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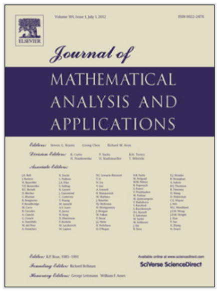

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Hardy and Rellich Type Inequalities on Complete Manifolds

Changyu Xia

Abstract

We prove some Hardy and Rellich type inequalities on complete non-compact Riemannian manifolds supporting a weight function which is not very far from the distance function in the Euclidean space.

1 Introduction

The classical Hardy inequality states that

$$\int_{\mathbb{R}^n} |\nabla u|^2 dx \geq \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} dx,$$

where $u \in C_0^\infty(\mathbb{R}^n)$ and $n \geq 3$. An extension of the Hardy's inequality is the following Rellich inequality :

$$\int_{\mathbb{R}^n} |\Delta u|^2 dx \geq \left(\frac{n(n-4)}{4}\right)^2 \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^4} dx,$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ and $n \geq 5$. The constants $\left(\frac{n-2}{2}\right)^2$ and $\left(\frac{n(n-4)}{4}\right)^2$ in the above inequalities are sharp, that is

$$\left(\frac{n-2}{2}\right)^2 = \inf_{0 \neq \phi \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 dx}{\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2} dx}$$

and

$$\left(\frac{n(n-4)}{4}\right)^2 = \inf_{0 \neq \phi \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\Delta u|^2 dx}{\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^4} dx}.$$

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For a bounded domain $\Omega \subset \mathbb{R}^n$, a much stronger version of Hardy inequality was obtained by Brézis and Vázquez [5]. More recently, Tertikas and Zographopoulos [21] obtained corresponding stronger version of Rellich's inequality as well as of the similar Rellich type inquality

$$\int_{\mathbb{R}^n} |\Delta u|^2 dx \geq \left(\frac{n(n-4)}{4}\right)^2 \int_{\mathbb{R}^n} \frac{|u|^2}{|x|^4} dx, \ \forall u \in C_0^\infty(\mathbb{R}^n), n \geq 3,$$

when $\mathbb{R}^n$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^n$.

In recent years, there are a very large number of papers dedicated to applications and generalizations of the Hardy, Rellich, Sobolev and Caffarelli-Kohn-Nirenberg inequalities in various contexts, e.g. in [1-3, 10-15], and the reference therein. In an interesting paper, Carron [7] studied weighted L^2-Hardy inequalities and obtained, among other results, the following inequality on a complete non-compact Riemannian manifold M:

$$\int_M \rho^\alpha |\nabla \phi|^2 \geq \left(\frac{C+\alpha-1}{2}\right)^2 \int_M \rho^{\alpha-2}\phi^2 \tag{1.1}$$

where $\alpha \in \mathbb{R}, C+\alpha-1 > 0, \phi \in C_0^\infty(M \setminus \rho^{-1}\{0\})$ and the weight function ρ is nonnegative and satisfies $|\nabla \rho| = 1$ and $\Delta \rho \geq \frac{C}{\rho}$ in the sense of distribution, being Δ the Laplace operator on M. Under the same geometric assumptions on the weight function ρ Kombe and Özaydin obtained in [16] an L^p version of (1.1) (where $1 < p < \infty$ and $C+1+\alpha-p > 0$):

$$\int_M \rho^\alpha |\nabla \phi|^p \geq \left(\frac{C+1+\alpha-p}{p}\right)^p \int_M \rho^{\alpha-p}|\phi|^p, \tag{1.2}$$

as well as a Rellich-type inequality (where $\alpha < 2, C+\alpha-3 > 0$):

$$\int_M \rho^\alpha (\Delta\phi)^2 \geq \frac{(C+\alpha-3)^2(C-\alpha+1)^2}{16} \int_M \rho^{\alpha-4}\phi^2. \tag{1.3}$$

Other kind of Hardy, Rellich, Heisenberg-Pauli-Weyl and Caffarelli-Kohn-Nirenberg type inequalities on complete non-compact manifolds have been also proved, e. g., in [4, 6, 8, 9, 17, 19, 22, 23], etc. Typical examples of manifolds supporting the above weight function ρ are n-dimensional Hadamard manifolds. In fact, the distance function r starting from any point in such a manifold satisfies $|\nabla r| = 1, \ \Delta r \geq \frac{n-1}{r}$ in the sense of distribution [20]. In this paper, we prove similar kind of Hardy, Rellich type inequalities for complete non-compact Riemannian manifolds which support a nonnegative weight function ρ satisfying $|\nabla \rho| = 1, \ \Delta \rho \leq C/\rho$ in the sense of distribution. Complete open manifolds of dimension n with nonnegative Ricci curvature are examples of our study since the distance function ρ starting from any point on these manifolds satisfies $|\nabla \rho| = 1, \ \Delta \rho \leq \frac{n-1}{\rho}$ in the sense of distribution [20].

2 Hardy and Rellich Inequalities on Complete Manifolds

In this section, we will prove several Hardy and Rellich type inequalities on complete non-compact Riemannian manifolds. Our first result is the following

Theorem 2.1. *Let M be an $n(\geq 2)$-dimensional complete noncompact Riemannian manifold. Let ρ be a nonnegative function on M such that $|\nabla\rho| = 1$ and $\Delta\rho \leq V + C/\rho$ in the sense of distribution where V is a continuous function on M and C is a constant. Then for any $p \in \mathbb{R}$, $q > 1 + C$ and all compactly supported smooth function $\phi \in C_0^\infty(M \setminus \rho^{-1}\{0\})$, the following inequality holds*

$$(q - C - 1)\int_M \frac{|\phi|^p}{\rho^q} \leq |p|\int_M \frac{|\phi|^{p-1}}{\rho^{q-1}}|\nabla\phi| + \int_M \frac{V|\phi|^p}{\rho^{q-1}}. \tag{2.1}$$

Proof. For a vector field X on M, we denote by div X the divergence of X. Observe that

$$\begin{aligned} \operatorname{div}\left(|\phi|^p\left(\frac{\nabla\rho}{\rho^{q-1}}\right)\right) &= p|\phi|^{p-1}\left\langle\nabla|\phi|, \frac{\nabla\rho}{\rho^{q-1}}\right\rangle + |\phi|^p\operatorname{div}\left(\frac{\nabla\rho}{\rho^{q-1}}\right) \\ &\leq p|\phi|^{p-1}\left\langle\nabla|\phi|, \frac{\nabla\rho}{\rho^{q-1}}\right\rangle + |\phi|^p\frac{(C - q + 1)}{\rho^q} + \frac{V|\phi|^p}{\rho^{q-1}}. \end{aligned} \tag{2.2}$$

Integrating (2.2) on M, we get

$$(q - C - 1)\int_M \frac{|\phi|^p}{\rho^q} \leq |p|\int_M \frac{|\phi|^{p-1}}{\rho^{q-1}}|\nabla\phi| + \int_M \frac{V|\phi|^p}{\rho^{q-1}}. \tag{2.3}$$

This proves (2.1).

In the special case that $V = 0$ in the above theorem, we have

Corollary 1.1. *Let M be an $n(\geq 2)$-dimensional complete noncompact Riemannian manifold. Let ρ be a nonnegative function on M such that $|\nabla\rho| = 1$ and $\Delta\rho \leq C/\rho$ in the sense of distribution where C is a constant. Then the following inequality holds*

$$\int_M \frac{|\phi|^p}{\rho^q} \leq \left(\frac{p}{q - C - 1}\right)^q \left(\int_M |\phi|^p\right)^{\frac{p-q}{p}} \left(\int_M |\nabla\phi|^p\right)^{\frac{q}{p}} \tag{2.4}$$

for all compactly supported smooth function $\phi \in C_0^\infty(M \setminus \rho^{-1}\{0\}), \max\{1, 1 + C\} < q \leq p < +\infty$.

Proof. From (2.1), we have

$$(q-C-1)\int_M \frac{|\phi|^p}{\rho^q} \le p\int_M \frac{|\phi|^{p-1}}{\rho^{q-1}}|\nabla\phi|. \tag{2.5}$$

It follows from Hölder's inequality that

$$\begin{aligned}\int_M \frac{|\phi|^{p-1}}{\rho^{q-1}}|\nabla\phi| &\le \left(\int_M |\nabla\phi|^p\right)^{\frac{1}{p}}\left(\int_M \frac{|\phi|^p}{\rho^{\frac{(q-1)p}{p-1}}}\right)^{\frac{p-1}{p}} \\ &\le \left(\int_M |\nabla\phi|^p\right)^{\frac{1}{p}}\left(\int_M \frac{|\phi|^p}{\rho^q}\right)^{\frac{q-1}{q}}\left(\int_M |\phi|^p\right)^{\left(1-\frac{p(q-1)}{q(p-1)}\right)\frac{p-1}{p}}.\end{aligned} \tag{2.6}$$

One can then easily obtain (2.4) by substituting (2.6) into (2.5). This completes the proof of Corollary 2.1.

The following result is a counterpart to (1.2) for manifolds admitting a nonnegative weight function ρ with $|\nabla\rho| = 1$ and $\Delta\rho \le C/\rho$ in the sense of distribution.

Theorem 2.2. *Let M be a complete noncompact Riemannian manifold of dimension $n \ge 2$ and let ρ be a nonnegative function on M such that $|\nabla\rho| = 1$ and $\Delta\rho \le V + C/\rho$ in the sense of distribution where V is a continuous function on M and C is a constant. Then for all compactly supported smooth function $\phi \in C_0^\infty(M \setminus \rho^{-1}\{0\})$, $1 < p < +\infty$ and $p-\alpha-C-1 > 0$, we have*

$$\begin{aligned}\int_M \rho^\alpha|\nabla\phi|^p &\ge \left(\frac{p-C-1-\alpha}{p}\right)^p\int_M \rho^{\alpha-p}|\phi|^p \\ &\quad -\left(\frac{p-C-1-\alpha}{p}\right)^{p-1}\int_M V\rho^{\alpha-p+1}|\phi|^p.\end{aligned} \tag{2.7}$$

Proof. Let $\gamma = (p-\alpha-C-1)/p$; then $\gamma > 0$. Let $\phi = \rho^\gamma f$ where $f \in C_0^\infty(M \setminus \rho^{-1}(0))$. When $a, b \in \mathbb{R}^n$, it holds that [18]

$$|a+b|^p \ge |a|^p + p|a|^{p-2}a\cdot b + \frac{|b|^p}{2^{p-1}-1} \text{ for } p \ge 2, \tag{2.8}$$

and

$$|a+b|^p \ge |a|^p + p|a|^{p-2}a\cdot b + \frac{c(p)|b|^2}{(|a|+|b|)^{2-p}}, \text{ for } 1 < p < 2, \text{ where } c(p) > 0. \tag{2.9}$$

Therefore, we have

$$\begin{aligned}\rho^\alpha|\nabla\phi|^p &= \rho^\alpha|\nabla(\rho^\gamma f)| \qquad (2.10)\\ &= \rho^\alpha|\gamma\rho^{\gamma-1}f\nabla\rho+\rho^\gamma\nabla f|\\ &\geq \gamma^p\rho^{\gamma p-p+\alpha}|f|^p+p\gamma^{p-1}\rho^{\alpha+\gamma p+1-p}|f|^{p-2}f\langle\nabla\rho,\nabla f\rangle\\ &= \gamma^p\rho^{\gamma p-p+\alpha}|f|^p+\frac{\gamma^{p-1}}{\alpha+\gamma p-p+2}\langle\nabla\rho^{\alpha+\gamma p+2-p},\nabla|f|^p\rangle,\ \text{when } C\neq 1.\end{aligned}$$

Similarly,

$$\rho^\alpha|\nabla\phi|^p\geq\gamma^p\rho^{\gamma p-p+\alpha}|f|^p+\gamma^{p-1}\langle\nabla\ln\rho,\nabla|f|^p\rangle \text{ when } C=1. \qquad (2.11)$$

In case $C\neq 1$, we integrate (2.10) on M and use the divergence theorem to get

$$\begin{aligned}&\int_M\rho^\alpha|\nabla\phi|^p \qquad (2.12)\\ \geq\ & \gamma^p\int_M\rho^{\gamma p-p+\alpha}|f|^p-\frac{\gamma^{p-1}}{\alpha+\gamma p-p+2}\int_M\Delta(\rho^{\alpha+\gamma p+2-p})|f|^p\\ \geq\ & \gamma^p\int_M\rho^{\gamma p-p+\alpha}|f|^p-\gamma^{p-1}\int_M(\alpha+\gamma p+C+1-p+\rho V)\rho^{\alpha+\gamma p-p}|f|^p\\ =\ & \gamma^p\int_M\rho^{\alpha+\gamma p-p}|f|^p-\gamma^{p-1}\int_M V\rho^{\alpha+\gamma p-p+1}|f|^p\\ =\ & \left(\frac{p-C-1-\alpha}{p}\right)^p\int_M\rho^{\alpha-p}|\phi|^p-\left(\frac{p-C-1-\alpha}{p}\right)^{p-1}\int_M V\rho^{\alpha-p+1}|\phi|^p.\end{aligned}$$

On the other hand, when $C=1$, we can integrate (2.11) on M to obtain

$$\begin{aligned}\int_M\rho^\alpha|\nabla\phi|^p &\geq \gamma^p\int_M\rho^{\alpha+\gamma p-p}|f|^p-\gamma^{p-1}\int_M|f|^p\Delta\ln\rho \qquad (2.13)\\ &\geq \gamma^p\int_M\rho^{\alpha+\gamma p-p}|f|^p-\gamma^{p-1}\int_M V|f|^p\rho^{-1}\\ &= \left(\frac{p-2-\alpha}{p}\right)^p\int_M\rho^{\alpha-p}|\phi|^p-\left(\frac{p-2-\alpha}{p}\right)^{p-1}\int_M V\rho^{\alpha-p+1}|\phi|^p.\end{aligned}$$

This shows that when $C=1$, the inequality (2.7) is also true. The proof of Theorem 2.2 is complete.

We now prove an improved Hardy inequality.

Theorem 2.3. *Let M be an n-dimensional complete noncompact Riemannian manifold and let ρ be a nonnegative function on M such that $|\nabla\rho|=1$ and $\Delta\rho\leq C/\rho$*

in the sense of distribution where C is a constant. Let Ω be a bounded domain with smooth boundary in M, $1 < q < 2,\ q - C - 1 - q\alpha/2 > 0,\ C + \alpha < 1,\ 0 < k < \min\{1, q(1 - C - \alpha)/(q - C - 1 - q\alpha/2)\},\ \phi \in C_0^\infty(\Omega \setminus \rho^{-1}\{0\})$. Then the following inequality is valid

$$\begin{aligned}\int_\Omega \rho^\alpha |\nabla\phi|^2 \ &\geq\ c_1(q)|\Omega|^{1-q/2}\left((1 + (q-1)k^q - qk^{q-1})\int_\Omega |\nabla\phi|^q \rho^{q\alpha/2}\right)^{2/q} \\ &\quad + \frac{k(q - C - 1 - \frac{q\alpha}{2})\left(q(1 - C - \alpha) - k\left(q - C - 1 - \frac{q\alpha}{2}\right)\right)}{q^2}\int_\Omega \rho^{\alpha-2}\phi^2,\end{aligned} \tag{2.14}$$

where $c_1(q)$ is a positive constant depending only on q and $|\Omega|$ is the volume of Ω.

Remark. It is easy to see that $1 + (q-1)k^q - qk^{q-1} > 0$ when $0 < k < 1 < q < 2$.

Proof of Theorem 2.3. Let $\beta > 0$. We have from the divergence theorem that

$$\begin{aligned}&\int_\Omega \left(|\nabla\phi|^2 - \langle\nabla(\frac{\phi^2}{\rho^\beta}), \nabla\rho^\beta\rangle\right)\rho^\alpha \\ =\ &\int_\Omega \rho^\alpha|\nabla\phi|^2 + \int_\Omega \frac{\phi^2}{\rho^\beta}\mathrm{div}(\rho^\alpha\nabla\rho^\beta) \\ \leq\ &\int_\Omega \rho^\alpha|\nabla\phi|^2 + \beta(\alpha + \beta + C - 1)\int_\Omega \rho^{\alpha-2}\phi^2 + \beta\int_\Omega V\rho^{\alpha-1}\phi^2.\end{aligned} \tag{2.15}$$

It is easy to see that

$$|\nabla\phi|^2 - \langle\nabla(\frac{\phi^2}{\rho^\beta}), \nabla\rho^\beta\rangle = \left|\nabla\phi - \frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^2, \tag{2.16}$$

and so

$$\begin{aligned}\int_\Omega \left(|\nabla\phi|^2 - \langle\nabla(\frac{\phi^2}{\rho^\beta}), \nabla\rho^\beta\rangle\right)\rho^\alpha \ &=\ \int_\Omega \left|\nabla\phi - \frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^2\rho^\alpha \\ &\geq\ |\Omega|^{1-2/q}\left(\int_\Omega \left|\nabla\phi - \frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^q \rho^{q\alpha/2}\right)^{2/q}.\end{aligned} \tag{2.17}$$

Now we use the following elementary inequality

$$c(q)|w_2|^q \geq |w_1 + w_2|^q - |w_1|^q - q|w_1|^{q-2}\langle w_1, w_2\rangle,\ 1 < q < 2,\ w_1, w_2 \in \mathbb{R}^n,$$

where $c(q) > 0$, Schwarz and Young's inequalities to obtain for any $\delta \in (0,1)$ that

$$\begin{aligned}
& c(q)\left|\nabla\phi - \frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^q \qquad (2.18)\\
\geq\ & |\nabla\phi|^q + (q-1)\left|\frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^q - q\left|\frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^{q-2}\cdot\langle\frac{\phi}{\rho^\beta}\nabla\rho^\beta, \nabla\phi\rangle\\
\geq\ & |\nabla\phi|^q + (q-1)\left|\frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^q - q\left|\frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^{q-1}\cdot|\nabla\phi|\\
\geq\ & |\nabla\phi|^q + (q-1)\left|\frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^q - q\left(\frac{(q-1)}{\delta q}\left|\frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^q + \frac{\delta^{q-1}|\nabla\phi|^q}{q}\right)\\
=\ & (1-\delta^{q-1})|\nabla\phi|^q + (q-1)\left(1-\frac{1}{\delta}\right)\left|\frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^q\\
=\ & (1-\delta^{q-1})|\nabla\phi|^q + (q-1)\left(1-\frac{1}{\delta}\right)\frac{\beta^q|\phi|^q}{\rho^q}.
\end{aligned}$$

Multiplying (2.18) by $\rho^{q\alpha/2}$ and integrating on Ω we have by using (2.7) that

$$\begin{aligned}
& c(q)\int_\Omega\left|\nabla\phi - \frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^q\rho^{q\alpha/2} \qquad (2.19)\\
\geq\ & \int_\Omega\left((1-\delta^{q-1})|\nabla\phi|^q + (q-1)\left(1-\frac{1}{\delta}\right)\frac{\beta^q|\phi|^q}{\rho^q}\right)\rho^{q\alpha/2}\\
\geq\ & \int_\Omega\left((1-\delta^{q-1}) + (q-1)\left(1-\frac{1}{\delta}\right)\left(\frac{\beta q}{q-C-1-\frac{q\alpha}{2}}\right)^q\right)|\nabla\phi|^q\rho^{q\alpha/2}.
\end{aligned}$$

Taking $\beta = \frac{k\left(q-C-1-\frac{q\alpha}{2}\right)}{q}, \delta = k$, we have

$$c(q)\int_\Omega\left|\nabla\phi - \frac{\phi}{\rho^\beta}\nabla\rho^\beta\right|^q \geq \left(1+(q-1)k^q - qk^{q-1}\right)\int_\Omega|\nabla\phi|^q\rho^{q\alpha/2}. \qquad (2.20)$$

Also, we deduce from (2.15) that

$$\begin{aligned}
& \int_\Omega\rho^\alpha|\nabla\phi|^2 \qquad (2.21)\\
\geq\ & \int_\Omega\left(|\nabla\phi|^2 - \langle\nabla(\frac{\phi^2}{\rho^\beta}), \nabla\rho^\beta\rangle\right)\rho^\alpha + \beta(-\beta-\alpha-C+1)\int_\Omega\rho^{\alpha-2}\phi^2\\
=\ & \int_\Omega\left(|\nabla\phi|^2 - \langle\nabla(\frac{\phi^2}{\rho^\beta}), \nabla\rho^\beta\rangle\right)\rho^\alpha\\
& + \frac{k(q-C-1-\frac{q\alpha}{2})\left(q(1-C-\alpha) - k\left(q-C-1-\frac{q\alpha}{2}\right)\right)}{q^2}\int_\Omega\rho^{\alpha-2}\phi^2.
\end{aligned}$$

One then gets (2.14) by combining (2.17), (2.20) and (2.21). This completes the proof of Theorem 1.3.

Our next result is a Rellich-type inequality which involves both first and second order derivatives.

Theorem 2.4. *Let M be a complete noncompact Riemannian manifold of dimension $n \geq 2$ and let ρ be a nonnegative function on M such that $|\nabla\rho| = 1$ and $\Delta\rho \leq C/\rho$ in the sense of distribution where $C < 1$. Then for all compactly supported smooth function $\phi \in C_0^\infty(M \setminus \rho^{-1}\{0\})$, $2 < \alpha < 3 - C$, we have*

$$\int_M \rho^\alpha (\Delta\phi)^2 \quad \geq \quad 2(\alpha-2)(3-\alpha-C)\int_M \rho^{\alpha-2}|\nabla\phi|^2. \tag{2.22}$$

Proof. It is easy to see that

$$\Delta\rho^{\alpha-2} \leq (\alpha-2)(C+\alpha-3)\rho^{\alpha-4}. \tag{2.23}$$

Multiplying the above inequality by ϕ^2 and integrating on M, we obtain

$$\begin{aligned}(\alpha-2)(C+\alpha-3)\int_M \rho^{\alpha-4}\phi^2 \quad &\geq \quad \int_M \rho^{\alpha-2}\Delta\phi^2 \\ &= \quad \int_M \rho^{\alpha-2}(2|\nabla\phi|^2 + 2\phi\Delta\phi).\end{aligned} \tag{2.24}$$

That is,

$$-2\int_M (\phi\Delta\phi)\rho^{\alpha-2} \geq 2\int_M \rho^{\alpha-2}|\nabla\phi|^2 - (\alpha-2)(C+\alpha-3)\int_M \rho^{\alpha-4}\phi^2.$$

Since

$$\begin{aligned}-2\int_M (\phi\Delta\phi)\rho^{\alpha-2} \quad \leq \quad &(\alpha-2)(3-C-\alpha)\int_M \rho^{\alpha-4}\phi^2 \\ &+\frac{1}{(\alpha-2)(3-C-\alpha)}\int_M \rho^\alpha(\Delta\phi)^2,\end{aligned}$$

we know that (2.22) is true.

The following is an improved Rellich inequality which is similar to the inequality (1.3).

Theorem 2.5. (Improved Rellich Inequality) *Let M be a complete noncompact Riemannian manifold of dimension $n > 1$. Let ρ be a nonnegative function on M such that $|\nabla\rho| = 1$ and $\Delta\rho \leq C/\rho$ in the sense of distribution. Let*

$1 < q < 2 < \alpha < 3 - C,\ 2q - C - 1 - q\alpha/2 > 0$. *Then there are positive constants* $C_1(q,\alpha,C)$, $C_2(q,\alpha,C)$ *depending only on* q,α,C, *such that for any* $\phi \in C_0^\infty(M \setminus \rho^{-1}\{0\})$, *the following inequality holds*

$$\int_M \rho^\alpha |\Delta\phi|^2 \geq C_1(q,\alpha,C)|S(\phi)|^{1-q/2}\left(\int_M |\nabla\phi|^q \rho^{q(\alpha-2)/2} dv\right)^{2/q} + C_2(q,\alpha,C)\int_M \rho^{\alpha-4}\phi^2, \tag{2.25}$$

where $|S(\phi)|$ *is the volume of the set* $S(\phi) =: \overline{\{x \in M : \phi(x) \neq 0\}}$.

Proof. Using the same discussions as in the proof of Theorem 2.4, we have

$$-2\int_M (\phi\Delta\phi)\rho^{\alpha-2} \geq 2\int_M \rho^{\alpha-2}|\nabla\phi|^2 - (\alpha-2)(C+\alpha-3)\int_M \rho^{\alpha-4}\phi^2. \tag{2.26}$$

Observe that

$$-2\int_M (\phi\Delta\phi)\rho^{\alpha-2} \leq \epsilon\int_M \rho^{\alpha-4}\phi^2 + \frac{1}{\epsilon}\int_M \rho^\alpha(\Delta\phi)^2, \tag{2.27}$$

where $\epsilon > 0$. For any $k \in (0,\ \min\{1, q(3-C-\alpha)/(2q-C-1-q\alpha/2)\})$, it follows from the improved Hardy inequality (2.14) that

$$\begin{aligned} & 2\int_M \rho^{\alpha-2}|\nabla\phi|^2 \\ \geq\ & \frac{2k(2q-C-1-q\alpha/2)(q(3-C-\alpha)-k(2q-C-1-q\alpha/2))}{q^2}\int_M \rho^{\alpha-4}\phi^2 \\ & + c_1(q)|S(\phi)|^{1-q/2}\left((1+(q-1)k^q - qk^{q-1})\int_M |\nabla\phi|^q \rho^{q(\alpha-2)/2}\right)^{2/q}, \end{aligned} \tag{2.28}$$

where $c_1(q)$ is a positive constant depending only on q. Substituting (2.26) and (2.28) into (2.27), we get

$$\begin{aligned} & \frac{1}{\epsilon}\int_M \rho^\alpha(\Delta\phi)^2 \\ \geq\ & \left(\frac{2k(2q-C-1-q\alpha/2)(q(3-C-\alpha)-k(2q-C-1-q\alpha/2))}{q^2} - \epsilon\right)\int_M \rho^{\alpha-4}\phi^2 \\ & + c_1(q)|\Omega|^{1-q/2}\left((1+(q-1)k^q - qk^{q-1})\int_M |\nabla\phi|^q \rho^{q(\alpha-2)/2} dv\right)^{2/q} \\ & + (\alpha-2)(3-C-\alpha)\int_M \rho^{\alpha-4}\phi^2. \end{aligned} \tag{2.29}$$

Taking

$$\epsilon = \frac{k(2q - C - 1 - q\alpha/2)(q(3 - C - \alpha) - k(2q - C - 1 - q\alpha/2))}{2q^2} + \frac{(\alpha - 2)(3 - C - \alpha)}{2},$$

we get (2.25). This completes the proof of Theorem 2.4.

Our final result in this paper is as follows.

Theorem 2.6. *Let M be a complete noncompact Riemannian manifold of dimension $n > 1$. Let ρ be a nonnegative function on M such that $|\nabla\rho| = 1$ and $\Delta\rho \leq C/\rho$ in the sense of distribution where $C > 0$. Then the following inequality holds*

$$\int_M \rho^{\alpha+p}|\langle\nabla\rho, \nabla\phi\rangle|^p \geq \left(\frac{|C + \alpha + 1|}{p}\right)^p \int_M \rho^\alpha|\phi|^p \tag{2.30}$$

for all $\phi \in C_0^\infty(M \setminus \rho^{-1}\{0\})$, $1 < p < \infty$, and $C + \alpha < -1$.

Proof. We have from the hypothesis that

$$\operatorname{div}(\rho\nabla\rho) \leq 1 + C. \tag{2.31}$$

Multiplying (2.31) by $\rho^\alpha|\phi|^p$ and integrating on M gives

$$\begin{aligned}
(C+1)\int_M \rho^\alpha|\phi|^p &\geq \int_M \operatorname{div}(\rho\nabla\rho)\rho^\alpha|\phi|^p \\
&= -\int_M \langle\rho\nabla\rho, \nabla(\rho^\alpha|\phi|^p)\rangle \\
&= -\alpha\int_M \rho^\alpha|\phi|^p + p\int_M |\phi|^{p-2}\phi\rho^{\alpha+1}\langle\nabla\rho, \nabla\phi\rangle.
\end{aligned} \tag{2.32}$$

Observe that $-(C+\alpha+1) > 0$. We obtain from (2.32) and the Hölder's and Young's inequalities that

$$\begin{aligned}
&|C + \alpha + 1|\int_M \rho^\alpha|\phi|^p \\
\leq\ & p\left|\int_M |\phi|^{p-2}\phi\rho^{\alpha+1}\langle\nabla\rho, \nabla\phi\rangle\right| \\
\leq\ & p\left(\int_M \rho^\alpha|\phi|^p\right)^{(p-1)/p}\left(\int_M \rho^{\alpha+p}|\langle\nabla\rho, \nabla\phi\rangle|^p\right)^{1/p} \\
\leq\ & (p-1)\epsilon^{-p/(p-1)}\int_M \rho^\alpha|\phi|^p + \epsilon^p\int_M \rho^{\alpha+p}|\langle\nabla\rho, \nabla\phi\rangle|^p
\end{aligned} \tag{2.33}$$

for any $\epsilon > 0$. Hence

$$\int_M \rho^{\alpha+p}|\langle\nabla\rho, \nabla\phi\rangle|^p \geq \epsilon^{-p}\left(|C+\alpha+1| - (p-1)\epsilon^{-p/(p-1)}\right)\int_M \rho^\alpha|\phi|^p. \quad (2.34)$$

The inequality (2.30) follows from (2.34) by taking

$$\epsilon = \left(\frac{p}{|C+\alpha+1|}\right)^{\frac{p-1}{p}},$$

This completes the proof of Theorem 2.6.

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References

[1] Adimurthi, M. Ramaswamy and N. Chaudhuri, An improved Hardy-Sobolev inequality and its applications, Proc. Amer. Math. Soc. **130** (2002), 489-505.

[2] G. Barbatis, S. Filippas and A. Tertikas, A unified approach to improved L_p Hardy inequalities with best constants, Trans. Amer. Math. Soc. **356** (2004), 2169-2196.

[3] G. Barbatis, Best constants for higher-order Rellich inequalities in $L_p(\Omega)$, Math. Z. **255** (2007), 877-896.

[4] Y. Bozhkov, A Caffarelli-Kohn-Nirenberg type inequality on Riemannian manifolds, App. Math. Lett. **23**(2010), 1166-1169.

[5] H. Brezis and J. L. Vázquez, Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complutense Madrid **10** (1997), 443-469.

[6] L. A. Caffarelli, R. Kohn and L. Nirenberg, First order interpolation inequalities with weights, Compositio Math. **53** (1984), 259-275.

[7] G. Carron, Inégalités de Hardy sur les variétés riemanniennes non-compactes, J. Math. Pures Appl. (9) 76 (1997), 883-891.

[8] F. Catrina and D. G. Costa, Sharp weighted-norm inequalities for functions with compact support in $\mathbb{R}^N \setminus \{0\}$, J. Diff. Equa. **246** (2009), 164-182.

[9] D.G. Costa, Some new and short proofs for a class of Caffarelli-Kohn-Nirenberg type inequalities, J. Math. Anal. Appl. **337** (2008) 311-317.

[10] C. Cowan, Optimal Hardy inequalities for general elliptic operators with improvements. Commun. Pure Appl. Anal. **9** (2010), 109-140.

[11] E. B. Davies, and A. M. Hinz, Explicit constants for Rellich inequalities in $L_p(\Omega)$, Math. Z. 227 (1998), no. 3, 511-523.

[12] G. B. Folland and A. Sitaram, The Uncertainty Principle: A Mathematical Survey, J. Fourier Anal. Appl. **3** (1997), 207-238.

[13] E. Fabes, C. Kenig and R. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm. in P.D.E., **7** (1982), 77-116.

[14] N. Ghoussoub, A. Moradifam. On the best possible remaining term in the Hardy inequality, Proc. Natl. Acad. Sci. USA **105** (2008), 13746-13751.

[15] N. Ghoussoub, A. Moradifam. Bessel pairs and optimal Hardy and Hardy-Rellich inequalities, Math. Ann. **349** (2011), 1-57.

[16] I. Kombe and M. Özaydin, Improved Hardy and Rellich inequalities on Riemannian manifolds, Trans. Amer. Math. Soc. **61** (2009), 6191-6203.

[17] I. Kombe and M. Özaydin, Hardy-Poncaré, Rellich and uncertainty principle inequalities on Riemannian manifolds, arXiv:1103.2747v1.

[18] P. Lindqvist, On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = 0$, Proc. Amer. Math. Soc. **109** (1990), 157-164.

[19] V. Minerbe, Weighted Sobolev inequalities and Ricci flat Manifolds, Geom. Funct. Anal. **18** (2009), 1696-1749.

[20] R.Schoen, S.T.Yau, Lectures on Differential Geometry, Cambridge, 2004, MA: International Press.

[21] A. Tertikas and N. Zographopoulos, Best constants in the Hardy-Rellich Inequalities and Related Improvements, Advances in Mathematics **209**, (2007), 407-459.

[22] C. Xia, The Gagliardo-Nirenberg inequalities and manifolds of non-negative Ricci curvature, *J. Funct. Anal.* 224 (2005), 230-241.

[23] C. Xia, The Caffarelli-Kohn-Nirenberg inequalities on complete manifolds, Math. Res. Lett. **14** (2007), 875-885.

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