_2}, \ldots, P_{n_N}$ from the sequence $\{P_n\}_n$ that satisfy

$$\varphi(T,P_{n_j}) < \delta \quad \text{and} \quad \|P_{n_j}P_{n_k}\| < \delta$$

for all indices $j \neq k$, $1 \le j,k \le N$. To simplify notation we will write P_j instead of P_{n_j}. Put $P := (P_1 \vee \cdots \vee P_N)$. Since $(I-P)(I-P_j) = (I-P)$ we have that $\|(I-P)TP_j\| \le \|(I-P_j)TP_j\| < \delta$ for all $j = 1,\ldots,N$. Hence $\|(I-P)T\left(\sum_j P_j\right)\| < N\delta$. Finally, Lemma 3.4 implies that $\sum_j P_j \ge \frac{1}{2}P$.

We denote the inverse of $\sum_j P_j$ on $P\mathcal{H}$ by Q. Then $\|Q\| \le 2$ and $P = \left(\sum_j P_j\right)Q$. This gives that $\|(I-P)TP\| < 2N\delta$. In the same way we show that $\|PT(I-P)\| < 2N\delta$, and, therefore, $\varphi(T,P) < 4N\delta < \varepsilon$ and by Lemma 3.4, $\operatorname{rank} P \ge N$. From Proposition 2.2 we conclude that T has a proper Følner sequence. □

With the preceding material we can now prove the main result of this section:

Proof of Theorem 3.2. If $T \in \mathcal{L}(\mathcal{H})$ has an orthogonal sum representation $T = A \oplus \widetilde{T}$ on $\mathcal{H} = \mathcal{H}_0 \oplus \widetilde{\mathcal{H}}$, where $\dim \mathcal{H}_0 < \infty$ and $\widetilde{T}$ is strongly non-Følner, then Proposition 3.6 implies that T has no proper Følner sequence.

To prove the other implication of the theorem suppose that T has no proper Følner sequence. By Proposition 2.2, there exist some $\varepsilon' > 0$ and $M \in \mathbb{N}$ such that

$$\forall P \in \mathcal{P}_{\mathrm{fin}}(\mathcal{H}), \quad \operatorname{rank} P > M \Longrightarrow \varphi(T, P) \geq \varepsilon'.$$

In particular, it follows that if T decomposes as $T = A \oplus \widetilde{T}$, where $A \in \mathcal{L}(\mathcal{H}_0)$, with $\mathcal{H}_0$ finite dimensional, then $\dim \mathcal{H}_0 \leq M$. Consider a decomposition $T = A \oplus \widetilde{T}$, where $A: \mathcal{H}_0 \to \mathcal{H}_0$, $\widetilde{T}: \widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}$, and $\ell := \dim \mathcal{H}_0$ is the largest possible. (The case where $\ell = 0$ is not excluded.) We prove next that $\widetilde{T}$ is a strongly non-Følner operator: by Proposition 3.6 and since $\ell \leq M$, we have that $\widetilde{T}$ has no proper Følner sequence. Therefore there exist $\varepsilon_1 > 0$ and $s \in \mathbb{N}$ such that

$$\forall \widetilde{P} \in \mathcal{P}_{\mathrm{fin}}(\widetilde{\mathcal{H}}), \quad \operatorname{rank} \widetilde{P} > s \Longrightarrow \varphi(\widetilde{T}, \widetilde{P}) \geq \varepsilon_1. \tag{3.3}$$

On the other hand, since ℓ is the largest possible, it follows that

$$\widetilde{P} \in \mathcal{P}_{\mathrm{fin}}(\widetilde{\mathcal{H}}) \quad \text{with } \widetilde{P} \neq 0 \Longrightarrow \varphi(\widetilde{T}, \widetilde{P}) \neq 0. \tag{3.4}$$

We claim that (3.4) implies that

$$\exists\, \varepsilon_2 > 0\ \forall \widetilde{P} \in \mathcal{P}_{\mathrm{fin}}(\widetilde{\mathcal{H}}) \quad 0 < \operatorname{rank} \widetilde{P} \leq s \Longrightarrow \varphi(\widetilde{T}, \widetilde{P}) \geq \varepsilon_2. \tag{3.5}$$

Then, putting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we will conclude that $\widetilde{T}$ is strongly non-Følner (cf. Definition 3.1). So it remains to deduce assertion (3.5).

Assume that (3.5) does not hold. Then there is some m, with $1 \leq m \leq s$, and a sequence of projections $\{P_n\}_n \subset \mathcal{P}_{\mathrm{fin}}(\widetilde{\mathcal{H}})$ of rank m that is a Følner sequence, i.e., we have

$$\lim_{n\to\infty} \varphi(\widetilde{T}, P_n) = 0. \tag{3.6}$$

From (3.4) and Proposition 3.7 applied to the operator $\widetilde{T}$ we conclude that $\widetilde{T}$ has a proper Følner sequence. But this contradicts (3.3) and, therefore, (3.5) must hold. $\square$

4. Relation between proper Følner sequences and finite operators

We show in this section a useful characterization of finite operators that involves proper Følner sequences. Recall the definitions and results stated at the end of Section 2.

Theorem 4.1. *Let $T \in \mathcal{L}(\mathcal{H})$. Then, T is a finite operator if and only if T is finite block reducible or T has a proper Følner sequence.*

Proof. (i) If T is finite block reducible, the T is a finite operator (cf. [42]). Moreover, if T has a proper Følner sequence, then the C*-algebra $C^*(T, \mathbb{1})$ has the same proper Følner sequence and, by Proposition 2.6, it also has an amenable trace. Then, by Williams' theorem (see also Theorem 4 in [42]) we conclude that T is finite.

(ii) To prove the other implication, assume T is a finite operator. We consider several cases. If there exists a (non-zero) $P \in \mathcal{P}_{\mathrm{fin}}(\mathcal{H})$ such that $\varphi(T, P) = 0$, then since $\varphi(T, P) = \frac{\|[T,P]\|_2}{\|P\|_2}$ we must have $[T, P] = 0$. This shows that T is finite block reducible. Consider next the situation where $\varphi(T, P) \neq 0$ for all $P \in \mathcal{P}_{\mathrm{fin}}(\mathcal{H})$. Since T is finite we can use again Theorem 4 in [42] to conclude that $C^*(T, \mathbb{1})$ has an amenable trace. Applying Proposition 2.6 (see also Theorem 1.1 in [6]) we conclude that there exists a Følner sequence of non-zero finite rank projections $\{P_n\}_n$, i.e., we have

$$\lim_{n\to\infty} \varphi(T, P_n) = 0.$$

(Note that P_n is not necessarily a proper Følner sequence in the sense of Definition 1.1; cf. Remark 2.10.) Two cases may appear: if $\dim P_n\mathcal{H} \leq m$ for some $m \in \mathbb{N}$, then choose a subsequence with constant rank and by Proposition 3.7 we conclude that T has a proper Følner sequence. If $\dim P_n\mathcal{H}$ is not bounded, then from Remark 2.3(i) we also have that T has a proper Følner sequence. $\square$

From Theorem 4.1 and taking into account the classification of operators described in Table 1 of the Introduction we have the following result.

Corollary 4.2. *Let $T \in \mathcal{L}(\mathcal{H})$. Then*

(i) *T is a finite operator if and only if T is in one of the following mutually disjoint classes: $\mathcal{W}_{0+}$, $\mathcal{W}_{0-}$, $\mathcal{W}_{1+}$.*
(ii) *T is not a finite operator (i.e., it is of class $\mathcal{S}$) if and only if T is strongly non-Følner.*
(iii) *The class of strongly non-Følner operators is open and dense in $\mathcal{L}(\mathcal{H})$.*

Proof. The characterization of finite operators and its complement stated in (i) and (ii) follows from Theorem 4.1 and Williams' theorem at the end of Section 2. To prove part (iii) we use that the class of finite operators is closed and nowhere dense (cf. [26]). Therefore the set of strongly non-Følner operators is an open and dense subset of $\mathcal{L}(\mathcal{H})$. $\square$

Remark 4.3. In fact, the assertion that the class of strongly non-Følner operators is open in the norm topology follows easily from our definition of this class. Indeed, let T be strongly non-Følner operator, so that there is an $\varepsilon > 0$ such that $\varphi(T, P) \geq \varepsilon$ for all $P \in \mathcal{P}_{\text{fin}}(\mathcal{H})$. It is easy to see that $|\varphi(T, P) - \varphi(T', P)| \leq 2\|T - T'\|$ for any operator T'. Hence any operator T' with $\|T - T'\| < \varepsilon/2$ is strongly non-Følner. So an application of Corollary 4.2 gives an alternative proof of the result by Williams [42] that the set of finite operators is closed.

5. Classes of non-normal operators with a proper Følner sequence

In the present section we single out several classes of operators such that any operator in these classes has a proper Følner sequence. The unilateral shift is a basic example that shows the difference between the notions of proper Følner sequences and quasidiagonality. It is a well-known fact that the unilateral shift S is not a quasidiagonal operator. (This was shown by Halmos in [24]; in fact, in this reference it is shown that S is not even quasitriangular.) In the setting of abstract C*-algebras it can also be shown that a C*-algebra containing a proper (i.e. non-unitary) isometry is not quasidiagonal (see, e.g., [8,10]).

On the other hand, S has a canonical proper Følner sequence. Indeed, let S be defined on $\mathcal{H} := \ell^2(\mathbb{N}_0)$ by $Se_i := e_{i+1}$, where $\{e_i \mid i = 0, 1, 2, \dots\}$ is the canonical basis of $\ell^2(\mathbb{N}_0)$. Then it is very easy to see that orthogonal projections P_n onto $\operatorname{span}\{e_i \mid i = 0, 1, 2, \dots, n\}$ form a proper Følner sequence for S. We will see later in this section that, in fact, any isometry has a proper Følner sequence.

We recall some standard definitions. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *hyponormal* if its self-commutator $[A^*, A]$ is nonnegative. T is called *essentially hyponormal* if the image in the Calkin algebra $\mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ of $[T^*, T]$ is a nonnegative element, that is, if $[T^*, T]$ is a sum of a nonnegative and a compact selfadjoint operator. Next, T is said to be *essentially normal* if $[T^*, T]$ is compact (that is, $[T^*, T]$ is zero as an element of the Calkin algebra). Finally, T is called *quasinormal* if T and T^*T commute. Any isometry is quasinormal, any quasinormal operator is subnormal and any subnormal operator is hyponormal, see [16, Chapter II].

Theorem 5.1. *Any essentially hyponormal operator $T \in \mathcal{L}(\mathcal{H})$ has a proper Følner sequence.*

Proof. Let T be essentially hyponormal. By Williams [42, Theorem 5] and the discussion that follows this theorem, T is a finite operator. Consider all finite-dimensional reducing subspaces $\mathcal{H}_0$ of T (including the zero one).

There are two possibilities.

(1) Suppose that among these subspaces there is one of largest dimension, say, $\mathcal{H}_0$, and $T = T_0 \oplus T_1$ with respect to the corresponding decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$. Then T_1 is essentially hyponormal and not finite block reducible. By Theorem 4.1, T_1 has a proper Følner sequence. Therefore T also has a proper Følner sequence.

(2) Now suppose that, to the opposite, the dimensions of these subspaces can be arbitrarily large. Then we deduce from Proposition 2.2 that in this case, too, T has a proper Følner sequence. $\square$

Corollary 5.2. *Every essentially normal operator (that is, an operator T such that $[T^*, T] \in \mathcal{K}(\mathcal{H})$) has a proper Følner sequence. Every hyponormal operator (in particular, any subnormal, any quasinormal and any isometry) also has a proper Følner sequence.*

Remark 5.3. For some of the above operator classes, one can give alternative direct proofs.

(i) Any isometry $V \in \mathcal{L}(\mathcal{H})$ has a proper Følner sequence. Indeed, without loss of generality we may assume that V is not unitary. By Wold's decomposition theorem (cf. [18, Section V.2]) we have that $V \cong S \oplus A$, where $\cong$ means unitary equivalence, S is the unilateral shift and $A = \left(\oplus_{i=0}^{n} S\right) \oplus U$ for some cardinal number n and unitary U. Since we showed above that S has a canonical proper Følner sequence we can apply Proposition 2.4 and the proof is concluded.

(ii) Any quasinormal operator has a proper Følner sequence. To see that, notice first that by Brown's theorem (see [16, Theorem II.3.2]), any quasinormal operator is unitarily equivalent to $N \oplus (A \otimes S)$, where N is normal, A is nonnegative and S is the unilateral shift. Next, as we mentioned already, S has an explicit proper Følner sequence and since A is selfadjoint it has a proper Følner sequence too. By Proposition 2.5, $A \otimes S$ has a proper Følner sequence. Finally, Proposition 2.4 implies that $N \oplus (A \otimes S)$ has a proper Følner sequence.

(iii) One can also give an alternative proof of the fact that any essentially normal operator has a proper Følner sequence by applying the Brown–Douglas–Fillmore theory (see [9,19]) and a model for essentially normal operators, similar to that given on p. 122 of the cited work.

The next result refers to the existence of a proper Følner sequence for the class of Toeplitz operators on the d-dimensional torus. Denote the unit torus by $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. We also recall that, given a function $F \in L^\infty(\mathbb{T}^d)$, the Toeplitz operator T_F on the classical Hardy space $H^2(\mathbb{T}^d)$ is defined by $T_F g = P_+(F \cdot g)$, $g \in H^2(\mathbb{T}^d)$, where P_+ stands for the orthogonal projection from $L^2(\mathbb{T}^d)$ onto $H^2(\mathbb{T}^d)$. Note that even for $d = 1$, there are Toeplitz operators which are not essentially normal (for instance, T_θ for any non-rational inner function θ). Using the same idea, it is easy to give an example of a Toeplitz operator T_F on H^2 such that neither T_F nor T_F^* is essentially hyponormal.

It is an easy consequence of Proposition 2.5 and Weierstraß' theorem that any Toeplitz operator with continuous symbol on $\mathbb{T}^d$ has a proper Følner sequence. The following more general result is also true. Its proof is similar to the one given in Example 7.17 of [22], where only the case $d = 1$ was considered. For convenience of the reader we give a brief sketch of the proof.

Proposition 5.4. *Any Toeplitz operator T_F on $H^2(\mathbb{T}^d)$ with any symbol F in $L^\infty(\mathbb{T}^d)$ has a proper Følner sequence.*

Proof. It is enough to consider the proof for $d = 2$ and other cases follow similarly. For any $F \in L^\infty(\mathbb{T}^2)$ we consider its decomposition

$$F = \sum_{k_1,k_2\in\mathbb{Z}} a_{k_1k_2}\, e_{k_1k_2} \quad \text{and} \quad \|F\|^2 = \sum_{k_1,k_2\in\mathbb{Z}} |a_{k_1k_2}|^2 < \infty \tag{5.1}$$

where $\{e_{k_1k_2}\}$ is the canonical basis of $L^2(\mathbb{T}^2) \,:\, e_{k_1k_2}(z_1,z_2) = z_1^{k_1}z_2^{k_2}$. Denote by P_N the orthogonal projection onto span $\{e_{k_1k_2} \mid k_1,k_2 = 0,\ldots,N-1\}$ and note that $\|P_N\|_2^2 = N^2$. If we choose b_N to be the smallest integer greater or equal than $\sqrt{N}$, we have

$$\frac{1}{N^2}\,\|(\mathbb{1}-P_N)\,T_F\,P_N\|_2^2 \le \frac{1}{N^2}\sum_{\substack{l_1,l_2=0,\ldots,N-1\\ k_1\ge N,k_2\ge 0}} \left|a_{k_1-l_1,k_2-l_2}\right|^2 + \frac{1}{N^2}\sum_{\substack{l_1,l_2=0,\ldots,N-1\\ k_1\ge 0,k_2\ge N}} \left|a_{k_1-l_1,k_2-l_2}\right|^2 =: A_1 + A_2.$$

Next, putting $s_j = k_j - l_j$, we get

$$\begin{aligned} A_1 &\le \frac{1}{N}\left(\sum_{s_1\ge 1,s_2\in\mathbb{Z}} \left|a_{s_1s_2}\right|^2 + \cdots + \sum_{s_1\ge N,s_2\in\mathbb{Z}} \left|a_{s_1s_2}\right|^2\right) \\ &\le \frac{1}{N}\left((N-b_N)\sum_{s_1> b_N,s_2\in\mathbb{Z}} \left|a_{s_1s_2}\right|^2 + b_N\sum_{s_1\ge 1,s_2\in\mathbb{Z}} \left|a_{s_1s_2}\right|^2\right) \\ &\le \sum_{s_1> b_N,s_2\in\mathbb{Z}} \left|a_{s_1s_2}\right|^2 + \frac{b_N}{N}\,\|F\|^2 \underset{N\to\infty}{\longrightarrow} 0 \end{aligned}$$

(see (5.1)). Similarly, $A_2 \to 0$ as $N \to \infty$. We get that $\frac{1}{N^2}\,\|(\mathbb{1}-P_N)\,T_F\,P_N\|_2^2 \to 0$. In the same way one proves that $\frac{1}{N^2}\,\|P_N\,T_F\,(\mathbb{1}-P_N)\|_2^2 \to 0$ as $N \to \infty$. $\square$

In particular, it follows that any Toeplitz operator on $H^2(\mathbb{T}^d)$ is finite.

For completeness we mention that all band-limited operators and uniform limits of them have a proper Følner sequence. Recall that a linear operator A on $\mathcal{H}$ is *band-limited* with respect to an orthonormal basis $\{e_n\}_{n=1}^\infty$ or $\{e_n\}_{n=-\infty}^\infty$ in $\mathcal{H}$ if there is $N \in \mathbb{N}$ such that the matrix elements of A satisfy $\langle Ae_j, e_k\rangle = 0$ for $|j-k| > N$ (see, e.g., [3,5,34]). This class of operators includes all (bounded) unilateral and bilateral weighted shifts.

Notice that not every weighted shift is essentially normal and not every weighted shift is quasidiagonal. A complete description of quasidiagonal weighted shifts was given in [36] (see also [31] for a generalization).

It is easy to see that band-limited operators can be generalized to what we can call *"acute wedge" operators*.

By definition, A has this property with respect to an orthonormal basis $\{e_n\}_{n=1}^\infty$ (or $\{e_n\}_{n=-\infty}^\infty$) in $\mathcal{H}$ if there exists a function $g(n)$ such that $\lim_{|n|\to\infty} \frac{g(n)}{|n|^{1/2}} = 0$ and $\langle Ae_j, e_k\rangle = 0$ for all j, k such that $|j-k| > g(j)$.

6. Examples of strongly non-Følner operators

Returning to Table 1 of the Introduction, it easy to give examples of operators of class $\mathcal{W}_{0+}$. Next, it is immediate to see that the unilateral shift is an example of an operator in the class $\mathcal{W}_{1+}$. In this Section, we will recall several examples of operators of class $\mathcal{S}$ (that is, strongly non-Følner operators) and will give a new example. We remark that for any strongly non-Følner operator T and any operator T_0 acting on a non-zero finite dimensional Hilbert space, the orthogonal sum $T_0 \oplus T$ is an example of an operator in $\mathcal{W}_{0-}$.

Halmos constructed in [25, Theorem 5] two operators $A, B \in \mathcal{L}(\mathcal{H})$, $\mathcal{H}$ infinite dimensional such that $W([A,B])$ is a vertical line segment in the open right half plane. It follows that A and B cannot be finite, hence both are examples of operators which are strongly non-Følner. In fact, this result was a motivational example for Williams' article [42].

It is also worth mentioning that Corollary 4 in [11] gives an example of a strongly non-Følner operator generating a type II_1 factor.

Now let us give one more example. We will use the amenable trace that appears in Proposition 2.6 as an obstruction. Recall the definition of the Cuntz algebra $\mathcal{O}_n$ (cf. [17,18]). It is the universal C*-algebra generated by $n \geq 2$ non-unitary isometries $S_1, \ldots, S_n$ with the property that their final projections add up to the identity, i.e.

$$\sum_{k=1}^{n} S_k S_k^* = \mathbb{1}. \tag{6.1}$$

This condition implies in particular that the range projections are pairwise orthogonal, i.e.,

$$S_l^* S_k = \delta_{lk} \mathbb{1}. \tag{6.2}$$

It is easy to realize the Cuntz algebra on the Hilbert space ℓ_2 of square summable sequences.

Proposition 6.1. *The Cuntz algebra $\mathcal{O}_n$, $n \geq 2$, is singly generated and its generator is strongly non-Følner.*

Proof. By Corollary 4 (or Theorem 9) in [32] any Cuntz algebra $\mathcal{O}_n$, $n \geq 2$, has a single generator C_n, i.e., $\mathcal{O}_n = C^*(C_n)$. We assert that C_n is strongly non-Følner. Indeed, assume that, to the contrary, it is not; then by Corollary 4.2(ii), C_n is finite. By Corollary 2.9, it would follow that $\mathcal{O}_n = C^*(C_n)$ has an amenable trace τ. But this gives a contradiction since applying τ to the Eqs. (6.1) and (6.2) we obtain $n = 1$. □

Corollary 6.2. *There exist invertible and contractible strongly non-Følner operators.*

Proof. From the previous examples of strongly non-Følner operators we can easily construct invertible or contractible strongly non-Følner operators. Just note that for any two complex constants $\lambda \neq 0$ and μ, an operator T is strongly non-Følner if and only if the operator $\lambda T + \mu \mathbb{1}$ is. □

It is also easy to see that for any operator T that has no Følner sequence, there is no orthogonal basis $\{e_n\}_{n=1}^{\infty}$ (nor $\{e_n\}_{n=-\infty}^{\infty}$) in $\mathcal{H}$ such that T is an "acute wedge" operator with respect to this basis (see the end of Section 5 for a definition). In [29], we will discuss these operators in more detail.

The above results allow one to present the classification of Table 1 in the following, more detailed way. Let $T \in \mathcal{L}(\mathcal{H})$, and put

$$\ell(T) := \sup_{R \subset \mathcal{H}} \dim R,$$

where the supremum is taken over all finite dimensional reducing subspaces R of T (including the zero one).

In the case when $\ell(T)$ is finite, there exists a unique reducing subspace $\mathcal{H}_0$ of T of dimension $\ell(T)$ (because $R_1 + R_2$ is a reducing subspace of T whenever R_1 and R_2 are). In this case, T decomposes as $T = T_0 \oplus T_1$ with respect to the orthogonal sum representation $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$. If $0 \leq \ell(T) < \infty$, put

$$\varepsilon(T) := \inf_{P \in \mathcal{P}_{\mathrm{fin}}(\mathcal{H}_0^{\perp})} \varphi(T_1, P).$$

Then we have the following cases.

Case 1: $0 \leq \ell(T) < \infty$. For these operators, Table 1 now looks as follows (see Table 2):

Table 2
Classification of operators in case 1.

	Operators with a proper Følner sequence ($\varepsilon(T) = 0$)	Operators with no proper Følner sequence ($\varepsilon(T) > 0$)
Finite block reducible ($0 < \ell(T) < \infty$)	$\mathcal{W}_{0+}$	$\mathcal{W}_{0-}$
Non finite block reducible ($\ell(T) = 0$)	$\mathcal{W}_{1+}$	$\mathcal{S}$

Case 2: $\ell(T) = \infty$. In this case, $\varepsilon(T)$ is not defined. These are exactly block diagonal operators, which are operators that can be decomposed into an infinite orthogonal sum of finite dimensional operators (see, e.g., [10, Chapter 16] or [40]). All these operators belong to $\mathcal{W}_{0+}$.

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Appendix. Single generators for C*-algebras

In the questions we are discussing, singly generated C*-algebras of operators have some relevance. See, e.g., the proof of Proposition 6.1, where it was crucial that the Cuntz algebra is singly generated. Moreover, from Theorem 5.1 and Proposition 2.1(i), any C^*-algebra generated by an essentially normal operator has a proper Følner sequence. However, we do not know in general whether any separable C*-subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ such that $[T, S] \in \mathcal{K}(\mathcal{H})$ for all $T, S \in \mathcal{A}$ (or, equivalently, all operators in $\mathcal{A}$ are essentially normal) has a proper Følner sequence. C*-algebras singly generated by an essentially normal operator were considered, e.g., in [20,21]. For a nice introduction to the single generator problem we refer to [37, Section 1] (see also [30,13]).

Here we will prove the following variation of a result in [32] which might be useful when addressing questions about Følner sequences for operators: roughly, every separable C*-algebra $\mathcal{A}$ of operators can be embedded into a singly generated C*-algebra acting on a larger Hilbert space. Notice that the word "separable" here refers to separability of the metric, associated to the operator norm. It is clear that a C*-subalgebra of $\mathcal{L}(\mathcal{H})$ is separable if and only if it is countably generated. In general if $\mathcal{A}$ is unital its image under the embedding need not be a unital C*-subalgebra of the larger algebra.

Proposition A.1. *Let $\mathcal{A}$ be a unital C*-subalgebra of $\mathcal{L}(\mathcal{H})$. Then the following two assertions are equivalent.*

(i) *$\mathcal{A}$ is separable.*

(ii) *There exists a separable Hilbert space $\mathcal{R} = \mathcal{H} \oplus \mathcal{H}'$, and a singly generated C*-subalgebra $\mathcal{B} \subset \mathcal{L}(\mathcal{R})$ such that $\mathcal{A} \oplus 0 \subset \mathcal{B}$, where $\mathcal{A} \oplus 0 = \{A \oplus 0 \in \mathcal{L}(\mathcal{R}) \mid A \in \mathcal{A}\}$.*

Proof. The implication (ii) $\Longrightarrow$ (i) is obvious. To show the reverse implication denote by $\mathcal{K}(\mathfrak{h})$ the C^*-algebra of compact operators on a separable Hilbert space $\mathfrak{h}$. Put $\mathcal{B} = \mathcal{A} \otimes \mathcal{K}(\mathfrak{h}) \subset \mathcal{L}(\mathcal{H} \otimes \mathfrak{h})$ (since $\mathcal{K}(\mathfrak{h})$ is nuclear, there is no ambiguity in the definition of the tensor product of these operator algebras; see, e.g., [27, vol. 2, Chapter 11]). Let $p \in \mathcal{K}(\mathfrak{h})$ be a rank one orthogonal projection onto the subspace $\langle e \rangle$ generated by a unit vector $e \in \mathfrak{h}$. The map $\Phi(a) := a \otimes p$ is an isometric $*$-isomorphism from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{K}(\mathfrak{h})$ and define $\mathcal{R} := \mathcal{H} \otimes \mathfrak{h}$. It is clear that $\Phi(\mathcal{A})$ has the form $\mathcal{A} \oplus 0$ in the decomposition $\mathcal{R} = (\mathcal{H} \otimes \langle e \rangle) \oplus \left(\mathcal{H} \otimes \langle e \rangle^{\perp}\right)$. Finally, by Theorem 8 in [32], $\mathcal{B}$ is singly generated. $\square$

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