



Topological conjugacy of piecewise monotonic functions of nonmonotonicity height ≥ 1 ☆



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ABSTRACT

The conjugacy problem is an important topic in the theory of dynamical systems and functional equations. In this paper, we investigate a class of piecewise monotone and continuous maps with nonmonotonicity height ≥ 1 . We give a sufficient and necessary condition under which any two of these maps are topologically conjugate, and construct a topological conjugacy with an extension method if such a conjugacy exists.

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1. Introduction

Let X, Y be topological spaces, $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be continuous maps. We say that $f : X \rightarrow X$ is *topologically conjugate* to $g : Y \rightarrow Y$ if there exists a homeomorphism $\varphi : X \rightarrow Y$ such that

$$\varphi \circ f = g \circ \varphi.$$

Such a homeomorphism φ is called a *topological conjugacy* from f to g . For convenience, write $f \sim g$ to denote that f is topologically conjugate to g . If $f \sim g$, then f and g have the same dynamical properties.

The conjugacy problem is an important topic in the theory of dynamical systems (e.g. linearization [14,17] and normal form [1,3]) and iterative functional equations [8,9] also including iterative roots. Some recent results concerning the conjugacy problem have been revisited in [4].

In 1944, U.T. Bödewadt [2] observed that the problem of iterative roots for a class of strictly monotone and continuous self-maps is equivalent to the conjugacy problem of this class of maps. For strictly monotone and continuous self-maps, Kuczma [7] in 1961 gave a complete description of iterative roots. However, it

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has been treated for a long time as a difficult problem to find iterative roots of non-monotone maps. An interesting notion is the so-called “characteristic interval” (see Section 2 for the definition), introduced by J. Zhang and L. Yang [23] in 1983, for the class of piecewise monotone and continuous functions (abbreviated as *PM functions* in [22]). They used a conjugacy relation to reduce some cases of non-monotonicity to monotone cases on the characteristic interval so that the results of monotone iterative roots are applicable. For one class of piecewise linear self-maps, G. Zhang [21] presented a sufficient and necessary condition of the conjugacy relation, and constructed a topological conjugacy to find iterative roots. However, to apply the construction method in his paper, one of the major requirements was that these maps must be piecewise expanding. Later, some results were developed in [20,18,19,15,16]. Recently, further investigations to construct continuous iterative roots of PM functions were made in [11,10]. But there still remain some open problems on iterative roots of PM functions, see [10].

In this paper we study a class of PM functions considered in [10], not necessary piecewise expanding. Our goal is to present a sufficient and necessary condition under which any two of these maps are topologically conjugate, and construct a topological conjugacy. Our extension method differs completely from the methods of [5], [6, Theorem 3.7], [13] and [12].

2. Nonmonotonicity height

In the section we introduce a notion of the nonmonotonicity height of a point for maps from the considered class and prove some properties of the notion.

Let I be a real interval $[a, b]$, where $a < b$. Without loss of generality we can assume that $a = 0$ and $b = 1$. A map $f : I \rightarrow I$ is said to be r -*modal* if it is continuous and there exists a partition $a = t_0 < t_1 < \dots < t_r < \dots < t_{r+1} = b$ such that f is alternately strictly increasing and strictly decreasing on $[t_i, t_{i+1}]$ for $i = 0, \dots, r$, $r \geq 0$. When $r = 0$, f is a continuous and strictly monotone map. When $r \geq 1$, each point t_i for $i = 1, \dots, r$ is referred to as a *turning point* of f .

Let $I_0 := [t_0, t_1]$ and $\mathcal{PM}_r(I)$ consist of all r -modal maps on I . Put $\mathcal{PM}(I) := \bigcup_{r=0}^{\infty} \mathcal{PM}_r(I)$. Denote $S(f)$ the set of all turning points of $f \in \mathcal{PM}(I)$, and $N(f)$ denote the number of turning points of f . Let f^n denote the n -th iterate of f . It is known that

$$S(f) \subseteq S(f^2) \subseteq \dots \subseteq S(f^n) \subseteq S(f^{n+1}) \subseteq \dots$$

and

$$0 = N(f^0) \leq N(f) \leq N(f^2) \leq \dots \leq N(f^n) \leq N(f^{n+1}) \leq \dots$$

Moreover, each element of $S(f^{n+1}) \setminus S(f^n)$ must be a preimage under f of a point from $S(f^n)$ (cf. [10, Lemma 2.3]). More precisely,

$$S(f^{n+1}) \setminus S(f^n) = f^{-1}(S(f^n)) \setminus S(f^n). \quad (2.1)$$

The nonmonotonicity height $H(f)$ of $f \in \mathcal{PM}(I)$ is defined as the least $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $N(f^n) = N(f^{n+1})$ if such an n exists and ∞ otherwise [10]. It is found that when $H(f) \leq 1$, namely, the number of turning points does not increase under iteration, there is a maximal sub-interval of I , which covers the range of f such that f is strictly monotone on it. Such a sub-interval is unique and called the *characteristic interval* of f .

Consider the following class of r -modal maps

$$\mathcal{M}_r(I) := \{f \in \mathcal{PM}_r(I) : f[I_0] \subseteq I_0 \text{ and } f(x) < x \text{ for } x \in (t_1, t_{r+1})\}.$$

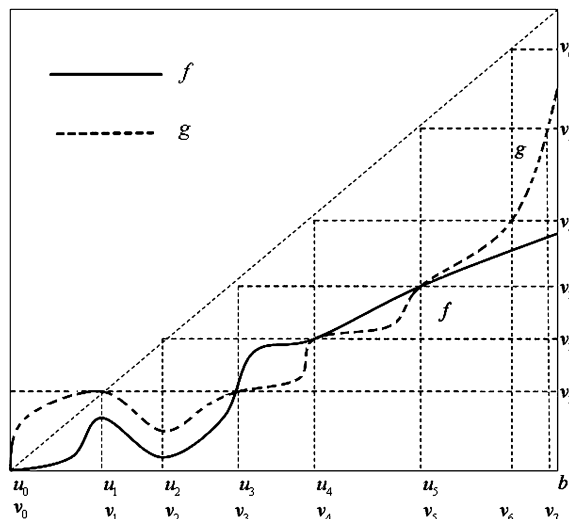


Fig. 1. $f \in \mathcal{M}_2^3(I)$, $g \in \mathcal{M}_2^4(I)$.

Put

$$\mathcal{M}_r^H(I) := \{f \in \mathcal{M}_r(I) : H(f) = H\}.$$

One can easily check that the functions f and g presented in Fig. 1 belong to $\mathcal{M}_2^3(I)$ and $\mathcal{M}_2^4(I)$, respectively.

For all $f \in \mathcal{M}_r(I)$ and $x \in I$ define the *nonmonotonicity height* of x under f by

$$H(f, x) := \inf\{n \in \mathbb{N}_0 : f^n(x) \in I_0\}.$$

Here we remark that the infimum of empty set is infinity.

Proposition 2.1. *For all $f \in \mathcal{M}_r(I)$ and $x \in I \setminus \{t_{r+1}\}$ we have $H(f, x) < \infty$.*

Proof. Since for each $x \in I \setminus (I_0 \cup \{t_{r+1}\})$, the sequence $(f^n(x))_{n \in \mathbb{N}_0}$ strictly decreasingly goes to the interval I_0 , i.e., $f^{n+1}(x) < f^n(x)$ for each $n \in \mathbb{N}_0$ such that $f^n(x) \notin I_0$ and for every $y > t_1$ there exists an $n_0 \in \mathbb{N}_0$ such that $f^n(x) < y$ for each $n \geq n_0$.

If t_1 is not a fixed point of f , then there exists an $\epsilon \in (0, t_1)$ such that $f(x) < x$ for $x \in (\epsilon, t_{r+1})$. Thus for every $y \in (\epsilon, t_1]$ there exists an $n \in \mathbb{N}$ such that $f^n(x) < y$. Consequently, $f^n(x) \in I_0$.

If t_1 is a fixed point of f , then $f|_{I_0}$ is strictly increasing and $f|_{[t_1, t_2]}$ is strictly decreasing. According to $f(x) < x$ for $x \in (t_1, t_{r+1})$, there exists an $n \in \mathbb{N}$ such that $f^n(x) < t_2$. Then either $f^n(x) \in I_0$ or $f^n(x) \in (t_1, t_2]$. In the latter case $f(f^n(x)) \in I_0$ since $f|_{[t_1, t_2]}$ is strictly decreasing. Hence $H(f, x) < \infty$. \square

Let us note that in the case where $f(t_{r+1}) < t_{r+1}$ the same reasoning as in the proof of Proposition 2.1 gives $H(f, t_{r+1}) < \infty$. If $f(t_{r+1}) = t_{r+1}$ we have $H(f, t_{r+1}) = \infty$.

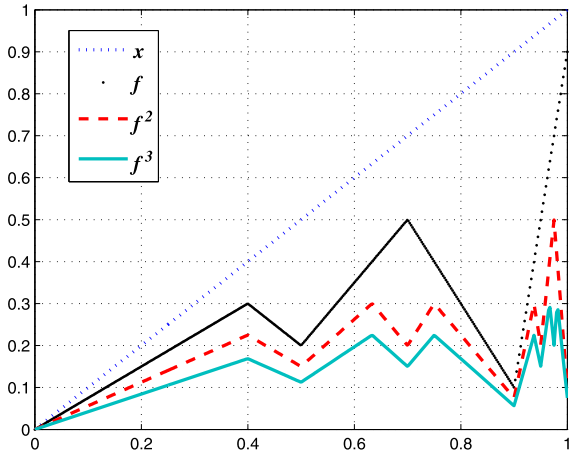
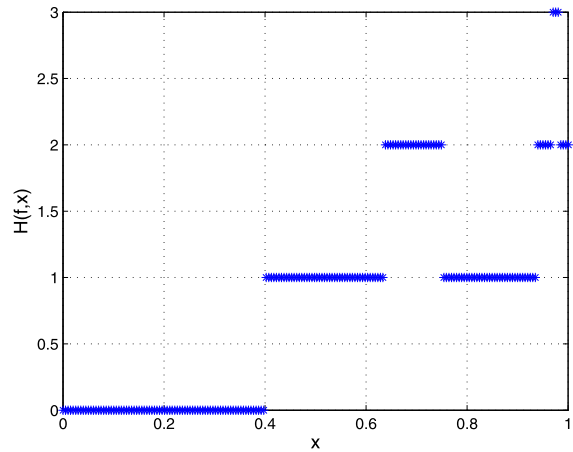
Proposition 2.2. *For all $f \in \mathcal{M}_r(I)$,*

$$H(f) = \sup\{H(f, x) : x \in I\}. \quad (2.2)$$

Moreover, if $H := H(f) < \infty$, then

$$H = \max\{H(f, p_i) : i = 1, \dots, N(f^H) + 1\}, \quad (2.3)$$

where p_i , $i = 1, \dots, N(f^H)$, are the turning points of f^H and $p_{N(f^H)+1} := t_{r+1}$.

Fig. 2. $f(x)$, $f^2(x)$ and $f^3(x)$.Fig. 3. $H(f, x)$ versus x .

Proof. If $H(f) = \infty$, then t_{r+1} is a fixed point of f . Consequently, $H(f, t_{r+1}) = \infty$ and $H(f, x) < \infty$ for every $x \in I \setminus \{t_{r+1}\}$. Therefore (2.2) holds.

If $H = H(f) < \infty$, one can see that $f^H(I) \subseteq I_0$ since $H(f^H) = 1$. According to the definition of $H(f, x)$, $H \geq H(f, x)$ for all $x \in I$.

Now, in order to prove (2.2), we need only prove (2.3).

Let us note that if $f \in \mathcal{M}_r(I)$ then $f^n \in \mathcal{M}_r(I)$ for each $n \in \mathbb{N}$.

If $H < \infty$, according to the above argument, $H \geq H(f, x)$ for all $x \in I$. In particular, $H \geq H(f, p_i)$ for $i = 1, \dots, N(f^H) + 1$. Denote by H' the maximum of the set $\{H(f, p_i) : i = 1, \dots, N(f^H) + 1\}$. Then $H \geq H'$.

On the other hand, for each $i = 1, \dots, N(f^H) + 1$ we have $H(f, p_i) \leq H'$, whence $f^{H'}(p_i) \in I_0$. Then $f^{H'}(x) \leq t_1$ for every $x \in I$, since $f^{H'} \in \mathcal{M}_r(I)$. Thus $f^{H'}(I) \subseteq I_0$. Since f is strictly monotone on I_0 , we have $S(f^{H'+1}) = S(f(f^{H'})) = S(f^{H'})$, which implies $H' \geq H$.

Therefore $H = H'$ and (2.3) holds. \square

Remark 2.1. Note that $H(f)$ is not necessarily equal to $\max\{H(f, t_i), i = 0, 1, \dots, r + 1\}$. For example, let $f : [0, 1] \rightarrow [0, 1]$ be defined by (see Figs. 2, 3)

$$f(x) := \begin{cases} 0.75x & \text{if } x \in [0, 0.4] \\ -x + 0.7 & \text{if } x \in [0.4, 0.5] \\ 1.5x - 0.55 & \text{if } x \in [0.5, 0.7] \\ -2x + 1.9 & \text{if } x \in [0.7, 0.9] \\ 8x - 7.1 & \text{if } x \in [0.9, 1] \end{cases}$$

One can see that $H(f, t_0) = H(f, t_1) = 0$, $H(f, t_2) = H(f, t_4) = 1$, and $H(f, t_3) = H(f, t_5) = 2$. However, $f(0.975) = t_3$, $f(t_3) = t_2$ and $f(t_2) \in I_0$. Hence $H(f, 0.975) = 3$.

For each $x \in I$, define a sequence of nonnegative integers $I_f(x) = (i_k(x))_{k \in \mathbb{N}_0}$ for $f \in \mathcal{M}_r(I)$ in such a way

$$i_k(x) := \begin{cases} l & \text{if } f^k(x) \in I_l \setminus \{t_1, \dots, t_r\}, l \in \{0, \dots, r\}, \\ m & \text{if } f^k(x) \in I_m \cap I_{m+1} = \{t_m\}, m \in \{0, \dots, r-1\}. \end{cases}$$

Since $f \in \mathcal{M}_r(I)$, we can see that (1) for each $x \in I$ the sequence $(i_k(x))_{k \in \mathbb{N}_0}$ is nonincreasing, (2) $i_k(x) = 0$ for $k \geq H(f, x)$, and (3) if $I_f(x)$ is constant, then $i_k(x) = 0$ for every k or $i_k(x) = r$ for every k . Note that $i_k(x) = r$ for every k if and only if $x = t_{r+1}$ and t_{r+1} is a fixed point of f .

For each $x \in I$, if $i_k(x) = 0$ for every k , define $H_0(f, x) = 0$; if $i_k(x) = r$ for every k , define $H_0(f, x) = \infty$; if there exists a positive integer k such that $i_k(x) \neq i_0(x)$, define $H_0(f, x)$ as such the least k .

3. Necessary conditions

In this section we will present some necessary conditions for the topological conjugacy relationship in the families $\mathcal{M}_r(I)$ for $r \in \mathbb{N}$.

Consider $f \in \mathcal{M}_{r_f}(I)$ and $g \in \mathcal{M}_{r_g}(J)$ with turning points and endpoints $\{t_i\}_{i=0}^{r_f+1}$ and $\{s_i\}_{i=0}^{r_g+1}$, respectively, where $r_f = N(f)$ and $r_g = N(g)$. Put

$$I_k := [t_k, t_{k+1}], \quad k = 0, \dots, r_f,$$

$$J_l := [s_l, s_{l+1}], \quad l = 0, \dots, r_g.$$

Lemma 3.1. *Assume that $f \in \mathcal{M}_{r_f}(I)$ and $g \in \mathcal{M}_{r_g}(J)$ and $f \sim g$ via φ . Then $r_f = r_g$, φ is strictly increasing and $\varphi(t_i) = s_i$ for $i \in \{0, \dots, r_f\}$.*

Proof. Since φ is a homeomorphism from I onto J , it is strictly increasing or strictly decreasing. Let us consider an interval (α, β) contained in I_k for a $k \in \{0, \dots, r_f\}$. Then $g|_{\varphi(\alpha, \beta)}$ is strictly monotone as the composition of strictly monotone functions $\varphi|_{f(\alpha, \beta)} \circ f|_{(\alpha, \beta)} \circ \varphi^{-1}|_{\varphi(\alpha, \beta)}$. Thus $\varphi(\alpha, \beta)$ does not contain any turning point of g . An analogous reasoning for any interval (γ, ζ) contained in J_l for an $l \in \{0, \dots, r_g\}$ gives that $\varphi^{-1}(\gamma, \zeta)$ does not contain any turning point of f . Consequently the homeomorphism φ maps the set $\{t_0, t_1, \dots, t_{r_f+1}\}$ onto the set $\{s_0, s_1, \dots, s_{r_g+1}\}$ and for each $k \in \{0, \dots, r_f\}$ we have $\varphi(I_k) = J_l$ for an $l \in \{0, \dots, r_g\}$. Thus $r_f = r_g$.

Let $r := r_f = r_g$. Then $\varphi(I_0) = J_0$ or $\varphi(I_0) = J_r$, since φ is strictly monotone. By the assumption that $f(I_0) \subseteq I_0$, we have $f^n(I_0) \subseteq I_0$ for every positive integer n . On the other hand, from the assumption that $g(x) < x$ for $x \in (s_1, s_{r+1})$, we get that for each $x \in J_r \setminus \{s_{r+1}\}$ there exists an $n \in \mathbb{N}$ such that $g^n(x) \notin J_r$. Thus $\varphi(I_0) = J_0$ and φ is strictly increasing. Consequently, $\varphi(t_i) = s_i$ for $i \in \{0, \dots, r\}$, since $\varphi(\{t_0, t_1, \dots, t_{r+1}\}) = \{s_0, s_1, \dots, s_{r+1}\}$. \square

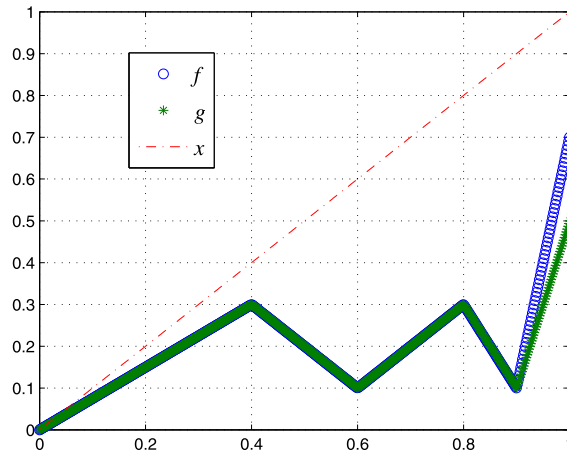
Corollary 3.1. *Assume that $f, g \in \mathcal{M}_r(I)$ and $f \sim g$ via φ . Then $\varphi(f^k(t_i)) = g^k(s_i)$ for $i \in \{0, 1, \dots, r+1\}$ and $k \in \mathbb{N}_0$. Moreover, $I_f(t_i) = I_g(s_i)$ for $i = 0, 1, \dots, r+1$. In particular, $H(f, t_i) = H(g, s_i)$ and $H_0(f, t_i) = H_0(g, s_i)$ for $i = 0, 1, \dots, r+1$.*

Proof. Due to the relationship $g^k = \varphi \circ f^k \circ \varphi^{-1}$ for $k \in \mathbb{N}$, we have that $\varphi(f^k(x)) = g^k(\varphi(x))$ for $k \in \mathbb{N}$ and $x \in I$. Putting $x = t_i$ we obtain, by Lemma 3.1, $\varphi(f^k(t_i)) = g^k(\varphi(t_i)) = g^k(s_i)$ for $k \in \mathbb{N}$ and $i = 0, 1, \dots, r+1$. The relation $I_f(t_i) = I_g(s_i)$ follows directly from the fact that $\varphi([t_j, t_{j+1}]) = [s_j, s_{j+1}]$ for $j = 0, 1, \dots, r$. \square

Corollary 3.2. *Assume that $f, g \in \mathcal{M}_r(I)$ and $f \sim g$. Then $H(f) = H(g)$.*

Proof. Let φ be a topological conjugacy from f to g . Then φ is also a topological conjugacy from f^i to g^i for each $i \in \mathbb{N}$. Hence, by Lemma 3.1, $N(f^i) = N(g^i)$ for every $i \in \mathbb{N}$. Thus we have $H(f) = H(g)$. \square

Remark 3.1. Let us note that the assumptions $H(f) = H(g)$, $H(f, t_i) = H(g, s_i)$ and $H_0(f, t_i) = H_0(g, s_i)$ for $i = 0, 1, \dots, r+1$ do not necessarily imply that $I_f(t_i) = I_g(s_i)$ for $i = 0, 1, \dots, r+1$. For example, let $f, g: [0, 1] \rightarrow [0, 1]$ be defined by (see Fig. 4)

Fig. 4. $I_f(t_5) \neq I_g(s_5)$.

$$f(x) := \begin{cases} 0.75x & \text{if } x \in [0, 0.4], \\ -x + 0.7 & \text{if } x \in [0.4, 0.6], \\ x - 0.5 & \text{if } x \in [0.6, 0.8], \\ -2x + 1.9 & \text{if } x \in [0.8, 0.9], \\ 6x - 5.3 & \text{if } x \in [0.9, 1] \end{cases}$$

and

$$g(x) := \begin{cases} 0.75x & \text{if } x \in [0, 0.4], \\ -x + 0.7 & \text{if } x \in [0.4, 0.6], \\ x - 0.5 & \text{if } x \in [0.6, 0.8], \\ -2x + 1.9 & \text{if } x \in [0.8, 0.9], \\ 4x - 3.5 & \text{if } x \in [0.9, 1]. \end{cases}$$

One can see that $f(x) = g(x)$ for $x \in [t_0, t_4]$, $I_f(t_i) = I_g(s_i)$ and $H_0(f, t_i) = H_0(g, s_i)$ for $i = 0, 1, 2, 3, 4$. Note that $H_0(f, t_5) = H_0(g, s_5) = 1$ and $H(f) = H(g) = 2$. However

$$\begin{aligned} I_f(t_5) &= (4, 2, 0, 0, \dots), \\ I_g(s_5) &= (4, 1, 0, 0, \dots). \end{aligned}$$

It follows from [Corollary 3.1](#) that f cannot be topologically conjugate to g .

4. A sufficient and necessary condition

In this section we will prove our main result. The topological conjugacy will be constructed with an extension method.

Let $r := N(f) = N(g)$, $H := H(f) = H(g)$. Consider $f, g \in \mathcal{M}_r^H(I)$ with turning points and endpoints $\{t_i\}_{i=0}^{r+1}$ and $\{s_i\}_{i=0}^{r+1}$, respectively. Put

$$I_k := [t_k, t_{k+1}], \quad f_k := f|_{I_k}$$

and

$$J_k := [s_k, s_{k+1}], \quad g_k := g|_{J_k}$$

for $k = 0, \dots, r$.

Remark 4.1. Let us note that the results of the previous section are only necessary conditions. In fact, let us consider $f, g \in \mathcal{M}_1^1(I)$ such that $f(x) < x$ for $x \in (t_0, t_2]$ and $g(x) < x$ for $x \in (s_0, s_2]$. Let $g(s_2) = s_0$ but $f(t_2) > t_0$. Then $f_0 := f|_{I_0}$ and $g_0 := g|_{J_0}$ are topologically conjugate, but the topological conjugacy φ_0 from f_0 to g_0 cannot be extended to a topological conjugacy φ from f to g . In fact, let us suppose, on the contrary, that such an extension φ exists. It follows from [Corollary 3.1](#) that $\varphi \circ f(t_2) = g(s_2) = s_0$. On the other hand, $\varphi \circ f(t_0) = \varphi(t_0) = s_0$, which implies that φ is not injective. This is a contradiction.

Before stating the main result let us point out that the values of each topological conjugacy φ from f to g on iterates of the turning points and endpoints are determined by conditions [\(4.4\)](#) and [\(4.5\)](#). However, the values of φ at other points depend uniquely on the choice of φ_0 .

Theorem 4.1. Suppose $f \in \mathcal{M}_r^H(I)$ and $g \in \mathcal{M}_r^H(J)$. Then $f \sim g$ if and only if $I_f(t_i) = I_g(s_i)$ for $i = 0, 1, \dots, r+1$ and there exists an increasing map $\varphi_0 : I_0 \rightarrow J_0$ such that φ_0 is a topological conjugacy from f_0 to g_0 with

$$\varphi_0 \circ f^{m_i}(t_i) = g^{m_i}(s_i), \quad i = 0, 1, \dots, r+1 \text{ if } H < \infty, \quad (4.4)$$

$$\varphi_0 \circ f^{m_i}(t_i) = g^{m_i}(s_i), \quad i = 0, 1, \dots, r \text{ if } H = \infty, \quad (4.5)$$

where $m_i := H(f, t_i) = H(g, s_i)$. Furthermore, there exists a unique extension φ from φ_0 such that φ is a topological conjugacy from f to g .

Proof. *Necessity.* Suppose that φ is a topological conjugacy from f to g . Then $\varphi_0 := \varphi|_{I_0}$ is an increasing topological conjugacy from f_0 to g_0 . From [Corollary 3.1](#), one can immediately obtain that $I_f(t_i) = I_g(s_i)$ and conditions [\(4.4\)](#), [\(4.5\)](#) hold.

Sufficiency. Since $\varphi_0 : I_0 \rightarrow J_0$ is an increasing topological conjugacy from f_0 to g_0 , one can see that

$$\varphi_0(I_0) = J_0, \quad \varphi_0 \circ f^m(t_j) = g^m(s_j) \quad \text{for } j = 0, 1, \quad \forall m \in \mathbb{N}_0.$$

Moreover, if there exists a positive integer m such that $f^m(t_i) \in [t_0, t_1]$ for $i = 2, \dots, r+1$, then by [\(4.4\)](#) and [\(4.5\)](#) we have

$$\varphi_0 \circ f^m(t_i) = g^m(s_i).$$

We will give a proof by induction. For some nonnegative integer $k < r$, assume that Ψ_k is an increasing topological conjugacy between $F_k := f|_{I_0 \cup \dots \cup I_k}$ and $G_k := g|_{J_0 \cup \dots \cup J_k}$, with

$$\Psi_k \left(\bigcup_{i=0}^k I_i \right) = \bigcup_{i=0}^k J_i, \quad \Psi_k \circ f^m(t_j) = g^m(s_j) \quad \text{for } j = 0, \dots, k+1, \quad \forall m \in \mathbb{N}_0. \quad (4.6)$$

Moreover, if there exists a positive integer m such that $f^m(t_i) \in [t_0, t_{k+1}]$ for $i = k+2, \dots, r+1$, then

$$\Psi_k \circ f^m(t_i) = g^m(s_i). \quad (4.7)$$

Now we shall show that for $k+1$ there exists a unique extension $\Psi_{k+1} : \bigcup_{i=0}^{k+1} I_i \rightarrow \bigcup_{i=0}^{k+1} J_i$ from Ψ_k such that Ψ_{k+1} is a topological conjugacy between F_{k+1} and G_{k+1} . We provide two steps to finish the proof.

Step 1: construct Ψ_{k+1} .

Since φ_0 is a strictly increasing topological conjugacy from f_0 to g_0 , f_0 and g_0 have the same monotonicity, which leads that f_i and g_i have the same monotonicity for each $i \in \{0, 1, \dots, r\}$. So we have two possibilities: (i) both f_{k+1} and g_{k+1} are strictly decreasing; (ii) both f_{k+1} and g_{k+1} are strictly increasing.

Case (i) In this case, we have

$$f_{k+1}[I_{k+1}] \subset \bigcup_{i=0}^k I_i, \quad g_{k+1}[J_{k+1}] \subset \bigcup_{i=0}^k J_i. \quad (4.8)$$

It follows from (4.7) that $g_{k+1}(s_{k+2}) = \Psi_k \circ f_{k+1}(t_{k+2})$. Thus $g_{k+1}[J_{k+1}] = \Psi_k \circ f_{k+1}[I_{k+1}]$ since Ψ_k is strictly increasing. Define $\Psi_{k+1} : \bigcup_{i=0}^{k+1} I_i \rightarrow \bigcup_{i=0}^{k+1} J_i$ by

$$\Psi_{k+1}(x) = \begin{cases} \Psi_k(x), & x \in \bigcup_{i=0}^k I_i, \\ g_{k+1}^{-1} \circ \Psi_k \circ f_{k+1}(x), & x \in I_{k+1}. \end{cases}$$

Since $g_{k+1}[J_{k+1}] = \Psi_k \circ f_{k+1}[I_{k+1}]$, thus Ψ_{k+1} is well defined and bijective. By (4.6) and (4.7), one has for every $m \in \mathbb{N}_0$

$$\Psi_{k+1} \left(\bigcup_{i=0}^{k+1} I_i \right) = \bigcup_{i=0}^{k+1} J_i, \quad \Psi_{k+1} \circ f^m(t_j) = g^m(s_j) \quad \text{for } j = 0, 1, \dots, k+2.$$

We shall show that if there exists a positive integer m such that $f^m(t_i) \in [t_0, t_{k+2}]$ for $i = k+3, \dots, r+1$, then

$$\Psi_{k+1} \circ f^m(t_i) = g^m(s_i). \quad (4.9)$$

In fact, if $f^m(t_i) \in I_{k+1}$, then $f^{m+1}(t_i) \in [t_0, t_{k+1}]$ and $g^m(s_i) \in J_{k+1}$ by $I_f(t_i) = I_g(s_i)$. By the definition of Ψ_{k+1} , we have

$$\Psi_k \circ f_{k+1} \circ f^m(t_i) = g_{k+1} \circ \Psi_{k+1} \circ f^m(t_i). \quad (4.10)$$

By (4.7), one can see that

$$\Psi_k \circ f_{k+1} \circ f^m(t_i) = g \circ g^m(s_i) = g_{k+1} \circ g^m(s_i),$$

which together with (4.10) implies (4.9).

Case (ii) Firstly for $j = 0, 1, \dots, k+1$ define $u_j = t_j$ and $v_j = s_j$, and for $j \geq k+2$ define

$$u_j = \begin{cases} f_{k+1}^{-1}(u_{j-1}), & \text{if } u_{j-1} \leq f(t_{k+1}), \\ t_{k+1}, & \text{if } u_{j-1} > f(t_{k+1}), \end{cases}$$

$$v_j = \begin{cases} g_{k+1}^{-1}(v_{j-1}), & \text{if } v_{j-1} \leq g(s_{k+1}), \\ s_{k+1}, & \text{if } v_{j-1} > g(s_{k+1}). \end{cases}$$

For every j , put $U_j := [u_j, u_{j+1}]$, $V_j := [v_j, v_{j+1}]$. It follows from $I_f(t_i) = I_g(s_i)$ that $H_0(f, t_{k+2}) = H_0(g, s_{k+2})$. Thus the cardinality of $\{U_j\}$ is equal to the cardinality of $\{V_j\}$. By (4.6) and (4.7), we have

$$\Psi_k \circ f^j(u_{k+j+1}) = g^j(v_{k+j+1}) \quad \text{for } j \geq 0. \quad (4.11)$$

Then define $\psi_j : U_{k+j} \rightarrow V_{k+j}$ by

$$\psi_j(x) = g_{k+1}^{-j} \circ \Psi_k \circ f_{k+1}^j(x), \quad x \in U_{k+j}, \quad j \geq 1.$$

It follows from (4.11) that $g_{k+1}^j[V_{k+j}] = \Psi_k \circ f_{k+1}^j[U_{k+j}]$. Thus the function ψ_j is well defined and bijective where $1 \leq j \leq H_0(f, t_{k+2})$.

If $H_0(f, t_{k+2}) < \infty$, define $\Psi_{k+1} : \bigcup_{i=0}^{k+1} I_i \rightarrow \bigcup_{i=0}^{k+1} J_i$ by

$$\Psi_{k+1}(x) = \begin{cases} \Psi_k(x), & x \in \bigcup_{i=0}^k U_i, \\ \psi_j(x), & x \in U_{k+j}, j = 1, 2, \dots, H_0(f, t_{k+2}). \end{cases}$$

Otherwise, define $\Psi_{k+1} : \bigcup_{i=0}^{k+1} I_i \rightarrow \bigcup_{i=0}^{k+1} J_i$ by

$$\Psi_{k+1}(x) = \begin{cases} \Psi_k(x), & x \in \bigcup_{i=0}^k U_i, \\ \psi_j(x), & x \in U_{k+j}, j = 1, 2, \dots, \\ s_{k+2}, & x = t_{k+2}. \end{cases}$$

By (4.6), (4.7) and (4.11), we have for every $m \in \mathbb{N}_0$

$$\begin{aligned} \Psi_{k+1} \circ f^m(t_i) &= \Psi_k \circ f^m(t_i) = g^m(s_i), \quad \text{for } i = 0, 1, \dots, k+1, \\ \Psi_{k+1} \circ f^m(t_{k+2}) &= \begin{cases} \Psi_k \circ f^m(t_{k+2}), & f^m(t_{k+2}) \in \bigcup_{i=0}^k I_i \\ g_{k+1}^{-j} \circ \Psi_k \circ f_{k+1}^j \circ f^m(t_{k+2}), & f^m(t_{k+2}) \in I_{k+1} \end{cases} \\ &= g^m(s_{k+2}), \\ \Psi_{k+1}(I_i) &= J_i \quad \text{for } i = 0, 1, \dots, k+1. \end{aligned}$$

Moreover, if there exists a positive integer m such that $f^m(t_i) \in [t_0, t_{k+2}]$, then

$$\begin{aligned} \Psi_{k+1} \circ f^m(t_i) &= \begin{cases} \Psi_k \circ f^m(t_i), & f^m(t_i) \in I_0 \cup \dots \cup I_k \\ g_{k+1}^{-j} \circ \Psi_k \circ f_{k+1}^j \circ f^m(t_i), & f^m(t_i) \in I_{k+1} \end{cases} \\ &= g^m(s_i) \quad \text{for } i = k+3, \dots, r+1. \end{aligned}$$

Step 2: show that Ψ_{k+1} is a topological conjugacy from F_{k+1} to G_{k+1} .

We only show that Ψ_{k+1} is a topological conjugacy from F_{k+1} to G_{k+1} for case (ii). The proof of case (i) is similar.

Since $g_{k+1}^{-j} \circ \Psi_k \circ f_{k+1}^j(x)$ maps U_{k+j} to V_{k+j} , we have $F_{k+1}(x) \in U_{k+j-1}$ for $x \in U_{k+j}$ with $1 \leq j \leq H_0(f, t_{k+2})$, and

$$\begin{aligned} \psi_j \circ F_{k+1}(x) &= g_{k+1}^{-j+1} \circ \Psi_k \circ f_{k+1}^{j-1} \circ f_{k+1}(x) \\ &= g_{k+1} \circ g_{k+1}^{-j} \circ \Psi_k \circ f_{k+1}^j(x) \\ &= g_{k+1} \circ \psi_j(x) \\ &= G_{k+1} \circ \psi_j(x). \end{aligned}$$

Hence $\Psi_{k+1} \circ F_{k+1} = G_{k+1} \circ \Psi_{k+1}$.

It suffices to show Ψ_{k+1} is a homeomorphism. Since ψ_j is continuous and strictly increasing, g_{k+1}^{-1} and f_{k+1} are continuous and have the same monotonicity, we see that Ψ_{k+1} is continuous on the subinterval U_{k+j} for each j , and strictly increasing on the interval I_{k+1} .

By (4.6), we have for $j \geq 1$

$$\begin{aligned} \psi_{j-1}(u_{k+j}) &= g_{k+1}^{-j+1} \circ \Psi_k \circ f_{k+1}^{j-1}(u_{k+j}) \\ &= g_{k+1}^{-j+1} \circ \Psi_k(u_{k+1}) \\ &= g_{k+1}^{-j+1}(v_{k+1}) \\ &= v_{k+j}, \end{aligned}$$

$$\begin{aligned}
\psi_j(u_{k+j}) &= g_{k+1}^{-j} \circ \Psi_k \circ f_{k+1}^j(u_{k+j}) \\
&= g_{k+1}^{-j} \circ \Psi_k \circ f_{k+1}(u_{k+1}) \\
&= g_{k+1}^{-j} \circ g_{k+1} \circ \Psi_k(u_{k+1}) \\
&= g_{k+1}^{-j+1}(v_{k+1}) \\
&= v_{k+j}.
\end{aligned}$$

Therefore Ψ_{k+1} is continuous at each junction u_{k+j} and strictly increasing on the domain $[t_0, t_{k+2})$. Now we prove that Ψ_{k+1} is continuous at the right endpoint t_{k+2} for $H_0(f, t_{k+2}) = \infty$. Since u_{k+j} and v_{k+j} monotonously approach t_{k+2} and s_{k+2} respectively as j tends toward $+\infty$, we have

$$\lim_{x \rightarrow t_{k+2}^-} \Psi_{k+1}(x) = \lim_{j \rightarrow +\infty} \psi_j(u_{k+j}) = \lim_{j \rightarrow +\infty} v_{k+j} = s_{k+2} = \Psi_{k+1}(t_{k+2}).$$

Hence Ψ_{k+1} is a topological conjugacy from F_{k+1} to G_{k+1} . Thus, by induction, $\Psi_r : I \rightarrow J$ is a topological conjugacy from f to g . \square

Remark 4.2. Now we present an example to show that the assumption $I_f(t_i) = I_f(s_i)$ for $i = 0, 1, \dots, r+1$ in Theorem 4.1 cannot be replaced by $H(f) = H(g)$.

Consider $f \in \mathcal{M}_2^3(I)$ where f_0 is strictly increasing. Let $u_3 := f_3^{-1}(t_2)$, $u_4 := f_3^{-1}(u_3)$, and $u_3 < f(t_3) < u_4$ and $H(f, t_3) = 3$. One can see that

$$\begin{aligned}
f^2(t_3) &\in (t_1, t_2), \\
f^3(t_3) &= f_1(f^2(t_3)) \in (t_0, t_1).
\end{aligned}$$

Hence $I_f(t_3) = (2, 2, 1, 0, \dots)$ and $H_0(f, t_3) = 3$. Now define $g \in \mathcal{M}_2^3(I)$ in such a way that $g(x) = f(x)$ for $x \in [t_0, u_3]$, g is strictly increasing in the interval $[u_3, t_3]$ and $g(t_3) = f_2^{-2}(f^3(t_3))$.

Since $f^3(t_3) \in (t_0, t_1)$, we have

$$\begin{aligned}
f_2^{-1}(f^3(t_3)) &\in (u_2, u_3), \\
g(t_3) &= f_2^{-1}(f_2^{-1}(f^3(t_3))) \in (u_4, t_3).
\end{aligned}$$

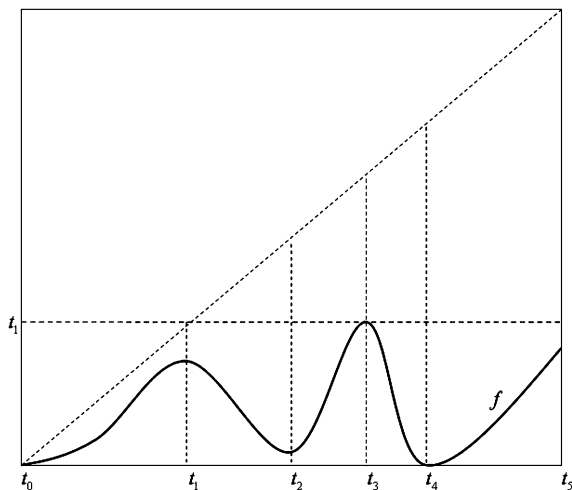
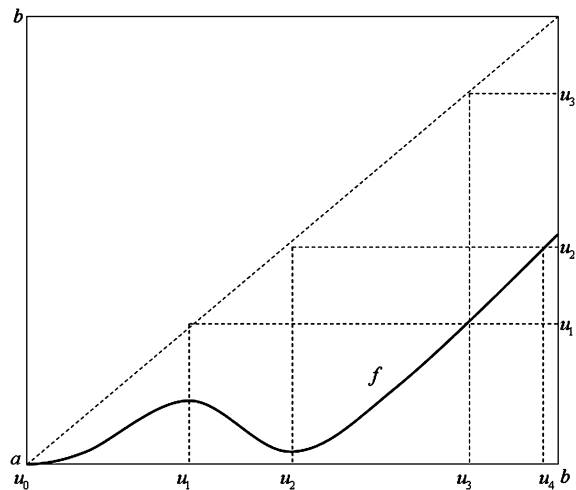
Thus $I_g(t_3) = (2, 2, 2, 0, \dots)$ and $H_0(g, t_3) = 4$. Therefore $I_f(t_3) \neq I_g(t_3)$. It follows from Corollary 3.1 that f cannot be topologically conjugate to g . Although $\varphi_0 = \text{id}$ is a topological conjugacy from f_0 to g_0 satisfying the condition $\varphi_0(f^3(t_i)) = g^3(t_i)$ for $i = 0, 1, 2, 3$.

5. Examples and applications

Finally, we give some applications of the main result. The presented examples show how the result works in simple cases.

Example 5.1. Suppose $f, g \in \mathcal{M}_r^1(I)$ (see Fig. 5), $f \sim g$ and $\varphi_0 : I_0 \rightarrow J_0$ is an increasing topological conjugacy from f_0 to g_0 with condition (4.4). Then a topological conjugacy from f to g is constructed by

$$\varphi(x) = \begin{cases} \varphi_0(x), & x \in I_0, \\ g_i^{-1} \circ \varphi_0 \circ f_i(x), & x \in I_i, \text{ for } 1 \leq i \leq 4. \end{cases}$$

Fig. 5. $f \in \mathcal{M}_4^1(I)$.Fig. 6. $f \in \mathcal{M}_2^H(I)$ with $1 < H < \infty$.

Example 5.2. Suppose $f, g \in \mathcal{M}_2^H(I)$ with $1 < H < \infty$, $f \sim g$, and f_0 and g_0 are strictly increasing self-maps, see Fig. 6. Suppose that $\varphi_0 : I_0 \rightarrow J_0$ is an increasing topological conjugacy from f_0 to g_0 with condition (4.4).

Define two sequences (u_j) and (v_j) respectively by

$$u_j = \begin{cases} t_j, & j = 0, 1, 2, \\ f_2^{-1}(u_{j-2}), & u_{j-2} \leq f_2(t_3), \quad j \geq 3, \\ t_3, & u_{j-2} > f_2(t_3), \quad j \geq 3, \end{cases}$$

$$v_j = \begin{cases} s_j, & j = 0, 1, 2, \\ g_2^{-1}(v_{j-2}), & v_{j-2} \leq g_2(s_3), \quad j \geq 3, \\ s_3, & v_{j-2} > g_2(s_3), \quad j \geq 3. \end{cases}$$

For every j , put $U_j := [u_j, u_{j+1}]$, $V_j := [v_j, v_{j+1}]$. Then by Theorem 4.1 a topological conjugacy from f to g can be constructed by

$$\varphi(x) = \begin{cases} \varphi_0(x), & x \in U_0, \\ g_1^{-1} \circ \varphi_0 \circ f_1(x), & x \in U_1, \\ g_2^{-j} \circ \varphi_0 \circ f_2^j(x), & x \in U_{2j}, \text{ for } 1 \leq j \leq H(f, t_3), \\ g_2^{-j} \circ g_1^{-1} \circ \varphi_0 \circ f_1 \circ f_2^j(x), & x \in U_{2j+1}, \text{ for } 1 \leq j \leq H(f, t_3). \end{cases}$$

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