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Sobolev spaces associated with Jacobi expansions ☆

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A B S T R A C T

We define and study Sobolev spaces associated with Jacobi expansions. We prove that these Sobolev spaces are isomorphic to Jacobi potential spaces. As a technical tool, we also show some approximation properties of Poisson–Jacobi integrals.

1. Introduction

Sobolev spaces associated with Hermite and Laguerre expansions were investigated not long ago in [5,6,13,28]. Recently Betancor et al. [4] studied Sobolev spaces in the context of ultraspherical expansions. Inspired by [4], in this paper we define and study Sobolev spaces in a more general situation of Jacobi expansions. Noteworthy, analysis related to Jacobi expansions received a considerable attention over the last fifty years. For the corresponding developments in the recent years, see for instance [2,3,8–11,14–16,18,21,22,24–26,31].

Our motivation is, first of all, to extend definitions and results from [4] to the framework of Jacobi expansions. Another motivation comes from a question of removing the restriction on the ultraspherical parameter of type λ imposed throughout [4]. In this paper we admit all possible Jacobi parameters of type α, β, thus also all possible λ. Finally, still another motivation originates in the very definition of the ultraspherical Sobolev spaces proposed in [4]. It is based on higher order 'derivatives' involving first order differential operators related to various parameters of type λ. Here we consider another, seemingly more natural, definition of Jacobi Sobolev spaces by means of higher order 'derivatives' linked to one fixed pair of the type parameters α, β. The concept of higher order 'derivative' we employ was postulated recently by

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Nowak and Stempak [27], and the question of its relevance to the theory of Sobolev spaces was posed there. Perhaps a bit unexpectedly, we show that the associated Sobolev spaces are not quite appropriate.

Given parameters $\alpha, \beta > -1$, consider the Jacobi differential operator

$$L_{\alpha,\beta} = -\frac{d^2}{d\theta^2} - \frac{1-4\alpha^2}{16\sin^2\frac{\theta}{2}} - \frac{1-4\beta^2}{16\cos^2\frac{\theta}{2}} = D^*_{\alpha,\beta} D_{\alpha,\beta} + A^2_{\alpha,\beta};$$

here $A_{\alpha,\beta} = (\alpha+\beta+1)/2$, $D_{\alpha,\beta} = \frac{d}{d\theta} - \frac{2\alpha+1}{4}\cot\frac{\theta}{2} + \frac{2\beta+1}{4}\tan\frac{\theta}{2}$ is the first order 'derivative' naturally associated with $L_{\alpha,\beta}$, and $D^*_{\alpha,\beta} = D_{\alpha,\beta} - 2\frac{d}{d\theta}$ is its formal adjoint in $L^2(0,\pi)$. It is well known that $L_{\alpha,\beta}$, defined initially on $C^2_c(0,\pi)$, has a non-negative self-adjoint extension to $L^2(0,\pi)$ whose spectral decomposition is discrete and given by the Jacobi functions $\phi^{\alpha,\beta}_n$, $n \geq 0$. The corresponding eigenvalues are $\lambda^{\alpha,\beta}_n = (n + A_{\alpha,\beta})^2$, and the system $\{\phi^{\alpha,\beta}_n : n \geq 0\}$ constitutes an orthonormal basis in $L^2(0,\pi)$; see Section 2 for more details. If $\alpha + 1/2 = \beta + 1/2 =: \lambda$, then the Jacobi context reduces to the ultraspherical situation considered in [4].

When $\alpha, \beta \geq -1/2$, the functions $\phi^{\alpha,\beta}_n$ belong to all $L^p(0,\pi)$, $1 \leq p \leq \infty$. However, if $\alpha < -1/2$ or $\beta < -1/2$, then $\phi^{\alpha,\beta}_n$ are in $L^p(0,\pi)$ if and only if $p < p(\alpha,\beta) := -1/\min(\alpha+1/2, \beta+1/2)$. This leads to the so-called pencil phenomenon (cf. [17,23]) manifesting in the restriction $p'(\alpha,\beta) < p < p(\alpha,\beta)$ for L^p mapping properties of various operators associated with $L_{\alpha,\beta}$ (here and elsewhere p' denotes the conjugate exponent of p, $1/p + 1/p' = 1$). Consequently, our main results are restricted to $p \in E(\alpha,\beta)$, where

$$E(\alpha,\beta) := \begin{cases} (1,\infty), & \alpha,\beta \geq -1/2, \\ (p'(\alpha,\beta), p(\alpha,\beta)), & \text{otherwise.} \end{cases}$$

Let $\sigma > 0$. For $\alpha+\beta \neq -1$, consider the potential operator $L^{-\sigma}_{\alpha,\beta}$. When $\alpha+\beta = -1$, zero is the eigenvalue of $L_{\alpha,\beta}$ and hence we consider instead the Bessel type potential operator $(\mathrm{Id} + L_{\alpha,\beta})^{-\sigma}$. In both cases the potentials are well defined spectrally and are bounded on $L^2(0,\pi)$ and possess integral representations valid not only in $L^2(0,\pi)$, but also far beyond that space. We will show that $L^{-\sigma}_{\alpha,\beta}$ and $(\mathrm{Id} + L_{\alpha,\beta})^{-\sigma}$ are one-to-one and bounded on $L^p(0,\pi)$, $p \in E(\alpha,\beta)$. Thus, given $s > 0$ and $p \in E(\alpha,\beta)$, it makes sense to define the Jacobi potential spaces as the ranges of the potential operators on $L^p(0,\pi)$,

$$\mathcal{L}^{p,s}_{\alpha,\beta} := \begin{cases} L^{-s/2}_{\alpha,\beta}(L^p(0,\pi)), & \alpha+\beta \neq -1, \\ (\mathrm{Id} + L_{\alpha,\beta})^{-s/2}(L^p(0,\pi)), & \alpha+\beta = -1. \end{cases}$$

Then the formula

$$\|f\|_{\mathcal{L}^{p,s}_{\alpha,\beta}} := \|g\|_{L^p(0,\pi)}, \qquad \begin{cases} f = L^{-s/2}_{\alpha,\beta} g, \quad g \in L^p(0,\pi), & \alpha+\beta \neq -1, \\ f = (\mathrm{Id} + L_{\alpha,\beta})^{-s/2} g, \quad g \in L^p(0,\pi), & \alpha+\beta = -1, \end{cases}$$

defines a norm on $\mathcal{L}^{p,s}_{\alpha,\beta}$ and it is straightforward to check that $\mathcal{L}^{p,s}_{\alpha,\beta}$ equipped with this norm is a Banach space.

According to a general concept, Sobolev spaces $\mathbb{W}^{p,m}_{\alpha,\beta}$, $m \geq 1$, associated with $L_{\alpha,\beta}$ should be defined by

$$\mathbb{W}^{p,m}_{\alpha,\beta} := \{ f \in L^p(0,\pi) : \mathbb{D}^{(k)} f \in L^p(0,\pi),\ k = 1,\ldots,m \}$$

and equipped with the norms

$$\|f\|_{\mathbb{W}^{p,m}_{\alpha,\beta}} := \sum_{k=0}^{m} \|\mathbb{D}^{(k)} f\|_{L^p(0,\pi)}.$$

Here $\mathbb{D}^{(k)}$ are suitably defined differential operators of orders k playing the role of higher order derivatives, and the differentiation is understood in a weak sense. Thus $\mathbb{W}^{p,m}_{\alpha,\beta}$ depends on a proper choice of $\mathbb{D}^{(k)}$, which is actually the heart of the matter. It was shown in [4] that even in the ultraspherical case seemingly the most natural choice $\mathbb{D}^{(k)} = D^k_{\alpha,\beta}$ is not appropriate since then the spaces $\mathbb{W}^{p,m}_{\alpha,\beta}$ and $\mathcal{L}^{p,m}_{\alpha,\beta}$ are not isomorphic in general.

On the other hand, the isomorphism between Sobolev and potential spaces is a crucial aspect of the classical theory (see e.g. [1,7] or [29, Chapter V]) that should be preserved in the present setting. With this motivation, inspired by [4], we introduce the higher order 'derivative'

$$D^{(k)} := D_{\alpha+k-1,\beta+k-1} \circ \ldots \circ D_{\alpha+1,\beta+1} \circ D_{\alpha,\beta}.$$

Then taking $\mathbb{D}^{(k)} = D^{(k)}$ we get Sobolev spaces satisfying the desired property. Denote

$$W^{p,m}_{\alpha,\beta} := \big\{ f \in L^p(0,\pi) : D^{(k)} f \in L^p(0,\pi),\ k = 1,\ldots,m \big\}.$$

We will prove the following.

Theorem A. *Let $\alpha,\beta > -1$, $p \in E(\alpha,\beta)$ and $m \geq 1$. Then*

$$W^{p,m}_{\alpha,\beta} = \mathcal{L}^{p,m}_{\alpha,\beta}$$

in the sense of isomorphism of Banach spaces.

Notice that the higher order 'derivative' $D^{(k)}$ has a philosophical disadvantage that is the dependence on the first order 'derivatives' related to variable parameters of type. Thus we ask if it is possible to overcome this inconvenience by introducing still another notion of higher order 'derivative'

$$\mathcal{D}^{(k)} := \underbrace{\ldots D_{\alpha,\beta} D^*_{\alpha,\beta} D_{\alpha,\beta} D^*_{\alpha,\beta} D_{\alpha,\beta}}_{k \text{ components}}.$$

This choice resulting from interlacing $D_{\alpha,\beta}$ with $D^*_{\alpha,\beta}$ was postulated in [27]. The operators $\mathcal{D}^{(k)}$ stem naturally from the symmetrization procedure proposed there, and their utility lies in a conjugacy scheme which is very close to the classical shape. Moreover, $\mathcal{D}^{(k)}$ are also supported by a good L^p-theory of the corresponding Riesz transforms, see [27] for the case $p = 2$ and [9,14] in case $1 \leq p < \infty$. Comparing to the other higher order Jacobi 'derivatives', $\mathcal{D}^{(k)}$ have a simpler and more symmetric structure, and thus a computational advantage, since $D^*_{\alpha,\beta} D_{\alpha,\beta} = L_{\alpha,\beta} - A^2_{\alpha,\beta}$. Furthermore, $\mathcal{D}^{(k)}$ depend only on $D_{\alpha,\beta}$ (with only one pair of type parameters involved), do not change much with k, and do not map far from the system $\{\phi^{\alpha,\beta}_n\}$. All these facts motivate the question posed in [27, p. 441] of the relevance of the interlacing 'derivatives' from the Sobolev spaces theory perspective. Unfortunately, $\mathcal{D}^{(k)}$ turn out to be unsuitable for defining the Sobolev spaces, leading in fact to essentially larger Sobolev spaces than $D^{(k)}$. Let

$$\mathcal{W}^{p,m}_{\alpha,\beta} := \big\{ f \in L^p(0,\pi) : \mathcal{D}^{(k)} f \in L^p(0,\pi),\ k = 1,\ldots,m \big\}.$$

Theorem B. *Let $\alpha,\beta > -1$, $p \in E(\alpha,\beta)$ and $m \geq 1$. Then*

$$\mathcal{L}^{p,m}_{\alpha,\beta} \subset \mathcal{W}^{p,m}_{\alpha,\beta}$$

in the sense of embedding of Banach spaces. However, the reverse inclusion does not hold for all parameters values. In particular, for each α,β satisfying $0 \neq \alpha,\beta < 1/p - 1/2$ there is $f \in \mathcal{W}^{p,2}_{\alpha,\beta}$ such that $f \notin \mathcal{L}^{p,2}_{\alpha,\beta}$.

Theorems A and B are the main results of the paper. Proving the first one requires actually more technical effort, but we follow a similar strategy to that in [4] aiming at demonstrating that the relevant norms are equivalent. Roughly, we shall show estimates of the form $\|D^{(k)}f\|_{L^p(0,\pi)} \sim \|L_{\alpha,\beta}^{k/2}f\|_{L^p(0,\pi)}$ or equivalently, $\|D^{(k)}L_{\alpha,\beta}^{-k/2}g\|_{L^p(0,\pi)} \sim \|g\|_{L^p(0,\pi)}$. Therefore we will need to prove essentially two things: first, L^p-boundedness of the operators $D^{(k)}L_{\alpha,\beta}^{-k/2}$ that may be regarded as analogues of the classical higher order Riesz transforms; second, existence of a certain inversion procedure that will enable us to write bounds of the form $\|g\|_{L^p(0,\pi)} \le C\|D^{(k)}L_{\alpha,\beta}^{-k/2}g\|_{L^p(0,\pi)}$. The latter task will require introducing some auxiliary operators and studying their L^p mapping properties. The main technical tool applied repeatedly will be a powerful multiplier-transplantation theorem due to Muckenhoupt [19]. The same result was used in [4], but here we apply it in a slightly simpler way. Another tool we shall need are some approximation properties of Poisson–Jacobi integrals. In particular, we will obtain certain results of independent interest for the corresponding maximal operators.

The paper is organized as follows. In Section 2 we describe in detail the setting and give some preparatory results, including the aforementioned multiplier-transplantation theorem and properties of the Poisson–Jacobi integrals. Sections 3 and 4 are devoted to the proofs of Theorems A and B, respectively. Finally, in Section 5 we study some elementary properties of the Sobolev spaces under consideration and their relation to classical Sobolev spaces on the interval $(0,\pi)$. We also discuss boundedness of the Poisson–Jacobi integral maximal operator on some of these spaces.

Throughout the paper we use a standard notation with all symbols referring to the measure space $((0,\pi), d\theta)$. In particular, we write L^p for $L^p(0,\pi)$ and $\|\cdot\|_p$ for the associated norm. Further, we set

$$p(\alpha,\beta) := \begin{cases} \infty, & \alpha,\beta \ge -1/2, \\ -1/\min(\alpha+1/2,\beta+1/2), & \text{otherwise} \end{cases}$$

and

$$\Psi^{\alpha,\beta}(\theta) := \left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}, \quad \theta\in(0,\pi).$$

2. Preliminaries and preparatory results

The Jacobi trigonometric functions are defined as

$$\phi_n^{\alpha,\beta}(\theta) := \Psi^{\alpha,\beta}(\theta)\,\mathcal{P}_n^{\alpha,\beta}(\theta), \quad \theta\in(0,\pi),$$

where $\mathcal{P}_n^{\alpha,\beta}$ are the normalized Jacobi trigonometric polynomials given by

$$\mathcal{P}_n^{\alpha,\beta}(\theta) := c_n^{\alpha,\beta}P_n^{\alpha,\beta}(\cos\theta);$$

here $c_n^{\alpha,\beta}$ are normalizing constants, and $P_n^{\alpha,\beta}$ denote the classical Jacobi polynomials as defined in Szegö's monograph [32].

Recall that the system $\{\phi_n^{\alpha,\beta} : n\ge 0\}$ is an orthonormal basis in L^2 consisting of eigenfunctions of the Jacobi operator,

$$L_{\alpha,\beta}\phi_n^{\alpha,\beta} = \lambda_n^{\alpha,\beta}\phi_n^{\alpha,\beta}, \quad \text{where } \lambda_n^{\alpha,\beta} := (n+A_{\alpha,\beta})^2 \text{ and } A_{\alpha,\beta} := \frac{\alpha+\beta+1}{2}.$$

Thus $L_{\alpha,\beta}$ has a non-negative self-adjoint extension which is natural in this context. It is given by the spectral series

$$L_{\alpha,\beta} f = \sum_{n=0}^{\infty} \lambda_n^{\alpha,\beta} a_n^{\alpha,\beta}(f) \phi_n^{\alpha,\beta}$$

on the domain consisting of those $f \in L^2$ for which this series converges in L^2. Here and elsewhere we denote by $a_n^{\alpha,\beta}(f)$ the Fourier–Jacobi coefficients of a function f whenever the defining integrals

$$a_n^{\alpha,\beta}(f) := \int_0^{\pi} f(\theta) \phi_n^{\alpha,\beta}(\theta)\, d\theta$$

exist. For further reference we also denote

$$S_{\alpha,\beta} := \operatorname{span}\{\phi_n^{\alpha,\beta} : n \ge 0\}.$$

According to [31, Lemma 2.3], $S_{\alpha,\beta}$ is a dense subspace of L^p provided that $1 \le p < p(\alpha,\beta)$.

The setting related to $L_{\alpha,\beta}$ was investigated recently in [21,25,31]. Its importance comes from the fact that it forms a natural environment for transplantation questions pertaining to expansions based on Jacobi polynomials, see for instance [10,19]. Actually, the following result plays a crucial role in our work. It is essentially a special case of the general weighted multiplier-transplantation theorem due to Muckenhoupt [19, Theorem 1.14], see [19, Corollary 17.11] and also [10, Theorem 2.5] together with the related comments on pp. 376–377 therein. Here and elsewhere we use the convention that $\phi_n^{\alpha,\beta} \equiv 0$ if $n < 0$.

Lemma 2.1 *(Muckenhoupt). Let $\alpha, \beta, \gamma, \delta > -1$ and let $d \in \mathbb{Z}$. Assume that $g(n)$ is a sequence satisfying for sufficiently large n the smoothness condition*

$$g(n) = \sum_{j=0}^{J-1} c_j\, n^{-j} + \mathcal{O}(n^{-J}),$$

where $J \ge \alpha + \beta + \gamma + \delta + 6$ and c_j are fixed constants.

Then for each p satisfying $p'(\gamma,\delta) < p < p(\alpha,\beta)$ the operator

$$f \mapsto \sum_{n=0}^{\infty} g(n) a_n^{\alpha,\beta}(f) \phi_{n+d}^{\gamma,\delta}(\theta), \quad f \in S_{\alpha,\beta},$$

extends to a bounded operator on $L^p(0,\pi)$.

Observe that for $f \in S_{\alpha,\beta}$ there are only finitely many non-zero terms in the last series. Moreover, since $S_{\alpha,\beta}$ is dense in L^p for $p < p(\alpha,\beta)$, the extension from Lemma 2.1 is unique.

The Poisson–Jacobi semigroup $\{\exp(-tL_{\alpha,\beta}^{1/2})\}_{t\ge 0}$ can be written in L^2 by means of the spectral theorem as

$$H_t^{\alpha,\beta} f = \sum_{n=0}^{\infty} \exp\Bigl(-t\sqrt{\lambda_n^{\alpha,\beta}}\Bigr) a_n^{\alpha,\beta}(f) \phi_n^{\alpha,\beta}.$$

This series converges in fact pointwise and, moreover, may serve as a pointwise definition of $H_t^{\alpha,\beta} f$, $t > 0$, for more general f. In particular, for $f \in L^p$, $p > p'(\alpha,\beta)$, the coefficients $a_n^{\alpha,\beta}(f)$ exist and grow polynomially in n (see [31, Theorem 2.1]) which together with the estimate (cf. [32, (7.32.2)])

$$\bigl|\phi_n^{\alpha,\beta}(\theta)\bigr| \le C \Psi^{\alpha,\beta}(\theta)(n+1)^{\alpha+\beta+2}, \quad \theta \in (0,\pi),\ n \ge 0, \tag{1}$$

implies pointwise convergence of the series in question (actually, the growth property of $a_n^{\alpha,\beta}(f)$ is a direct consequence of (1)). Further, $H_t^{\alpha,\beta}$ has an integral representation

$$H_t^{\alpha,\beta} f(\theta) = \int_0^\pi H_t^{\alpha,\beta}(\theta,\varphi) f(\varphi)\, d\varphi, \quad t > 0,\ \theta \in (0,\pi),$$

valid for $f \in L^p$ with p as before. The integral kernel here is directly related to the Poisson–Jacobi kernel $\mathcal{H}_t^{\alpha,\beta}(\theta,\varphi)$ in the context of expansions into $\mathcal{P}_n^{\alpha,\beta}$ (see [25, Section 2]),

$$H_t^{\alpha,\beta}(\theta,\varphi) = \Psi^{\alpha,\beta}(\theta)\Psi^{\alpha,\beta}(\varphi)\mathcal{H}_t^{\alpha,\beta}(\theta,\varphi). \tag{2}$$

Thus sharp estimates of $\mathcal{H}_t^{\alpha,\beta}(\theta,\varphi)$ obtained in [24, Theorem 5.2] and [26, Theorem 6.1] imply readily sharp estimates for $H_t^{\alpha,\beta}(\theta,\varphi)$. Further, known results on the maximal operator associated with $\mathcal{H}_t^{\alpha,\beta}(\theta,\varphi)$ imply the following.

Proposition 2.2. *Let $\alpha,\beta > -1$ and let $p \in E(\alpha,\beta)$. Then the maximal operator*

$$H_*^{\alpha,\beta} f := \sup_{t>0} \left| H_t^{\alpha,\beta} f \right|$$

is bounded on $L^p(0,\pi)$.

The proof of this result refers to the Muckenhoupt class of A_p weights related to the measure $d\mu_{\alpha,\beta}(\theta) = \Psi^{2\alpha+1/2,2\beta+1/2}(\theta)\, d\theta$ in $(0,\pi)$. Denoted by $A_p^{\alpha,\beta}$, this is the class of all nonnegative functions w such that

$$\sup_{I\in\mathcal{I}} \left[\frac{1}{\mu_{\alpha,\beta}(I)} \int_I w(\theta)\, d\mu_{\alpha,\beta}(\theta)\right] \left[\frac{1}{\mu_{\alpha,\beta}(I)} \int_I w(\theta)^{-p'/p}\, d\mu_{\alpha,\beta}(\theta)\right]^{p/p'} < \infty$$

when $1 < p < \infty$, or

$$\sup_{I\in\mathcal{I}} \frac{1}{\mu_{\alpha,\beta}(I)} \int_I w(\theta)\, d\mu_{\alpha,\beta}(\theta) \operatorname*{ess\,sup}_{\theta\in I} \frac{1}{w(\theta)} < \infty$$

if $p = 1$; here $\mathcal{I}$ is the family of all subintervals of $(0,\pi)$.

Proof of Proposition 2.2. Let $1 < p < \infty$. By [24, Corollary 2.5] and [26, Corollary 5.2], the maximal operator

$$\mathcal{H}_*^{\alpha,\beta} f(\theta) := \sup_{t>0} \left| \int_0^\pi \mathcal{H}_t^{\alpha,\beta}(\theta,\varphi) f(\varphi) \Psi^{2\alpha+1/2,2\beta+1/2}(\varphi)\, d\varphi \right|$$

is bounded on $L^p(w\Psi^{2\alpha+1/2,2\beta+1/2})$ for $w \in A_p^{\alpha,\beta}$. Letting $w_{r,s}(\theta) := \Psi^{r-1/2,s-1/2}(\theta)$ be a double-power weight, the condition $w_{r,s} \in A_p^{\alpha,\beta}$ is equivalent to saying that $-(2\alpha+2) < r < (2\alpha+2)(p-1)$ and $-(2\beta+2) < s < (2\beta+2)(p-1)$. The conclusion follows by combining the boundedness of $\mathcal{H}_*^{\alpha,\beta}$ in double-power weighted L^p with the relation, see (2), $H_*^{\alpha,\beta} f = \Psi^{\alpha,\beta}\mathcal{H}_*^{\alpha,\beta}(\Psi^{-\alpha-1,-\beta-1} f)$. $\square$

By standard arguments, Proposition 2.2 leads to norm and almost everywhere boundary convergence of the Poisson–Jacobi semigroup; see the proof of Theorem 2.3 below.

Closely related to the Poisson–Jacobi semigroup is the Poisson–Jacobi integral

$$U_r^{\alpha,\beta}(f) := \sum_{n=0}^{\infty} r^n a_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}, \quad 0 < r < 1,$$

and its 'spectral' variant

$$\widetilde{U}_r^{\alpha,\beta}(f) := \sum_{n=0}^{\infty} r^{|n+A_{\alpha,\beta}|} a_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}, \quad 0 < r < 1.$$

The following result extends [30, Theorem 2.2] in the ultraspherical setting. Some parallel results obtained recently by different methods can be found in [8].

Theorem 2.3. *Let $\alpha, \beta > -1$ and let $p \in E(\alpha,\beta)$. Then*

(a) *the maximal operators*

$$f \mapsto \sup_{0<r<1} \left|U_r^{\alpha,\beta} f\right| \quad \text{and} \quad f \mapsto \sup_{0<r<1} \left|\widetilde{U}_r^{\alpha,\beta} f\right|$$

are bounded on $L^p(0,\pi)$;

(b) *given any $f \in L^p(0,\pi)$,*

$$U_r^{\alpha,\beta} f(\theta) \to f(\theta) \quad \text{and} \quad \widetilde{U}_r^{\alpha,\beta} f(\theta) \to f(\theta) \quad \text{for a.a. } \theta \in (0,\pi),$$

as $r \to 1^-$;

(c) *there exists $C > 0$ depending only on α, β and p such that*

$$\left\|U_r^{\alpha,\beta} f\right\|_{L^p(0,\pi)} + \left\|\widetilde{U}_r^{\alpha,\beta} f\right\|_{L^p(0,\pi)} \le C\|f\|_{L^p(0,\pi)}$$

for all $0 < r < 1$ and $f \in L^p(0,\pi)$;

(d) *for each $f \in L^p(0,\pi)$,*

$$\left\|U_r^{\alpha,\beta} f - f\right\|_{L^p(0,\pi)} \to 0 \quad \text{and} \quad \left\|\widetilde{U}_r^{\alpha,\beta} f - f\right\|_{L^p(0,\pi)} \to 0$$

as $r \to 1^-$.

Proof. Item (c) is an obvious consequence of (a). Then (b) and (d) follow from (a) and (c) and the density of $S_{\alpha,\beta}$ in L^p. Thus it remains to prove (a).

In case of $\widetilde{U}_r^{\alpha,\beta}$, the conclusion follows immediately from Proposition 2.2 because $\widetilde{U}_r^{\alpha,\beta} = H_t^{\alpha,\beta}$, where $t = -\log r$. To treat the other case, we split the supremum according to $r \le 1/2$ and $r > 1/2$, and denote the resulting maximal operators by $U_{*,0}^{\alpha,\beta}$ and $U_{*,1}^{\alpha,\beta}$, respectively. Then using (1) and Hölder's inequality we get

$$\begin{aligned}\left|U_{*,0}^{\alpha,\beta}(f)(\theta)\right| &\le \sum_{n=0}^{\infty} 2^{-n}\left|a_n^{\alpha,\beta}(f)\right|\left|\phi_n^{\alpha,\beta}(\theta)\right| \\ &\le C\|f\|_p\left\|\Psi^{\alpha,\beta}\right\|_{p'}\Psi^{\alpha,\beta}(\theta)\sum_{n=0}^{\infty} 2^{-n}(n+1)^{2(\alpha+\beta+2)} \\ &\le C\|f\|_p\Psi^{\alpha,\beta}(\theta).\end{aligned}$$

This implies the boundedness of $U^{\alpha,\beta}_{*,0}$. To deal with $U^{\alpha,\beta}_{*,1}$, we write

$$U^{\alpha,\beta}_r(f) = r^{-A_{\alpha,\beta}}\widetilde{U}^{\alpha,\beta}_r(f) + \big(1 - r^{|A_{\alpha,\beta}|-A_{\alpha,\beta}}\big)a^{\alpha,\beta}_0(f)\phi^{\alpha,\beta}_0$$

and use the L^p-boundedness of $\widetilde{U}^{\alpha,\beta}_r$ and Hölder's inequality. It follows that $U^{\alpha,\beta}_{*,1}$ is L^p-bounded. □

We remark that a more detailed analysis of the maximal operators of the Poisson–Jacobi integrals and of boundary convergence of those integrals is possible via the above mentioned sharp estimates for $H^{\alpha,\beta}_t(\theta,\varphi)$. Assuming for instance that $\alpha,\beta \geq -1/2$, one can easily check by means of [26, Theorem 6.1] that the integral kernel $U^{\alpha,\beta}_r(\theta,\varphi)$ of $U^{\alpha,\beta}_r$ satisfies

$$0 < U^{\alpha,\beta}_r(\theta,\varphi) \leq C\frac{1-r}{(1-r)^2 + (\theta-\varphi)^2}, \quad \theta,\varphi \in (0,\pi),\ 0<r<1. \tag{3}$$

This extends the estimate for the Poisson-ultraspherical kernel used in [30], see also [20, Lemma 1, p. 27]. Moreover, (3) shows that when $\alpha,\beta \geq -1/2$, the maximal operators from Theorem 2.3(a) are controlled by the centered Hardy–Littlewood maximal operator restricted to $(0,\pi)$. Consequently, (b) of Theorem 2.3 holds for $f \in L^1$. Independently, (3) gives also (c), and so (d), of Theorem 2.3 for $f \in L^1$, still under the assumption $\alpha,\beta \geq -1/2$.

Finally, we gather some facts about potential operators associated with $L_{\alpha,\beta}$. When $\alpha+\beta \neq -1$, we consider the Riesz type potentials defined for $f \in L^2$ by

$$L^{-\sigma}_{\alpha,\beta}f = \sum_{n=0}^{\infty}\big(\lambda^{\alpha,\beta}_n\big)^{-\sigma}a^{\alpha,\beta}_n(f)\phi^{\alpha,\beta}_n.$$

In case $\alpha+\beta=-1$ we have $\lambda^{\alpha,\beta}_0 = 0$ and thus consider instead the Bessel type potentials given for $f \in L^2$ by

$$(\mathrm{Id} + L_{\alpha,\beta})^{-\sigma}f = \sum_{n=0}^{\infty}\big(1+\lambda^{\alpha,\beta}_n\big)^{-\sigma}a^{\alpha,\beta}_n(f)\phi^{\alpha,\beta}_n$$

(notice that this definition makes actually sense for all $\alpha,\beta > -1$). Clearly, these potentials are bounded on L^2. Further, both $L^{-\sigma}_{\alpha,\beta}$ and $(\mathrm{Id} + L_{\alpha,\beta})^{-\sigma}$ possess integral representations that are valid not only for $f \in L^2$, but also for $f \in L^p$ provided that $p > p'(\alpha,\beta)$, see [21] for more details.

The following result ensures that the definition of the potential spaces $\mathcal{L}^{p,s}_{\alpha,\beta}$ from Section 1 is indeed correct.

Proposition 2.4. *Let $\alpha,\beta > -1$ and let $\sigma > 0$. Assume that $p \in E(\alpha,\beta)$. Then*

(a) *$L^{-\sigma}_{\alpha,\beta}$ is bounded and one-to-one on $L^p(0,\pi)$ when $\alpha+\beta \neq -1$;*
(b) *$(\mathrm{Id} + L_{\alpha,\beta})^{-\sigma}$ is bounded and one-to-one on $L^p(0,\pi)$.*

To prove this we need a simple auxiliary property.

Lemma 2.5. *Let α,β and p be as in Proposition 2.4. Assume that $f \in L^p(0,\pi)$. If $a^{\alpha,\beta}_n(f) = 0$ for all $n \geq 0$, then $f \equiv 0$.*

Proof. It is enough to observe that the lemma holds for $f \in S_{\alpha,\beta}$, and then recall that such functions form a dense subspace in the dual space $(L^p)^* = L^{p'}$. □

Proof of Proposition 2.4. The L^p-boundedness in (a) and (b) is contained in [21], see [21, Theorem 2.4] together with comments on Bessel–Jacobi potentials in [21, Section 1]. It can also be obtained with the aid of Lemma 2.1. To show the remaining assertions we focus on $L_{\alpha,\beta}^{-\sigma}$; the case of $(\mathrm{Id}+L_{\alpha,\beta})^{-\sigma}$ is analogous.

As in the proof of [4, Proposition 1], notice that for $f \in S_{\alpha,\beta}$

$$a_n^{\alpha,\beta}\big(L_{\alpha,\beta}^{-\sigma} f\big) = \big(\lambda_n^{\alpha,\beta}\big)^{-\sigma} a_n^{\alpha,\beta}(f), \quad n \geq 0. \tag{4}$$

Since, by Hölder's inequality and the L^p-boundedness of $L_{\alpha,\beta}^{-\sigma}$, the functionals

$$f \mapsto a_n^{\alpha,\beta}\big(L_{\alpha,\beta}^{-\sigma} f\big) \quad \text{and} \quad f \mapsto a_n^{\alpha,\beta}(f)$$

are bounded from L^p to $\mathbb{C}$, and $S_{\alpha,\beta}$ is dense in L^p, we infer that (4) holds for $f \in L^p$. Now, if $L_{\alpha,\beta}^{-\sigma} f \equiv 0$ for some $f \in L^p$, then $a_n^{\alpha,\beta}(f) = 0$ for all $n \geq 0$ and hence Lemma 2.5 implies $f \equiv 0$. Therefore $L_{\alpha,\beta}^{-\sigma}$ is one-to-one on L^p. □

We finish this section by formulating an important consequence of Proposition 2.4 and the fact that $S_{\alpha,\beta}$ coincides with its images under the action of the potential operators.

Corollary 2.6. *Let $\alpha, \beta > -1$ and let $s > 0$. Assume that $p \in E(\alpha,\beta)$. Then $S_{\alpha,\beta}$ is a dense subspace of $\mathcal{L}_{\alpha,\beta}^{p,s}$.*

3. Sobolev spaces defined by variable index derivatives

The aim of this section is to prove Theorem A. Thus we let $\mathbb{D}^{(k)} = D^{(k)}$ be the higher order 'derivatives' defined by means of the first order 'derivatives' related to variable parameters of type. In what follows we shall generalize the line of reasoning from [4, Section 3] elaborated in the ultraspherical case.

To begin with, we look at the action of $D^{(k)}$ and its formal adjoint in L^2 on the Jacobi functions.

Lemma 3.1. *Let $\alpha, \beta > -1$. Then for any $k, n \geq 0$*

$$D^{(k)} \phi_n^{\alpha,\beta} = (-1)^k \sqrt{(n-k+1)_k (n+\alpha+\beta+1)_k}\, \phi_{n-k}^{\alpha+k,\beta+k},$$
$$\big(D^{(k)}\big)^* \phi_n^{\alpha+k,\beta+k} = (-1)^k \sqrt{(n+1)_k (n+k+\alpha+\beta+1)_k}\, \phi_{n+k}^{\alpha,\beta},$$

where $(z)_k$ is the Pochhammer symbol, $(z)_k = z(z+1)\ldots(z+k-1)$ when $k \neq 0$ and $(z)_0 = 1$.

Proof. To get the first identity it is enough to iterate the formula (see [32, (4.21.7)])

$$D_{\alpha,\beta} \phi_n^{\alpha,\beta} = -\sqrt{n(n+\alpha+\beta+1)}\, \phi_{n-1}^{\alpha+1,\beta+1}. \tag{5}$$

To prove the second identity, observe that by (5) and the relation

$$D_{\alpha,\beta}^* D_{\alpha,\beta} \phi_n^{\alpha,\beta} = \big(L_{\alpha,\beta} - A_{\alpha,\beta}^2\big) \phi_n^{\alpha,\beta} = \big(\lambda_n^{\alpha,\beta} - \lambda_0^{\alpha,\beta}\big)\, \phi_n^{\alpha,\beta}$$

we have

$$D_{\alpha,\beta}^* \phi_{n-1}^{\alpha+1,\beta+1} = -\sqrt{n(n+\alpha+\beta+1)} \phi_n^{\alpha,\beta}, \quad n \geq 1. \tag{6}$$

Applying this repeatedly we get the desired conclusion. □

For $k \geq 0$ and $1 \leq j \leq k$, denote by $D^{(k,j)}$ the operator emerging from $D^{(k)}$ by replacing k by j, and then α by $\alpha + k - j$ and β by $\beta + k - j$, i.e.

$$D^{(k,j)} := D_{\alpha+k-1,\beta+k-1} \circ D_{\alpha+k-2,\beta+k-2} \circ \ldots \circ D_{\alpha+k-j,\beta+k-j}.$$

Then by the second identity of Lemma 3.1 it follows that

$$\left(D^{(k,j)}\right)^* \phi_n^{\alpha+k,\beta+k} = (-1)^j \sqrt{(n+1)_j (n+2k-j+\alpha+\beta+1)_j}\, \phi_{n+j}^{\alpha+k-j,\beta+k-j}. \tag{7}$$

Next, we state some factorization identities for $D^{(k)}$ and its adjoint. It is elementary to check that

$$D_{\alpha,\beta} f(\theta) = \Psi^{\alpha,\beta}(\theta) \frac{d}{d\theta}\left(\frac{1}{\Psi^{\alpha,\beta}(\theta)} f(\theta)\right), \tag{8}$$

$$D^*_{\alpha,\beta} f(\theta) = -\frac{1}{\Psi^{\alpha,\beta}(\theta)} \frac{d}{d\theta}\left(\Psi^{\alpha,\beta}(\theta) f(\theta)\right). \tag{9}$$

Then with a bit more effort we see that

$$D^{(k)} f(\theta) = \Psi^{\alpha,\beta}(\theta)(\sin\theta)^k \left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^k \left(\frac{1}{\Psi^{\alpha,\beta}(\theta)} f(\theta)\right),$$

$$\left(D^{(k)}\right)^* f(\theta) = \frac{(-1)^k}{\Psi^{\alpha,\beta}(\theta)} \sin\theta \left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^k \left((\sin\theta)^{k-1}\Psi^{\alpha,\beta}(\theta) f(\theta)\right). \tag{10}$$

Notice that the last identity implies

$$\left(D^{(k,j)}\right)^* f(\theta) = \frac{(-1)^j}{\Psi^{\alpha,\beta}(\theta)} (\sin\theta)^{-k+j+1} \left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^j \left((\sin\theta)^{k-1}\Psi^{\alpha,\beta}(\theta) f(\theta)\right). \tag{11}$$

Our next objective is to demonstrate that $S_{\alpha,\beta}$ is a dense subspace of the Sobolev spaces.

Proposition 3.2. *Let $\alpha, \beta > -1$ and let $m \geq 1$. Assume that $p \in E(\alpha, \beta)$. Then $S_{\alpha,\beta}$ is a dense subspace of $W^{p,m}_{\alpha,\beta}$.*

To prove this we will need the following auxiliary technical result.

Lemma 3.3. *Let α, β, m and p be as in Proposition 3.2. Then for each $f \in W^{p,m}_{\alpha,\beta}$*

$$a_n^{\alpha+k,\beta+k}\left(D^{(k)} f\right) = \int_0^\pi f(\theta)\left(D^{(k)}\right)^* \phi_n^{\alpha+k,\beta+k}(\theta)\, d\theta, \quad 0 \leq k \leq m,\ n \geq 0.$$

Proof. For $k = 0$ there is nothing to prove, so assume that $k \geq 1$. Choose a sequence of smooth and compactly supported functions $\{\gamma_l : l \geq 1\}$ on $(0, \pi)$ satisfying (see the proof of [4, Proposition 2])

(i) $\operatorname{supp}\gamma_l \subset (\frac{1}{2l}, \pi - \frac{1}{2l})$, $\gamma_l(\theta) = 1$ for $\theta \in (\frac{1}{l}, \pi - \frac{1}{l})$, $0 \leq \gamma_l(\theta) \leq 1$ for $\theta \in (0, \pi)$,
(ii) for each $r \geq 0$ there exists $C_r > 0$ such that

$$\left|\frac{d^r}{d\theta^r}\gamma_l(\theta)\right| \leq C_r (\sin\theta)^{-r}, \quad \theta \in (0, \pi),\ l \geq 1.$$

By assumption $D^{(k)}f \in L^p$ and so, by Hölder's inequality, the product $D^{(k)}f\phi_n^{\alpha+k,\beta+k}$ is integrable over $(0,\pi)$. Since $\gamma_l \to 1$ pointwise as $l \to \infty$, the dominated convergence theorem leads to

$$\begin{aligned} a_n^{\alpha+k,\beta+k}\big(D^{(k)}f\big) &= \lim_{l\to\infty} \int_0^\pi D^{(k)}f(\theta)\gamma_l(\theta)\phi_n^{\alpha+k,\beta+k}(\theta)\,d\theta \\ &= \lim_{l\to\infty} \int_0^\pi f(\theta)\big(D^{(k)}\big)^*\big[\gamma_l(\theta)\phi_n^{\alpha+k,\beta+k}(\theta)\big]\,d\theta. \end{aligned} \tag{12}$$

We now analyze the last integral. An application of (10) and the Leibniz rule yield

$$\begin{aligned} &\big(D^{(k)}\big)^*\big[\gamma_l(\theta)\phi_n^{\alpha+k,\beta+k}(\theta)\big] \\ &= \frac{(-1)^k}{\Psi^{\alpha,\beta}(\theta)}\sin\theta\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^k\big((\sin\theta)^{k-1}\Psi^{\alpha,\beta}(\theta)\gamma_l(\theta)\phi_n^{\alpha+k,\beta+k}(\theta)\big) \\ &= \frac{(-1)^k}{\Psi^{\alpha,\beta}(\theta)}\sin\theta\sum_{j=0}^k\binom{k}{j}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^j\big((\sin\theta)^{k-1}\Psi^{\alpha,\beta}(\theta)\phi_n^{\alpha+k,\beta+k}(\theta)\big)\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^{k-j}\gamma_l(\theta). \end{aligned}$$

This combined with (11) gives

$$\big(D^{(k)}\big)^*\big[\gamma_l(\theta)\phi_n^{\alpha+k,\beta+k}(\theta)\big] = \sum_{j=0}^k\binom{k}{j}(-1)^{k-j}(\sin\theta)^{k-j}\big(D^{(k,j)}\big)^*\phi_n^{\alpha+k,\beta+k}(\theta)\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^{k-j}\gamma_l(\theta).$$

Furthermore, by a straightforward analysis and (ii) we have

$$\left|\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\right)^r\gamma_l(\theta)\right| \le C_r\sum_{i=1}^r\left|\frac{1}{(\sin\theta)^{2r-i}}\frac{d^i}{d\theta^i}\gamma_l(\theta)\right| \le C_r\frac{1}{(\sin\theta)^{2r}}, \quad \theta\in(0,\pi),\ l\ge 1,$$

where $r \ge 1$. We conclude that

$$\begin{aligned} &\left|\big(D^{(k)}\big)^*\big[\gamma_l(\theta)\phi_n^{\alpha+k,\beta+k}(\theta)\big] - \gamma_l(\theta)\big(D^{(k)}\big)^*\phi_n^{\alpha+k,\beta+k}(\theta)\right| \\ &\quad\le C_k\left|\sum_{j=0}^{k-1}\frac{1}{(\sin\theta)^{k-j}}\big(D^{(k,j)}\big)^*\phi_n^{\alpha+k,\beta+k}(\theta)\right|, \quad \theta\in(0,\pi). \end{aligned}$$

In view of (7), the right-hand side here is controlled by a constant multiple of $\Psi^{\alpha,\beta}(\theta)$, uniformly in $l \ge 1$, and $\Psi^{\alpha,\beta} \in L^{p'}$ since $p \in E(\alpha,\beta)$. On the other hand, the left-hand side tends to 0 pointwise, by the choice of γ_l. Thus the dominated convergence theorem implies

$$\lim_{l\to\infty}\int_0^\pi f(\theta)\big(D^{(k)}\big)^*\big[\gamma_l(\theta)\phi_n^{\alpha+k,\beta+k}(\theta)\big]\,d\theta = \lim_{l\to\infty}\int_0^\pi f(\theta)\gamma_l(\theta)\big(D^{(k)}\big)^*\phi_n^{\alpha+k,\beta+k}(\theta)\,d\theta,$$

the integrable majorant being $c\Psi^{\alpha,\beta}f$. Taking into account (12), this together with another application of the dominated convergence theorem finishes the proof. $\square$

Proof of Proposition 3.2. We will demonstrate that any function from $W_{\alpha,\beta}^{p,m}$ can be approximated in the $W_{\alpha,\beta}^{p,m}$-norm by partial sums of its Poisson–Jacobi integral. The latter functions belong to $S_{\alpha,\beta}$, which is by

Lemma 3.1 a subspace of $W^{p,m}_{\alpha,\beta}$. For this purpose we need to reveal an interaction between $D^{(k)}$ and $\widetilde{U}^{\alpha,\beta}_r$. It turns out that these operators, roughly speaking, almost commute, see (13) below. This observation is crucial. Then the proof proceeds with the aid of Theorem 2.3(d).

Let $f \in W^{p,m}_{\alpha,\beta}$ be fixed and let $0 \le k \le m$. Combining Lemma 3.3 with the second identity of Lemma 3.1 we see that

$$a_n^{\alpha+k,\beta+k}\big(D^{(k)}f\big) = (-1)^k \sqrt{(n+1)_k(n+k+\alpha+\beta+1)_k}\, a^{\alpha,\beta}_{n+k}(f).$$

Using this and the first identity of Lemma 3.1 we can write

$$\begin{aligned} D^{(k)}\widetilde{U}^{\alpha,\beta}_r(f)(\theta) &= \sum_{n=k}^{\infty} r^{|n+A_{\alpha,\beta}|} a_n^{\alpha,\beta}(f)(-1)^k\sqrt{(n-k+1)_k\,(n+\alpha+\beta+1)_k}\,\phi^{\alpha+k,\beta+k}_{n-k}(\theta) \\ &= \sum_{n=k}^{\infty} r^{|n+A_{\alpha,\beta}|} a^{\alpha+k,\beta+k}_{n-k}\big(D^{(k)}f\big)\phi^{\alpha+k,\beta+k}_{n-k}(\theta) \\ &= \sum_{n=0}^{\infty} r^{|n+A_{\alpha+k,\beta+k}|} a^{\alpha+k,\beta+k}_{n}\big(D^{(k)}f\big)\phi^{\alpha+k,\beta+k}_{n}(\theta) \\ &= \widetilde{U}^{\alpha+k,\beta+k}_r\big(D^{(k)}f\big)(\theta), \end{aligned} \tag{13}$$

where $0 \le r < 1$ and $\theta \in (0,\pi)$. Exchanging the order of $D^{(k)}$ and the summation in the first equality of the above chain is indeed legitimate, as easily verified with the aid of (1).

Analogous arguments apply to tails of the Poisson–Jacobi integral,

$$\widetilde{U}^{\alpha,\beta}_{r,l}(f)(\theta) := \sum_{n=l+1}^{\infty} r^{|n+A_{\alpha,\beta}|} a_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}(\theta),$$

producing

$$D^{(k)}\widetilde{U}^{\alpha,\beta}_{r,l}(f)(\theta) = \widetilde{U}^{\alpha+k,\beta+k}_{r,l-k}\big(D^{(k)}f\big)(\theta), \quad l \ge k,$$

where r and θ are as before. This identity combined with Hölder's inequality and (1) leads to the estimates

$$\begin{aligned} \big\|D^{(k)}\widetilde{U}^{\alpha,\beta}_{r,l}(f)\big\|_p &\le \big\|D^{(k)}f\big\|_p \sum_{n=l+1-k}^{\infty} r^{|n+A_{\alpha+k,\beta+k}|}\big\|\phi_n^{\alpha+k,\beta+k}\big\|_p\big\|\phi_n^{\alpha+k,\beta+k}\big\|_{p'} \\ &\le \big\|D^{(k)}f\big\|_p \sum_{n=l+1-k}^{\infty} r^{|n+A_{\alpha+k,\beta+k}|} n^{2(\alpha+\beta+2k+2)}. \end{aligned} \tag{14}$$

Now, choose an arbitrary $\varepsilon > 0$. By (13) and Theorem 2.3(d)

$$\big\|D^{(k)}\big[\widetilde{U}^{\alpha,\beta}_{r_0}(f) - f\big]\big\|_p < \varepsilon, \quad 0 \le k \le m,$$

for some $0 < r_0 < 1$. Further, by (14), there exists l_0 depending on r_0 such that

$$\big\|D^{(k)}\widetilde{U}^{\alpha,\beta}_{r_0,l_0}(f)\big\|_p < \varepsilon, \quad 0 \le k \le m.$$

Thus

$$\big\|\widetilde{U}^{\alpha,\beta}_{r_0}(f) - \widetilde{U}^{\alpha,\beta}_{r_0,l_0}(f) - f\big\|_{W^{p,m}_{\alpha,\beta}} < 2(m+1)\varepsilon.$$

Since $\widetilde{U}^{\alpha,\beta}_{r_0}(f) - \widetilde{U}^{\alpha,\beta}_{r_0,l_0}(f)$ belongs to $S_{\alpha,\beta}$, the conclusion follows. $\square$

We continue by showing L^p-boundedness of some variants of higher order Riesz–Jacobi transforms and certain related operators.

Proposition 3.4. *Let $\alpha, \beta > -1$ and let $k \geq 0$. Assume that $p \in E(\alpha, \beta)$. Then the operators*

$$R^{k,1}_{\alpha,\beta} = D^{(k)} L^{-k/2}_{\alpha,\beta}, \quad \alpha + \beta \neq -1,$$
$$\widetilde{R}^{k,1}_{\alpha,\beta} = D^{(k)} (\mathrm{Id} + L_{\alpha,\beta})^{-k/2},$$

defined initially on $S_{\alpha,\beta}$, extend to bounded operators on $L^p(0, \pi)$.

Proof. We first focus on $R^{k,1}_{\alpha,\beta}$. Using Lemma 3.1 we get

$$R^{k,1}_{\alpha,\beta} f = \sum_{n=k}^{\infty} g(n) a_n^{\alpha,\beta}(f) \phi_{n-k}^{\alpha+k,\beta+k}, \quad f \in S_{\alpha,\beta},$$

where

$$g(n) = (-1)^k \sqrt{(n-k+1)_k (n+\alpha+\beta+1)_k} |n + A_{\alpha,\beta}|^{-k} = (-1)^k \sqrt{\frac{w(n)}{(n + A_{\alpha,\beta})^{2k}}},$$

and here w is a polynomial of degree $2k$.

Consider now the function

$$h(x) = g\left(\frac{1}{x}\right) = (-1)^k \sqrt{\frac{x^{2k} w(\frac{1}{x})}{(1 + x A_{\alpha,\beta})^{2k}}}.$$

Here the numerator and the denominator of the fraction under the square root are polynomials, each of them having value 1 at $x = 0$. Thus $h(x)$ is analytic in a neighborhood of $x = 0$. In particular, for any fixed $J \geq 1$ we have the representation

$$g(n) = h\left(\frac{1}{n}\right) = \sum_{j=0}^{J-1} c_j \left(\frac{1}{n}\right)^j + \mathcal{O}\left(\left(\frac{1}{n}\right)^J\right),$$

provided that n is sufficiently large. Therefore g satisfies the assumptions of Lemma 2.1 and the L^p-boundedness of $R^{k,1}_{\alpha,\beta}$ follows.

The case of $\widetilde{R}^{k,1}_{\alpha,\beta}$ is analogous and is left to the reader. □

The next result states that operators playing the role of conjugates of $R^{k,1}_{\alpha,\beta}$ and $\widetilde{R}^{k,1}_{\alpha,\beta}$ are also bounded on L^p.

Proposition 3.5. *Let $\alpha, \beta > -1$ and let $k \geq 0$. Assume that $p \in E(\alpha, \beta)$. Then the operators*

$$R^{k,2}_{\alpha,\beta} = \left(D^{(k)}\right)^* L^{-k/2}_{\alpha+k,\beta+k}, \quad \alpha + \beta \neq -1,$$
$$\widetilde{R}^{k,2}_{\alpha,\beta} = \left(D^{(k)}\right)^* (\mathrm{Id} + L_{\alpha+k,\beta+k})^{-k/2},$$

defined initially on $S_{\alpha+k,\beta+k}$, extend to bounded operators on $L^p(0, \pi)$.

Proof. Using the second identity of Lemma 3.1 we obtain

$$R^{k,2}_{\alpha,\beta} f = \sum_{n=0}^{\infty} g(n) a_n^{\alpha+k,\beta+k}(f)\phi_{n+k}^{\alpha,\beta}, \quad f \in S_{\alpha+k,\beta+k},$$

where

$$g(n) = \frac{(-1)^k}{(n+k+A_{\alpha,\beta})^k}\sqrt{(n+1)_k(n+k+\alpha+\beta+1)_k}.$$

As in the proof of Proposition 3.4 one verifies that $g(n)$ satisfies the assumptions of Lemma 2.1 and the conclusion follows. The treatment of $\widetilde{R}^{k,2}_{\alpha,\beta}$ relies on the same argument. $\square$

A straightforward computation reveals that for $f \in S_{\alpha,\beta}$

$$R^{k,2}_{\alpha,\beta} R^{k,1}_{\alpha,\beta} f = \sum_{n=k}^{\infty} (n-k+1)_k (n+\alpha+\beta+1)_k (n+A_{\alpha,\beta})^{-2k} a_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta},$$

$$\widetilde{R}^{k,2}_{\alpha,\beta} \widetilde{R}^{k,1}_{\alpha,\beta} f = \sum_{n=k}^{\infty} (n-k+1)_k (n+\alpha+\beta+1)_k \big(1+(n+A_{\alpha,\beta})^2\big)^{-k} a_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta},$$

where in the first case we tacitly assume that $\alpha+\beta \neq -1$. The operators that appear in the proposition below are the inverses of $R^{k,2}_{\alpha,\beta} R^{k,1}_{\alpha,\beta}$ and $\widetilde{R}^{k,2}_{\alpha,\beta} \widetilde{R}^{k,1}_{\alpha,\beta}$, respectively, on the subspace

$$\big\{f \in S_{\alpha,\beta} : a_n^{\alpha,\beta}(f) = 0 \text{ for } n \leq k-1\big\} \subset S_{\alpha,\beta}.$$

Proposition 3.6. *Let $\alpha, \beta > -1$ and let $k \geq 0$. Assume that $p \in E(\alpha,\beta)$. Then the operators*

$$T^k_{\alpha,\beta} f = \sum_{n=k}^{\infty} \frac{(n+A_{\alpha,\beta})^{2k}}{(n-k+1)_k (n+\alpha+\beta+1)_k} a_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}, \quad \alpha+\beta \neq -1,$$

$$\widetilde{T}^k_{\alpha,\beta} f = \sum_{n=k}^{\infty} \frac{[1+(n+A_{\alpha,\beta})^2]^k}{(n-k+1)_k (n+\alpha+\beta+1)_k} a_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta},$$

defined initially on $S_{\alpha,\beta}$, extend to bounded operators on $L^p(0,\pi)$.

Proof. The reasoning is based on a direct application of Lemma 2.1, see the proofs of Propositions 3.4 and 3.5. $\square$

Finally, we are in a position to prove Theorem A.

Proof of Theorem A. Recall that $S_{\alpha,\beta}$ is a dense subspace of $W^{p,m}_{\alpha,\beta}$ (Proposition 3.2) and of $\mathcal{L}^{p,m}_{\alpha,\beta}$ (Corollary 2.6). Moreover, if $f_n \to f$, either in $W^{p,m}_{\alpha,\beta}$ or in $\mathcal{L}^{p,m}_{\alpha,\beta}$, then also $f_n \to f$ in L^p. This implication is trivial in case of convergence in $W^{p,m}_{\alpha,\beta}$, and in the other case it follows by Proposition 2.4. Hence the two spaces have the same elements and to prove that they coincide as Banach spaces it suffices to show that the norms in $W^{p,m}_{\alpha,\beta}$ and $\mathcal{L}^{p,m}_{\alpha,\beta}$ are equivalent on $S_{\alpha,\beta}$, i.e. there is $C > 0$ such that

$$C^{-1}\|f\|_{W^{p,m}_{\alpha,\beta}} \leq \|f\|_{\mathcal{L}^{p,m}_{\alpha,\beta}} \leq C\|f\|_{W^{p,m}_{\alpha,\beta}}, \quad f \in S_{\alpha,\beta}.$$

To proceed, we assume that $\alpha+\beta \neq -1$. The complementary case requires only minor modifications (including replacements of $R^{m,1}_{\alpha,\beta}$, $R^{m,2}_{\alpha,\beta}$ and $T^m_{\alpha,\beta}$ by their tilded counterparts) and is left to the reader.

Let $f \in S_{\alpha,\beta}$ and take $g \in S_{\alpha,\beta}$ such that $f = L_{\alpha,\beta}^{-m/2} g$. We write $g = g_1 + g_2$, where $g_1 = \sum_{n=0}^{m-1} a_n^{\alpha,\beta}(g)\phi_n^{\alpha,\beta} = \sum_{n=0}^{m-1} |n+A_{\alpha,\beta}|^m a_n^{\alpha,\beta}(f)\phi_n^{\alpha,\beta}$. Then observing that $R_{\alpha,\beta}^{m,1} g_2 = R_{\alpha,\beta}^{m,1} g$ and using Propositions 3.5 and 3.6 we obtain

$$\begin{aligned}\|f\|_{\mathcal{L}_{\alpha,\beta}^{p,m}} &= \|g\|_p \le \|g_1\|_p + \|g_2\|_p \\ &\le \|f\|_p \sum_{n=0}^{m-1} |n + A_{\alpha,\beta}|^m \big\|\phi_n^{\alpha,\beta}\big\|_p \big\|\phi_n^{\alpha,\beta}\big\|_{p'} + \big\|T_{\alpha,\beta}^m R_{\alpha,\beta}^{m,2} R_{\alpha,\beta}^{m,1} g\big\|_p \\ &\le C\big(\|f\|_p + \big\|R_{\alpha,\beta}^{m,1} g\big\|_p\big) = C\big(\|f\|_p + \big\|D^{(m)} f\big\|_p\big) \le C\|f\|_{W_{\alpha,\beta}^{p,m}}.\end{aligned}$$

To prove the reverse estimate we apply Propositions 3.4 and 2.4 and get

$$\begin{aligned}\|f\|_{W_{\alpha,\beta}^{p,m}} &= \sum_{k=0}^m \big\|D^{(k)} f\big\|_p = \sum_{k=0}^m \big\|D^{(k)} L_{\alpha,\beta}^{-m/2} g\big\|_p = \sum_{k=0}^m \big\|R_{\alpha,\beta}^{k,1} L_{\alpha,\beta}^{-(m-k)/2} g\big\|_p \\ &\le C \sum_{k=0}^m \big\|L_{\alpha,\beta}^{-(m-k)/2} g\big\|_p \le C\|g\|_p = C\|f\|_{\mathcal{L}_{\alpha,\beta}^{p,m}}.\end{aligned}$$

The proof of Theorem A is complete. □

4. Sobolev spaces defined by interlacing derivatives

In this section we prove Theorem B. Thus the higher-order ‘derivative’ under consideration is $\mathbb{D}^{(k)} = \mathcal{D}^{(k)}$, the operator emerging from interlacing $D_{\alpha,\beta}$ and its adjoint. We start with a simple result describing the action of $\mathcal{D}^{(k)}$ on the Jacobi functions.

Lemma 4.1. *Let $\alpha, \beta > -1$. Then for any $k, n \ge 0$,*

$$\mathcal{D}^{(k)} \phi_n^{\alpha,\beta} = (-1)^k \big[n(n + \alpha + \beta + 1)\big]^{k/2} \begin{cases} \phi_n^{\alpha,\beta}, & k \text{ even}, \\ \phi_{n-1}^{\alpha+1,\beta+1}, & k \text{ odd}. \end{cases}$$

Proof. A direct computation based on (5) and (6). □

Next, we show that higher-order Riesz–Jacobi transforms defined by means of $\mathcal{D}^{(k)}$ are bounded on L^p.

Proposition 4.2. *Let $\alpha, \beta > -1$ and let $k \ge 0$. Assume that $p \in E(\alpha, \beta)$. Then the operators*

$$\begin{aligned}\mathcal{R}_{\alpha,\beta}^k f &= \mathcal{D}^{(k)} L_{\alpha,\beta}^{-k/2} f, \quad \alpha + \beta \ne -1, \\ \widetilde{\mathcal{R}}_{\alpha,\beta}^k f &= \mathcal{D}^{(k)} (\mathrm{Id} + L_{\alpha,\beta})^{-k/2} f,\end{aligned}$$

defined initially on $S_{\alpha,\beta}$, extend to bounded operators on $L^p(0, \pi)$.

Proof. Consider first the case of k even. According to Lemma 4.1, we have

$$\mathcal{R}_{\alpha,\beta}^k f = \sum_{n=0}^\infty g(n) a_n^{\alpha,\beta}(f) \phi_n^{\alpha,\beta}, \quad f \in S_{\alpha,\beta},$$

where the multiplier sequence is given by

$$g(n) = \sqrt{\frac{[n(n+\alpha+\beta+1)]^k}{(n+A_{\alpha,\beta})^{2k}}}.$$

As easily verified (see the proof of Proposition 3.4), the sequence $g(n)$ satisfies the assumptions of Lemma 2.1 and hence the L^p-boundedness of $\mathcal{R}^k_{\alpha,\beta}$ follows.

The case of k odd, as well as the treatment of $\widetilde{\mathcal{R}}^k_{\alpha,\beta}$, is analogous. The conclusion is again a consequence of Lemma 2.1. $\square$

We are now ready to prove Theorem B.

Proof of Theorem B. To show the inclusion we assume that $\alpha+\beta \neq -1$. The opposite case requires essentially the same reasoning and thus is left to the reader.

Let $f \in \mathcal{L}^{p,m}_{\alpha,\beta}$. By the definition of the potential space, there exists $g \in L^p$ such that $f = L^{-m/2}_{\alpha,\beta} g$ and $\|f\|_{\mathcal{L}^{p,m}_{\alpha,\beta}} = \|g\|_p$. Using Proposition 4.2 and then Proposition 2.4 we see that, for each $k = 0, 1, \ldots, m$,

$$\big\|\mathcal{D}^{(k)} f\big\|_p = \big\|\mathcal{D}^{(k)} L^{-m/2}_{\alpha,\beta} g\big\|_p = \big\|\mathcal{R}^k_{\alpha,\beta} L^{-(m-k)/2}_{\alpha,\beta} g\big\|_p \le C \big\|L^{-(m-k)/2}_{\alpha,\beta} g\big\|_p \le C\|g\|_p = \|f\|_{\mathcal{L}^{p,m}_{\alpha,\beta}}.$$

It follows that $\mathcal{L}^{p,m}_{\alpha,\beta}$ is continuously included in $\mathcal{W}^{p,m}_{\alpha,\beta}$.

To demonstrate that the reverse inclusion does not hold in general, we give an explicit counterexample. For $\alpha, \beta > -1$, consider the function

$$f(\theta) = \Psi^{-\alpha,-\beta}(\theta) = \left(\sin\frac{\theta}{2}\right)^{-\alpha+1/2} \left(\cos\frac{\theta}{2}\right)^{-\beta+1/2}.$$

Assume for simplicity that $\alpha \neq 0$ and $\beta \neq 0$. A direct analysis based on formulas (8) and (9), and the elementary estimates $\sin\frac{\theta}{2} \simeq \theta, \cos\frac{\theta}{2} \simeq \pi - \theta$, $\theta \in (0,\pi)$, show that

$$\begin{aligned}
f(\theta) &\le C\theta^{-\alpha+1/2}(\pi-\theta)^{-\beta+1/2},\\
\big|D_{\alpha,\beta} f(\theta)\big| &= \big|\alpha\Psi^{-\alpha-1,-\beta+1}(\theta) - \beta\Psi^{-\alpha+1,-\beta-1}(\theta)\big| \le C\theta^{-\alpha-1/2}(\pi-\theta)^{-\beta-1/2},\\
\big|D^*_{\alpha,\beta} D_{\alpha,\beta} f(\theta)\big| &= |\alpha+\beta|\Psi^{-\alpha,-\beta}(\theta) \le C\theta^{-\alpha+1/2}(\pi-\theta)^{-\beta+1/2},\\
\big|D_{\alpha+1,\beta+1} D_{\alpha,\beta} f(\theta)\big| + 1 &= \big|\alpha(1+\alpha)\Psi^{-\alpha-2,-\beta+2}(\theta) - 2\alpha\beta\Psi^{-\alpha,-\beta}(\theta) + \beta(1+\beta)\Psi^{-\alpha+2,-\beta-2}(\theta)\big| + 1\\
&\ge C\theta^{-\alpha-3/2}(\pi-\theta)^{-\beta-3/2},
\end{aligned}$$

where $C > 0$ is independent of $\theta \in (0,\pi)$. It is now clear that if $p \in E(\alpha,\beta)$, and $\alpha \neq 0$ and $\beta \neq 0$ are such that $\alpha < 1/p - 1/2$ and $\beta < 1/p - 1/2$, then $f \in \mathcal{W}^{p,2}_{\alpha,\beta}$. On the other hand, $D^{(2)} f \notin L^p$, so in view of Theorem A one has $f \notin \mathcal{L}^{p,2}_{\alpha,\beta}$. $\square$

Combining Theorem B with Theorem A reveals that the Sobolev spaces defined by means of $\mathbb{D}^{(k)} = D^{(k)}$ are contained in those related to $\mathbb{D}^{(k)} = \mathcal{D}^{(k)}$, but they do not coincide in general.

5. Final comments and remarks

We first point out some natural monotonicity properties of the potential spaces. Analogous facts are easily seen to be true also for the Sobolev spaces.

Proposition 5.1. *Let $\alpha, \beta > -1$. Assume that $p, q \in E(\alpha, \beta)$. Then*

(a) *if $p \le q$, then $\mathcal{L}^{q,s}_{\alpha,\beta} \subset \mathcal{L}^{p,s}_{\alpha,\beta}$ for all $s \ge 0$;*
(b) *if $0 \le s \le t$, then $\mathcal{L}^{p,t}_{\alpha,\beta} \subset \mathcal{L}^{p,s}_{\alpha,\beta}$ for all p.*

Moreover, the embeddings in (a) and (b) are continuous.

Proof. To get (a) it is enough to use the fact that $\|\cdot\|_p$ is dominated by a constant times $\|\cdot\|_q$ when $p \le q$. Item (b) is a consequence of Proposition 2.4, see the proof of [13, Proposition 6.3] for the analogous argument in the Laguerre case. $\square$

Next, we comment on the relation between the Sobolev spaces defined in this paper and the classical Sobolev spaces $W^{p,m}(a,b)$ related to the interval (a,b). The result below shows that there is only a local connection, and in general $W^{p,m}_{\alpha,\beta}$ and $\mathcal{W}^{p,m}_{\alpha,\beta}$ cannot be compared with $W^{p,m}(0,\pi)$ in terms of inclusion.

Proposition 5.2. *Let $\alpha, \beta > -1$, $p \in E(\alpha, \beta)$ and $m \ge 1$. Assume that f is in $\mathbb{W}^{p,m}_{\alpha,\beta}$, the Sobolev space defined either by means of $\mathbb{D}^{(k)} = D^{(k)}$ or by means of $\mathbb{D}^{(k)} = \mathcal{D}^{(k)}$. Then*

(a) *$f \in W^{p,m}(a,b)$ whenever $0 < a < b < \pi$;*
(b) *$f \in W^{p,m}(0,\pi)$, provided that $\operatorname{supp} f \subset\subset (0,\pi)$.*

Furthermore, none of the inclusions $\mathbb{W}^{p,m}_{\alpha,\beta} \subset W^{p,m}(0,\pi)$ and $W^{p,m}(0,\pi) \subset \mathbb{W}^{p,m}_{\alpha,\beta}$ is true in general. In particular, for $(\alpha,\beta) \ne (-1/2,-1/2)$ there exists $f \in W^{p,m}(0,\pi)$ such that $f \notin \mathbb{W}^{p,m}_{\alpha,\beta}$, and for each α, β satisfying $-1/2 \ne \alpha \le 1/2 - 1/p$ or $-1/2 \ne \beta \le 1/2 - 1/p$ there is $f \in \mathbb{W}^{p,m}_{\alpha,\beta}$ such that $f \notin W^{p,m}(0,\pi)$.

Proof. It is not hard to check that $f \in \mathbb{W}^{p,m}_{\alpha,\beta}$ implies $\frac{d^k}{d\theta^k} f \in L^p(K)$ for each compact set $K \subset (0,\pi)$ and $0 \le k \le m$. Thus (a) and (b) follow.

To prove the remaining part, we give explicit counterexamples. Let $f(\theta) \equiv 1$. Clearly, $f \in W^{p,m}(0,\pi)$. However, a simple computation shows that $D_{\alpha,\beta} f \notin L^p$ unless $\alpha = \beta = -1/2$. Thus $f \notin \mathbb{W}^{p,m}_{\alpha,\beta}$ when $(\alpha,\beta) \ne (-1/2,-1/2)$.

To disprove the other inclusion, consider $g = \Psi^{\alpha,\beta}$. Since g is up to a constant factor the Jacobi function $\phi_0^{\alpha,\beta}$, we know that $g \in \mathbb{W}^{p,m}_{\alpha,\beta}$. On the other hand, $\frac{d}{d\theta} g \notin L^p$ if $-1/2 \ne \alpha \le 1/2-1/p$ or $-1/2 \ne \beta \le 1/2-1/p$, so $g \notin W^{p,m}(0,\pi)$ for the indicated α and β. $\square$

Another issue that we would like to clarify pertains to the Sobolev spaces $\mathcal{W}^{p,m}_{\alpha,\beta}$ defined by means of the interlacing 'derivatives' $\mathcal{D}^{(k)}_{\alpha,\beta}$ and the Sobolev spaces based on the iterated 'derivatives' $D^k_{\alpha,\beta}$. Let us denote the latter spaces by $\boldsymbol{W}^{p,m}_{\alpha,\beta}$. We already know that neither $\mathcal{W}^{p,m}_{\alpha,\beta}$ nor $\boldsymbol{W}^{p,m}_{\alpha,\beta}$ is suitable from a general theory perspective, since in general these spaces are not isomorphic with the potential spaces $\mathcal{L}^{p,m}_{\alpha,\beta}$. Nevertheless, it is interesting to ask if, perhaps, $\mathcal{W}^{p,m}_{\alpha,\beta} = \boldsymbol{W}^{p,m}_{\alpha,\beta}$. This, however, turns out to be false. In fact, the two kinds of spaces are not related by inclusions in general, as shown below.

To see that the inclusion $\mathcal{W}^{p,m}_{\alpha,\beta} \subset \boldsymbol{W}^{p,m}_{\alpha,\beta}$ does not hold in general assume that $p \in E(\alpha,\beta)$, and $\alpha \ne 0$ and $\beta \ne 0$ are such that $-1/2 \ne \alpha < 1/p - 1/2$ and $-1/2 \ne \beta < 1/p - 1/2$. From the proof of Theorem B we know that the function $f = \Psi^{-\alpha,-\beta}$ belongs to $\mathcal{W}^{p,2}_{\alpha,\beta}$. On the other hand, by an elementary analysis we get

$$\left|D^2_{\alpha,\beta} f(\theta)\right| + 1 \ge C\theta^{-\alpha-3/2}(\pi-\theta)^{-\beta-3/2}, \quad \theta \in (0,\pi),$$

and consequently $f \notin \boldsymbol{W}^{p,2}_{\alpha,\beta}$. To disprove the reverse inclusion $\boldsymbol{W}^{p,m}_{\alpha,\beta} \subset \mathcal{W}^{p,m}_{\alpha,\beta}$ we take into account $p \in E(\alpha,\beta)$, where $\alpha,\beta > -1$ are such that $-1/2 \neq \alpha \leq 1/2 - 1/p$ or $-1/2 \neq \beta \leq 1/2 - 1/p$. Similarly as in the proof of Theorem B we find that the function $g = \Psi^{\alpha+1,\beta+1}$ satisfies the bounds

$$g(\theta) + \left|D_{\alpha,\beta} g(\theta)\right| + \left|D^2_{\alpha,\beta} g(\theta)\right| \leq C\theta^{\alpha+1/2}(\pi-\theta)^{\beta+1/2},$$
$$\left|D^*_{\alpha,\beta} D_{\alpha,\beta} g(\theta)\right| + 1 \geq C\theta^{\alpha-1/2}(\pi-\theta)^{\beta-1/2},$$

with $C > 0$ independent of $\theta \in (0,\pi)$. It follows that $g \in \boldsymbol{W}^{p,2}_{\alpha,\beta}$, but $g \notin \mathcal{W}^{p,2}_{\alpha,\beta}$.

Finally, we observe that the tools established in this paper allow to generalize the results proved in [4, Section 5] in the context of ultraspherical expansions.

Theorem 5.3. *Let $\alpha,\beta > -1$ and assume that $p \in E(\alpha,\beta)$. Then the maximal operator*

$$f \mapsto \sup_{0 \leq r < 1} \left|U^{\alpha,\beta}_r(f)\right|$$

is bounded on the Sobolev space $\mathbb{W}^{p,1}_{\alpha,\beta}$ defined by means of $\mathbb{D}^{(1)} = D^{(1)} = \mathcal{D}^{(1)} = D_{\alpha,\beta}$.

Proof. We argue in the same way as in the proof of [4, Theorem 3], replacing the relevant ultraspherical results by their Jacobi counterparts. More precisely, instead of [30, Theorem 2.2] one should use Theorem 2.3. Further, (13) should be applied in place of [4, (19)], and Proposition 5.2 in place of [4, Proposition 3]. Finally, smoothness of the Poisson–Jacobi integral can be justified directly. Indeed, in view of (1) and (5), the defining series can be differentiated term by term arbitrarily many times. □

Following the ideas of [4, Section 5], we also note that (13) together with [12, Theorem F] allow one to conclude boundedness on $W^{p,1}_{\alpha,\beta} = \mathcal{W}^{p,1}_{\alpha,\beta}$ of the maximal operator associated with partial sums of Jacobi expansions, at least when $\alpha,\beta \geq -1/2$.

Theorem 5.4. *Let $\alpha,\beta \geq -1/2$ and let $1 < p < \infty$. Then the maximal operator*

$$f \mapsto \sup_{N \geq 0} \left|\sum_{n=0}^{N} a^{\alpha,\beta}_n(f)\phi^{\alpha,\beta}_n\right|$$

is bounded on the Sobolev space appearing in Theorem 5.3.

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