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Exponential stability of uniform Euler–Bernoulli beams with non-collocated boundary controllers ☆

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ABSTRACT

We study the stability of a robot system composed of two Euler–Bernoulli beams with non-collocated controllers. By the detailed spectral analysis, we prove that the asymptotical spectra of the system are distributed in the complex left-half plane and there is a sequence of the generalized eigenfunctions that forms a Riesz basis in the energy space. Since there exist at most finitely many spectral points of the system in the right half-plane, to obtain the exponential stability, we show that one can choose suitable feedback gains such that all eigenvalues of the system are located in the left half-plane. Hence the Riesz basis property ensures that the system is exponentially stable. Finally we give some simulation for spectra of the system.

1. Introduction

In recent years there has been an increased interest in systems composed of multiple robots since they exhibit cooperative behaviors. Such systems are of interest for several reasons: on one hand, the tasks may be inherently too complex for a single robot to accomplish, or performance benefits can be gained from using multiple robots; on the other hand, building and using several simple robots can be easier, cheaper, more flexible and more fault-tolerant than having a single powerful robot for each separate task. Therefore, numerous cooperation problems emerge in the engineering. Thus the research on single robot systems was naturally extended to that on multiple-robot systems since ultimately a single robot, no matter how capable, is spatially limited. The main questions are how to cooperate to achieve the predictable aim.

The multiple-robot systems are different from other distributed systems because of their implicit "real-world" environment, they have presumably more difficulty in modeling than traditional components of distributed system environments (i.e., computers, databases, networks). In the present paper, we consider how two robot-arms should cooperate to be qualified in one stable movement. Our system shown as in Fig. 1 consists of two robot arms, their common task is to grab stably the huge mass M. To complete this task, we make the exterior forces F_i, $i = 1, 2$ act on two root ends such that the other ends can clamp the mass, and we observe the velocity of the mass end so that the system completes the performance. To model this system, we regard the mass as a tip mass and suppose that the arms are the same and uniform with length one, whose positions are denoted by $w_i(x, t)$ with the root ends $x = 0$ and the arm motions described by the Euler–Bernoulli beams. To complete the performance, we observe the rotation angles $w_{ix}(0, t)$, rotation angle velocities $w_{ixt}(0, t)$ and the velocity of other ends $w_{it}(1, t)$. As feedback controls we assume the control forces of the form

$$F_i = F_i(w_{ixt}(0, t), w_{ix}(0, t), w_{it}(1, t)), \quad i = 1, 2.$$

The performance requirement is to make the displacement continuous at the other end $x = 1$, that is,

$$w_1(1, t) = w_2(1, t), \quad \forall t > 0,$$

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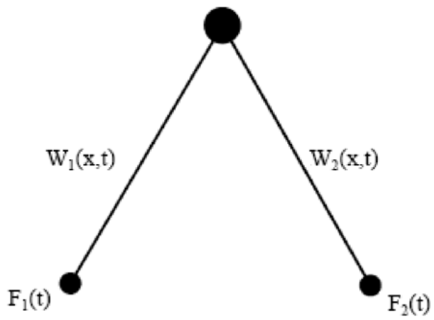

Fig. 1. Two robot arms.

here we neglect the size of mass. Thus, the motion of the system is governed by the partial differential equations:

$$\begin{cases} \dfrac{\partial^2 w_i(x,t)}{\partial t^2} + \dfrac{\partial^4 w_i(x,t)}{\partial x^4} = 0, \quad i = 1,2,\ x \in (0,1),\ t > 0, \\ M\dfrac{\partial^2 w(1,t)}{\partial t^2} = \displaystyle\sum_{i=1}^{2} \dfrac{\partial^3 w_i(1,t)}{\partial x^3}, \quad t > 0, \\ w_1(0,t) = w_2(0,t) = 0, \quad t > 0, \\ \dfrac{\partial^2 w_i(0,t)}{\partial x^2} = F_i(w_{ixt}(0,t), w_{ix}(0,t), w_{it}(1,t)), \quad i = 1,2,\ t > 0, \\ w_1(1,t) = w_2(1,t) = w(1,t), \\ \dfrac{\partial w_1(1,t)}{\partial x} = \dfrac{\partial w_2(1,t)}{\partial x}, \\ \displaystyle\sum_{i=1}^{2} \dfrac{\partial^2 w_i(1,t)}{\partial x^2} = 0, \quad t > 0, \\ w_i(x,0) = w_{i0}, \qquad w_{it}(x,0) = w_{i1}, \quad i = 1,2, \end{cases}$$

where x stands for the position and t the time and M is the tip mass at $x = 1$.

First, we consider the steady state of the system, $\widetilde{w}_i$, namely $\widetilde{w}_i$ satisfy differential equation

$$\begin{cases} \dfrac{d^4 \widetilde{w}_i(x)}{dx^4} = 0, \quad x \in (0,1),\ i = 1,2, \\ \widetilde{w}_1(0) = \widetilde{w}_2(0) = 0, \\ \dfrac{d^2 \widetilde{w}_i(0)}{dx^2} = F_i(0, \widetilde{w}_{ix}(0), 0), \quad i = 1,2, \\ \widetilde{w}_1(1) = \widetilde{w}_2(1), \qquad \dfrac{d\widetilde{w}_1(1)}{dx} = \dfrac{d\widetilde{w}_2(1)}{dx}, \\ \displaystyle\sum_{i=1}^{2} \dfrac{d^2 \widetilde{w}_i(1)}{dx^2} = 0, \\ \displaystyle\sum_{i=1}^{2} \dfrac{d^3 \widetilde{w}_i(1)}{dx^3} = 0. \end{cases}$$

From this we can determine the control forces $\widetilde{F}_i = F(0, w_{ix}, 0)$, $i = 1, 2$, that are constant forces.

Next we determine the forces F_i, $i = 1, 2$. Let $F_i(z, u, v)$, $i = 1, 2$ be twice continuously differentiable functions. Linearizing F_i at fixed point $\widetilde{W}_i = (0, \widetilde{w}_{ix}(0), 0)$ yields

$$\begin{aligned} F_i(w_{ixt}(0,t), w_{ix}(0,t), w_{it}(1,t)) &= F_i(0, \widetilde{w}_{ix}(0), 0) + \left.\frac{\partial F_i}{\partial z}\right|_{\widetilde{W}_i} w_{ixt}(0,t) \\ &\quad + \left.\frac{\partial F_i}{\partial u}\right|_{\widetilde{W}_i} (w_{ix}(0,t) - \widetilde{w}_{ix}(0)) + \left.\frac{\partial F_i}{\partial v}\right|_{\widetilde{W}_i} w_{it}(1,t) + o(w_i - \widetilde{w}_i). \end{aligned}$$

Let $y_i(x,t) = w_i(x,t) - \widetilde{w}_i(x)$ and denote by

$$\alpha_i = \left.\frac{\partial F_i}{\partial z}\right|_{\widetilde{W}_i}, \qquad \tau_i = \left.\frac{\partial F_i}{\partial u}\right|_{\widetilde{W}_i}, \qquad \beta_i = \left.\frac{\partial F_i}{\partial v}\right|_{\widetilde{W}_i}, \quad i = 1, 2.$$

Then the behavior of the error system is determined by the following linearized equations

$$\begin{cases} \dfrac{\partial^2 y_i(x,t)}{\partial t^2} + \dfrac{\partial^4 y_i(x,t)}{\partial x^4} = 0, \quad i = 1,2, x \in (0,1), \\ M\dfrac{d^2 y(1,t)}{dt^2} = \displaystyle\sum_{i=1}^{2} \dfrac{\partial^3 y_i(1,t)}{\partial x^3}, \\ y_1(0,t) = y_2(0,t) = 0, \\ \dfrac{\partial^2 y_i(0,t)}{\partial x^2} = u_i(t), \quad i = 1,2, \\ y_1(1,t) = y_2(1,t) = y(1,t), \\ \dfrac{\partial y_1(1,t)}{\partial x} = \dfrac{\partial y_2(1,t)}{\partial x}, \\ \displaystyle\sum_{i=1}^{2} \dfrac{\partial^2 y_i(1,t)}{\partial x^2} = 0, \\ y_i(x,0) = w_{i0} - \widetilde{w}_i, \qquad y_{it}(x,0) = w_{i1} - \widetilde{w}_i, \quad i = 1,2, \end{cases} \tag{1.1}$$

where

$$u_i(t) = \tau_i \frac{\partial y_i(0,t)}{\partial x} + \alpha_i \frac{\partial^2 y_i(0,t)}{\partial x \partial t} + \beta_i \frac{\partial y_i(1,t)}{\partial t}, \quad i = 1,2.$$

For simplification, we can take $F_i = \widetilde{F}_i + u_i(t)$. In control input $u_i(t)$, the terms $\frac{\partial y_i(0,t)}{\partial x}$ and $\frac{\partial^2 y_i(0,t)}{\partial x \partial t}$ are collocated to the force $\widetilde{F}_i$, but the term $\frac{\partial y_i(1,t)}{\partial t}$ is from the tip mass end. In the performance, we ensure the position term $y_1(1,t) = y_2(1,t)$ mainly based on the term $\frac{\partial y_i(1,t)}{\partial t}$, so $\beta_j \neq 0$ is necessary. Therefore the system (1.1) can be regarded as a linear system with the non-collocated feedback control law (namely, the actuators and the sensor are located at different positions) and $\alpha_i, \beta_i, \tau_i$ can be regarded as the feedback gain constants.

Finally, under guaranteeing the performance, we seek for the conditions on the coefficients τ_i, α_i and β_i that ensure the system (1.1) is exponentially stable. From the description above we see that guaranteeing the performance of the system requests the coefficients $\beta_j \neq 0, j = 1,2$. Under these restrictions, the stability of the original nonlinear system is equivalent to that of (1.1) by determining the parameters τ_i, α_i and β_i. Therefore, in the present paper, we mainly pay our attention to determine the parameters τ_i, α_i and β_i.

Non-collocated control has been widely used in engineering practice due to its more feasiblity (e.g. see, [1,4,18,11, 14,5,20,22,27,29]); however, there is few theoretical study including stability analysis on such systems from the view of mathematical control. The first difficulty for the non-collocated control comes from the non-minimum-phase of the open-loop form, a small increment of the feedback gain will lead to an unstable closed-loop system. The second difficulty arises from the non-dissipativity for closed-loop form, which gives rise to difficulty in applying the traditional Lyapunov methods or the energy multiplier methods to analyze the stability. Comparing with the huge works on the stabilization of collocated PDEs in existed literature, the study for non-collocated PDEs is fairly scarce. To analyze the stability of the system with non-collocated feedback, some authors used the finite-dimensional approximations of the observers for infinite-dimensional systems (for analytic systems), we refer the reader to the literature [2,9,10,12,13]. There were some authors using non-collocated feedback controllers to stabilize the system (e.g. see [17,8,7]). To remove the negative effect of non-collocated term, however, these authors must use more complex feedback control signal. The method used by them, however, does not fit our model. Recently our paper [3] adopt direct feedback manner to stabilize exponentially a wave system with variable coefficients, in which the key point is to choose the suitable feedback gains. In the present paper, we shall use the idea in [3] to determine the parameters τ_i, α_i and $\beta_i, i = 1,2$.

In what follows, we describe the method used in this paper. Consider the energy functional of the system (1.1), which is defined as

$$E(t) = \frac{1}{2}\sum_{j=1}^{2}\int_0^1 [|y_{jxx}(x,t)|^2 + |y_{jt}^2(x,t)|^2]dx + \frac{1}{2}My_t^2(1,t) + \sum_{j=1}^{2}\frac{1}{2}\tau_j|y_{jx}(0,t)|^2.$$

Formally differentiating with respect to t yields

$$\begin{aligned} \frac{dE(t)}{dt} &= \sum_{j=1}^{2}\int_0^1 [y_{jxx}y_{jxxt} + y_{jt}y_{jtt}]dx + My_t(1,t)y_{tt}(1,t) + \sum_{j=1}^{2}\tau_j y_{jx}(0,t)y_{jxt}(0,t) \\ &= \sum_{j=1}^{2}[-\alpha_j y_{jxt}^2(0,t) - \beta_j y_{jt}(1,t)y_{jxt}(0,t)]. \end{aligned}$$

Clearly, the energy of the system is not dissipative for all α_j, β_j, which means a non-minimum-phase system. Our aim is to choose suitable α_j, β_j such that the energy of the system decays exponentially, at least decays, i.e., $\frac{dE(t)}{dt} < 0$. In the present paper, the main task is to prove that this can be done.

To analyze the stability of (1.1), we employ the asymptotic analysis technique to obtain the asymptotic frequency of the system (the asymptotic spectrum of the system operator). Furthermore we prove the Riesz basis property of the eigenvectors of this system. Note that the Riesz basis property implies the spectrum determined growth condition. Hence we can assert the stability of the system can be determined via spectrum of the system operator. This approach has been extensively used in the system analysis, for instance, [21,6,19,24,23] for single beam or string system and [25] for the serially connected Timoshenko beams. Our model is different from the literature mentioned above, however, there may exist finitely many eigenvalues of the system in the right half-plane, although its asymptotic spectra are in the left half-plane. So we shall choose suitable parameters α_i and β_i such that all spectra are located in the left half-plane.

The rest is organized as follows. In Section 2, we discuss the well-posedness of the system (1.1). In Section 3, we carry out a complete asymptotic analysis for the spectrum of the system operator. In Section 4, we prove the Riesz basis property of the eigenvectors and generalized eigenvectors of the system operator. In Section 5, we discuss the exponential stability of the system under some conditions and give some simulation for the spectrum of the system. Section 6 concludes.

2. Well-posedness of the system

In this section, we shall study the well-posedness of system (1.1). We begin with formulating the system into an appropriate Hilbert state space.

Let $H^k(0,1)$ be the usual Sobolev space. Set

$$Y := \{y = (y_1, y_2) \in H^2(0,1) \times H^2(0,1) | y_1(0) = y_2(0) = 0, y_1(1) = y_2(1), y_1'(1) = y_2'(1)\}$$

and $Z = \{z = (z_1, z_2) \in L^2[0,1] \times L^2[0,1]\}$. Let state space

$$\mathcal{H} = Y \times Z \times \mathbb{C}, \tag{2.1}$$

equipped the inner product, for any $(y, z, p), (f, g, h) \in \mathcal{H}$, via

$$\langle (y,z,p),(f,g,h)\rangle_{\mathcal{H}} = \sum_{j=1}^{2}\int_0^1 \left[y_j''(x)\overline{f_j''(x)} + z_j(x)\overline{g_j(x)}\right]dx + \sum_{j=1}^{2}\tau_j y_j'(0)\overline{f_j'(0)} + Mp\overline{h},$$

by which the induced norm is

$$\|(y,v,p)\|^2 = \sum_{i=1}^{2}\int_0^1 [|y_i''(x)|^2 + |v_i(x)|^2]dx + \sum_{i=1}^{2}\tau_i|y_i'(0)|^2 + M|p|^2. \tag{2.2}$$

Obviously, $(\mathcal{H}, \|\cdot\|)$ is a Hilbert space.

Define the operator $\mathcal{A}$ in $\mathcal{H}$ by

$$\mathcal{A}(y,v,p) = \left(v, -y^{(4)}, \frac{1}{M}\sum_{j=1}^{2} y_j'''(1)\right) \tag{2.3}$$

with domain

$$D(\mathcal{A}) = \left\{\begin{array}{l} (y,v,p) \in Y \cap [H^4(0,1)]^2 \times Y \times \mathbb{C}, \\ y_j''(0) = \tau_j y_j'(0) + \alpha_j v_j'(0) + \beta_j v_j(1), \quad j = 1, 2, \\ p = v_1(1) = v_2(1), \qquad \displaystyle\sum_{j=1}^{2} y_j''(1) = 0, \end{array}\right\}. \tag{2.4}$$

With the operator $\mathcal{A}$ at hand, we can write system (1.1) into an evolutionary equation in $\mathcal{H}$

$$\begin{cases} \dfrac{dY(t)}{dt} = \mathcal{A}Y(t), \quad t > 0, \\ Y(0) = Y_0 \end{cases} \tag{2.5}$$

where $Y(t) = (y(\cdot,t), \frac{\partial y}{\partial t}(\cdot,t), \frac{\partial y}{\partial t}(1,t))$ and $Y_0 = (U_0, U_1, U_2) \in \mathcal{H}$ is the initial data given.

To discuss the well-posedness of system (1.1), we need the following definition of dissipative operator.

Let X be a Banach space and X^* be its dual space, and denote by $\langle x^*, x\rangle$ the value of $x^* \in X^*$ at $x \in X$. For every $x \in X$, the duality set of x, $F(x) \subseteq X^*$, is defined as

$$F(x) = \{x^* \in X^* | \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}.$$

Definition 2.1 (*[15]*)**.** A linear operator $\mathcal{A}$ is dissipative if for every $x \in \mathcal{D}(\mathcal{A})$, there is a $x^* \in F(x)$ such that $\Re\langle Ax, x^*\rangle \le 0$.

Lemma 2.1. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined by (2.1), (2.3) and (2.4) respectively. Then $\mathcal{A}$ is a closed and densely defined linear operator in $\mathcal{H}$. If $\beta_j \le \alpha_j\gamma_j, j = 1, 2$, then $\mathcal{A} - (\frac{1}{M}\sum_{j=1}^{2}\beta_j\gamma_j)I$ is a dissipative operator in $\mathcal{H}$.*

Proof. It is easy to check that $\mathcal{A}$ is a densely defined and closed linear operator. Here we only prove $\mathcal{A} - (\frac{1}{M}\sum_{j=1}^{2}\beta_j\gamma_j)I$ is dissipative.

For any $F = (y, v, p) \in D(\mathcal{A})$, a directly calculation gives

$$\begin{aligned}\langle \mathcal{A}F, F\rangle_{\mathcal{H}} &= \sum_{i=1}^{2}\int_0^1 \left[v_i''(x)\overline{y_i''(x)} - y_i^{(4)}(x)\overline{v_i(x)}\right]dx + \sum_{i=1}^{2} y_i'''(1)\overline{p} + \sum_{i=1}^{2}\tau_i v_i'(0)\overline{y_i'(0)}\\ &= \sum_{i=1}^{2}\left[-\overline{v_i'(0)}y_i''(0) + \tau_i v_i'(0)\overline{y_i'(0)}\right] + \sum_{i=1}^{2}\left[\int_0^1 v_i''(x)\overline{y_i''(x)}dx - \int_0^1 y_i''(x)\overline{v_i''(x)}dx\right].\end{aligned}$$

So we have

$$2\Re\langle \mathcal{A}F, F\rangle_{\mathcal{H}} = \sum_{i=1}^{2} -\left[\overline{v_i'(0)}y_i''(0) + v_i'(0)\overline{y_i''(0)}\right] + \sum_{i=1}^{2}\tau_i\left[v_i'\overline{y_i'(0)} + y_i'(0)\overline{v_i'(0)}\right].$$

Using the connective and boundary conditions, we get that

$$\begin{aligned}\Re\langle \mathcal{A}F, F\rangle_{\mathcal{H}} &= -\sum_{i=1}^{2}\left[\alpha_i|v_i'(0)|^2 + \beta_i\Re(v_i(1)\overline{v_i'(0)})\right]\\ &\le -\sum_{i=1}^{2}\alpha_i|v_i'(0)|^2 + \sum_{i=1}^{2}\beta_i\left(\gamma_i|v_i(1)|^2 + \frac{1}{\gamma_i}|v_i'(0)|^2\right)\\ &\le \sum_{i=1}^{2}\left[\left(-\alpha_i + \frac{\beta_i}{\gamma_i}\right)|v_i'(0)|^2 + \frac{\beta_i\gamma_i}{M}M|p|^2\right]\\ &\le \sum_{i=1}^{2}\left[\left(-\alpha_i + \frac{\beta_i}{\gamma_i}\right)|v_i'(0)|^2 + \frac{\beta_i\gamma_i}{M}\langle F, F\rangle_{\mathcal{H}}\right]\end{aligned}$$

where $\gamma_i > 0$. So it holds that

$$\Re\left\langle\left(\mathcal{A} - \frac{1}{M}\sum_{i=1}^{2}\beta_i\gamma_i I\right)F, F\right\rangle_{\mathcal{H}} \le \sum_{i=1}^{2}\left(-\alpha_i + \frac{\beta_i}{\gamma_i}\right)|v_i'(0)|^2 \le 0$$

if $\alpha_i\gamma_i \ge \beta_i$, $i = 1, 2$. The desired result follows. □

Lemma 2.2. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before. Then $0 \in \rho(\mathcal{A})$ and $\mathcal{A}^{-1}$ is compact on $\mathcal{H}$, and hence the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ consists of all isolated eigenvalues of finite algebraic multiplicities. In particular, all eigenvalues appear in conjugate pairs on the complex plane.*

Proof. For any $F = (f, g, h) \in \mathcal{H}$, we consider the solvability of equation

$$\mathcal{A}Y = F, \quad Y = (y, v, p) \in \mathcal{D}(\mathcal{A}),$$

that is, y, v and p satisfy the equations

$$v(x) = f(x), \tag{2.6}$$

$$-y_i^{(4)}(x) = g_i(x), \quad i = 1, 2, \tag{2.7}$$

$$\frac{1}{M}\sum_{i=1}^{2} y_i'''(1) = h, \tag{2.8}$$

and the boundary conditions

$$\begin{cases} y_1(0) = y_2(0) = 0, \qquad y_1(1) = y_2(1),\\ y_i''(0) = \tau_i y_i'(0) + \alpha_i v_i'(0) + \beta_i v_i(1), \quad i = 1, 2,\\ p = v_1(1) = v_2(1), \qquad y_1'(1) = y_2'(1),\\ \displaystyle\sum_{i=1}^{2} y_i''(1) = 0. \end{cases} \tag{2.9}$$

Solving the differential equation (2.7) yields

$$y_i(x) = y_i(0) + xy_i'(0) + \frac{x^2}{2}y_i''(0) + \frac{x^3}{3!}y_i'''(0) - \int_0^x \frac{(x-r)^3}{3!}g_i(r)dr. \tag{2.10}$$

For simplification, we denote

$$G_i(x) = \int_0^x \frac{(x-r)^3}{3!}g_i(r)dr, \quad i = 1, 2,$$

and

$$\begin{aligned} &k_1 = G_1(x) - G_2(x)|_{x=1}, \qquad k_2 = G_{1x}(x) - G_{2x}(x)|_{x=1}, \\ &k_3 = \tau_1 G_1'(x) - G_1''(x)|_{x=0} - \alpha_1 f_{1x}(0) - \beta_1 f_1(1), \\ &k_4 = \tau_2 G_2'(x) - G_2''(x)|_{x=0} - \alpha_2 f_{2x}(0) - \beta_2 f_2(1), \\ &k_5 = G_1''(x) + G_2''(x)|_{x=1}, \qquad k_6 = G_1'''(x) + G_2'''(x)|_{x=1} + Mh. \end{aligned}$$

Set

$$\begin{cases} X = [y_1'(0), y_1''(0), y_1'''(0), y_2'(0), y_2''(0), y_2'''(0)]^T, \\ K = [k_1, \dots, k_6]^T. \end{cases} \tag{2.11}$$

Substituting (2.10) into the boundary conditions (2.9) leads to the following algebraic equations

$$CX = K \tag{2.12}$$

where the coefficient matrix is

$$C = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3!} & -1 & -\frac{1}{2} & -\frac{1}{3!} \\ 1 & 1 & \frac{1}{2} & -1 & -1 & -\frac{1}{2} \\ \tau_1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tau_2 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

A direct calculation shows $\det[C] = \frac{1}{2}\tau_1 + (1 + \frac{\tau_1}{\tau_2}) \neq 0$. So Eqs. (2.12) have a unique solution X and hence we can determine uniquely functions $y_i(x) \in H^4(0, 1)$, $i = 1, 2$. Clearly, $(y, v, p) = (y, f, f_1(1)) \in D(\mathcal{A})$ and $\mathcal{A}(y, v, p) = (f, g, h)$. The inverse operator theorem asserts that $\mathcal{A}^{-1}$ is bounded on $\mathcal{H}$, which indicates that $0 \in \rho(\mathcal{A})$. In addition, the fact that $D(\mathcal{A}) \subset Y \cap [H^4(0, 1)]^2 \times Y \times \mathbb{C} \subset \mathcal{H}$ shows that $\mathcal{A}^{-1}$ is compact due to Sobolev's Embedding theorem. The spectral theory of linear compact operator shows that $\sigma(\mathcal{A})$ consists of all isolated eigenvalues of finite algebraic multiplicities. Note that $\mathcal{A}$ is a real linear operator on $\mathcal{H}$, so its spectrum distributes symmetrically with respect to the real axis. This completes the proof. □

Thanks to the above lemmas and Lumer–Phillips theorem (see, [15]), we have the following result.

Theorem 2.1. *Let $\mathcal{A}$ and $\mathcal{H}$ be defined as before. Then $\mathcal{A}$ generates a C_0 semigroup $T(t)$ on $\mathcal{H}$. Hence, the system (1.1) is well-posed, that is, for any $Y_0 \in \mathcal{H}$, the system (1.1) has a unique solution $Y(t) = T(t)Y_0$.*

3. Asymptotic eigenvalue problem

To investigate the properties of the semigroup $T(t)$ generated by $\mathcal{A}$, we need to find out some spectral properties of $\mathcal{A}$. We know from Lemma 2.2 that the spectrum of $\mathcal{A}$ consists of all isolated eigenvalues, so we need only to discuss the eigenvalue of $\mathcal{A}$ and its asymptotical distribution. In this section, we shall calculate the asymptotic values of eigenvalues of $\mathcal{A}$.

Let $\lambda \in \mathbb{C}$, we consider the existence of a nonzero solution of the equation

$$\mathcal{A}(y, v, p) = \lambda(y, v, p), \quad (y, v, p) \in \mathcal{D}(\mathcal{A}).$$

This is equivalent to $v = \lambda y$, $p = v_1(1) = v_2(1) = \lambda y(1)$ and y_i satisfies the following boundary eigenvalue problem

$$\begin{cases} \lambda^2 y_i(x) + y_i^{(4)}(x) = 0, \\ y_1(0) = y_2(0) = 0, \\ y_i''(0) - (\tau_i + \lambda\alpha_i) y_i'(0) - \lambda\beta_i y_i(1) = 0, \\ y_1(1) = y_2(1) = y(1), \qquad y_1'(1) = y_2'(1), \\ \displaystyle\sum_{i=1}^{2} y_i''(1) = 0, \\ \displaystyle\sum_{i=1}^{2} y_i'''(1) - M\lambda^2 y(1) = 0. \end{cases} \tag{3.1}$$

To solve the above equations, let $\lambda = \mu^2$, we rewrite above equations into the equations about parameter μ as follows:

$$\begin{cases} \mu^4 y_i(x) + y_i^{(4)}(x) = 0, \\ y_1(0) = y_2(0) = 0, \\ y_i''(0) - (\tau_i + \mu^2\alpha_i) y_i'(0) - \mu^2\beta_i y_i(1) = 0, \quad i = 1, 2, \\ y_1(1) = y_2(1) = y(1), \qquad y_1'(1) = y_2'(1), \\ \displaystyle\sum_{i=1}^{2} y_i''(1) = 0, \\ \displaystyle\sum_{i=1}^{2} y_i'''(1) - \frac{1}{2} M\mu^4 \sum_{i=1}^{2} y_i(1) = 0. \end{cases} \tag{3.2}$$

Here we use the equality $y(1) = y_1(1) = y_2(1) = \frac{1}{2}(y_1(1) + y_2(1))$. Obviously, the differential equation has general solution

$$y_i(x) = \sum_{j=1}^{4} a_{i,j} e^{\omega_j \mu x}$$

where all ω_j are the distinct root of equation $\omega^4 = -1$. Substituting these expressions into the boundary conditions in (3.2) leads to

$$\begin{cases} a_{11} + a_{12} + a_{13} + a_{14} = 0, \\ a_{21} + a_{22} + a_{23} + a_{24} = 0, \\ c_{11}(\mu)a_{11} + c_{12}(\mu)a_{12} + c_{13}(\mu)a_{13} + c_{14}(\mu)a_{14} = 0, \\ c_{21}(\mu)a_{21} + c_{22}(\mu)a_{22} + c_{23}(\mu)a_{23} + c_{24}(\mu)a_{24} = 0, \\ e^{\omega_1\mu}(a_{11} - a_{21}) + e^{\omega_2\mu}(a_{12} - a_{22}) + e^{\omega_3\mu}(a_{13} - a_{23}) + e^{\omega_4\mu}(a_{14} - a_{24}) = 0, \\ \omega_1 e^{\omega_1\mu}(a_{11} - a_{21}) + \omega_2 e^{\omega_2\mu}(a_{12} - a_{22}) + \omega_3 e^{\omega_3\mu}(a_{13} - a_{23}) + \omega_4 e^{\omega_4\mu}(a_{14} - a_{24}) = 0, \\ \omega_1^2 e^{\omega_1\mu}(a_{11} + a_{21}) + \omega_2^2 e^{\omega_2\mu}(a_{12} + a_{22}) + \omega_3^2 e^{\omega_3\mu}(a_{13} + a_{23}) + \omega_4^2 e^{\omega_4\mu}(a_{14} + a_{24}) = 0, \\ \left(\omega_1^3 - \dfrac{M}{2}\mu\right) e^{\omega_1\mu}(a_{11} + a_{21}) + \left(\omega_2^3 - \dfrac{M}{2}\mu\right) e^{\omega_2\mu}(a_{12} + a_{22}) + \left(\omega_3^3 - \dfrac{M}{2}\mu\right) e^{\omega_3\mu}(a_{13} + a_{23}) \\ \quad + \left(\omega_4^3 - \dfrac{M}{2}\mu\right) e^{\omega_4\mu}(a_{14} + a_{24}) = 0 \end{cases} \tag{3.3}$$

where

$$c_{ij}(\mu) = (\omega_j)^2 - \frac{\tau_i\omega_j}{\mu} - \alpha_i\omega_j\mu - \beta_i e^{\omega_j\mu}.$$

Denote by $d_j(\mu) = \omega_j^3 - \frac{M}{2}\mu$ and

$$D(\mu) = \det \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ c_{11}(\mu) & c_{12}(\mu) & c_{13}(\mu) & c_{14}(\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{21}(\mu) & c_{22}(\mu) & c_{23}(\mu) & c_{24}(\mu) \\ e^{\omega_1\mu} & e^{\omega_2\mu} & e^{\omega_3\mu} & e^{\omega_4\mu} & -e^{\omega_1\mu} & -e^{\omega_2\mu} & -e^{\omega_3\mu} & -e^{\omega_4\mu} \\ \omega_1 e^{\omega_1\mu} & \omega_2 e^{\omega_2\mu} & \omega_3 e^{\omega_3\mu} & \omega_4 e^{\omega_4\mu} & -\omega_1 e^{\omega_1\mu} & -\omega_2 e^{\omega_2\mu} & -\omega_3 e^{\omega_3\mu} & -\omega_4 e^{\omega_4\mu} \\ \omega_1^2 e^{\omega_1\mu} & \omega_2^2 e^{\omega_2\mu} & \omega_3^2 e^{\omega_3\mu} & \omega_4^2 e^{\omega_4\mu} & \omega_1^2 e^{\omega_1\mu} & \omega_2^2 e^{\omega_2\mu} & \omega_3^2 e^{\omega_3\mu} & \omega_4^2 e^{\omega_4\mu} \\ d_1(\mu) e^{\omega_1\mu} & d_2(\mu) e^{\omega_2\mu} & d_3(\mu) e^{\omega_3\mu} & d_4(\mu) e^{\omega_4\mu} & d_1(\mu) e^{\omega_1\mu} & d_2(\mu) e^{\omega_2\mu} & d_3(\mu) e^{\omega_3\mu} & d_4(\mu) e^{\omega_4\mu} \end{bmatrix}. \tag{3.4}$$

Then we have the following result.

Lemma 3.1. *Let $\mathcal{A}$ be defined as before and $D(\mu)$ be defined as (3.4), then $\lambda = \mu^2 \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}$ if and only if $D(\mu) = 0$, i.e.,*

$$\sigma(\mathcal{A}) = \{\lambda = \mu^2 | D(\mu) = 0, \mu \in \mathbb{C}\}.$$

We are now in a position to determine the zeros of $D(\mu)$. Due to the symmetry of the spectrum of $\mathcal{A}$ with respect to the real axis, we only need to discuss the case that $\arg\lambda \in (0, \pi)$ (complex up-half-plane), or equivalently $\arg\mu \in (0, \frac{\pi}{2})$.

First, we consider the cases that $\arg\lambda \in (0, \frac{\pi}{4})$ and $\arg\lambda \in (\frac{3\pi}{4}, \pi)$.

When $\arg\lambda \in (0, \frac{\pi}{4})$, we have $\arg\mu \in (0, \frac{\pi}{8})$. We order ω_j as follows

$$\omega_1 = e^{\frac{3\pi}{4}i}, \qquad \omega_2 = e^{\frac{5\pi}{4}i}, \qquad \omega_3 = e^{\frac{\pi}{4}i}, \qquad \omega_4 = e^{\frac{7\pi}{4}i}$$

so that

$$\Re(\omega_1\mu) \le \Re(\omega_2\mu) < 0 < \Re(\omega_3\mu) \le \Re(\omega_4\mu), \quad \forall \arg\mu \in \left(0, \frac{\pi}{8}\right).$$

When $\arg\lambda \in (\frac{3\pi}{4}, \pi)$, we have $\arg\mu \in (\frac{3\pi}{8}, \frac{\pi}{2})$. Taking

$$\omega_1 = e^{\frac{3\pi}{4}i}, \qquad \omega_2 = e^{\frac{\pi}{4}i}, \qquad \omega_3 = e^{\frac{5\pi}{4}i}, \qquad \omega_4 = e^{\frac{7\pi}{4}i},$$

we have

$$\Re(\omega_1\mu) \le \Re(\omega_2\mu) < 0 < \Re(\omega_3\mu) \le \Re(\omega_4\mu), \quad \forall \arg\mu \in \left(\frac{3\pi}{8}, \frac{\pi}{2}\right).$$

In both cases, we always have $\max_\mu \Re(\omega_2\mu) \le 0 \le \min_\mu \Re(\omega_3\mu)$. Therefore when $|\mu| \to \infty$ with $\arg\mu \in (0, \frac{\pi}{8}) \cup (\frac{3\pi}{8}, \frac{\pi}{2})$, we find out

$$\lim_{|\mu|\to\infty} \frac{D(\mu)}{\mu^3 e^{2(\omega_3+\omega_4)\mu}} = \det\begin{vmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ -\omega_1\alpha_1 & -\omega_2\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_1\alpha_2 & -\omega_2\alpha_2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & \omega_3 & \omega_4 & 0 & 0 & -\omega_3 & -\omega_4 \\ 0 & 0 & \omega_3^2 & \omega_4^2 & 0 & 0 & \omega_3^2 & \omega_4^2 \\ 0 & 0 & -\dfrac{M}{2} & -\dfrac{M}{2} & 0 & 0 & -\dfrac{M}{2} & -\dfrac{M}{2} \end{vmatrix} \neq 0.$$

So there is at most finitely many eigenvalues of $\mathcal{A}$ in the region $\{\lambda \in \mathbb{C} | \arg\lambda \in (0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)\}$.

For region $S = \{\lambda \in \mathbb{C} | \arg\lambda \in (\frac{\pi}{4}, \frac{3\pi}{4})\}$, corresponding μ-plane domain is $\arg\mu \in (\frac{\pi}{8}, \frac{3\pi}{8})$, we range ω_j as follows

$$\omega_1 = e^{\frac{3\pi}{4}i}, \qquad \omega_2 = e^{\frac{5\pi}{4}i}, \qquad \omega_3 = e^{\frac{\pi}{4}i}, \qquad \omega_4 = e^{\frac{7\pi}{4}i}, \quad \text{for } \arg\mu \in \left(\frac{\pi}{8}, \frac{\pi}{4}\right)$$

and

$$\omega_1 = e^{\frac{3\pi}{4}i}, \qquad \omega_2 = e^{\frac{\pi}{4}i}, \qquad \omega_3 = e^{\frac{5\pi}{4}i}, \qquad \omega_4 = e^{\frac{7\pi}{4}i}, \quad \text{for } \arg\mu \in \left(\frac{\pi}{4}, \frac{3\pi}{8}\right),$$

so that

$$\Re(\omega_1\mu) \le \Re(\omega_2\mu) < 0 < \Re(\omega_3\mu) \le \Re(\omega_4\mu), \quad \forall \arg\mu \in \left(\frac{\pi}{8}, \frac{\pi}{8}\right).$$

Note that, in that case, it always holds that $\omega_2 = -\omega_3$ and $\omega_1 = -\omega_4$. Thus,

$$D(\mu) = \det\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ c_{11}(\mu) & c_{12}(\mu) & c_{13}(\mu) & c_{14}(\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{21}(\mu) & c_{22}(\mu) & c_{23}(\mu) & c_{24}(\mu) \\ e^{\omega_1\mu} & e^{\omega_2\mu} & e^{\omega_3\mu} & e^{\omega_4\mu} & -e^{\omega_1\mu} & -e^{\omega_2\mu} & -e^{\omega_3\mu} & -e^{\omega_4\mu} \\ \omega_1 e^{\omega_1\mu} & \omega_2 e^{\omega_2\mu} & \omega_3 e^{\omega_3\mu} & \omega_4 e^{\omega_4\mu} & -\omega_1 e^{\omega_1\mu} & -\omega_2 e^{\omega_2\mu} & -\omega_3 e^{\omega_3\mu} & -\omega_4 e^{\omega_4\mu} \\ \omega_1^2 e^{\omega_1\mu} & \omega_2^2 e^{\omega_2\mu} & \omega_3^2 e^{\omega_3\mu} & \omega_4^2 e^{\omega_4\mu} & \omega_1^2 e^{\omega_1\mu} & \omega_2^2 e^{\omega_2\mu} & \omega_3^2 e^{\omega_3\mu} & \omega_4^2 e^{\omega_4\mu} \\ d_1(\mu)e^{\omega_1\mu} & d_2(\mu)e^{\omega_2\mu} & d_3(\mu)e^{\omega_3\mu} & d_4(\mu)e^{\omega_4\mu} & d_1(\mu)e^{\omega_1\mu} & d_2(\mu)e^{\omega_2\mu} & d_3(\mu)e^{\omega_3\mu} & d_4(\mu)e^{\omega_4\mu} \end{vmatrix}$$

$$= \det\begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ c_{11}(\mu) & c_{12}(\mu) & c_{13}(\mu) & c_{14}(\mu) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{21}(\mu) & c_{22}(\mu) & c_{23}(\mu) & c_{24}(\mu) \\ e^{\omega_1\mu} & e^{\omega_2\mu} & e^{-\omega_2\mu} & e^{-\omega_1\mu} & -e^{\omega_1\mu} & -e^{\omega_2\mu} & -e^{-\omega_2\mu} & -e^{-\omega_1\mu} \\ \omega_1 e^{\omega_1\mu} & \omega_2 e^{\omega_2\mu} & -\omega_2 e^{-\omega_2\mu} & -\omega_1 e^{-\omega_1\mu} & -\omega_1 e^{\omega_1\mu} & -\omega_2 e^{\omega_2\mu} & \omega_2 e^{-\omega_2\mu} & \omega_1 e^{-\omega_1\mu} \\ \omega_1^2 e^{\omega_1\mu} & \omega_2^2 e^{\omega_2\mu} & \omega_2^2 e^{-\omega_2\mu} & \omega_1^2 e^{-\omega_1\mu} & \omega_1^2 e^{\omega_1\mu} & \omega_2^2 e^{\omega_2\mu} & \omega_2^2 e^{-\omega_2\mu} & \omega_1^2 e^{-\omega_1\mu} \\ d_1(\mu)e^{\omega_1\mu} & d_2(\mu)e^{\omega_2\mu} & d_3(\mu)e^{\omega_3\mu} & d_4(\mu)e^{\omega_4\mu} & d_1(\mu)e^{\omega_1\mu} & d_2(\mu)e^{\omega_2\mu} & d_3(\mu)e^{-\omega_2\mu} & d_4(\mu)e^{-\omega_1\mu} \end{vmatrix}.$$

A direct but complicated calculation gives

$$D(\mu) = -2i\mu^3 e^{2\mu\omega_4}\Big\{\Big(-4\sqrt{2}\alpha_1\alpha_2 M\Big)[ie^{2\mu\omega_2}+e^{-2\mu\omega_2}+i+1]+\frac{-8i\alpha_1\alpha_2-8i\alpha_1 M-8i\alpha_2 M}{\mu}e^{2\mu\omega_2}$$
$$+\frac{-8i\alpha_1\alpha_2+8i\alpha_1 M+8i\alpha_2 M}{\mu}e^{-2\mu\omega_2}-\frac{16i\alpha_1\alpha_2+8M(\alpha_1+\alpha_2)}{\mu}+O(\mu^{-2})+O\left(e^{-c|\omega_1\mu|}\right)\Big\}, \tag{3.5}$$

where c is some positive constant. Therefore, the asymptotic zeros of $D(\mu)$ in the region $\arg\mu\in(\frac{\pi}{8},\frac{3\pi}{8})$ are determined by

$$\Big(-4\sqrt{2}\alpha_1\alpha_2 M\Big)[ie^{2\mu\omega_2}+e^{-2\mu\omega_2}+i+1]+\frac{-8i\alpha_1\alpha_2-8i\alpha_1 M-8i\alpha_2 M}{\mu}e^{2\mu\omega_2}$$
$$+\frac{-8i\alpha_1\alpha_2+8i\alpha_1 M+8i\alpha_2 M}{\mu}e^{-2\mu\omega_2}-\frac{16i\alpha_1\alpha_2+8M(\alpha_1+\alpha_2)}{\mu}=0. \tag{3.6}$$

In the next theorem, we shall give the asymptotic expression of eigenvalues for $\mathcal{A}$.

Theorem 3.1. *Let $D(\mu)$ be defined as (3.4) in the sector S with $\lambda=\mu^2$. Then the zeros of $D(\mu)$ in S have two branches $\mu_{1,n}$ and $\mu_{2,n}$, they have asymptotic expansion*

$$\mu_{1,n}=\frac{\left(n+\frac{1}{2}\right)\pi i}{\omega_2}+O\left(\frac{1}{n}\right)=\frac{m_1\pi i}{\omega_2}+O\left(\frac{1}{n}\right)$$

and

$$\mu_{2,n}=\frac{\left(n+\frac{1}{4}\right)\pi i}{\omega_2}+O\left(\frac{1}{n}\right)=\frac{m_2\pi i}{\omega_2}+O\left(\frac{1}{n}\right),$$

where $\omega_2=e^{\frac{\pi i}{4}}$. Hence all eigenvalues of $\mathcal{A}$ can be written as $\{\lambda_{j,n},\overline{\lambda_{j,n}}\mid \lambda_{j,n}=\mu_{j,n}^2, j=1,2\}$, they have asymptotic expressions

$$\lambda_{1,n}=\mu_{1,n}^2=-\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}+m_1^2\pi^2 i+O\left(\frac{1}{n}\right),\qquad m_1=n+\frac{1}{2},$$
$$\lambda_{2,n}=\mu_{2,n}^2=-\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}+\left(m_2^2\pi^2+\frac{2}{M}\right)i+O\left(\frac{1}{n}\right),\qquad m_2=n+\frac{1}{4}.$$

Proof. Let $D(\mu)=0$, $\lambda=\mu^2\in S$. According to (3.5) its asymptotic zeros are determined by (3.6), or equivalently,

$$\Big(-4\sqrt{2}\alpha_1\alpha_2 M\Big)[ie^{2\mu\omega_2}+e^{-2\mu\omega_2}+i+1]+O\left(\frac{1}{\mu}\right)=0,\qquad \arg\mu\in\left(\frac{\pi}{8},\frac{3\pi}{8}\right).$$

Solving function equation $ie^{2\mu\omega_2}+e^{-2\mu\omega_2}+i+1=0$ yields

$$\widehat{\mu}_{1,n}=\frac{\left(n+\frac{1}{2}\right)\pi i}{\omega_2}=\frac{m_1\pi i}{\omega_2},\qquad \widehat{\mu}_{2,n}=\frac{\left(n+\frac{1}{4}\right)\pi i}{\omega_2}=\frac{m_2\pi i}{\omega_2}.$$

Let $\mu_{j,n}=\widehat{\mu}_{j,n}+\varepsilon_{j,n}$, $j=1,2$ be the zeros of (3.6). For $j=1$, substituting it into (3.6) yields

$$-4\sqrt{2}M\alpha_1\alpha_2[-ie^{2\omega_2\varepsilon_{1,n}}-e^{-2\omega_2\varepsilon_{1,n}}+(i+1)]-\frac{\omega_2(-8M\alpha_1 i-8M\alpha_2 i-8\alpha_1\alpha_2 i)}{m_1\pi i+\omega_2\varepsilon_{1,n}}e^{2\omega_2\varepsilon_{1,n}}$$
$$-\frac{\omega_2(8M\alpha_1 i+8M\alpha_2 i-8\alpha_1\alpha_2 i)}{m_1\pi i+\omega_2\varepsilon_{1,n}}e^{-2\omega_2\varepsilon_{1,n}}+\frac{\omega_2[-16\alpha_1\alpha_2 i-8M(\alpha_1+\alpha_2)]}{m_1\pi i+\omega_2\varepsilon_{1,n}}=0.$$

Expanding the exponential function leads to

$$8\sqrt{2}M\alpha_1\alpha_2[(i-1)\omega_2\varepsilon_{1,n}+O(\varepsilon_{1,n})^2]-\frac{\omega_2(-8M\alpha_1 i-8M\alpha_2 i-8\alpha_1\alpha_2 i)}{m_1\pi+\omega_2\varepsilon_{1,n}}(1+2\omega_2\varepsilon_{1,n}+O(\varepsilon_{1,n}^2))$$
$$-\frac{\omega_2(8M\alpha_1 i+8M\alpha_2 i-8\alpha_1\alpha_2 i)}{m_1\pi i+\omega_2\varepsilon_{1,n}}(1-2\omega_2\varepsilon_{1,n}+O(\varepsilon_{1,n}^2))+\frac{\omega_2[-16\alpha_1\alpha_2 i-8M(\alpha_1+\alpha_2)]}{m_1\pi i+\omega_2\varepsilon_{1,n}}=0$$

which implies

$$\varepsilon_{1,n}=\frac{\alpha_1+\alpha_2}{\sqrt{2}(i-1)\alpha_1\alpha_2 m_1\pi i}+O\left(\frac{1}{n^2}\right).$$

Hence

$$\mu_{1,n} = \widehat{\mu}_{1,n} + \varepsilon_{1,n} = \frac{m_1\pi i}{\omega_2} + \frac{\alpha_1+\alpha_2}{\sqrt{2}(i-1)\alpha_1\alpha_2 m_1\pi i} + O\left(\frac{1}{n^2}\right).$$

Similarly, for $j = 2$, substituting $\mu_{2,n}$ into (3.6) yields

$$\begin{aligned}
&-4\sqrt{2}M\alpha_1\alpha_2\left[ie^{2\left(n+\frac{1}{4}\right)\pi i+2\omega_2\varepsilon_{2,n}} + e^{-\left[2\left(n+\frac{1}{4}\right)\pi i+2\omega_2\varepsilon_{2,n}\right]} + (i+1)\right]\\
&\quad+\frac{\omega_2(-8M\alpha_1 i - 8M\alpha_2 i - 8\alpha_1\alpha_2 i)}{m_2\pi i+\omega_2\varepsilon_{2,n}}e^{2\left(n+\frac{1}{4}\right)\pi i+2\omega_2\varepsilon_{2,n}} + \frac{\omega_2(8M\alpha_1 i + 8M\alpha_2 i - 8\alpha_1\alpha_2 i)}{m_2\pi i+\omega_2\varepsilon_{2,n}}e^{-\left[2\left(n+\frac{1}{4}\right)\pi i+2\omega_2\varepsilon_{2,n}\right]}\\
&\quad+\frac{\omega_2[-16\alpha_1\alpha_2 i - 8M(\alpha_1+\alpha_2)]}{m_2\pi i+\omega_2\varepsilon_{2,n}} = 0.
\end{aligned}$$

By simplification, we obtain

$$\begin{aligned}
&8\sqrt{2}M\alpha_1\alpha_2(1-i)\omega_2\varepsilon_{2,n} + \frac{\omega_2 i(-8M\alpha_1 i - 8M\alpha_2 i - 8\alpha_1\alpha_2 i)}{m_2\pi i+\omega_2\varepsilon_{2,n}} + \frac{\omega_2(-i)(-8\alpha_1\alpha_2 i + 8M\alpha_1 i + 8M\alpha_2 i)}{m_2\pi i+\omega_2\varepsilon_{2,n}}\\
&\quad+\frac{\omega_2[-16\alpha_1\alpha_2 i - 8M(\alpha_1+\alpha_2)]}{m_2\pi i+\omega_2\varepsilon_{2,n}} + O(\varepsilon_{2,n}^2) = 0,
\end{aligned}$$

which implies

$$\varepsilon_{2,n} = \frac{2\alpha_1\alpha_2 i - M(\alpha_1+\alpha_2)}{\sqrt{2}(1-i)M\alpha_1\alpha_2 m_2\pi i} + O\left(\frac{1}{n^2}\right).$$

So we have

$$\mu_{2,n} = \widehat{\mu}_{2,n} + \varepsilon_{2,n} = \frac{m_2\pi i}{\omega_2} + \frac{2\alpha_1\alpha_2 i - M(\alpha_1+\alpha_2)}{\sqrt{2}M(1-i)\alpha_1\alpha_2 m_2\pi i} + O\left(\frac{1}{n^2}\right).$$

Therefore, we have

$$\lambda_{1,n} = \mu_{1,n}^2 = -\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2} + m_1^2\pi^2 i + O\left(\frac{1}{n}\right)$$

and

$$\lambda_{2,n} = \mu_{2,n}^2 = -\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2} + \left(m_2^2\pi^2 + \frac{2}{M}\right)i + O\left(\frac{1}{n}\right).$$

This completes the proof. □

4. Completeness of (generalized) eigenvectors of $\mathcal{A}$ and Riesz basis property

In this section, we shall discuss the completeness of (generalized) eigenvectors of $\mathcal{A}$ and its Riesz basis property. First we establish the completeness of (generalized) eigenvectors of $\mathcal{A}$. For this purpose, we need the following lemma.

Lemma 4.1. *Suppose $\mathcal{H}$ is defined by (2.1). Let $\mathcal{A}_0$ be a new operator in $\mathcal{H}$ defined as*

$$\mathcal{A}_0(y, v, p) = \left(v, -y^{(4)}, \frac{1}{M}\sum_{i=1}^{2} y_i'''(1)\right), \tag{4.1}$$

with domain

$$D(\mathcal{A}_0) = \left\{\begin{array}{l}(y, v, p) \in Y \cap [H^4(0,1)]^2 \times Y \times \mathbb{C} :\\ y_i''(0) - \tau_i y_i'(0) = 0, \quad i = 1, 2,\\ p = v_1(1) = v_2(1), \qquad \sum_{i=1}^{2} y_i''(1) = 0\end{array}\right\}. \tag{4.2}$$

Then $\mathcal{A}_0$ is a skew self-adjoint operator in $\mathcal{H}$, i.e., $\mathcal{A}_0^* = -\mathcal{A}_0$.

The proof is a direct verification, we omit the details.

Theorem 4.1. *The (generalized) eigenvectors of $\mathcal{A}$ is complete in $\mathcal{H}$, i.e.,*

$$\mathcal{H} = \mathrm{Sp}(\mathcal{A}) = \overline{\mathrm{span}\{E(\lambda_k, \mathcal{A})\mathcal{H}, \forall \lambda_k \in \sigma(\mathcal{A})\}}$$

where $E(\lambda_k, \mathcal{A})$ is the Riesz projector corresponding to λ_k.

Proof. Let $\mathcal{H}$ and $\mathcal{A}$ be defined as before and $\mathcal{A}_0$ be defined as (4.1) and (4.2). To prove $\mathrm{Sp}(\mathcal{A}) = \mathcal{H}$, let $U_0 = (w_0, z_0, v_0) \in \mathcal{H}$ and $(w_0, z_0, v_0) \perp \mathrm{Sp}(\mathcal{A})$, we shall show $U_0 = 0$.

Since $U_0 \perp \mathrm{Sp}(\mathcal{A})$, $R^*(\lambda, \mathcal{A})U_0$ can extend to an entire function on the complex plane $\mathbb{C}$. Thus for any $F = (f_1, f_2, f_3) \in \mathcal{H}$, the function defined by

$$G(\lambda) = \langle F, R^*(\lambda, \mathcal{A})U_0\rangle_{\mathcal{H}}, \quad \lambda \in \mathbb{C}$$

also is an entire function. Since $\mathcal{A}$ is the generator of semigroup $T(t)$, we have $\lim_{\Re\lambda \to +\infty} |G(\lambda)| = 0$. In addition, for $\lambda \in \rho(\mathcal{A})$,

$$G(\lambda) = (R(\lambda, \mathcal{A})F, U_0)_{\mathcal{H}}.$$

Now for $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{A}_0)$, we write

$$Y_1 = R(\lambda, \mathcal{A})F, \qquad Y_2 = R(\lambda, \mathcal{A}_0)F, \tag{4.3}$$

where $Y_1 \in \mathcal{D}(\mathcal{A})$, $Y_2 = (z, u, q) \in \mathcal{D}(\mathcal{A}_0)$. Let $Y = Y_1 - Y_2 = (y, v, p)$, then $R(\lambda, \mathcal{A})F = R(\lambda, \mathcal{A}_0)F + Y$ and $Y = (y, v, p)$ satisfy the following equations

$$\begin{cases} v(x) = \lambda y(x), \qquad p = v_1(1) = v_2(1), \\ \lambda^2 y_i(x) + y_i^{(4)}(x) = 0, \quad i = 1, 2, \\ y_1(0) = y_2(0) = 0, \\ y_i''(0) - (\alpha_i\lambda + \tau_i)y_i'(0) - \lambda\beta_i y_i(1) = \alpha_i u_i'(0) + \beta_i u_i(1), \quad i = 1, 2, \\ y_1(1) = y_2(1), \qquad y_1'(1) = y_2'(1), \\ \displaystyle\sum_{i=1}^{2} y_i''(1) = 0, \qquad \sum_{i=1}^{2} y_i'''(1) - M\lambda^2 y_1(1) = 0. \end{cases} \tag{4.4}$$

In what follows, we shall estimate the norm of Y for real λ, i.e.,

$$\begin{aligned} \|Y\|^2 = \|(y, v, p)\|^2 &= \sum_{i=1}^{2} \int_0^1 [|y_i''(x)|^2 + |v_i(x)|^2]dx + \sum_{i=1}^{2} \tau_i |y_i'(0)|^2 + M|p|^2 \\ &= \sum_{i=1}^{2} \int_0^1 [|y_i''(x)|^2 + |\lambda y_i(x)|^2]dx + \sum_{i=1}^{2} \tau_i |y_i'(0)|^2 + M|\lambda y_1(1)|^2. \end{aligned}$$

Since

$$\begin{aligned} \int_0^1 |y_i''(x)|^2 dx &= \int_0^1 y_i''(x)\overline{y_i''(x)}dx = y_i''(x)\overline{y_i'(x)}\,|_0^1 - y_i'''(x)\overline{y_i(x)}\,|_0^1 + \int_0^1 y_i^{(4)}(x)\overline{y_i(x)}dx \\ &= y_i''(1)\overline{y_i'(1)} - y_i''(0)\overline{y_i'(0)} - y_i'''(1)\overline{y_i(1)} - \lambda^2 \int_0^1 |y_i(x)|^2 dx, \end{aligned}$$

for real $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} &\sum_{i=1}^{2} \int_0^1 |y_i''(x)|^2 dx + \lambda^2 \sum_{i=1}^{2} \int_0^1 |y_i(x)|^2 dx + \sum_{i=1}^{2} \tau_i |y_i'(0)|^2 + M|\lambda y_1(1)|^2 \\ &\quad = \sum_{i=1}^{2} y_i''(1)\overline{y_i'(1)} - \sum_{i=1}^{2} (y_i''(0) - \tau_i y_i'(0))\overline{y_i'(0)} - \sum_{i=1}^{2} y_i'''(1)\overline{y_i(1)} + \lambda^2 M y_1^2(1) \\ &\quad = -\sum_{i=1}^{2} (\alpha_i \lambda y_i'(0) + \alpha_i u_i'(0) + \lambda\beta_i y_i(1) + \beta_i u_i(1))\overline{y_i'(0)} \\ &\quad = -\lambda \sum_{i=1}^{2} \alpha_i |y_i'(0)|^2 - \sum_{i=1}^{2} \alpha_i u_i'(0)\overline{y_i'(0)} - \lambda \sum_{i=1}^{2} \beta_i y_i(1)\overline{y_i'(0)} - \sum_{i=1}^{2} \beta_i u_i(1)\overline{y_i'(0)}. \end{aligned}$$

To estimate $\|Y\|^2$, we split it into three steps:

Step 1. There exists positive constant C such that

$$\sum_{i=1}^{2}\alpha_i|y_i'(0)|^2 \le \frac{C}{|\lambda|}\sum_{i=1}^{2}(|u_i'(0)|^2+|u_i(1)|^2)$$

and

$$\sum_{i=1}^{2}\beta_i|y_i(1)|^2 \le \frac{C}{|\lambda|}\sum_{i=1}^{2}[|u_i'(0)|^2+|u_i(1)|^2],$$

where $C=\max\{C_1,C_2,C_3\}$.

According to the analysis in Section 3, we can set

$$y_i(x)=\sum_{j=1}^{4}a_{i,j}e^{\omega_j\mu x}$$

where ω_j are the distinct root of equation $\omega^4=-1$. Eq. (4.4) becomes

$$HF=E \tag{4.5}$$

where

$$F=[a_{11},a_{12},a_{13},a_{14},a_{21},a_{22},a_{23},a_{24}]^\tau,$$
$$\frac{y_i'(0)}{\mu}=a_{i1}\omega_1+a_{i2}\omega_2+a_{i3}\omega_3+a_{i4}\omega_4,\quad i=1,2,$$
$$E=\left[0,0,\frac{\alpha_1u_1'(0)+\beta_1u_1(1)}{\mu^3},\frac{\alpha_2u_2'(0)+\beta_2u_2(1)}{\mu^3},0,0,0,0\right]^\tau$$

where $H=(h_{ij})_{8\times 8}$ is the same as $D(\mu)_{ij}$,

$$H^{-1}=\frac{1}{\det H}\begin{pmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{18}\\ A_{21} & A_{22} & \cdots & \cdots & A_{28}\\ A_{31} & A_{32} & \cdots & \cdots & A_{38}\\ A_{41} & A_{42} & \cdots & \cdots & A_{48}\\ A_{51} & A_{52} & \cdots & \cdots & A_{58}\\ A_{61} & A_{62} & \cdots & \cdots & A_{68}\\ A_{71} & A_{72} & \cdots & \cdots & A_{78}\\ A_{81} & A_{82} & \cdots & \cdots & A_{88}\end{pmatrix},$$

where $\{A_{ij}|i,j=1,\dots,8\}$ is the algebraic complement of matrix H. Directly calculating we find out

$$\begin{aligned}\det H &= D(\mu)\\ &\approx \mathcal{O}\left(-\mu 2ie^{2\mu\omega_4}\left(-4\sqrt{2}\alpha_1\alpha_2M\right)[ie^{2\mu\omega_2}+e^{-2\mu\omega_2}+i+1]\right),\end{aligned}$$

and

$$\max_{i,j=1,\dots,8}|A_{ij}|\approx\mathcal{O}\left(\mu e^{2\mu\omega_4}e^{\mu\omega_2}[4\alpha_2M(1-i)e^{\mu\omega_2}+\left(\sqrt{2}-4\right)i\alpha_2Me^{-\mu\omega_2}]\right).$$

Hence

$$\|H^{-1}\|\le\max_{i,j=1,\dots,8}\frac{|A_{ij}|}{|\det H|}\le\widehat{\alpha},$$

where $\widehat{\alpha}$ is a positive constant. We have $\|F\|\le\|H^{-1}\|\,\|E\|$. Thus we obtain

$$\sum_{i=1}^{2}\alpha_i|y_i'(0)|^2\le\frac{C_1}{|\lambda|}\sum_{i=1}^{2}(|u_i'(0)|^2+|u_i(1)|^2).$$

Since

$$\left|\frac{y_i''(0)}{\mu^2}\right|=|a_{ij}\omega_j^2|\le C_2\frac{1}{\mu^3}\sum_{i=1}^{2}(|u_i'(0)|^2+|u_i(1)|^2)$$

and

$$y_i''(0)-(\alpha_i\lambda+\tau_i)y_i'(0)-\lambda\beta_iy_i(1)=\alpha_iu_i'(0)+\beta_iu_i(1),\quad i=1,2,$$

so we can obtain

$$\sum_{i=1}^{2}\beta_i|y_i(1)|^2 \le \frac{C_3}{|\lambda|}\sum_{i=1}^{2}[|u_i'(0)|^2+|u_i(1)|^2].$$

Step 2. There exists positive constant C_4 such that

$$\sum_{i=1}^{2}\alpha_i|u_i'(0)|^2 \le \frac{C_4}{|\lambda^2|\,|\Re\lambda|}\|F\|^2, \qquad |u_i(1)| \le \frac{C_4}{|\Re\lambda|}\|F\|, \quad i=1,2.\ \forall\lambda\in\rho(\mathcal{A}_0)\cap\mathbb{R}.$$

Since $\mathcal{A}_0$ is a skew self-adjoint operator in $\mathcal{H}$, we have

$$\|R(\lambda,\mathcal{A}_0)\| \le \frac{1}{|\Re\lambda|}, \quad \forall\lambda\in\rho(\mathcal{A}_0),$$

Step 3. There exists positive constant C_5 such that

$$\|Y\| \le \frac{C_5}{|\lambda|}\|F\|, \quad \lambda\in\mathbb{R}.$$

In fact, for $\lambda\in\mathbb{R}\cap\rho(\mathcal{A})\cap\rho(\mathcal{A}_0)$,

$$\begin{aligned}
\|Y\|^2 &= \sum_{i=1}^{2}\int_0^1|y_i''(x)|^2dx+\lambda^2\sum_{i=1}^{2}\int_0^1|y_i(x)|^2dx+\sum_{i=1}^{2}\tau_i|y_i'(0)|^2+M|\lambda y_1(1)|^2\\
&= -\lambda\sum_{i=1}^{2}\alpha_i|y_i'(0)|^2-\sum_{i=1}^{2}\alpha_iu_i'(0)\overline{y_i'(0)}-\lambda\sum_{i=1}^{2}\beta_iy_i(1)\overline{y_i'(0)}-\sum_{i=1}^{2}\beta_iu_i(1)\overline{y_i'(0)}\\
&\le |\lambda|\sum_{i=1}^{2}\alpha_i|y_i'(0)|^2+\frac{1}{2}\left(\sum_{i=1}^{2}\alpha_i|u_i'(0)|^2+\sum_{i=1}^{2}\alpha_i|y_i'(0)|^2\right)\\
&\quad+\frac{|\lambda|}{2}\left(\sum_{i=1}^{2}\beta_i|y_i(1)|^2+\sum_{i=1}^{2}\beta_i|y_i'(0)|^2\right)+\frac{1}{2}\left(\sum_{i=1}^{2}\beta_i|u_i(1)|^2+\sum_{i=1}^{2}\beta_i|y_i'(0)|^2\right)\\
&\le \left(2C+\frac{C}{2|\lambda|}+\frac{1}{2}+C+\frac{\beta_i}{|\lambda|}\right)\sum_{i=1}^{2}\alpha_i|u_i'(0)|^2+\beta_i|u_i(1)|^2 \quad \text{by Step 1}\\
&\le \left(2C+\frac{C}{2|\lambda|}+\frac{1}{2}C+\frac{\beta_i}{|\lambda|}\right)\left(\frac{C_4}{|\lambda|^3}+\frac{C_4^2}{|\lambda|^2}\sum_{i=1}^{2}\beta_i\right)\|F\|^2 \quad \text{by Step 2}\\
&\le \frac{C_5^2}{|\lambda|^2}\|F\|^2
\end{aligned}$$

where $C_5^2=\sup_{|\lambda|>\delta}\left(2C+\frac{C}{2|\lambda|}+\frac{1}{2}C+\frac{\beta_i}{|\lambda|}\right)\left(\frac{C_4}{|\lambda|}+C_4^2\sum_{i=1}^{2}\beta_i\right)$.

Since

$$\|Y_2\|=\|R(\lambda,\mathcal{A}_0)F\| \le \frac{1}{|\lambda|}\|F\|,$$

we find out

$$\begin{aligned}
\|Y_1\| &= \|R(\lambda,\mathcal{A})F\| \le (\|R(\lambda,\mathcal{A}_0)F\|+\|Y\|)\\
&\le \left[\frac{C_5}{|\lambda|}+\frac{1}{|\lambda|}\right]\|F\|, \quad \forall\lambda\in\rho(\mathcal{A})\cap\rho(\mathcal{A}_0)\cap\mathbb{R}_-.
\end{aligned}$$

Therefore $\lim_{\lambda\to-\infty}\|R(\lambda,\mathcal{A})F\|=0$.

Note that $G(\lambda)$ is an entire function of finite exponential type, $G(\lambda)$ is uniformly bounded along the line $\Re\lambda=\alpha>0$. The Phragmén-Linderlöf theorem (cf. [28]) asserts that

$$|G(\lambda)| \le M, \quad \forall\lambda\in\mathbb{C}.$$

The Liouville theorem says that $G(\lambda)\equiv 0$. Observe that $G(\lambda)=(F,R^*(\lambda,\mathcal{A})U_0)_{\mathcal{H}}$ for any $F\in\mathcal{H}$, so it must be $R^*(\lambda,\mathcal{A})U_0=0$ which implies $U_0=0$. Therefore $\mathrm{Sp}(\mathcal{A})=\mathcal{H}$. $\square$

In what follows, we shall study the Riesz basis generation of the (generalized) eigenvectors of $\mathcal{A}$, we need the following result that comes from [26].

Lemma 4.2. *Let $\mathcal{H}$ be a separable Hilbert space, and $\mathcal{A}$ be the generator of a C_0 semigroup $T(t)$ on $\mathcal{H}$. Suppose that the following conditions are satisfied:*

(1) $\sigma(\mathcal{A}) = \sigma_1(\mathcal{A}) \cup \sigma_2(\mathcal{A})$, *where* $\sigma_2(\mathcal{A}) = \{\lambda_k\}_{k=1}^{\infty}$ *consists of isolated eigenvalues of $\mathcal{A}$ with finite multiplicity;*
(2) $\sup_{k\geq 1} m_a(\lambda_k) < \infty$, *where* $m_a(\lambda_k) = \dim E(\lambda_k, \mathcal{A})\mathcal{H}$ *and* $E(\lambda_k, \mathcal{A})$ *is the Riesz projector associated with* λ_k*;*
(3) *There is a constant α such that*

$$\sup\{\Re\lambda \mid \lambda \in \sigma_1(\mathcal{A})\} \leq \alpha \leq \inf\{\Re\lambda \mid \lambda \in \sigma_2(\mathcal{A})\}$$

and

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0.$$

Then the following assertions are true.

(i) *There exist two $T(t)$-invariant closed subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ with the property that $\sigma(\mathcal{A}|_{\mathcal{H}_1}) = \sigma_1(\mathcal{A})$, $\sigma(\mathcal{A}|_{\mathcal{H}_2}) = \sigma_2(\mathcal{A})$, $E(\lambda_k, \mathcal{A})\mathcal{H}_2$ forms a subspace Riesz basis for $\mathcal{H}_2$ and*
$$\mathcal{H} = \overline{\mathcal{H}_1 \bigoplus \mathcal{H}_2}.$$
(ii) *If $\sup_{k\geq 1} \|E(\lambda_k, \mathcal{A})\| < \infty$, then*
$$\mathcal{D}(\mathcal{A}) \subset \mathcal{H}_1 \oplus \mathcal{H}_2 \subset \mathcal{H}. \tag{4.6}$$
(iii) *$\mathcal{H}$ has the decomposition*
$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \quad \text{(topological direct sum)},$$
if and only if
$$\sup_{n\geq 1} \left\| \sum_{k=1}^{n} E(\lambda_k, \mathcal{A}) \right\| < \infty. \tag{4.7}$$

Applying Theorem 4.1 and Lemma 4.2 to our problem, we have the following result.

Theorem 4.2. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as (2.1) and (2.3), respectively. Then there is a sequence of eigenvectors and generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis for $\mathcal{H}$. In particular, the system associated with $\mathcal{A}$ satisfies the spectrum determined growth condition.*

Proof. Set $\sigma_1(\mathcal{A}) = \{-\infty\}$, $\sigma_2(\mathcal{A}) = \sigma(\mathcal{A})$. Lemma 2.2 shows that $\mathcal{A}$ generates a C_0 semigroup and the condition (1) in Lemma 4.2 holds. Theorem 3.1 shows that, for sufficient large n, $\lambda_{j,n}$ are separable and $m_a(\lambda_{j,n}) = 1$, which means that the conditions (2) and (3) in Lemma 4.2 are fulfilled. Therefore, according to Lemma 4.2, there is a sequence of eigenvectors and generalized eigenvectors of $\mathcal{A}$ that forms a Riesz basis for $\mathcal{H}_2$ due to $m_a(\lambda_{j,n}) = 1$ for sufficient large n. Theorem 4.1 shows that the eigenvectors and generalized eigenvectors are complete in $\mathcal{H}$, that is, $\mathcal{H}_2 = \mathcal{H}$. Therefore the sequence is also a Riesz basis for $\mathcal{H}$.

Note that when n is sufficient large, all $\lambda_{j,n}$ are simple eigenvalues of $\mathcal{A}$. There are probably finitely many eigenvalues of multiplicity two. The Riesz basis property of the eigenvectors and generalized eigenvectors of $\mathcal{A}$ ensures that the system associated with $\mathcal{A}$ satisfies the spectrum determined growth condition. The desired result follows. $\square$

5. Stability analysis of the cooperation system

5.1. Stability analysis

In this section, we shall discuss the stability of the system (1.1). It is well known that if $(\beta_1, \beta_2) = 0$, then the system is exponentially stable. However, the performance of the nonlinear system requires $\beta_j \neq 0$. So the requirement is inherited to the system (1.1). So we only need to consider the cases of $(\beta_1, \beta_2) > 0$.

We know from Theorem 4.2 that the growth rate of the system is determined via the maximal real part of spectrum of $\mathcal{A}$. We denote by

$$s(\alpha, \beta) = \sup_{\lambda\in\sigma(A)} \Re\lambda, \tag{5.1}$$

where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$. Then we have the following result.

Theorem 5.1. *Let $\mathcal{H}$ and $\mathcal{A}$ be defined as (2.1) and (2.3), respectively. If we can choose the gain coefficients α and β such that $s(\alpha, \beta) < 0$, then the closed loop system (2.1) is exponentially stable.*

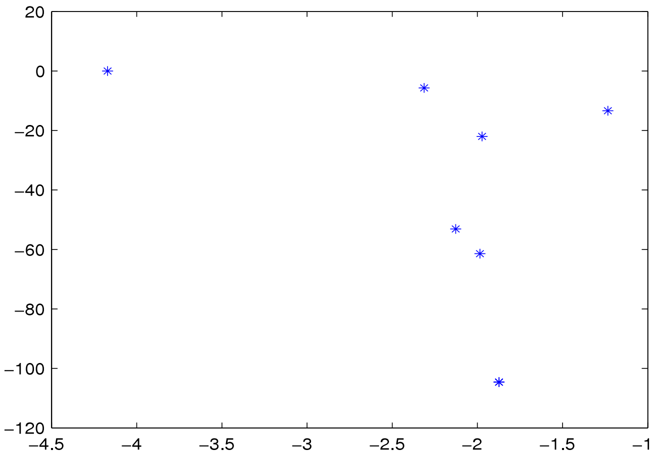

Fig. 2. The spectrum graph.

For sufficient small $\delta > 0$, denote by γ the negative real

$$\gamma = -\frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2} + \delta.$$

According to the asymptotic expression of the eigenvalues of $\mathcal{A}$ in Theorem 3.1, we see that there are at most finitely many number of eigenvalues of $\mathcal{A}$ in the half plane $\Re\lambda > \gamma$. Since the asymptotic values of eigenvalues are not obviously dependent on β, we can assert that only the finite eigenvalues of $\mathcal{A}$ depend upon the value of β strongly. Therefore, we have to check whether there exist $\beta > 0$ such that $s(\alpha, \beta) < 0$. Obviously, if $\beta = 0$, From the proof of Lemma 2.1 we can see that $\mathcal{A}$ is a dissipative operator in $\mathcal{H}$. In this case, we have $s(\alpha, 0) < 0$.

To obtain the stability of (1.1) for $\beta > 0$, we need the following result, which comes from [16, Theorem 2.1].

Lemma 5.1. *Suppose that $B \subset \mathbb{R}^n$ is an open and connected set, $h(\lambda, \overrightarrow{r})$ is continuous in $(\lambda, \overrightarrow{r}) \in \mathbb{C} \times B$ and analytic in $\lambda \in \mathbb{C}$, and zeros of $h(\lambda, \overrightarrow{r})$ in the right half plane $\{\lambda \in \mathbb{C} | \Re\lambda \geq 0\}$ are uniformly bounded. If for any $\overrightarrow{r} \in B_1 \subset B$, where B_1 is a bounded, closed and connected set, $h(\lambda, \overrightarrow{r})$ has no zero on the imaginary axis, then the sum of the orders of the zero of $h(\lambda, \overrightarrow{r})$ in the open right half plane $(\Re\lambda > 0)$ is a fixed number for B_1, that is, it is independent of the parameter $\overrightarrow{r} \in B_1$.*

To apply Lemma 5.1 to our case, we take

$$h(\lambda, \overrightarrow{r}) = D(\mu, \alpha, \beta)$$

where $D(\mu, \alpha, \beta) = D(\mu)$ is defined as in (3.4) and $\lambda = \mu^2$. Here we only consider the case that $\alpha > 0$ is fixed and β varies in neighbor of the original point, $B_1 \subset \mathcal{O}(r) \subset \mathbb{R}^2$, that is, $\overrightarrow{r} = \beta$. Obviously, $h(\lambda, \overrightarrow{r})$ is analytic in λ and continuous in $\overrightarrow{r}$. Theorem 3.1 ensures that zeros of $h(\lambda, \overrightarrow{r})$ in the right half-plane are uniformly bounded. By Lemma 5.1, there exists a small neighbor $B_1 \subset \mathcal{O}(r)$ in which the zeros of $h(\lambda, \overrightarrow{r})$ are constants in the right half-plane. Since $\overrightarrow{r} = \beta = 0$, there are no zeros of $h(\lambda, \overrightarrow{r})$ in the right half-plane. Therefore, there is no zero of $h(\lambda, \overrightarrow{r})$ in the right half-plane for all $\overrightarrow{r} = \beta \in B_1$.

Applying Lemma 5.1 to our model, using the continuity of $s(\alpha, \beta)$ with respect to β, there exists $\delta(\alpha) > 0$ such that $0 < \|\beta\| < \delta(\alpha)$, it holds that $s(\alpha, \beta) < 0$ since $s(\alpha, 0) < 0$. Therefore, we have the following result.

Theorem 5.2. *Let $\mathcal{A}$ be defined by (2.3) and (2.4). Then for each $\alpha > 0$, there exists $\delta(\alpha) > 0$ such that $s(\alpha, \beta) < 0$ provided that $0 < \|\beta\| < \delta(\alpha)$. Hence the closed loop system (1.1) is exponentially stable.*

From Theorem 5.2 we know that one can choose smaller β such that the system (1.1) is exponentially stable. However we cannot obtain an estimate for $\delta(\alpha)$ or the relation between α and β. Therefore, how to choose the gain parameters is still an important question in practice. In the next subsection, we shall give some simulations to show the relationship between α and β that make the closed-loop system (1.1) stable exponentially.

5.2. Simulation

In this subsection, we give some simulations for the eigenvalues of the system (1.1) to show the relation between α and β.

(1) The first simulation is to check the stability of the system for $(\alpha_1, \alpha_2) < (\beta_1, \beta_2)$.

Set $\alpha = (1, 1)$ and $\beta = (5, 5)$ or set $\alpha = (0.1, 0.1)$ and $\beta = (2, 2)$. By Matlab scientific calculation, we obtain the finitely many number spectral points of the system (1.1), whose distributions are shown in Figs. 2 and 3, respectively. In Figs. 2 and 3, the notation $*$ denotes all spectral points.

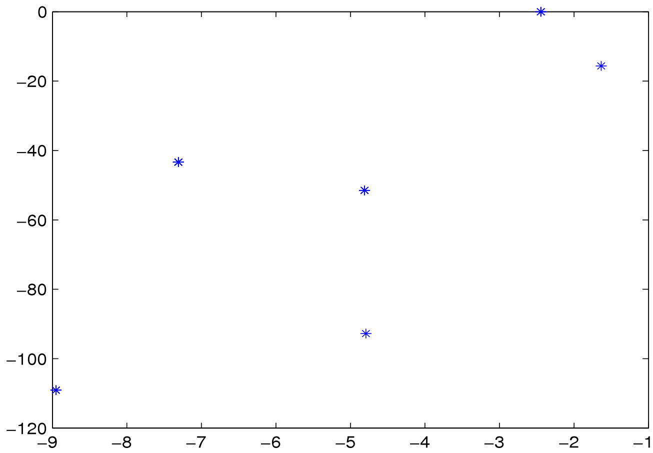

Fig. 3. The spectrum graph.

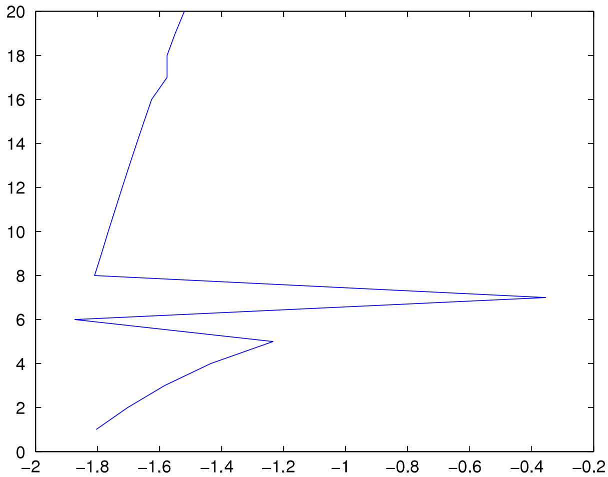

Fig. 4. The relation between β and the spectrum.

The simulation results (see Figs. 2 and 3) show that all eigenvalues are located on the left hand side of the imaginary axis. Hence the system (1.1) is exponentially stable with such feedback gains. These results also show that the feedback controllers are feasible for (β_1, β_2) reasonable large.

(2) The second simulation is to describe change of $s(\alpha, \beta)$ with $\beta = (\beta_1, \beta_2)$.

In order to study the behavior of the rightmost eigenvalue of the system with the change of gain parameter β, we keep $\alpha = (\alpha_1, \alpha_2)$ fixed and change the value of β.

Set $\alpha = (1, 1)$, we take β of the form $\beta = (1, 1)\xi$ where $\xi \in \mathbb{R}$. Let ξ change from 0 to 20 and take step length $h = 0.1$, calculate the value of $s(\alpha, \beta)$. We obtain the calculation results whose graphics are shown in Fig. 4. But we cannot obtain an analytic expression between β and the maximal real part of the spectrum since the figure is so complex.

The simulation result (see, Fig. 4) shows that all eigenvalues of the system (1.1) are in the left half-plane. In this case, the maximal real part of spectrum of $\mathcal{A}$ is still negative. This implies that the system is still exponentially stable. However, when ξ changes in interval $[6, 8]$, $s(\alpha, \beta)$ has a singular variety. The rightmost eigenvalue approximates to -0.2 when ξ is about 7.

6. Conclusion

We studied the stability problem of a robot system composed of two Euler–Bernoulli beams with non-collocated feedback controllers. The advantage of this class of the non-collocated feedback controllers is simpler and more feasible in practice. With the help of the Riesz basis approach and the asymptotic analysis technique, we proved the exponential stability of uniform Euler–Bernoulli beam equations under non-collocated boundary feedback controllers with suitable choice of the feedback gains. The key point of this kind of controllers is the suitable choice of the feedback gains. Although we do not obtain an analytic expression of relation between (α_1, α_2) and (β_1, β_2), we have proved that such a relation exists.

The simulation result (see, Fig. 4) shows that for the variables α and β of the form $\alpha = (1, 1)$ and $\beta = (1, 1)\xi$, $\xi \in \mathbb{R}$, all eigenvalues of the system (1.1) are in the left half-plane. Hence the corresponding closed loop systems are exponentially stable. We observe that β and α need not have this form. Our further work will determine (α_1, α_2) for certain fixed $\beta = (\beta_1, \beta_2)$.

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