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Multiply warped products with a quarter-symmetric connection

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ABSTRACT

In this paper, we study the Einstein warped products and multiply warped products with a quarter-symmetric connection. We also study warped products and multiply warped products with a quarter-symmetric connection with constant scalar curvature. Then we apply our results to generalized Robertson–Walker space–times with a quarter-symmetric connection and generalized Kasner space–times with a quarter-symmetric connection.

1. Introduction

The (singly) warped product $B \times_b F$ of two pseudo-Riemannian manifolds (B, g_B) and (F, g_F) with a smooth function $b : B \to (0, \infty)$ is a product manifold of form $B \times F$ with the metric tensor $g = g_B \oplus b^2 g_F$. Here, (B, g_B) is called the base manifold and (F, g_F) is called the fiber manifold and b is called the warping function. The concept of warped products was first introduced by Bishop and O'Neill [2] to construct examples of Riemannian manifolds with negative curvature. In [3], F. Dobarro and E. Dozo had studied the problem of showing when a Riemannian metric of constant scalar curvature can be produced on a product manifolds by a warped product construction from the viewpoint of partial differential equations and variational methods. In [5], Ehrlich, Jung and Kim got explicit solutions to warping function to have a constant scalar curvature for generalized Robertson–Walker space–times. In [1], explicit solutions were also obtained for the warping function to make the space–time as Einstein when the fiber is also Einstein.

One can generalize (singly) warped products to multiply warped products. A multiply warped product (M, g) is the product manifold $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, \cdots, m\}$, $b_i : B \to (0, \infty)$ is smooth and (F_i, g_{F_i}) is a pseudo-Riemannian

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manifold. In particular, when $B = (c, d)$, the metric $g_B = -dt^2$ is negative and (F_i, g_{F_i}) is a Riemannian manifold, we call M as the multiply generalized Robertson–Walker space–time.

Singly warped products have a natural generalization. A twisted product (M, g) is a product manifold of form $M = B \times_b F$, with a smooth function $b : B \times F \to (0, \infty)$, and the metric tensor $g = g_B \oplus b^2 g_F$. In [6], it was shown that mixed Ricci-flat twisted products could be expressed as warped products. As a consequence, any Einstein twisted products are warped products. Similar to the definition of multiply warped product, a multiply twisted product (M, g) is a product manifold of the form $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, \cdots, m\}$, $b_i : B \times F_i \to (0, \infty)$ is smooth. So in this paper, we define the multiply twisted products as generalizations of twisted products and multiply warped products.

The definition of a semi-symmetric metric connection was given by H. Hayden in [8]. In 1970, K. Yano [15] considered a semi-symmetric metric connection and studied some of its properties. Then in 1975, Golab [7] introduced the idea of a quarter-symmetric linear connection in differentiable manifold which is a generalization of semi-symmetric connection. A linear connection ∇ on an n-dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection if its torsion tensor T of the connection ∇ satisfies $T(X, Y) = \pi(Y)\phi X - \pi(X)\phi Y$, where π is a 1-form and ϕ is a $(1, 1)$ tensor field. In particular, if $\phi(X) = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection.

In [4], Dobarro and Ünal studied Ricci-flat and Einstein–Lorentzian multiply warped products and considered the case of having constant scalar curvature for multiply warped products and applied their results to generalized Kasner space–times. In [10], S. Sular and C. Özgür studied warped product manifolds with a semi-symmetric metric connection, they computed curvature of semi-symmetric metric connection and considered Einstein warped product manifolds with a semi-symmetric metric connection. In [11], they studied warped product manifolds with a semi-symmetric non-metric connection. In [14], we considered multiply warped products with a semi-symmetric metric connection, then apply our results to generalized Robertson–Walker spacetimes with a semi-symmetric metric connection and generalized Kasner spacetimes with a semi-symmetric metric connection. In [13], we studied curvature of multiply warped products with a semi-symmetric non-metric connection. In this paper, we will generalize our result to warped and multiply warped products with a special quarter-symmetric connection which satisfies equations (2.5) and (2.6) in [12]. This special quarter-symmetric connection is defined by equation (3). All the work we do is about it.

This paper is arranged as follows. In Section 2, we get a special quarter-symmetric connection and its curvature, then give the formula of the Levi-Civita connection and curvature of singly warped and multiply twisted product. In Section 3, we first compute curvature of a singly warped product with this quarter-symmetric connection, then study the generalized Robertson–Walker space–times with this quarter-symmetric connection. In Section 4, firstly we compute curvature of multiply twisted products with this quarter-symmetric connection, secondly we study the special multiply warped product with this quarter-symmetric connection, finally we consider the generalized Kasner space–times with this quarter-symmetric connection.

2. Preliminaries

Let M be a Riemannian manifold with Riemannian metric g. A linear connection $\overline{\nabla}$ on a Riemannian manifold M is called a quarter-symmetric connection if the torsion tensor T of the connection $\overline{\nabla}$

$$T(X, Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] \tag{1}$$

satisfies

$$T(X, Y) = \pi(Y)\phi X - \pi(X)\phi Y \tag{2}$$

where π is a 1-form associated with the vector field P on M defined by $\pi(X) = g(X, P)$ and ϕ is a $(1, 1)$ tensor field. $\overline{\nabla}$ is called a quarter-symmetric metric connection if it satisfies $\overline{\nabla}g = 0$. $\overline{\nabla}$ is called a quarter-symmetric non-metric connection if it satisfies $\overline{\nabla}g \neq 0$.

If ∇ is the Levi-Civita connection of M, in equation (2.4) in [12], let $\varphi_1 = \lambda_1 \mathrm{id}$, $\varphi_2 = 0$, $U = P$, $f_1 = 0$, $f_2 = \lambda_2 - \lambda_1$, $U_2 = P$, $\lambda_1 \in \mathbb{R} \setminus \{0\}$, $\lambda_2 \in \mathbb{R} \setminus \{0\}$, then we get a linear connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X Y = \nabla_X Y + \lambda_1 \pi(Y) X - \lambda_2 g(X, Y) P. \tag{3}$$

It is easy to see that:

(a) when $\lambda_1 = \lambda_2 = 1$, $\overline{\nabla}$ is a semi-symmetric metric connection;
(b) when $\lambda_1 = \lambda_2 \neq 1$, $\overline{\nabla}$ is a quarter-symmetric metric connection;
(c) when $\lambda_1 \neq \lambda_2$, $\overline{\nabla}$ is a quarter-symmetric non-metric connection.

In [14], we considered case (a). In this paper, we will consider cases (b) and (c).

Let R and $\overline{R}$ be the curvature tensors of ∇ and $\overline{\nabla}$, respectively. By equation (3.13) in [12], we can get

$$\begin{aligned}\overline{R}(X,Y)Z = {} & R(X,Y)Z + \lambda_1 g(Z, \nabla_X P)Y - \lambda_1 g(Z, \nabla_Y P)X \\ & + \lambda_2 g(X,Z)\nabla_Y P - \lambda_2 g(Y,Z)\nabla_X P \\ & + \lambda_1\lambda_2 \pi(P)[g(X,Z)Y - g(Y,Z)X] \\ & + \lambda_2^2 [g(Y,Z)\pi(X) - g(X,Z)\pi(Y)]P \\ & + \lambda_1^2 \pi(Z)[\pi(Y)X - \pi(X)Y]\end{aligned} \tag{4}$$

for any vector fields X, Y, Z on M.

Remark 1. When $\lambda_1 = \lambda_2 = 1$, we can get equation (4) in [14].

2.1. Warped product

Let (B, g_B) and (F, g_F) be two Riemannian manifolds and $f : B \to (0, \infty)$ be a smooth function. The warped product is the product manifold $B \times F$ with the metric tensor $g = g_B \oplus f^2 g_F$. The function f is called the warping function of the warped product, and the Hessian of f is defined by $H^f(X, Y) = XYf - (\nabla_X Y)f$.

We need the following two lemmas from [9], for later use:

Lemma 2.1. *Let $M = B \times_f F$ be a warped product, ∇, ∇^B and ∇^F denote the Levi-Civita connection on M, B and F, respectively. If $X, Y \in \Gamma(TB)$ and $U, W \in \Gamma(TF)$, then:*

(1) $\nabla_X Y = \nabla^B_X Y$;
(2) $\nabla_X U = \nabla_U X = \frac{Xf}{f} U$;
(3) $\nabla_U W = -\frac{g(U,W)}{f} \mathrm{grad}_B f + \nabla^F_U W$.

Lemma 2.2. *Let $M = B \times_f F$ be a warped product with curvature R. If $X, Y, Z \in \Gamma(TB)$ and $U, V, W \in \Gamma(TF)$, then:*

(1) $R(X, Y)Z = R^B(X, Y)Z$;
(2) $R(V, X)Y = -\frac{H^f_B(X,Y)}{f} V$;
(3) $R(X, Y)V = R(V, W)X = 0$;

(4) $R(X,V)W = -\frac{g(V,W)}{f}\nabla_X^B grad_B f$;
(5) $R(V,W)U = R^F(V,W)U - \frac{|grad_B f|_B^2}{f^2}[g(W,U)V - g(V,U)W]$.

2.2. Multiply twisted product

A multiply twisted product (M,g) is a product manifold of form $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$, where for each $i \in \{1, \cdots, m\}$, $b_i : B \times F_i \to (0, \infty)$ is smooth. Similarly the Hessian of b_i is defined by $H^{b_i}(X,Y) = XYb_i - (\nabla_X Y)b_i$.

We need the following two lemmas from [14], for later use:

Lemma 2.3. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply twisted product and let $X, Y \in \Gamma(TB)$ and $U \in \Gamma(TF_i), W \in \Gamma(TF_j)$, then:*

(1) $\nabla_X Y = \nabla_X^B Y$;
(2) $\nabla_X U = \nabla_U X = \frac{Xb_i}{b_i}U$;
(3) $\nabla_U W = 0$ *if* $i \neq j$;
(4) $\nabla_U W = U(lnb_i)W + W(lnb_i)U - \frac{g_{F_i}(U,W)}{b_i}grad_{F_i} b_i - b_i g_{F_i}(U,W)grad_B b_i + \nabla_U^{F_i} W$ *if* $i = j$.

Lemma 2.4. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply twisted product and let $X, Y, Z \in \Gamma(TB)$ and $V \in \Gamma(TF_i), W \in \Gamma(TF_j), U \in \Gamma(TF_k)$, then:*

(1) $R(X,Y)Z = R^B(X,Y)Z$;
(2) $R(V,X)Y = -\frac{H_B^{b_i}(X,Y)}{b_i}V$;
(3) $R(X,V)W = R(V,W)X = R(V,X)W = 0$ *if* $i \neq j$;
(4) $R(X,Y)V = 0$;
(5) $R(V,W)X = VX(lnb_i)W - WX(lnb_i)V$ *if* $i = j$;
(6) $R(V,W)U = 0$ *if* $i = j \neq k$ *or* $i \neq j \neq k$;
(7) $R(U,V)W = -g(V,W)\frac{g_B(grad_B b_i, grad_B b_k)}{b_i b_k}U$ *if* $i = j \neq k$;
(8) $R(X,V)W = -\frac{g(V,W)}{b_i}\nabla_X^B(grad_B b_i) + [WX(lnb_i)]V - g_{F_i}(W,V)grad_{F_i}X(lnb_i)$ *if* $i = j$;
(9) $R(V,W)U = g(V,U)grad_B(W(lnb_i)) - g(W,U)grad_B(V(lnb_i)) + R^{F_i}(V,W)U - \frac{|grad_B b_i|_B^2}{b_i^2}[g(W,U)V - g(V,U)W]$ *if* $i = j = k$.

Remark 2. It is easy to see that Lemmas 2.1 and 2.2 are corollaries of Lemma 2.3 and 2.4, respectively.

Finally we define the curvature, Ricci curvature and scalar curvature as follows:

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$
$$Ric(X,Y) = tr\, R(X,\cdot,Y,\cdot),$$
$$S = tr(Ric).$$

In this paper, we use the equivalent representations of Ricci curvature and scalar curvature as follows:

$$Ric(X,Y) = \sum_k \varepsilon_k \langle R(X,E_k)Y, E_k\rangle,$$
$$S = \sum_k \varepsilon_k Ric(E_k, E_k),$$

where E_k is an orthonormal base of M with $\langle E_k, E_k\rangle = \varepsilon_k$, $\varepsilon_k = \pm 1$.

3. Warped product with a quarter-symmetric connection

In this section, we firstly compute curvature, Ricci curvature and scalar curvature of singly warped product with a quarter-symmetric connection, then study the generalized Robertson–Walker space–times with a quarter-symmetric connection.

3.1. Connection and curvature

By Lemma 2.1 and equation (3), we have the following two propositions:

Proposition 3.1. *Let $M = B \times_f F$ be a warped product. If $X, Y \in \Gamma(TB)$, $U, W \in \Gamma(TF)$ and $P \in \Gamma(TB)$, then:*

(1) $\overline{\nabla}_X Y = \overline{\nabla}^B_X Y$;
(2) $\overline{\nabla}_X U = \frac{Xf}{f} U$;
(3) $\overline{\nabla}_U X = \left[\frac{Xf}{f} + \lambda_1 \pi(X)\right] U$;
(4) $\overline{\nabla}_U W = -f g_F(U, W) grad_B f + \nabla^F_U W - \lambda_2 g(U, W) P$.

Proposition 3.2. *Let $M = B \times_f F$ be a warped product. If $X, Y \in \Gamma(TB)$, $U, W \in \Gamma(TF)$ and $P \in \Gamma(TF)$, then:*

(1) $\overline{\nabla}_X Y = \nabla^B_X Y - \lambda_2 g(X, Y) P$;
(2) $\overline{\nabla}_X U = \frac{Xf}{f} U + \lambda_1 \pi(U) X$;
(3) $\overline{\nabla}_U X = \frac{Xf}{f} U$;
(4) $\overline{\nabla}_U W = -\frac{g(U,W)}{f} grad_B f + f^2 \overline{\nabla}^F_U W + (1 - f^2) \nabla^F_U W$.

By Lemmas 2.1, 2.2 and equation (4), we have the following two propositions:

Proposition 3.3. *Let $M = B \times_f F$ be a warped product. If $X, Y, Z \in \Gamma(TB)$, $U, V, W \in \Gamma(TF)$ and $P \in \Gamma(TB)$, then:*

(1) $\overline{R}(X, Y)Z = \overline{R}^B(X, Y)Z$;
(2) $\overline{R}(V, X)Y = -\left[\frac{H^f_B(X,Y)}{f} + \lambda_2 \frac{Pf}{f} g(X, Y) + \lambda_1\lambda_2 \pi(P) g(X, Y) + \lambda_1 g(Y, \nabla_X P) - \lambda_1^2 \pi(X)\pi(Y)\right] V$;
(3) $\overline{R}(X, Y)V = 0$;
(4) $\overline{R}(V, W)X = 0$;
(5) $\overline{R}(X, V)W = -g(V, W)\left[\frac{\nabla^B_X(grad_B f)}{f} + \lambda_1 \frac{Pf}{f} X + \lambda_2 \nabla_X P + \lambda_1\lambda_2 \pi(P) X - \lambda_2^2 \pi(X) P\right]$;
(6) $\overline{R}(U, V)W = R^F(U, V)W - \left[\frac{|grad_B f|^2_B}{f^2} + (\lambda_1 + \lambda_2)\frac{Pf}{f} + \lambda_1\lambda_2 \pi(P)\right]\left[g(V, W)U - g(U, W)V\right]$.

Proposition 3.4. *Let $M = B \times_f F$ be a warped product. If $X, Y, Z \in \Gamma(TB)$, $U, V, W \in \Gamma(TF)$ and $P \in \Gamma(TF)$, then:*

(1) $\overline{R}(X, Y)Z = R^B(X, Y)Z + \lambda_2\left[g(X, Z)\frac{Yf}{f} - g(Y, Z)\frac{Xf}{f}\right] P + \lambda_1\lambda_2 \pi(P)[g(X, Z)Y - g(Y, Z)X]$;
(2) $\overline{R}(V, X)Y = -\frac{H^f_B(X,Y)}{f} V - \lambda_1 \pi(V)\frac{Yf}{f} X - \lambda_2 g(X, Y)\nabla_V P - g(X, Y)[\lambda_1\lambda_2 \pi(P) V - \lambda_2^2 \pi(V) P]$;
(3) $\overline{R}(X, Y)V = \lambda_1 \pi(V)\left[\frac{Xf}{f} Y - \frac{Yf}{f} X\right]$;
(4) $\overline{R}(V, W)X = \lambda_1 \frac{Xf}{f}[\pi(W)V - \pi(V)W]$;

(5) $\overline{R}(X,V)W = -g(V,W)\frac{\nabla^B_X(grad_B f)}{f} + \lambda_1 \frac{Xf}{f}\pi(W)V - \lambda_1 g(W,\nabla_V P)X - \lambda_2 g(V,W)\frac{Xf}{f}P - \lambda_1\lambda_2 g(V,W) \times \pi(P)X + \lambda_1^2 \pi(W)\pi(V)X$;

(6) $\overline{R}(U,V)W = R^F(U,V)W - \frac{|grad_B f|_B^2}{f^2}[g(V,W)U - g(U,W)V] + \lambda_1[g(W,\nabla_U P)V - g(W,\nabla_V P)U] + \lambda_2[g(U,W)\nabla_V P - g(V,W)\nabla_U P] + \lambda_1\lambda_2\pi(P)[g(U,W)V - g(V,W)U] + \lambda_2^2[g(V,W)\pi(U) - g(U,W)\pi(V)]P + \lambda_1^2\pi(W)[\pi(V)U - \pi(U)V]$.

By Propositions 3.3 and 3.4 and the definition of the Ricci curvature tensor, we have the following two propositions:

Proposition 3.5. *Let $M = B \times_f F$ be a warped product, $\dim B = n_1, \dim F = n_2$ and $\dim M = \bar{n} = n_1 + n_2$. If $X, Y \in \Gamma(TB)$, $V, W \in \Gamma(TF)$ and $P \in \Gamma(TB)$, then:*

(1) $\overline{Ric}(X,Y) = \overline{Ric}^B(X,Y) + n_2\Big[\frac{H^f_B(X,Y)}{f} + \lambda_2\frac{Pf}{f}g(X,Y) + \lambda_1\lambda_2\pi(P)g(X,Y) + \lambda_1 g(Y,\nabla_X P) - \lambda_1^2\pi(X)\pi(Y)\Big]$;

(2) $\overline{Ric}(X,V) = \overline{Ric}(V,X) = 0$;

(3) $\overline{Ric}(V,W) = Ric^F(V,W) + \Big\{\frac{\Delta_B f}{f} + (n_2-1)\frac{|grad_B f|_B^2}{f^2} + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi(P) + \lambda_2 div_B P + [(\bar{n}-1)\lambda_1 + (n_2-1)\lambda_2]\frac{Pf}{f}\Big\}g(V,W)$,

where $div_B P = \sum\limits_{k=1}^{n_1}\varepsilon_k\langle\nabla_{E_k}P, E_k\rangle$, and $E_k, 1 \le k \le n_1$ is an orthonormal base of B with $\varepsilon_k = g(E_k, E_k)$.

Proposition 3.6. *Let $M = B \times_f F$ be a warped product, $\dim B = n_1, \dim F = n_2$ and $\dim M = \bar{n} = n_1 + n_2$. If $X, Y \in \Gamma(TB)$, $V, W \in \Gamma(TF)$ and $P \in \Gamma(TF)$, then:*

(1) $\overline{Ric}(X,Y) = Ric^B(X,Y) + n_2\frac{H^f_B(X,Y)}{f} + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi(P)g(X,Y) + \lambda_2 g(X,Y) div_F P$;

(2) $\overline{Ric}(X,V) = [(\bar{n}-1)\lambda_1 - \lambda_2]\pi(V)\frac{Xf}{f}$;

(3) $\overline{Ric}(V,X) = [\lambda_2 - (\bar{n}-1)\lambda_1]\pi(V)\frac{Xf}{f}$;

(4) $\overline{Ric}(V,W) = Ric^F(V,W) + g(V,W)\Big\{\frac{\Delta_B f}{f} + (n_2-1)\frac{|grad_B f|_B^2}{f^2} + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi(P)\Big\} + [(\bar{n}-1)\lambda_1 - \lambda_2]g(W,\nabla_V P) + [\lambda_2^2 + (1-\bar{n})\lambda_1^2]\pi(V)\pi(W) + \lambda_2 g(V,W) div_F P$.

By Proposition 3.5 and the definition of the scalar curvature, we have the following:

Proposition 3.7. *Let $M = B \times_f F$ be a warped product, $\dim B = n_1, \dim F = n_2$ and $\dim M = \bar{n} = n_1 + n_2$. If $P \in \Gamma(TB)$, then the scalar curvature $\overline{S}$ has the following expression:*

$$\overline{S} = \overline{S}^B + 2n_2\frac{\Delta_B f}{f} + \frac{S^F}{f^2} + n_2(n_2-1)\frac{|grad_B f|_B^2}{f^2} + n_2(\bar{n}-1)(\lambda_1+\lambda_2)\frac{Pf}{f}$$
$$+ [n_2(\bar{n}+n_1-1)\lambda_1\lambda_2 - n_2(\lambda_1^2+\lambda_2^2)]\pi(P) + n_2(\lambda_1+\lambda_2) div_B P.$$

By Proposition 3.6 and the definition of the scalar curvature, we have the following:

Proposition 3.8. *Let $M = B \times_f F$ be a warped product, $\dim B = n_1, \dim F = n_2$ and $\dim M = \bar{n} = n_1 + n_2$. If $P \in \Gamma(TF)$, then the scalar curvature $\overline{S}$ has the following expression:*

$$\overline{S} = S^B + 2n_2\frac{\Delta_B f}{f} + \frac{S^F}{f^2} + n_2(n_2-1)\frac{|grad_B f|_B^2}{f^2} + [\bar{n}(\bar{n}-1)\lambda_1\lambda_2 + (1-\bar{n})(\lambda_1^2+\lambda_2^2)]\pi(P)$$
$$+ (\bar{n}-1)(\lambda_1+\lambda_2) div_F P.$$

3.2. Generalized Robertson–Walker space–times with a quarter-symmetric connection

Theorem 3.9. *Let $M = I \times_f F$ be a warped product, where I is an open interval in $\mathbb{R}$, $\dim I = 1$ and $\dim F = \bar{n} - 1$ ($\bar{n} \geq 3$). Then $(M, \overline{\nabla})$ is Einstein if and only if (F, ∇^F) is Einstein for $P = \frac{\partial}{\partial t}$ or f is a constant on I for $P \in \Gamma(TF)$, $\lambda_2 \neq (\bar{n}-1)\lambda_1$.*

Proof. (a) Assume that $P = \frac{\partial}{\partial t}$. Let $f = e^{\frac{q}{2}}$, by Proposition 3.5, we can write

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = (1-\bar{n})\Big[\frac{1}{4}(q')^2 + \frac{1}{2}q'' - \frac{1}{2}\lambda_2 q' + \lambda_1\lambda_2 - \lambda_1^2\Big] g_I\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big); \tag{5}$$

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, V\Big) = 0;$$

$$\begin{aligned}\overline{Ric}(V,W) = Ric^F(V,W) + e^q\Big\{&\frac{\bar{n}-1}{4}(q')^2 + \frac{1}{2}\Big[(\bar{n}-1)\lambda_1 + (\bar{n}-2)\lambda_2\Big]q' \\ &+ \lambda_2^2 + (1-\bar{n})\lambda_1\lambda_2 + \frac{1}{2}q''\Big\} g_F(V,W)\end{aligned} \tag{6}$$

for any $V, W \in \Gamma(TF)$.

Since M is Einstein manifold, we have

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = \alpha g_I\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big); \tag{7}$$

$$\overline{Ric}(V,W) = \alpha e^q g_F(V,W). \tag{8}$$

From equations (5) and (7), we get

$$\alpha = (1-\bar{n})\Big[\frac{1}{4}(q')^2 + \frac{1}{2}q'' - \frac{1}{2}\lambda_2 q' + \lambda_1\lambda_2 - \lambda_1^2\Big]. \tag{9}$$

Similarly, from equations (6) and (8), and by the use of (9), we obtain

$$Ric^F(V,W) = \Big[\frac{1-\bar{n}}{2}(q')^2 - \frac{\bar{n}}{2}q'' + \Big(\frac{1-\bar{n}}{2}\lambda_1 + \frac{1}{2}\lambda_2\Big)q' + (\bar{n}-1)\lambda_1^2 - \lambda_2^2\Big] e^q g_F(V,W)$$

which implies that (F, ∇^F) is Einstein manifold.

(b) Assume that $P \in \Gamma(TF)$, by Proposition 3.6, we have

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, V\Big) = \frac{1}{2}q'[(\bar{n}-1)\lambda_1 - \lambda_2]\pi(V); \tag{10}$$

$$\overline{Ric}\Big(V, \frac{\partial}{\partial t}\Big) = \frac{1}{2}q'[\lambda_2 - (\bar{n}-1)\lambda_1]\pi(V) \tag{11}$$

for any $V \in \Gamma(TF)$.

Since M is an Einstein manifold, we can write

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, V\Big) = \alpha g\Big(\frac{\partial}{\partial t}, V\Big) = 0 = \alpha g\Big(V, \frac{\partial}{\partial t}\Big) = \overline{Ric}\Big(V, \frac{\partial}{\partial t}\Big) \tag{12}$$

where $\frac{\partial}{\partial t} \in \Gamma(TB)$ and $V \in \Gamma(TF)$.

Since $\lambda_2 \neq (\bar{n}-1)\lambda_1, \pi(V) \neq 0$, using equations (10), (11), (12), then we can get $q' = 0$, which means q is a constant on I, then f is a constant on I. $\square$

Theorem 3.10. *Let $M = B \times_f I$ be a warped product, where I is an open interval in $\mathbb{R}$, $\dim I = 1$ and $\dim B = \bar{n} - 1$ ($\bar{n} \geq 3$). Then*

(1) *If $(M, \overline{\nabla})$ is an Einstein manifold, $P \in \Gamma(TB)$, $\nabla^B P = 0$, and f is a constant on B, then:*

$$\overline{S}^B = [(\bar{n}-1)(\bar{n}-2)\lambda_1\lambda_2 + \lambda_1^2 + (1-\bar{n})\lambda_2^2]\pi(P).$$

Furthermore, if $(\bar{n}-1)(\bar{n}-2)\lambda_1\lambda_2 + \lambda_1^2 + (1-\bar{n})\lambda_2^2 = 0$, then $\overline{S}^B = 0$.

(2) *If $(M, \overline{\nabla})$ is Einstein manifold, $P \in \Gamma(TI)$, and $\lambda_2 \neq (\bar{n}-1)\lambda_1$ then f is a constant on B.*

(3) *If f is a constant on B, (B, ∇^B) is Einstein, $P \in \Gamma(TI)$, then $(M, \overline{\nabla})$ is Einstein manifold; furthermore, if $\lambda_2 = (\bar{n}-1)\lambda_1$, then $\alpha = \alpha_B$, where α and α_B denote the Einstein constant on M and B, respectively.*

Proof. (1) Assume that $(M, \overline{\nabla})$ is an Einstein manifold, $P \in \Gamma(TB)$, then

$$\overline{Ric}(X,Y) = \frac{\bar{s}}{\bar{n}} g(X,Y) \tag{13}$$

for any $X, Y \in \Gamma(TB)$. Consider that f is a constant on B and by Proposition 3.7, we can get

$$\overline{Ric}(X,Y) = \frac{1}{\bar{n}} g(X,Y)\{\overline{S}^B + [(2\bar{n}-2)\lambda_1\lambda_2 - (\lambda_1^2 + \lambda_2^2)]\pi(P)\}.$$

Define $\widehat{S} = \sum_{k=1}^{\bar{n}-1} \frac{1}{\bar{n}} \overline{Ric}(E_k, E_k)$, where E_k is an orthonormal base of B, then we have

$$\widehat{S} = \frac{\bar{n}-1}{\bar{n}} \Big\{\overline{S}^B + [(2\bar{n}-2)\lambda_1\lambda_2 - (\lambda_1^2 + \lambda_2^2)]\pi(P)\Big\}. \tag{14}$$

On the other hand, using Proposition 3.5, we can get

$$\overline{Ric}(X,Y) = \overline{Ric}^B(X,Y) + [\lambda_1\lambda_2\pi(P)g(X,Y) - \lambda_1^2\pi(X)\pi(Y)],$$

then we have

$$\widehat{S} = \overline{S}^B + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_1^2]\pi(P). \tag{15}$$

From equations (14) and (15), we can get

$$\overline{S}^B = [(\bar{n}-1)(\bar{n}-2)\lambda_1\lambda_2 + \lambda_1^2 + (1-\bar{n})\lambda_2^2]\pi(P).$$

(2) Assume that $(M, \overline{\nabla})$ is Einstein manifold, $P \in \Gamma(TI)$, by Proposition 3.6, we obtain

$$\overline{Ric}(X,P) = [(\bar{n}-1)\lambda_1 - \lambda_2]\pi(P)\frac{Xf}{f}; \tag{16}$$

$$\overline{Ric}(P,X) = [\lambda_2 - (\bar{n}-1)\lambda_1]\pi(P)\frac{Xf}{f}. \tag{17}$$

Using the similar proof of Theorem 3.9(b), we can get $Xf = 0$, which means f is a constant on B.

(3) If f is a constant on B, and (B, ∇^B) is Einstein, and $P \in \Gamma(TI)$, then

$$Ric^B(X,Y) = \alpha_B g(X,Y). \tag{18}$$

By Theorem 3.6, we have

$$\overline{Ric}(X,Y) = \overline{Ric}^B(X,Y) + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi(P)g(X,Y). \tag{19}$$

So by equations (18) and (19), we can easily get

$$\overline{Ric}(X,Y) = \{\alpha_B + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi(P)\}g(X,Y)$$

which means $(M, \overline{\nabla})$ is an Einstein manifold.

Furthermore, if $\lambda_2 = (\bar{n}-1)\lambda_1$, then $\overline{Ric}(X,Y) = \alpha_B g(X,Y)$, and $\alpha = \alpha_B$. □

Now we specially study $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, I is an open interval in $\mathbb{R}$. Let begin with the following theorem:

Theorem 3.11. *Let $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F, P = \frac{\partial}{\partial t}, \dim F = l$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if the following conditions are satisfied*

(1) *(F, ∇^F) is Einstein with the Einstein constant α_F;*
(2) $l\Big(\lambda_2 \frac{f'}{f} - \frac{f''}{f} + \lambda_1^2 - \lambda_1\lambda_2\Big) = \alpha$;
(3) $\alpha_F - ff'' + (1-l)(f')^2 + [\lambda_2^2 - l\lambda_1\lambda_2 - \alpha]f^2 + [l\lambda_1 + (l-1)\lambda_2]ff' = 0.$

Proof. By Proposition 3.5, we have

$$\begin{aligned}
&\overline{Ric}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = -l\Big(\lambda_2 \frac{f'}{f} - \frac{f''}{f} + \lambda_1^2 - \lambda_1\lambda_2\Big);\\
&\overline{Ric}\Big(\frac{\partial}{\partial t}, V\Big) = \overline{Ric}\Big(V, \frac{\partial}{\partial t}\Big) = 0;\\
&\overline{Ric}(V,W) = Ric^F(V,W) + g_F(V,W)\{-ff'' - (l-1){f'}^2 + (\lambda_2^2 - l\lambda_1\lambda_2)f^2\\
&\qquad\qquad + [l\lambda_1 + (l-1)\lambda_2]ff'\}.
\end{aligned}$$

Then by the Einstein condition, we get Theorem 3.11. □

Remark 3. When $\lambda_1 = \lambda_2 = 1$, we can get Corollary 21 in [14].

Considering the dimension of F, we get Corollaries 3.12 and 3.13 of Theorem 3.11:

Corollary 3.12. *Let $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F, P = \frac{\partial}{\partial t}, \dim F = 1$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if $f'' = \lambda_2 f' + (\lambda_1^2 - \lambda_1\lambda_2)f - \alpha f$.*

Remark 4.

(1) From Theorem 3.11, we can also get: if $\dim F = 1$, then $\alpha_F = 0$;
(2) When $\lambda_1 = \lambda_2 = 1$, we can get Corollary 23 in [14].

Corollary 3.13. *Let $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F, P = \frac{\partial}{\partial t}, \dim F = l > 1$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if the following conditions are satisfied:*

(1) *(F, ∇^F) is Einstein with the Einstein constant α_F;*

(2) $f'' = \lambda_2 f' + (\lambda_1^2 - \lambda_1\lambda_2 - \frac{\alpha}{l})f$;
(3) $\frac{\alpha_F}{1-l} + (f')^2 + \left[\frac{\alpha}{l} + \lambda_1\lambda_2 + \frac{\lambda_2^2-\lambda_1^2}{1-l}\right]f^2 + \left[\frac{l}{1-l}\lambda_1 + \frac{(l-2)}{1-l}\lambda_2\right]ff' = 0$.

Remark 5. When $\lambda_1 = \lambda_2 = 1$, we can get Corollary 24 in [14].

By Corollary 3.12 and elementary methods for ordinary differential equations, we get:

Theorem 3.14. *Let $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F, P = \frac{\partial}{\partial t}, \dim F = 1$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if*

(1) $\alpha < (\lambda_1 - \frac{1}{2}\lambda_2)^2, f(t) = c_1 e^{((\lambda_2+\sqrt{(2\lambda_1-\lambda_2)^2-4\alpha})/2)t} + c_2 e^{((\lambda_2-\sqrt{(2\lambda_1-\lambda_2)^2-4\alpha})/2)t}$,
(2) $\alpha = (\lambda_1 - \frac{1}{2}\lambda_2)^2, f(t) = c_1 e^{(\lambda_2/2)t} + c_2 t e^{(\lambda_2/2)t}$,
(3) $\alpha > (\lambda_1-\frac{1}{2}\lambda_2)^2, f(t) = c_1 e^{(\lambda_2/2)t}\cos((\sqrt{4\alpha-(2\lambda_1-\lambda_2)^2}/2)t) + c_2 e^{(\lambda_2/2)t}\sin((\sqrt{4\alpha-(2\lambda_1-\lambda_2)^2}/2)t)$.

Remark 6. When $\lambda_1 = \lambda_2 = 1$, we can get Corollary 25 in [14].

As a corollary of Theorem 3.14, we have

Corollary 3.15. *Let $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F, P = \frac{\partial}{\partial t}$, $\dim F = 1$, and $\lambda_2 = 2\lambda_1$, then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if*

(1) $\alpha < 0, f(t) = c_1 e^{(\lambda_1+\sqrt{-\alpha})t} + c_2 e^{(\lambda_1-\sqrt{-\alpha})t}$,
(2) $\alpha = 0, f(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}$,
(3) $\alpha > 0, f(t) = c_1 e^{\lambda_1 t}\cos(\sqrt{\alpha}t) + c_2 e^{\lambda_1 t}\sin(\sqrt{\alpha}t)$.

Theorem 3.16. *Let $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F, P = \frac{\partial}{\partial t}, \dim F = l > 1$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if one of the following conditions is satisfied:*

(1) $\alpha = (\lambda_1^2 - \lambda_1\lambda_2)l$, $\alpha_F = c_2^2(l\lambda_1^2 - \lambda_2^2)$, $f(t) = c_2$;
(2) $\lambda_1 = \lambda_2$, $\alpha = 0$, $\alpha_F = (l-1)c_2^2\lambda_1^2$, $f(t) = c_1 e^{\lambda_1 t} + c_2$;
(3) $\lambda_2^2 - 2l\lambda_1^2 + l\lambda_1\lambda_2 \neq 0$, $\lambda_2 \neq l\lambda_1$, $\alpha = \frac{(3l^2+l)\lambda_1^2\lambda_2^2-(l^2+l)\lambda_1\lambda_1^3-2l^2\lambda_1^3\lambda_2}{(l\lambda_1-\lambda_2)^2}$, $\alpha_F = 0$, $f(t) = c_0 e^{(l\lambda_1^2-\lambda_2^2)t/(l\lambda_1-\lambda_2)}$;
(4) $\lambda_2^2 - 2l\lambda_1^2 + l\lambda_1\lambda_2 = 0$, $\alpha = l(\lambda_1 - \frac{1}{2}\lambda_2)^2$, $\alpha_F = 0$, $f(t) = c_1 e^{\frac{\lambda_2}{2}t}$.

Proof. Let $\frac{\alpha}{l} = d_0$, $\frac{\alpha_F}{1-l} = \overline{d_0}$, $a_0 = \frac{\lambda_2+\sqrt{(2\lambda_1-\lambda_2)^2-4d_0}}{2}$, $b_0 = \frac{\lambda_2-\sqrt{(2\lambda_1-\lambda_2)^2-4d_0}}{2}$, then $a_0 + b_0 = \lambda_2$, $a_0 b_0 = d_0 + \lambda_1\lambda_2 - \lambda_1^2$.

(a) $d_0 < (\lambda_1 - \frac{1}{2}\lambda_2)^2$, then $f(t) = c_1 e^{a_0 t} + c_2 e^{b_0 t}$. By Corollary 3.13(3),

$$\begin{aligned}
&\overline{d_0} + c_1^2\Big(a_0^2 + a_0 b_0 + \lambda_1^2 + \frac{\lambda_2^2-\lambda_1^2}{1-l} + \frac{l}{1-l}\lambda_1 a_0 + \frac{l-2}{1-l}\lambda_2 a_0\Big)e^{2a_0 t} \\
&+ c_2^2\Big(b_0^2 + a_0 b_0 + \lambda_1^2 + \frac{\lambda_2^2-\lambda_1^2}{1-l} + \frac{l}{1-l}\lambda_1 b_0 + \frac{l-2}{1-l}\lambda_2 b_0\Big)e^{2b_0 t} \\
&+ c_1 c_2\Big[4a_0 b_0 + \frac{2\lambda_2^2 - 2l\lambda_1^2}{1-l} + \Big(\frac{l}{1-l}\lambda_1 + \frac{l-2}{1-l}\lambda_2\Big)(a_0+b_0)\Big]e^{(a_0+b_0)t} = 0. \qquad (20)
\end{aligned}$$

1) $b_0 = 0$, we have $d_0 = \lambda_1^2 - \lambda_1\lambda_2 < (\lambda_1 - \frac{1}{2}\lambda_2)^2$, $a_0 = \lambda_2$, $\alpha = (\lambda_1^2 - \lambda_1\lambda_2)l$. By equation (20), we get

$$\overline{d_0} + c_1^2\frac{l\lambda_1(\lambda_2-\lambda_1)}{1-l}e^{2\lambda_2 t} + c_2^2\frac{\lambda_2^2 - l\lambda_1^2}{1-l} + c_1 c_2\frac{l(\lambda_2+2\lambda_1)(\lambda_2-\lambda_1)}{1-l}e^{\lambda_2 t} = 0.$$

Since $e^{2\lambda_2 t}$ and $e^{\lambda_2 t}$ are linearly independent, we have

$$\begin{cases} c_1^2\frac{l\lambda_1(\lambda_2-\lambda_1)}{1-l}=0 \\ \overline{d_0}+c_2^2\frac{\lambda_2^2-l\lambda_1^2}{1-l}=0 \\ c_1c_2\frac{l(\lambda_2+2\lambda_1)(\lambda_2-\lambda_1)}{1-l}=0 \end{cases}$$

1′. $c_1=0$, $c_2\neq 0$. We have $\underline{\alpha=(\lambda_1^2-\lambda_1\lambda_2)l, \alpha_F=c_2^2(l\lambda_1^2-\lambda_2^2), f(t)=c_2}$.
2′. $c_1\neq 0$, $c_2=0$. We have $\underline{\lambda_1=\lambda_2, \alpha=\alpha_F=0, f(t)=c_1e^{\lambda_1 t}}$.
3′. $c_1\neq 0$, $c_2\neq 0$. We have $\underline{\lambda_1=\lambda_2, \alpha=0, \alpha_F=(l-1)c_2^2\lambda_1^2, f(t)=c_1e^{\lambda_1 t}+c_2}$.

It is easy to see that the conclusion of 2′ is a special case of 3′. Then we get Theorem 3.16(1) and Theorem 3.16(2).

2) $b_0\neq 0$, 1′. $c_1=0$, $c_2\neq 0$. Since e^{2a_0t}, e^{2b_0t} and $e^{(a_0+b_0)t}$ are linearly independent, we have $\overline{d_0}=0$, $b_0^2+a_0b_0+\lambda_1^2+\frac{\lambda_2^2-\lambda_1^2}{1-l}+\frac{l}{1-l}\lambda_1b_0+\frac{l-2}{1-l}\lambda_2b_0=0$, then $\alpha_F=0$, $\frac{l\lambda_1-\lambda_2}{1-l}b_0=\frac{l\lambda_1^2-\lambda_2^2}{1-l}$. If $\lambda_2=l\lambda_1$, we have $0=l\lambda_1^2$, this is a contradiction to $\lambda_1\neq 0$. So $\lambda_2\neq l\lambda_1$ and $b_0=\frac{l\lambda_1^2-\lambda_2^2}{l\lambda_1-\lambda_2}$. Considering that $b_0<\frac{\lambda_2}{2}$, we get $\frac{\lambda_2^2+l\lambda_1\lambda_2-2l\lambda_1^2}{l\lambda_1-\lambda_2}>0$. $a_0=\frac{l\lambda_1(\lambda_2-\lambda_1)}{l\lambda_1-\lambda_2}$, $d_0=\frac{(3l+1)\lambda_1^2\lambda_2^2-(l+1)\lambda_1\lambda_2^3-2l\lambda_1^3\lambda_2}{(l\lambda_1-\lambda_2)^2}$, $\alpha=\frac{(3l^2+l)\lambda_1^2\lambda_2^2-(l^2+l)\lambda_1\lambda_2^3-2l^2\lambda_1^3\lambda_2}{(l\lambda_1-\lambda_2)^2}$ and d_0 satisfies $d_0<(\lambda_1-\frac{1}{2}\lambda_2)^2$. So in this case, we obtain

$$\underline{\frac{\lambda_2^2+l\lambda_1\lambda_2-2l\lambda_1^2}{l\lambda_1-\lambda_2}>0,\quad \alpha=\frac{(3l^2+l)\lambda_1^2\lambda_2^2-(l^2+l)\lambda_1\lambda_2^3-2l^2\lambda_1^3\lambda_2}{(l\lambda_1-\lambda_2)^2},\quad \alpha_F=0,\quad f(t)=c_2e^{\frac{l\lambda_1^2-\lambda_2^2}{l\lambda_1-\lambda_2}t}}.$$

2′. $c_1\neq 0$, $c_2=0$. Using the same method we can get $a_0=\frac{l\lambda_1^2-\lambda_2^2}{l\lambda_1-\lambda_2}$, $b_0=\frac{l\lambda_1(\lambda_2-\lambda_1)}{l\lambda_1-\lambda_2}$ and

$$\underline{\frac{\lambda_2^2+l\lambda_1\lambda_2-2l\lambda_1^2}{l\lambda_1-\lambda_2}<0,\quad \alpha=\frac{(3l^2+l)\lambda_1^2\lambda_2^2-(l^2+l)\lambda_1\lambda_2^3-2l^2\lambda_1^3\lambda_2}{(l\lambda_1-\lambda_2)^2},\quad \alpha_F=0,\quad f(t)=c_1e^{\frac{l\lambda_1^2-\lambda_2^2}{l\lambda_1-\lambda_2}t}}.$$

So by 1′ and 2′, we get Theorem 3.16(3).

3′. $c_1\neq 0$, $c_2\neq 0$. Since e^{2a_0t}, e^{2b_0t} and $e^{(a_0+b_0)t}$ are linearly independent, we have

$$a_0^2+a_0b_0+\lambda_1^2+\frac{\lambda_2^2-\lambda_1^2}{1-l}+\frac{l}{1-l}\lambda_1a_0+\frac{l-2}{1-l}\lambda_2a_0=0; \tag{21a}$$

$$b_0^2+a_0b_0+\lambda_1^2+\frac{\lambda_2^2-\lambda_1^2}{1-l}+\frac{l}{1-l}\lambda_1b_0+\frac{l-2}{1-l}\lambda_2b_0=0. \tag{21b}$$

By (21a) − (21b) we get $(a_0-b_0)\lambda_2+\frac{l\lambda_1}{1-l}(a_0-b_0)+\frac{l-2}{1-l}\lambda_2(a_0-b_0)=0$, since $a_0\neq b_0$ we have $\lambda_2+\frac{l\lambda_1}{1-l}+\frac{l-2}{1-l}\lambda_2=0$, then $\lambda_2=l\lambda_1$, using (21a) again we get $0=l\lambda_1^2$, this is a contradiction to $\lambda_1\neq 0$. So in case 3′ we have no solution.

(b) $d_0=(\lambda_1-\frac{1}{2}\lambda_2)^2$, then $f(t)=c_1e^{\frac{\lambda_2}{2}t}+c_2te^{\frac{\lambda_2}{2}t}$. By Corollary 3.13(3), we get

$$\begin{aligned}&\overline{d_0}+\Big(\frac{\lambda_2}{2}c_1+c_2+\frac{\lambda_2}{2}c_2t\Big)^2e^{\lambda_2t}+\frac{(5-l)\lambda_2^2-4l\lambda_1^2}{4(1-l)}(c_1+c_2t)^2e^{\lambda_2t}\\&+\Big(\frac{l}{1-l}\lambda_1+\frac{l-2}{1-l}\lambda_2\Big)(c_1+c_2t)\Big(\frac{\lambda_2}{2}c_1+c_2+\frac{\lambda_2}{2}c_2t\Big)e^{\lambda_2t}=0\end{aligned}$$

1′. $c_1=0$, $c_2\neq 0$. The coefficient of $e^{\lambda_2 t}$ is $c_2^2=0$, this is a contradiction to $c_2\neq 0$.
2′. $c_1\neq 0$, $c_2=0$. Then $\overline{d_0}=0$ and from the coefficient of e^{λ_2t} we can get $\underline{\lambda_2^2+l\lambda_1\lambda_2-2l\lambda_1^2=0}$, so $\underline{\alpha=d_0l=l(\lambda_1-\frac{1}{2}\lambda_2)^2, \alpha_F=0, f(t)=c_1e^{\frac{\lambda_2}{2}t}}$. Then we obtain Theorem 3.16(4).

$3'$. $c_1 \neq 0$, $c_2 \neq 0$. From the coefficient of $t^2 e^{\lambda_2 t}$ we get

$$\lambda_2^2 + l\lambda_1\lambda_2 - 2l\lambda_1^2 = 0, \tag{22}$$

by equation (22), the coefficient of $te^{\lambda_2 t}$ becomes

$$\frac{(l-3)\lambda_2 + 2l\lambda_1}{2(1-l)} c_2 - \frac{\lambda_2^2}{4} c_1 = 0, \tag{23}$$

the coefficient of $e^{\lambda_2 t}$ is $\frac{l\lambda_1 - \lambda_2}{1-l} c_1 c_2 + c_2^2 = 0$, then $c_2 = \frac{\lambda_2 - l\lambda_1}{1-l} c_1$, so by equation (23) we get $(2l - 7 - l^2)\lambda_2 = (4l^2 - 10l)\lambda_1$. Considering that $2l - 7 - l^2 \neq 0, 4l^2 - 10l \neq 0$, we obtain $\lambda_2 = \frac{(4l^2-10l)}{(2l-7-l^2)}\lambda_1$. Using equation (22) again, we have $(l-1)^2(3l^2 - 15l + 49) = 0$, but $l - 1 \neq 0, 3l^2 - 15l + 49 \neq 0$, so we have no solution in this case.

(c) $d_0 > (\lambda_1 - \frac{1}{2}\lambda_2)^2$. Let $h_0 = \frac{1}{2}\sqrt{4d_0 - (2\lambda_1 - \lambda_2)^2}$, then $f(t) = e^{\frac{\lambda_2}{2}t}(c_1 \cos(h_0 t) + c_2 \sin(h_0 t))$. By Corollary 3.13(3), we get

$$\begin{aligned}
&\overline{d_0} + \left[\left(\frac{\lambda_2}{2}c_1 + c_2 h_0\right)\cos(h_0 t) + \left(\frac{\lambda_2}{2}c_2 - c_1 h_0\right)\sin(h_0 t)\right]^2 e^{\lambda_2 t} \\
&\quad + \left(d_0 + \lambda_1\lambda_2 + \frac{\lambda_2^2 - \lambda_1^2}{1-l}\right)\left(c_1\cos(h_0 t) + c_2 \sin(h_0 t)\right)^2 e^{\lambda_2 t} + \left(\frac{l}{1-l}\lambda_1 + \frac{l-2}{1-l}\lambda_2\right) \\
&\quad \left(c_1\cos(h_0 t) + c_2\sin(h_0 t)\right)\left[\left(\frac{\lambda_2}{2}c_1 + c_2 h_0\right)\cos(h_0 t) + \left(\frac{\lambda_2}{2}c_2 - c_1 h_0\right)\sin(h_0 t)\right] e^{\lambda_2 t} = 0.
\end{aligned}$$

The coefficient of $\cos^2(h_0 t)e^{\lambda_2 t}$ is

$$\left[d_0 + \frac{(l+1)\lambda_2^2 + (4-2l)\lambda_1\lambda_2 - 4\lambda_1^2}{4(1-l)}\right]c_1^2 + c_2^2 h_0^2 - \frac{l\lambda_1 - \lambda_2}{1-l}c_1 c_2 h_0 = 0. \tag{24}$$

Similarly the coefficient of $\sin^2(h_0 t)e^{\lambda_2 t}$ is

$$\left[d_0 + \frac{(l+1)\lambda_2^2 + (4-2l)\lambda_1\lambda_2 - 4\lambda_1^2}{4(1-l)}\right]c_2^2 + c_1^2 h_0^2 - \frac{l\lambda_1 - \lambda_2}{1-l}c_1 c_2 h_0 = 0. \tag{25}$$

If $c_1 = 0$, $c_2 \neq 0$, by equation (24) we get $c_2^2 h_0^2 = 0$, then $h_0 = 0$, this is a contradiction to $h_0 \neq 0$.

If $c_1 \neq 0$, $c_2 = 0$, by equation (25) we get $c_1^2 h_0^2 = 0$, then $h_0 = 0$, this is a contradiction to $h_0 \neq 0$. So $c_1 \neq 0$, $c_2 \neq 0$, (24) + (25) we get

$$d_0 = \frac{[(2l-4)\lambda_1 - l\lambda_2](\lambda_2 - \lambda_1)}{4(1-l)}. \tag{26}$$

Considering $d_0 > (\lambda_1 - \frac{1}{2}\lambda_2)^2$, we have

$$\lambda_2^2 - 2l\lambda_1^2 + l\lambda_1\lambda_2 > 0. \tag{27}$$

The coefficient of $\sin(h_0 t)\cos(h_0 t)e^{\lambda_2 t}$ is

$$\frac{\lambda_2^2 - 2l\lambda_1^2 + l\lambda_1\lambda_2}{1-l}c_1 c_2 + \frac{l\lambda_1 - \lambda_2}{1-l}h_0(c_2^2 - c_1^2) = 0. \tag{28}$$

Using (24)–(25), we get

$$\frac{\lambda_2^2 - 2l\lambda_1^2 + l\lambda_1\lambda_2}{2(1-l)}(c_1^2 - c_2^2) + \frac{2(l\lambda_1 - \lambda_2)}{1-l}h_0 c_1 c_2. \tag{29}$$

Let $a = \frac{\lambda_2^2-2l\lambda_1^2+l\lambda_1\lambda_2}{2(1-l)} < 0$, $b = \frac{l\lambda_1-\lambda_2}{1-l}h_0$, $x = c_1^2 - c_2^2$, $y = c_1c_2 \neq 0$, from equations (28) and (29) we obtain

$$ax + 2by = 0, \tag{30a}$$

$$2ay - bx = 0. \tag{30b}$$

Through (30a) $\times\, b +$ (30b) $\times\, a$, we have $a^2 + b^2 = 0$, which means $\left[\frac{\lambda_2^2-2l\lambda_1^2+l\lambda_1\lambda_2}{2(1-l)}\right]^2 + \left[\frac{l\lambda_1-\lambda_2}{1-l}h_0\right]^2 = 0$, then we get

$$d_0 = \frac{(3l+1)\lambda_1^2\lambda_2^2 - (l+1)\lambda_1\lambda_2^3 - 2l\lambda_1^3\lambda_2}{(l\lambda_1 - \lambda_2)^2}. \tag{31}$$

Using equations (26) and (31), we have $(\lambda_1 - \lambda_2)[(2-l)\lambda_1 - \lambda_2](\lambda_2^2 + l\lambda_1\lambda_2 - 2l\lambda_2^2) = 0$. By inequality (27), we get $\lambda_1 = \lambda_2$ or $\lambda_2 = (2-l)\lambda_1$.

1) $\lambda_1 = \lambda_2$, by equation (26) we get $d_0 = 0$, this is a contradiction to $d_0 > (\lambda_1 - \frac{1}{2}\lambda_2)^2 = \frac{1}{4}\lambda_1^2 > 0$.
2) $\lambda_2 = (2-l)\lambda_1$, by equation (26) we get $d_0 = \frac{l^2-4}{4}\lambda_1^2 < \frac{l^2}{4}\lambda_1^2$. On the other hand, $d_0 > (\lambda_1 - \frac{1}{2}\lambda_2)^2 = \frac{l^2}{4}\lambda_1^2$, this is a contradiction.

In a word, we have no solution in the case of $d_0 > (\lambda_1 - \frac{1}{2}\lambda_2)^2$. □

Remark 7. When $\lambda_1 = \lambda_2 = 1$, we get Theorem 26 in [14].

Proposition 3.17. *Let $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}$. Then $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$ if and only if (F, ∇^F) has constant scalar curvature S^F and*

$$\overline{S} = \frac{S^F}{f^2} - 2l\frac{f''}{f} - l(l-1)\frac{(f')^2}{f^2} + l^2(\lambda_1 + \lambda_2)\frac{f'}{f} + l[\lambda_1^2 + \lambda_2^2 - (l+1)\lambda_1\lambda_2]. \tag{32}$$

Proof. Considering that $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$ and $P = \frac{\partial}{\partial t}$, by Proposition 3.7 we get equation (32). With the fact that S^F is a function defined on F and f is a function defined on I, using separation of variables we complete the proof of this proposition. □

Proposition 3.18. *Let $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P \in \Gamma(TF)$. If $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$, then*

(1) *If $\lambda_1 + \lambda_2 \neq 0$ and $\lambda_1^2 + \lambda_2^2 - \bar{n}\lambda_1\lambda_2 = 0$, and $div_F P$ is a constant, then S^F is a constant;*
(2) *If $\lambda_1 = -\lambda_2 \neq 0$ and $g_F(P,P)$ is a constant, then S^F is a constant;*
(3) *If $\lambda_1 + \lambda_2 \neq 0$ and $\lambda_1^2 + \lambda_2^2 - \bar{n}\lambda_1\lambda_2 \neq 0$, and $div_F P$, $g_F(P,P)$ are constants, then S^F is a constant.*

Proof. Considering that $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P \in \Gamma(TF)$, by Proposition 3.8 we can get

$$\overline{S} = \frac{S^F}{f^2} - 2l\frac{f''}{f} - l(l-1)\frac{(f')^2}{f^2} + [l\bar{n}\lambda_1\lambda_2 - l(\lambda_1^2 + \lambda_2^2)]f^2 g_F(P,P) + l(\lambda_1 + \lambda_2)div_F P. \tag{33}$$

Then by separation of variables, we have:

(1) If $\lambda_1 + \lambda_2 \neq 0$ and $\lambda_1^2 + \lambda_2^2 - \bar{n}\lambda_1\lambda_2 = 0$, and $div_F P$ is a constant, we can obtain $\overline{S} = \frac{S^F}{f^2} - 2l\frac{f''}{f} - l(l-1)\frac{(f')^2}{f^2} + l(\lambda_1+\lambda_2)div_F P$. Then S^F is a constant.
(2) If $\lambda_1 = -\lambda_2 \neq 0$ and $g_F(P,P)$ is a constant, which means $\lambda_1 + \lambda_2 = 0$ and $\lambda_1^2 + \lambda_2^2 - \bar{n}\lambda_1\lambda_2 \neq 0$, and $g_F(P,P)$ is a constant, we can obtain $\overline{S} = \frac{S^F}{f^2} - 2l\frac{f''}{f} - l(l-1)\frac{(f')^2}{f^2} + [l\bar{n}\lambda_1\lambda_2 - l(\lambda_1^2+\lambda_2^2)]f^2 g_F(P,P)$. Then S^F is a constant.
(3) It is obvious.
(4) If $\lambda_1 + \lambda_2 = 0$ and $\lambda_1^2 + \lambda_2^2 - \bar{n}\lambda_1\lambda_2 = 0$, then we can get $\lambda_1 = \lambda_2 = 0$, which is a contradiction. $\square$

In (32), we make the change of variable $f(t) = \sqrt{v(t)}$ and have the following equation:

$$v''(t) + \frac{l-3}{4}\frac{v'(t)^2}{v(t)} - \frac{l}{2}(\lambda_1+\lambda_2)v'(t) + \Big[(l+1)\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2 + \frac{\overline{S}}{l}\Big]v(t) - \frac{S^F}{l} = 0. \tag{34}$$

Remark 8. When $\lambda_1 = \lambda_2 = 1$, equations (32), (33), (34) respectively become (20), (21), (22) in [14], then by Proposition 3.17, we can get Corollary 27 in [14], and by Proposition 3.18(3), we can get Corollary 28 in [14].

Theorem 3.19. *Let $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F, P = \frac{\partial}{\partial t}$, and $\dim F = l = 3$. Then $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$ if and only if (F, ∇^F) has constant scalar curvature S^F and*

(1) $\overline{S} < \frac{27}{16}(\lambda_1+\lambda_2)^2 + 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$ *and* $\overline{S} \neq 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$,

$$\begin{aligned} v(t) &= c_1 e^{((\frac{3}{2}(\lambda_1+\lambda_2)+\sqrt{\frac{9}{4}(\lambda_1+\lambda_2)^2-\frac{4}{3}\overline{S}+4\lambda_1^2+4\lambda_2^2-16\lambda_1\lambda_2})/2)t} \\ &\quad + c_2 e^{((\frac{3}{2}(\lambda_1+\lambda_2)-\sqrt{\frac{9}{4}(\lambda_1+\lambda_2)^2-\frac{4}{3}\overline{S}+4\lambda_1^2+4\lambda_2^2-16\lambda_1\lambda_2})/2)t} + \frac{S^F}{12\lambda_1\lambda_2 - 3\lambda_1^2 - 3\lambda_2^2 + \overline{S}}; \end{aligned}$$

(2) $\overline{S} = \frac{27}{16}(\lambda_1+\lambda_2)^2 + 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$,

$$v(t) = c_1 e^{\frac{3}{4}(\lambda_1+\lambda_2)t} + c_2 t e^{\frac{3}{4}(\lambda_1+\lambda_2)t} + \frac{S^F}{12\lambda_1\lambda_2 - 3\lambda_1^2 - 3\lambda_2^2 + \overline{S}};$$

(3) $\overline{S} > \frac{27}{16}(\lambda_1+\lambda_2)^2 + 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$,

$$\begin{aligned} v(t) &= c_1 e^{\frac{3}{4}(\lambda_1+\lambda_2)t}\cos((\Big(\sqrt{\frac{4}{3}\overline{S} - 4\lambda_1^2 - 4\lambda_2^2 + 16\lambda_1\lambda_2 - \frac{9}{4}(\lambda_1+\lambda_2)^2}\Big)/2)t) \\ &\quad + c_2 e^{\frac{3}{4}(\lambda_1+\lambda_2)t}\sin((\Big(\sqrt{\frac{4}{3}\overline{S} - 4\lambda_1^2 - 4\lambda_2^2 + 16\lambda_1\lambda_2 - \frac{9}{4}(\lambda_1+\lambda_2)^2}\Big)/2)t) \\ &\quad + \frac{S^F}{12\lambda_1\lambda_2 - 3\lambda_1^2 - 3\lambda_2^2 + \overline{S}}; \end{aligned}$$

(4) $\overline{S} = 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$, *and* $\lambda_1 + \lambda_2 \neq 0$, $v(t) = c_1 - \frac{2S^F}{9(\lambda_1+\lambda_2)} + c_2 e^{\frac{3}{2}(\lambda_1+\lambda_2)t}$;
(5) $\overline{S} = 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$, *and* $\lambda_1 + \lambda_2 = 0$, $v(t) = \frac{S^F}{6}t^2 + c_1 t + c_2$.

Proof. If $l = 3$, then we have a simple differential equation as follows:

$$v''(t) - \frac{3}{2}(\lambda_1+\lambda_2)v'(t) + (4\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2 + \frac{\overline{S}}{3})v(t) - \frac{S^F}{3} = 0. \tag{35}$$

(a) If $\overline{S} \neq 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$, putting $h(t) = (4\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2 + \frac{\overline{S}}{3})v(t) - \frac{S^F}{3}$, we get $h''(t) - \frac{3}{2}(\lambda_1 + \lambda_2)h'(t) + (4\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2 + \frac{\overline{S}}{3})h(t) = 0$. The above solutions (1)–(3) follow directly from elementary methods for ordinary differential equations.

(b) If $\overline{S} = 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$, and $\lambda_1 + \lambda_2 \neq 0$, then $v''(t) - \frac{3}{2}(\lambda_1 + \lambda_2)v'(t) - \frac{S^F}{3} = 0$, and we get solution (4).

(c) If $\overline{S} = 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$, and $\lambda_1 + \lambda_2 = 0$, then $v''(t) - \frac{S^F}{3} = 0$, and we get solution (5). □

Remark 9. When $\lambda_1 = \lambda_2 = 1$, we get Theorem 29 in [14].

Theorem 3.20. *Let $M = I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F, P = \frac{\partial}{\partial t}$, and $\dim F = l \neq 3$ and $S^F = 0$. Then $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$ if and only if*

(1) $\overline{S} < \frac{l^3}{4(l+1)}(\lambda_1 + \lambda_2)^2 - l[(l+1)\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2]$,

$$v(t) = \Big[c_1 e^{\left(\left(\frac{l}{2}(\lambda_1+\lambda_2)+\sqrt{\frac{l^2}{4}(\lambda_1+\lambda_2)^2-(l+1)\left[(l+1)\lambda_1\lambda_2-\lambda_1^2-\lambda_2^2+\frac{\overline{S}}{l}\right]}\right)/2\right)t} + c_2 e^{\left(\left(\frac{l}{2}(\lambda_1+\lambda_2)-\sqrt{\frac{l^2}{4}(\lambda_1+\lambda_2)^2-(l+1)\left[(l+1)\lambda_1\lambda_2-\lambda_1^2-\lambda_2^2+\frac{\overline{S}}{l}\right]}\right)/2\right)t}\Big]^{\frac{4}{l+1}};$$

(2) $\overline{S} = \frac{l^3}{4(l+1)}(\lambda_1 + \lambda_2)^2 - l[(l+1)\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2]$, $v(t) = \Big[c_1 e^{\frac{l}{4}(\lambda_1+\lambda_2)t} + c_2 t e^{\frac{l}{4}(\lambda_1+\lambda_2)t}\Big]^{\frac{4}{l+1}}$;

(3) $\overline{S} > \frac{l^3}{4(l+1)}(\lambda_1 + \lambda_2)^2 - l[(l+1)\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2]$,

$$v(t) = \Big[c_1 e^{\frac{l}{4}(\lambda_1+\lambda_2)t}\cos\Big(\Big(\sqrt{(l+1)\big[(l+1)\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2 + \frac{\overline{S}}{l}\big] - \frac{l^2}{4}(\lambda_1+\lambda_2)^2}\Big)/2\Big)t + c_2 e^{\frac{l}{4}(\lambda_1+\lambda_2)t}\sin\Big(\Big(\sqrt{(l+1)\big[(l+1)\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2 + \frac{\overline{S}}{l}\big] - \frac{l^2}{4}(\lambda_1+\lambda_2)^2}\Big)/2\Big)t\Big]^{\frac{4}{l+1}}.$$

Proof. In this case, (34) is changed into the simpler form

$$v''(t) + \frac{l-3}{4}\frac{v'(t)^2}{v(t)} - \frac{l}{2}(\lambda_1 + \lambda_2)v'(t) + \Big[(l+1)\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2 + \frac{\overline{S}}{l}\Big]v(t) = 0. \tag{36}$$

Putting $v(t) = w(t)^{\frac{4}{l+1}}$, $w(t)$ satisfies the equation $w''(t) - \frac{l}{2}(\lambda_1 + \lambda_2)w'(t) + \frac{l+1}{4}\big[(l+1)\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2 + \frac{\overline{S}}{l}\big]w(t) = 0$. By the elementary methods for ordinary differential equations, we prove the above theorem. □

Remark 10. When $\lambda_1 = \lambda_2 = 1$, we get Theorem 30 in [14].

When $\dim F = l \neq 3$, and $S^F \neq 0$, then putting $v(t) = w(t)^{\frac{4}{l+1}}$, $w(t)$ satisfies the following equation:

$$w''(t) - \frac{l}{2}(\lambda_1 + \lambda_2)w'(t) + \frac{l+1}{4}\Big[(l+1)\lambda_1\lambda_2 - \lambda_1^2 - \lambda_2^2 + \frac{\overline{S}}{l}\Big]w(t) - \frac{l+1}{4l}S^F w^{1-\frac{4}{l+1}} = 0. \tag{37}$$

4. Multiply warped product with a quarter-symmetric connection

In this section, firstly we compute curvature of multiply twisted product with a quarter-symmetric connection, secondly we study the special multiply warped product with a quarter-symmetric connection, finally we consider the generalized Kasner space–times with a quarter-symmetric connection.

4.1. Connection and curvature

By Lemma 2.3 and equation (3), we have the following two propositions:

Proposition 4.1. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply twisted product. If $X, Y \in \Gamma(TB)$, $U \in \Gamma(TF_i), W \in \Gamma(TF_j)$ and $P \in \Gamma(TB)$, then:*

(1) $\overline{\nabla}_X Y = \overline{\nabla}^B_X Y$;
(2) $\overline{\nabla}_X U = \frac{Xb_i}{b_i} U$;
(3) $\overline{\nabla}_U X = \Big[\frac{Xb_i}{b_i} + \lambda_1 \pi(X)\Big] U$;
(4) $\overline{\nabla}_U W = 0$ *if* $i \neq j$;
(5) $\overline{\nabla}_U W = U(lnb_i)W + W(lnb_i)U - \frac{g_{F_i}(U,W)}{b_i} grad_{F_i} b_i - b_i g_{F_i}(U,W) grad_B b_i + \nabla^{F_i}_U W - \lambda_2 g(U,W)P$ *if* $i = j$.

Proposition 4.2. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply twisted product. If $X, Y \in \Gamma(TB)$, $U \in \Gamma(TF_i), W \in \Gamma(TF_j)$ and $P \in \Gamma(TF_r)$ for a fixed r, then:*

(1) $\overline{\nabla}_X Y = \nabla^B_X Y - \lambda_2 g(X,Y)P$;
(2) $\overline{\nabla}_X U = \frac{Xb_i}{b_i} U + \lambda_1 \pi(U) X$;
(3) $\overline{\nabla}_U X = \frac{Xb_i}{b_i} U$;
(4) $\overline{\nabla}_U W = \lambda_1 g(W,P) U$ *if* $i \neq j$;
(5) $\overline{\nabla}_U W = U(lnb_i)W + W(lnb_i)U - \frac{g_{F_i}(U,W)}{b_i} grad_{F_i} b_i - b_i g_{F_i}(U,W) grad_B b_i + b_i^2 \overline{\nabla}^{F_i}_U W + (1 - b_i^2)\nabla^{F_i}_U W$ *if* $i = j$.

By Lemmas 2.3, 2.4 and equation (4), we have the following two propositions:

Proposition 4.3. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply twisted product. If $X, Y, Z \in \Gamma(TB)$, $V \in \Gamma(TF_i), W \in \Gamma(TF_j), U \in \Gamma(TF_k)$ and $P \in \Gamma(TB)$, then:*

(1) $\overline{R}(X,Y)Z = \overline{R}^B(X,Y)Z$;
(2) $\overline{R}(V,X)Y = -\Big[\frac{H^{b_i}_B(X,Y)}{b_i} + \lambda_2 \frac{Pb_i}{b_i} g(X,Y) + \lambda_1\lambda_2 \pi(P) g(X,Y) + \lambda_1 g(Y, \nabla_X P) - \lambda_1^2 \pi(X)\pi(Y)\Big] V$;
(3) $\overline{R}(X,Y)V = 0$;
(4) $\overline{R}(V,W)X = VX(lnb_i)W - WX(lnb_i)V$ *if* $i = j$;
(5) $\overline{R}(V,W)U = 0$ *if* $i = j \neq k$ *or* $i \neq j \neq k$;
(6) $\overline{R}(X,V)W = \overline{R}(V,W)X = \overline{R}(V,X)W = 0$ *if* $i \neq j$;
(7) $\overline{R}(X,V)W = WX(lnb_i)V - g(V,W)\Big[\frac{\nabla^B_X(grad_B b_i)}{b_i} + grad_{F_i}\frac{X(lnb_i)}{b_i^2} + \lambda_1 \frac{Pb_i}{b_i} X + \lambda_2 \nabla_X P + \lambda_1\lambda_2\pi(P)X - \lambda_2^2 \pi(X)P\Big]$ *if* $i = j$;
(8) $\overline{R}(U,V)W = -g(V,W)\Big[\frac{g_B(grad_B b_i, grad_B b_k)}{b_i b_k} + \lambda_1 \frac{Pb_i}{b_i} + \lambda_2 \frac{Pb_k}{b_k} + \lambda_1\lambda_2\pi(P)\Big] U$ *if* $i = j \neq k$;
(9) $\overline{R}(U,V)W = g(U,W) grad_B(V(lnb_i)) - g(V,W) grad_B(U(lnb_i)) + R^{F_i}(U,V)W - \Big[\frac{|grad_B b_i|_B^2}{b_i^2} + (\lambda_1 + \lambda_2)\frac{Pb_i}{b_i} + \lambda_1\lambda_2\pi(P)\Big][g(V,W)U - g(U,W)V]$ *if* $i = j = k$.

Proposition 4.4. *Let $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply twisted product. If $X, Y, Z \in \Gamma(TB)$, $V \in \Gamma(TF_i), W \in \Gamma(TF_j), U \in \Gamma(TF_k)$ and $P \in \Gamma(TF_r)$ for a fixed r, then:*

(1) $\overline{R}(X,Y)Z = R^B(X,Y)Z + \lambda_2\Big[g(X,Z)\frac{Yb_r}{b_r} - g(Y,Z)\frac{Xb_r}{b_r}\Big] P + \lambda_1\lambda_2\pi(P)[g(X,Z)Y - g(Y,Z)X]$;

(2) $\overline{R}(V,X)Y = -\frac{H_B^{b_i}(X,Y)}{b_i}V - \lambda_1\lambda_2\pi(P)g(X,Y)V$ *if* $i \neq r$;
(3) $\overline{R}(V,X)Y = -\frac{H_B^{b_i}(X,Y)}{b_i}V - \lambda_1\pi(V)\frac{Yb_i}{b_i}X - \lambda_2 g(X,Y)\nabla_V P - g(X,Y)[\lambda_1\lambda_2\pi(P)V - \lambda_2^2\pi(V)P]$ *if* $i = r$;
(4) $\overline{R}(X,Y)V = \lambda_1\pi(V)[\frac{Xb_r}{b_r}Y - \frac{Yb_r}{b_r}X]$;
(5) $\overline{R}(V,W)X = -\lambda_1\delta_i^r\frac{Xb_i}{b_i}\pi(V)W + \lambda_1\delta_j^r\frac{Xb_j}{b_j}\pi(W)V$ *if* $i \neq j$;
(6) $\overline{R}(V,W)X = VX(lnb_i)W - WX(lnb_i)V - \lambda_1\delta_i^r\frac{Xb_i}{b_i}[\pi(V)W - \pi(W)V]$ *if* $i = j$;
(7) $\overline{R}(V,W)U = 0$ *if* $i = j \neq k$ *or* $i \neq j \neq k$;
(8) $\overline{R}(X,V)W = \lambda_1\frac{Xb_r}{b_r}\pi(W)V$ *if* $i \neq j$;
(9) $\overline{R}(X,V)W = WX(lnb_i)V - g(V,W)\frac{\nabla_X^B(grad_B b_i)}{b_i} - grad_{F_i}(Xlnb_i)g_{F_i}(V,W) + \lambda_1\frac{Xb_r}{b_r}\pi(W)V - \lambda_1 g(W,\nabla_V P)X - \lambda_2 g(V,W)\frac{Xb_r}{b_r}P - \lambda_1\lambda_2 g(V,W)\pi(P)X + \lambda_1^2\pi(W)\pi(V)X$ *if* $i = j$;
(10) $\overline{R}(U,V)W = -g(V,W)\frac{g_B(grad_B b_i, grad_B b_k)}{b_i b_k}U - \lambda_1 g(W,\nabla_V P)U - \lambda_2 g(V,W)\nabla_U P - \lambda_1\lambda_2\pi(P)g(V,W)U + \lambda_2^2 g(V,W)\pi(U)P + \lambda_1^2\pi(W)[\pi(V)U - \pi(U)V]$ *if* $i = j \neq k$;
(11) $\overline{R}(U,V)W = g(U,W)grad_B(V(lnb_i)) - g(V,W)grad_B(U(lnb_i)) + R^{F_i}(U,V)W - \frac{|grad_B b_i|_B^2}{b_i^2}[g(V,W)U - g(U,W)V] + \lambda_1\lambda_2\pi(P)[g(U,W)V - g(V,W)U]$ *if* $i = j = k \neq r$;
(12) $\overline{R}(U,V)W = g(U,W)grad_B(V(lnb_i)) - g(V,W)grad_B(U(lnb_i)) + R^{F_i}(U,V)W - \frac{|grad_B b_i|_B^2}{b_i^2}[g(V,W)U - g(U,W)V] + \lambda_1[g(W,\nabla_U P)V - g(W,\nabla_V P)U] + \lambda_2[g(U,W)\nabla_V P - g(V,W)\nabla_U P] + \lambda_1\lambda_2\pi(P)\times[g(U,W)V - g(V,W)U] + \lambda_2^2[g(V,W)\pi(U) - g(U,W)\pi(V)]P + \lambda_1^2\pi(W)[\pi(V)U - \pi(U)V]$ *if* $i = j = k = r$, *where* δ_i^r *denotes the Kronecker symbol.*

By Propositions 4.3 and 4.4 and the definition of the Ricci curvature tensor, we have the following two propositions:

Proposition 4.5. *Let* $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product,* $\dim M = \bar{n}$, $\dim B = n$, $\dim F_i = l_i$. *If* $X, Y, Z \in \Gamma(TB)$, $V \in \Gamma(TF_i), W \in \Gamma(TF_j)$ *and* $P \in \Gamma(TB)$, *then:*

(1) $\overline{Ric}(X,Y) = \overline{Ric}^B(X,Y) + \sum_{i=1}^{m} l_i\Big[\frac{H_B^{b_i}(X,Y)}{b_i} + \lambda_2\frac{Pb_i}{b_i}g(X,Y) + \lambda_1\lambda_2\pi(P)g(X,Y) + \lambda_1 g(Y,\nabla_X P) - \lambda_1^2\pi(X)\pi(Y)\Big]$;
(2) $\overline{Ric}(X,V) = \overline{Ric}(V,X) = (l_i - 1)[VX(lnb_i)]$;
(3) $\overline{Ric}(V,W) = 0$ *if* $i \neq j$;
(4) $\overline{Ric}(V,W) = Ric^{F_i}(V,W) + \Big\{\frac{\Delta_B b_i}{b_i} + (l_i - 1)\frac{|grad_B b_i|_B^2}{b_i^2} + \sum_{s\neq i} l_s\frac{g_B(grad_B b_i, grad_B b_s)}{b_i b_s} + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi(P) + \lambda_2 div_B P + \lambda_2\sum_{s\neq i} l_s\frac{Pb_s}{b_s} + [(\bar{n}-1)\lambda_1 + (l_i - 1)\lambda_2]\frac{Pb_i}{b_i}\Big\}g(V,W)$ *if* $i = j$,

where $div_B P = \sum_{k=1}^{n}\varepsilon_k\langle\nabla_{E_k}P, E_k\rangle$, *and* $E_k, 1 \leq k \leq n$ *is an orthonormal base of* B *with* $\varepsilon_k = g(E_k, E_k)$.

As a corollary of Proposition 4.5, we have:

Corollary 4.6. *Let* $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product, and* $\dim F_i = l_i > 1$, $P \in \Gamma(TB)$, *then* $(M, \overline{\nabla})$ *is mixed Ricci-flat if and only if* M *can be expressed as a multiply warped product. In particular, if* $(M, \overline{\nabla})$ *is Einstein, then* M *can be expressed as a multiply warped product.*

Proof. By Proposition 4.5(2) and (3), similarly to the proof of Theorem 1 in [6], we get this corollary. □

Proposition 4.7. *Let* $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product,* $\dim M = \bar{n}$, $\dim B = n$, $\dim F_i = l_i$. *If* $X, Y, Z \in \Gamma(TB)$, $V \in \Gamma(TF_i), W \in \Gamma(TF_j)$ *and* $P \in \Gamma(TF_r)$ *for a fixed* r, *then:*

(1) $\overline{Ric}(X,Y) = Ric^B(X,Y) + \sum_{i=1}^{m} l_i \frac{H_B^{b_i}(X,Y)}{b_i} + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi(P)g(X,Y) + \lambda_2 g(X,Y)div_{F_r}P;$
(2) $\overline{Ric}(X,V) = (l_i-1)[VX(lnb_i)] + [(\bar{n}-1)\lambda_1 - \lambda_2]\pi(V)\frac{Xb_r}{b_r};$
(3) $\overline{Ric}(V,X) = (l_i-1)[VX(lnb_i)] + [\lambda_2 - (\bar{n}-1)\lambda_1]\pi(V)\frac{Xb_r}{b_r};$
(4) $\overline{Ric}(V,W) = 0$ *if* $i \neq j$;
(5) $\overline{Ric}(V,W) = Ric^{F_i}(V,W) + g(V,W)\Big\{\frac{\Delta_B b_i}{b_i} + (l_i-1)\frac{|grad_B b_i|_B^2}{b_i^2} + \sum_{s\neq i} l_s \frac{g_B(grad_B b_i, grad_B b_s)}{b_i b_s} + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]\pi(P)\Big\} + [(\bar{n}-1)\lambda_1 - \lambda_2]g(W,\nabla_V P) + [\lambda_2^2 + (1-\bar{n})\lambda_1^2]\pi(V)\pi(W) + \lambda_2 g(V,W)div_{F_r}P$ *if* $i = j$.

As a corollary of Proposition 4.7, we have:

Corollary 4.8. *Let* $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product, and* $\dim F_i = l_i > 1$, $P \in \Gamma(TF_r)$, *then* $(M, \overline{\nabla})$ *is mixed Ricci-flat if and only if one of the following two conditions is satisfied:*

(1) $\lambda_2 = (\bar{n}-1)\lambda_1$, *and* M *can be expressed as a multiply warped product;*
(2) $\lambda_2 \neq (\bar{n}-1)\lambda_1$, M *can be expressed as a multiply warped product and* b_r *is only dependent on* F_r.

In particular, if $(M, \overline{\nabla})$ *is Einstein, then* M *can be expressed as a multiply warped product.*

Proof. By Proposition 4.7(2) and (3), we have that $(M, \overline{\nabla})$ is mixed Ricci-flat if and only if $VX(lnb_i) = 0$, $[(\bar{n}-1)\lambda_1 - \lambda_2]\pi(V)\frac{Xb_r}{b_r} = 0$. Similarly to the proof of Corollary 4.6, we get that

(*a*) $\lambda_2 = (\bar{n}-1)\lambda_1$, and M can be expressed as a multiply warped product.
(*b*) $\lambda_2 \neq (\bar{n}-1)\lambda_1$, M can be expressed as a multiply warped product. When $i \neq r$, $\pi(V) = 0$. When $i = r$, by $\pi(V)\frac{Xb_r}{b_r} = 0$, then b_r depends only on F_r. □

By Proposition 4.5 and the definition of the scalar curvature, we have the following:

Proposition 4.9. *Let* $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product,* $\dim M = \bar{n}$, $\dim B = n$, $\dim F_i = l_i$. *If* $P \in \Gamma(TB)$, *then the scalar curvature* $\overline{S}$ *has the following expression:*

$$\overline{S} = \overline{S}^B + 2\sum_{i=1}^{m} l_i \frac{\Delta_B b_i}{b_i} + \sum_{i=1}^{m} \frac{S^{F_i}}{b_i^2} + \sum_{i=1}^{m} l_i(l_i-1)\frac{|grad_B b_i|_B^2}{b_i^2} + \sum_{i=1}^{m}\sum_{s\neq i} l_i l_s \frac{g_B(grad_B b_i, grad_B b_s)}{b_i b_s}$$
$$+ \sum_{i=1}^{m} l_i[(\bar{n}-1)\lambda_1 + (n+l_i-1)\lambda_2]\frac{Pb_i}{b_i} + \lambda_2 \sum_{i=1}^{m}\sum_{s\neq i} l_i l_s \frac{Pb_s}{b_s} + \sum_{i=1}^{m} l_i[(\bar{n}+n-1)\lambda_1\lambda_2$$
$$- (\lambda_1^2 + \lambda_2^2)]\pi(P) + (\lambda_1 + \lambda_2)\sum_{i=1}^{m} l_i div_B P.$$

By Proposition 4.7 and the definition of the scalar curvature, we have the following:

Proposition 4.10. *Let* $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ *be a multiply twisted product,* $\dim M = \bar{n}$, $\dim B = n$, $\dim F_i = l_i$. *If* $P \in \Gamma(TF_r)$ *for a fixed* r, *then the scalar curvature* $\overline{S}$ *has the following expression:*

$$\overline{S} = S^B + 2\sum_{i=1}^{m} l_i \frac{\Delta_B b_i}{b_i} + \sum_{i=1}^{m} \frac{S^{F_i}}{b_i^2} + \sum_{i=1}^{m} l_i(l_i-1)\frac{|grad_B b_i|_B^2}{b_i^2} + \sum_{i=1}^{m}\sum_{s\neq i} l_i l_s \frac{g_B(grad_B b_i, grad_B b_s)}{b_i b_s}$$
$$+ [\bar{n}(\bar{n}-1)\lambda_1\lambda_2 + (1-\bar{n})(\lambda_1^2 + \lambda_2^2)]\pi(P) + (\bar{n}-1)(\lambda_1 + \lambda_2)div_{F_r}P.$$

Remark 11. (1) It is easy to see that Propositions 3.1–3.8 are corollaries of Propositions 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, 4.9, 4.10, respectively.

(2) When $\lambda_1 = \lambda_2 = 1$, we get Propositions 1, 2, 4, 5, 7, 9, 12, 13 in [14], by Propositions 4.1, 4.2, 4.3, 4.4, 4.5, 4.7, 4.9, 4.10, respectively.

4.2. Special multiply warped product with a quarter-symmetric connection

Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product with the metric tensor $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$ and I is an open interval in $\mathbb{R}$, and $b_i \in C^\infty(I)$, $\dim M = \bar{n}$, $\dim I = 1$, $\dim F_i = l_i$.

Similarly to the proof method of Theorem 3.11, we have:

Theorem 4.11. *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product with the metric tensor $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$, $P = \frac{\partial}{\partial t}$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if the following conditions are satisfied*

(1) *(F_i, ∇^{F_i}) is Einstein with the Einstein constant α_i, $i \in \{1, \dots, m\}$;*

(2) $\sum_{i=1}^{m} l_i \Big(\lambda_2 \frac{b_i'}{b_i} - \frac{b_i''}{b_i} + \lambda_1^2 - \lambda_1\lambda_2\Big) = \alpha$;

(3) $\alpha_i - b_i b_i'' + (1 - l_i)(b_i')^2 + (\lambda_2 b_i^2 - b_i b_i') \sum_{s \neq i} l_s \frac{b_s'}{b_s} + [\lambda_2^2 + (1 - \bar{n})\lambda_1\lambda_2] b_i^2 + [(\bar{n} - 1)\lambda_1 + (l_i - 1)\lambda_2] b_i b_i' = \alpha b_i^2$.

Theorem 4.12. *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product with the metric tensor $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$, $P \in \Gamma(TF_r)$ with $g_{F_r}(P, P) = 1$ and $\bar{n} > 2$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if the following conditions are satisfied for any $i \in \{1, \dots, m\}$:*

(1) *$(F_i, \nabla^{F_i})(i \neq r)$ is Einstein with the Einstein constant $\alpha_i, i \in \{1, \dots, m\}$;*

(2) *b_r is a constant and $\sum_{i=1}^{m} l_i \frac{b_i''}{b_i} = \mu_0$; $div_{F_r} P = \mu_1$; $\mu_0 - \lambda_2\mu_1 + \alpha = [(\bar{n} - 1)\lambda_1\lambda_2 - \lambda_2^2] b_r^2$, where μ_0, μ_1 are constants;*

(3) *$Ric^{F_r}(V, W) + \bar{\alpha} g_{F_r}(V, W) = [(\bar{n}-1)\lambda_1^2 - \lambda_2^2]\pi(V)\pi(W) - [(\bar{n}-1)\lambda_1 - \lambda_2] g(W, \nabla_V P)$, for $V, W \in \Gamma(TF_r)$, where $\bar{\alpha} = b_r^2\{[(\bar{n} - 1)\lambda_1\lambda_2 - \lambda_2^2] b_r^2 + \lambda_2\mu_1 - \alpha\}$;*

(4) $\alpha_i - b_i b_i'' + [(\bar{n} - 1)\lambda_1\lambda_2 - \lambda_2^2] b_i^2 b_r^2 - b_i b_i' \sum_{s \neq i} l_s \frac{b_s'}{b_s} - (l_i - 1)(b_i')^2 = (\alpha - \lambda_2\mu_1) b_i^2$.

Proof. By Proposition 4.7(2) and $g_{F_r}(P, P) = 1$, we have that b_r is a constant. By Proposition 4.7(1), we have

$$\overline{Ric}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = \sum_{i=1}^{m} l_i \frac{b_i''}{b_i} + [\lambda_2^2 + (1 - \bar{n})\lambda_1\lambda_2] b_r^2 - \lambda_2\, div_{F_r} P = -\alpha.$$

By separation of variables, we have

$$\sum_{i=1}^{m} l_i \frac{b_i''}{b_i} = \mu_0;\ div_{F_r} P = \mu_1;\ \mu_0 - \lambda_2\mu_1 + \alpha = [(\bar{n} - 1)\lambda_1\lambda_2 - \lambda_2^2] b_r^2,$$

then we get (2). By Proposition 4.7(3), we have

$$\overline{Ric}(V, W) = Ric^{F_i}(V, W) + b_i^2 g_{F_i}(V, W)\Big\{-\frac{b_i''}{b_i} + (l_i - 1)\frac{-(b_i')^2}{b_i^2} + \sum_{s \neq i} l_s \frac{-b_i' b_s'}{b_i b_s} + [(\bar{n} - 1)\lambda_1\lambda_2 - \lambda_2^2]\pi(P)\Big\}$$
$$+ [(\bar{n} - 1)\lambda_1 - \lambda_2] g(W, \nabla_V P) + [\lambda_2^2 + (1 - \bar{n})\lambda_1^2]\pi(V)\pi(W) + \lambda_2 g(V, W) div_{F_r} P.$$

When $i \neq r$, then $\nabla_V P = \pi(V) = 0$, so

$$\overline{Ric}(V,W) = Ric^{F_i}(V,W) + b_i^2 g_{F_i}(V,W)\Big\{-\frac{b_i''}{b_i} + (l_i-1)\frac{-(b_i')^2}{b_i^2} + \sum_{s\neq i} l_s \frac{-b_i' b_s'}{b_i b_s} + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]b_r^2\Big\} + \lambda_2\mu_1 b_i^2 g_{F_i}(V,W) = \alpha b_i^2 g_{F_i}(V,W).$$

By separation of variables, we have that (F_i, ∇^{F_i}) $(i \neq r)$ is Einstein with the Einstein constant α_i and

$$\alpha_i - b_i b_i'' + [(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]b_i^2 b_r^2 - b_i b_i' \sum_{s\neq i} l_s \frac{b_s'}{b_s} - (l_i-1)(b_i')^2 = (\alpha - \lambda_2\mu_1)b_i^2.$$

Then we get (1) and (4).

When $i = r$ and b_r is a constant, then

$$\begin{aligned} &Ric^{F_r}(V,W) + b_r^2\big\{[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]b_r^2 + \lambda_2\mu_1 - \alpha\big\}g_{F_r}(V,W) \\ &= [(\bar{n}-1)\lambda_1^2 - \lambda_2^2]\pi(V)\pi(W) - [(\bar{n}-1)\lambda_1 - \lambda_2]g(W, \nabla_V P), \end{aligned}$$

let $\bar{\alpha} = b_r^2\big\{[(\bar{n}-1)\lambda_1\lambda_2 - \lambda_2^2]b_r^2 + \lambda_2\mu_1 - \alpha\big\}$, we get (3). □

Theorem 4.13. *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product and $P = \frac{\partial}{\partial t}$. If $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$, then each (F_i, ∇^{F_i}) has constant scalar curvature S^{F_i}.*

Proof. By Proposition 4.9, we have

$$\begin{aligned} \overline{S} = &-2\sum_{i=1}^m l_i \frac{b_i''}{b_i} + \sum_{i=1}^m \frac{S^{F_i}}{b_i^2} + \sum_{i=1}^m l_i(l_i-1)\frac{-(b_i')^2}{b_i^2} + \sum_{i=1}^m \sum_{s\neq i} l_i l_s \frac{-b_i' b_s'}{b_i b_s} \\ &+ \sum_{i=1}^m l_i[(\bar{n}-1)\lambda_1 + l_i\lambda_2]\frac{b_i'}{b_i} + \lambda_2 \sum_{i=1}^m \sum_{s\neq i} l_i l_s \frac{b_s'}{b_s} \\ &- \sum_{i=1}^m l_i[\bar{n}\lambda_1\lambda_2 - (\lambda_1^2 + \lambda_2^2)]. \end{aligned} \tag{38}$$

Note that each S^{F_i} is function defined on F_i, using separation of variables we complete this proof. □

Theorem 4.14. *Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product and $P \in \Gamma(TF_r)$. If $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$, then*

(1) *each (F_i, ∇^{F_i}) $(i \neq r)$ has constant scalar curvature S^{F_i};*
(2) *If $\lambda_1 + \lambda_2 \neq 0$ and $\lambda_1^2 + \lambda_2^2 - \bar{n}\lambda_1\lambda_2 = 0$, and $div_{F_r} P$ is a constant, then S^{F_r} is a constant;*
(3) *If $\lambda_1 = -\lambda_2 \neq 0$ and $g_{F_r}(P,P)$ is a constant, then S^{F_r} is a constant;*
(4) *If $\lambda_1 + \lambda_2 \neq 0$ and $\lambda_1^2 + \lambda_2^2 - \bar{n}\lambda_1\lambda_2 \neq 0$, and $div_{F_r} P, g_{F_r}(P,P)$ are constants, then S^{F_r} is a constant.*

Proof. By Proposition 4.10, we have

$$\begin{aligned} \overline{S} = &-2\sum_{i=1}^m l_i \frac{b_i''}{b_i} + \sum_{i=1}^m \frac{S^{F_i}}{b_i^2} + \sum_{i=1}^m l_i(l_i-1)\frac{-(b_i')^2}{b_i^2} + \sum_{i=1}^m \sum_{s\neq i} l_i l_s \frac{-b_i' b_s'}{b_i b_s} \\ &+ [\bar{n}(\bar{n}-1)\lambda_1\lambda_2 + (1-\bar{n})(\lambda_1^2 + \lambda_2^2)]b_r^2 g_{F_r}(P,P) + (\bar{n}-1)(\lambda_1+\lambda_2) div_{F_r} P \end{aligned} \tag{39}$$

then similarly to the proof of Proposition 3.18, we complete this proposition. □

Remark 12. When $\lambda_1 = \lambda_2 = 1$, we get Theorems 15, 16 and Propositions 18, 19 in [14] by Theorems 4.11–4.14, respectively.

4.3. Generalized Kasner space–times with a quarter-symmetric connection

In this section, we consider the Einstein and scalar curvature of generalized Kasner space–times with a quarter-symmetric connection. We recall the definition of generalized Kasner space–times in [4].

Definition 4.15. A generalized Kasner space–time (M, g) is a Lorentzian multiply warped product of the form $M = I \times_{\phi^{p_1}} F_1 \times_{\phi^{p_2}} F_2 \cdots \times_{\phi^{p_m}} F_m$ with the metric tensor $g = -dt^2 \oplus \phi^{2p_1} g_{F_1} \oplus \phi^{2p_2} g_{F_2} \cdots \oplus \phi^{2p_m} g_{F_m}$, where $\phi : I \to (0, \infty)$ is smooth and $p_i \in \mathbb{R}$, for any $i \in \{1, \ldots, m\}$ and also $I = (t_1, t_2)$.

We introduce the following parameters $\zeta = \sum_{i=1}^{m} l_i p_i$ and $\eta = \sum_{i=1}^{m} l_i p_i^2$ for generalized Kasner space–times. By Theorem 4.11 and direct computations, we get the following:

Proposition 4.16. *Let $M = I \times_{\phi^{p_1}} F_1 \times_{\phi^{p_2}} F_2 \cdots \times_{\phi^{p_m}} F_m$ be a generalized Kasner space–time and $P = \frac{\partial}{\partial t}$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if the following conditions are satisfied for any $i \in \{1, \ldots, m\}$:*

(1) (F_i, ∇^{F_i}) *is Einstein with the Einstein constant* $\alpha_i, i \in \{1, \ldots, m\}$*;*
(2) $\zeta\left(\lambda_2 \frac{\phi'}{\phi} - \frac{\phi''}{\phi}\right) - (\eta - \zeta)\frac{(\phi')^2}{\phi^2} + (\lambda_1^2 - \lambda_1\lambda_2)(\bar{n} - 1) = \alpha$*;*
(3) $\frac{\alpha_i}{\phi^{2p_i}} - p_i \frac{\phi''}{\phi} - (\zeta - 1)p_i \frac{(\phi')^2}{\phi^2} + \{\lambda_2\zeta + [(\bar{n} - 1)\lambda_1 - \lambda_2]p_i\}\frac{\phi'}{\phi} = \alpha - \lambda_2^2 + (\bar{n} - 1)\lambda_1\lambda_2.$

By equation (38), we obtain the following:

Proposition 4.17. *Let $M = I \times_{\phi^{p_1}} F_1 \times_{\phi^{p_2}} F_2 \cdots \times_{\phi^{p_m}} F_m$ be a generalized Kasner space–time and $P = \frac{\partial}{\partial t}$. Then $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$ if and only if each (F_i, ∇^{F_i}) has constant scalar curvature S^{F_i} and*

$$\overline{S} = \sum_{i=1}^{m} \frac{S^{F_i}}{\phi^{2p_i}} - 2\zeta\frac{\phi''}{\phi} - (\eta + \zeta^2 - 2\zeta)\frac{(\phi')^2}{\phi^2} + (\lambda_1 + \lambda_2)\zeta(\bar{n} - 1)\frac{\phi'}{\phi} + (\bar{n} - 1)(\lambda_1^2 + \lambda_2^2 - \bar{n}\lambda_1\lambda_2). \tag{40}$$

Next, we first give a classification of four-dimensional generalized Kasner space–times with a quarter-symmetric connection and then consider Ricci tensors and scalar curvatures of them.

Definition 4.18. Let $M = I \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ be a multiply warped product with the metric tensor $g = -dt^2 \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \cdots \oplus b_m^2 g_{F_m}$. Then:

(1) (M, g) is said to be of type (I) if $m = 1$ and $\dim F = 3$;
(2) (M, g) is said to be of type (II) if $m = 2$ and $\dim F_1 = 1$ and $\dim F_2 = 2$;
(3) (M, g) is said to be of type (III) if $m = 3$ and $\dim F_1 = 1$, $\dim F_2 = 1$, $\dim F_3 = 1$.

4.3.1. Classification of Einstein type (I) generalized Kasner space–times with a quarter-symmetric connection

By Theorem 3.16, we have given a classification of Einstein type (I) generalized Kasner space–times with a quarter-symmetric connection.

4.3.2. Type (I) generalized Kasner space–times with a quarter-symmetric connection with constant scalar curvature

By Theorem 3.19, we have given a classification of type (I) generalized Kasner space–times with a quarter-symmetric connection with constant scalar curvature.

4.3.3. Classification of Einstein type (II) generalized Kasner space–times with a quarter-symmetric connection

Let $M = I \times_{\phi^{p_1}} F_1 \times_{\phi^{p_2}} F_2$ be an Einstein type (II) generalized Kasner space–time and $P = \frac{\partial}{\partial t}$. Then $\alpha_1 = 0$ because of $\dim F_1 = 1$. $\zeta = p_1 + 2p_2$, $\eta = p_1^2 + 2p_2^2$. By Proposition 4.16, we have

$$\zeta\Big(\lambda_2\frac{\phi'}{\phi} - \frac{\phi''}{\phi}\Big) - (\eta-\zeta)\frac{(\phi')^2}{\phi^2} + 3(\lambda_1^2 - \lambda_1\lambda_2) = \alpha, \tag{41a}$$

$$-p_1\frac{\phi''}{\phi} - (\zeta-1)p_1\frac{(\phi')^2}{\phi^2} + [\lambda_2\zeta + (3\lambda_1-\lambda_2)p_1]\frac{\phi'}{\phi} = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2, \tag{41b}$$

$$\frac{\alpha_2}{\phi^{2p_2}} - p_2\frac{\phi''}{\phi} - (\zeta-1)p_2\frac{(\phi')^2}{\phi^2} + [\lambda_2\zeta + (3\lambda_1-\lambda_2)p_2]\frac{\phi'}{\phi} = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2, \tag{41c}$$

where α_2 is a constant. Consider the following two cases:

4.3.3.1. When $\underline{\zeta = 0}$ In this case, $p_2 = -\frac{1}{2}p_1$, $\eta = \frac{3}{2}p_1^2$. Then by equations (41a)–(41c), we have:

$$-\eta\frac{(\phi')^2}{\phi^2} + 3(\lambda_1^2 - \lambda_1\lambda_2) = \alpha, \tag{42a}$$

$$p_1\Big[-\frac{\phi''}{\phi} + \frac{(\phi')^2}{\phi^2} + (3\lambda_1-\lambda_2)\frac{\phi'}{\phi}\Big] = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2, \tag{42b}$$

$$\frac{\alpha_2}{\phi^{-p_1}} - \frac{1}{2}p_1\Big[-\frac{\phi''}{\phi} + \frac{(\phi')^2}{\phi^2} + (3\lambda_1-\lambda_2)\frac{\phi'}{\phi}\Big] = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2. \tag{42c}$$

(*a*) $\underline{\eta = 0.}$ We have $p_i = 0$, by equation (42a), we get $\alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2$; by equation (42b), we get $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2$; then we have $\lambda_2^2 = 3\lambda_1^2$ and by equation (42c), we get $\alpha_2 = 0$. So we have

$$\underline{\lambda_2^2 = 3\lambda_1^2,\ p_i = 0,\ \alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2 = \lambda_2^2 - 3\lambda_1\lambda_2,\ \alpha_1 = \alpha_2 = 0}.\ [\mathbf{A}]$$

(*b*) $\underline{\eta \neq 0}$. We have $p_i \neq 0$.

1) $\underline{\alpha_2 = 0}$. By equations (42b), (42c), we get $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2$ and

$$-\frac{\phi''}{\phi} + \frac{(\phi')^2}{\phi^2} + (3\lambda_1-\lambda_2)\frac{\phi'}{\phi} = 0; \tag{43a}$$

$$\frac{(\phi')^2}{\phi^2} = \frac{3\lambda_1^2 - \lambda_2^2}{\eta}. \tag{43b}$$

1′. $3\lambda_1^2 - \lambda_2^2 < 0$, we have no solution;

2′. $3\lambda_1^2 - \lambda_2^2 = 0$, we have $\phi = c$, then by equation (42b), we get $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2$.

So we have

$$\underline{\lambda_2^2 = 3\lambda_1^2,\ p_1 \neq 0,\ p_2 \neq 0,\ \alpha = \lambda_2^2 - 3\lambda_1\lambda_2,\ \alpha_1 = \alpha_2 = 0,\ \phi = c}.\ [\mathbf{B}]$$

3′. $3\lambda_1^2-\lambda_2^2>0$, we have $\phi=c_0e^{\pm\sqrt{\frac{3\lambda_1^2-\lambda_2^2}{\eta}}t}$, by equation (43a), we get $\lambda_2=3\lambda_1$, considering that $3\lambda_1^2-\lambda_2^2>0$, we have $\lambda_1^2<0$, which is a contradiction.

2) $\underline{\alpha_2\neq 0}$. By equations (42b), (42c), we get $\frac{\alpha_2}{\phi^{-p_1}}=\frac{3}{2}(\alpha-\lambda_2^2+3\lambda_1\lambda_2)$, so $\phi=c$; then by equation (42b), we get $\alpha-\lambda_2^2+3\lambda_1\lambda_2=0$, then $\alpha_2=0$, this is a contradiction.

4.3.3.2. When $\underline{\zeta\neq 0}$ Then $\eta\neq 0$. Putting $\phi=\psi^{\frac{\zeta}{\eta}}$, we have $\psi''-\lambda_2\psi'+(\alpha+3\lambda_1\lambda_2-3\lambda_1^2)\frac{\eta}{\zeta^2}\psi=0$. Hence:

(1) $\alpha<\frac{\lambda_2^2\zeta^2}{4\eta}+3\lambda_1^2-3\lambda_1\lambda_2$, $\psi=c_1e^{\frac{\lambda_2+\sqrt{\lambda_2^2-4(\alpha+3\lambda_1\lambda_2-3\lambda_1^2)\frac{\eta}{\zeta^2}}}{2}t}+c_2e^{\frac{\lambda_2-\sqrt{\lambda_2^2-4(\alpha+3\lambda_1\lambda_2-3\lambda_1^2)\frac{\eta}{\zeta^2}}}{2}t}$;
(2) $\alpha=\frac{\lambda_2^2\zeta^2}{4\eta}+3\lambda_1^2-3\lambda_1\lambda_2$, $\psi=c_1e^{\frac{\lambda_2}{2}t}+c_2te^{\frac{\lambda_2}{2}t}$;
(3) $\alpha>\frac{\lambda_2^2\zeta^2}{4\eta}+3\lambda_1^2-3\lambda_1\lambda_2$,

$$\psi=c_1e^{\frac{\lambda_2}{2}t}\cos\Big(\frac{\sqrt{4(\alpha+3\lambda_1\lambda_2-3\lambda_1^2)\frac{\eta}{\zeta^2}-\lambda_2^2}}{2}t\Big)+c_2e^{\frac{\lambda_2}{2}t}\sin\Big(\frac{\sqrt{4(\alpha+3\lambda_1\lambda_2-3\lambda_1^2)\frac{\eta}{\zeta^2}-\lambda_2^2}}{2}t\Big).$$

We make equations (41a)–(41c) into

$$\frac{\zeta^2}{\eta}\frac{\lambda_2\psi'-\psi''}{\psi}=\alpha+3\lambda_1\lambda_2-3\lambda_1^2; \tag{44a}$$

$$-\frac{p_1}{\zeta}\frac{(\phi^\zeta)''}{\phi^\zeta}+\frac{\lambda_2\zeta+(3\lambda_1-\lambda_2)p_1}{\zeta}\frac{(\phi^\zeta)'}{\phi^\zeta}=\alpha-\lambda_2^2+3\lambda_1\lambda_2; \tag{44b}$$

$$\frac{\alpha_2}{\phi^{2p_2}}-\frac{p_2}{\zeta}\frac{(\phi^\zeta)''}{\phi^\zeta}+\frac{\lambda_2\zeta+(3\lambda_1-\lambda_2)p_2}{\zeta}\frac{(\phi^\zeta)'}{\phi^\zeta}=\alpha-\lambda_2^2+3\lambda_1\lambda_2. \tag{44c}$$

When $p_1=p_2$, type (II) spaces turn into type (I) spaces, so we assume $p_1\neq p_2$. By (44b) $\times p_2-$ (44c) $\times p_1$, we get

$$\psi'=\frac{p_1\alpha_2}{\lambda_2(p_2-p_1)}\frac{\eta}{\zeta^2}\psi^{1-\frac{2p_2\zeta}{\eta}}+\frac{\alpha-\lambda_2^2+3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}\psi. \tag{45}$$

Now we consider the following three cases to solve this problem:

4.3.3.2.1 $\alpha<\frac{\lambda_2^2\zeta^2}{4\eta}+3\lambda_1^2-3\lambda_1\lambda_2$. Then $\psi=c_1e^{at}+c_2e^{bt}$, where $a=\frac{\lambda_2+\sqrt{\lambda_2^2-4(\alpha+3\lambda_1\lambda_2-3\lambda_1^2)\frac{\eta}{\zeta^2}}}{2}$, $b=\frac{\lambda_2-\sqrt{\lambda_2^2-4(\alpha+3\lambda_1\lambda_2-3\lambda_1^2)\frac{\eta}{\zeta^2}}}{2}$. By equation (45), we get

$$ac_1e^{at}+bc_2e^{bt}=\frac{p_1\alpha_2}{\lambda_2(p_2-p_1)}\frac{\eta}{\zeta^2}\big(c_1e^{at}+c_2e^{bt}\big)^{1-\frac{2p_2\zeta}{\eta}}+\frac{\alpha-\lambda_2^2+3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}\big(c_1e^{at}+c_2e^{bt}\big). \tag{46}$$

1) $\underline{c_1=0}$. We have

$$\Big[b-\frac{\alpha-\lambda_2^2+3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}\Big]c_2e^{bt}=\frac{p_1\alpha_2}{\lambda_2(p_2-p_1)}\frac{\eta}{\zeta^2}\big(c_2e^{bt}\big)^{1-\frac{2p_2\zeta}{\eta}} \tag{47}$$

1′. $\underline{b\neq 0, p_1\alpha_2\neq 0}$. By equation (47), we get $p_2=0$, so $\zeta=p_1$, $\eta=p_1^2$ and $\frac{\zeta^2}{\eta}=1$. Then $b=\frac{\lambda_2-\sqrt{\lambda_2^2-4(\alpha+3\lambda_1\lambda_2-3\lambda_1^2)}}{2}$, so

$$b^2-\lambda_2b=3\lambda_1^2-3\lambda_2\lambda_2-\alpha. \tag{48}$$

On the other hand, $\phi^\zeta = \psi = c_2 e^{bt}$, so by equations (44b), (44c), we get

$$-b^2 + 3\lambda_1 b = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2; \tag{49a}$$

$$\alpha_2 + \lambda_2 b = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2. \tag{49b}$$

By (48) + (49a), we get $(3\lambda_1 - \lambda_2)b = 3\lambda_1^2 - \lambda_2^2$, if $\lambda_2 = 3\lambda_1$, then $0 = -6\lambda_1^2$, this is a contradiction to $\lambda_1 \neq 0$, so $\lambda_2 \neq 3\lambda_1$, $b = \frac{3\lambda_1^2-\lambda_2^2}{3\lambda_1-\lambda_2}$. Then by equation (48), we have $\alpha = \frac{18\lambda_1^4-36\lambda_1^3\lambda_2+24\lambda_1^2\lambda_2^2-6\lambda_1\lambda_2^3}{(3\lambda_1-\lambda_2)^2} = \frac{6\lambda_1(\lambda_1-\lambda_2)(3\lambda_1^2-3\lambda_1\lambda_2+\lambda_2^2)}{(3\lambda_1-\lambda_2)^2}$, by equation (49b), we get $\alpha_2 = \frac{18\lambda_1^4-18\lambda_1^3\lambda_2+6\lambda_1\lambda_2^3-2\lambda_2^4}{(3\lambda_1-\lambda_2)^2} = \frac{2(3\lambda_1^2-\lambda_2^2)(3\lambda_1^2-3\lambda_1\lambda_2+\lambda_2^2)}{(3\lambda_1-\lambda_2)^2}$, since $3\lambda_1^2-3\lambda_1\lambda_2+\lambda_2^2 \neq 0$, and $\alpha_2 \neq 0$, we have $3\lambda_1^2 \neq \lambda_2^2$. Considering that $b < \frac{\lambda_2}{2}$, we get $\frac{6\lambda_1^2-3\lambda_1\lambda_2-\lambda_2^2}{3\lambda_1-\lambda_2} < 0$.

So we obtain

$$\underline{3\lambda_1^2 \neq \lambda_2^2,\ \frac{6\lambda_1^2-3\lambda_1\lambda_2-\lambda_2^2}{3\lambda_1-\lambda_2} < 0,\ p_1 \neq 0,\ p_2 = 0,\ \alpha = \frac{18\lambda_1^4-36\lambda_1^3\lambda_2+24\lambda_1^2\lambda_2^2-6\lambda_1\lambda_2^3}{(3\lambda_1-\lambda_2)^2},}$$

$$\underline{\alpha_1 = 0,\ \alpha_2 = \frac{18\lambda_1^4-18\lambda_1^3\lambda_2+6\lambda_1\lambda_2^3-2\lambda_2^4}{(3\lambda_1-\lambda_2)^2},\ \phi = c_0 e^{\frac{3\lambda_1^2-\lambda_2^2}{(3\lambda_1-\lambda_2)p_1}t}}.\ [\mathbf{C}]$$

2′. $\underline{b \neq 0, p_1\alpha_2 = 0}$.

1°. $\underline{p_1 = 0}$. Then $\zeta = 2p_2$, $\eta = 2p_2^2$, $\frac{\eta}{\zeta^2} = \frac{1}{2}$. By equation (47), we get $b = \frac{\alpha-\lambda_2^2+3\lambda_1\lambda_2}{2\lambda_2}$, on the other hand, $b = \frac{\lambda_2-\sqrt{\lambda_2^2-2(\alpha+3\lambda_1\lambda_2-3\lambda_1^2)}}{2}$, so we get

$$\alpha^2 + (6\lambda_1\lambda_2 - 2\lambda_2^2)\alpha + 3\lambda_1^2\lambda_2^2 - 6\lambda_1^3\lambda_2 + 3\lambda_1^4 = 0. \tag{50}$$

When $\lambda_1 = \lambda_2$, we get $\alpha^2 + 4\lambda_1^2\alpha = 0$, then $\alpha = 0$ or $\alpha = -4\lambda_1^2$. If $\alpha = 0$, we have $b = \lambda_1 = \lambda_2 < 0$, by equation (44c), we get $\alpha_2 = 0$; $\phi = \psi^{\frac{\zeta}{\eta}} = c_0 e^{\frac{\lambda_1}{p_2}t}$. If $\alpha = -4\lambda_1^2$, then $\lambda_1 = \lambda_2 > 0$, $b = -\lambda_1$, and by equation (44c), we get $\lambda_1 = 0$, which is a contradiction.

When $\lambda_1 \neq \lambda_2$, $\Delta = 8\lambda_2^2(3\lambda_1^2 - \lambda_2^2)$, if $\lambda_2^2 > 3\lambda_1^2$, we have no solution; if $\lambda_2^2 = 3\lambda_1^2$, we have $b = 0$, which is a contradiction; if $\lambda_2^2 < 3\lambda_1^2$, we have $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 + \sqrt{2\lambda_2^2(3\lambda_1^2 - \lambda_2^2)}$ or $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 - \sqrt{2\lambda_2^2(3\lambda_1^2 - \lambda_2^2)}$. By equation (44c), we have $\alpha_2 = 0$ and

$$-2b^2 + (3\lambda_1 + \lambda_2)b = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2. \tag{51}$$

When $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 + \sqrt{2\lambda_2^2(3\lambda_1^2 - \lambda_2^2)}$, we have $b = \frac{\sqrt{2\lambda_2^2(3\lambda_1^2-\lambda_2^2)}}{2\lambda_2}$, by equation (51), we have $\lambda_1 = \lambda_2$, which is a contradiction. When $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 - \sqrt{2\lambda_2^2(3\lambda_1^2 - \lambda_2^2)}$, we have the same contradiction.

So we get

$$\underline{\lambda_1 = \lambda_2 < 0,\ p_1 = 0,\ p_2 \neq 0,\ \alpha = \alpha_1 = \alpha_2 = 0,\ \phi = c_0 e^{\frac{\lambda_1}{p_2}t}}.\ [\mathbf{D}]$$

2°. $\underline{\alpha_2 = 0}$. Since $p_1 = 0$ we have discussed, so we assume that $p_1 \neq 0$. By equation (47), we have $b = \frac{\alpha-\lambda_2^2+3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}$, $\phi^\zeta = ce^{b\frac{\zeta^2}{\eta}t}$, then by equation (44b), we have $\alpha(\alpha - \lambda_2^2 + 3\lambda_1\lambda_2) = 0$. So when $\lambda_2 = 3\lambda_1$, we have $\alpha = 0$; when $\lambda_2 \neq 3\lambda_1$, we have $\alpha = 0$ or $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 \neq 0$.

When $\lambda_2 = 3\lambda_1$, $\alpha = 0$, then $b = 0$, which is a contradiction;

When $\lambda_2 \neq 3\lambda_1$, $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 \neq 0$, then $b = 0$, which is a contradiction;

When $\lambda_2 \neq 3\lambda_1$, $\alpha = 0$, then $b = (3\lambda_1 - \lambda_2)\frac{\eta}{\zeta^2}$, by equation (44a), we get $\frac{\eta}{\zeta^2} = \frac{3\lambda_1^2-\lambda_2^2}{(3\lambda_1-\lambda_2)^2}$, then $b = \frac{3\lambda_1^2-\lambda_2^2}{3\lambda_1-\lambda_2}$. Since $b < \frac{\lambda_2}{2}$, we get $\frac{6\lambda_1^2-3\lambda_1\lambda_2-\lambda_2^2}{3\lambda_1-\lambda_2} < 0$, by $\alpha < \frac{\lambda_2^2\zeta^2}{4\eta} + 3\lambda_1^2 - 3\lambda_1\lambda_2$, we get $\lambda_2^2 < 3\lambda_1^2$. $\phi = \psi^{\frac{\zeta}{\eta}} = ce^{\frac{3\lambda_1-\lambda_2}{\zeta}t}$.

So we have

$$\underline{\lambda_2^2 < 3\lambda_1^2,\ \frac{6\lambda_1^2-3\lambda_1\lambda_2-\lambda_2^2}{3\lambda_1-\lambda_2} < 0,\ p_1 \neq 0,\ p_2 \neq -\tfrac{1}{2}p_1,\ \alpha = \alpha_1 = \alpha_2 = 0,\ \phi = ce^{\frac{3\lambda_1-\lambda_2}{\zeta}t}}.\ [\mathbf{E}]$$

$3'$. $\underline{b=0}$. Then $\psi = c_2$, by equation (44a), we have $\alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2$; by equation (44b), we have $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2$; so $\lambda_2^2 = 3\lambda_1^2$, by equation (44c), we have $\alpha_2 = 0$.

So we have

$$\underline{\lambda_2^2 = 3\lambda_1^2,\ \zeta \neq 0,\ \eta \neq 0,\ \alpha = \lambda_2^2 - 3\lambda_1\lambda_2 = 3\lambda_1^2 - 3\lambda_1\lambda_2,\ \alpha_1 = \alpha_2 = 0,\ \phi = c.}\ [\mathbf{F}]$$

2) $\underline{c_2 = 0}$. We have

$$\left[a - \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}\right]c_1e^{at} = \frac{p_1\alpha_2}{\lambda_2(p_2-p_1)}\frac{\eta}{\zeta^2}\left(c_1e^{at}\right)^{1-\frac{2p_2\zeta}{\eta}} \tag{52}$$

$1'$. $\underline{p_1\alpha_2 \neq 0}$. By equation (52), we get $p_2 = 0$, so $\zeta = p_1$, $\eta = p_1^2$ and $\frac{\zeta^2}{\eta} = 1$. Then $a = \frac{\lambda_2 + \sqrt{\lambda_2^2 - 4(\alpha + 3\lambda_1\lambda_2 - 3\lambda_1^2)}}{2}$, using equation (52) again, we get $a - \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2} = \frac{p_1\alpha_2}{-\lambda_2 p_1} = -\frac{\alpha_2}{\lambda_2}$, so $\alpha_2 = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2 - \lambda_2 a$. By equation (44b), we get $(3\lambda_1 - \lambda_2)^2\alpha = 18\lambda_1^4 - 6\lambda_1\lambda_2^3 + 24\lambda_1^2\lambda_2^2 - 36\lambda_1^3\lambda_2$, if $3\lambda_1 = \lambda_2$, then $0 = -36\lambda_1^2$, which is a contradiction, so $3\lambda_1 \neq \lambda_2$ and $\alpha = \frac{18\lambda_1^4 - 6\lambda_1\lambda_2^3 + 24\lambda_1^2\lambda_2^2 - 36\lambda_1^3\lambda_2}{(3\lambda_1-\lambda_2)^2}$, so $a = \frac{\lambda_2 + \sqrt{\frac{(6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2)^2}{(3\lambda_1-\lambda_2)^2}}}{2}$.

If $\frac{6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2}{3\lambda_1 - \lambda_2} > 0$, then $a = \frac{3\lambda_1^2 - \lambda_2^2}{3\lambda_1 - \lambda_2}$, $\alpha_2 = \frac{18\lambda_1^4 - 2\lambda_2^4 + 6\lambda_1\lambda_2^3 - 18\lambda_1^3\lambda_2}{(3\lambda_1-\lambda_2)^2} = \frac{(3\lambda_1^2-\lambda_2^2)(3\lambda_1^2 - 3\lambda_1\lambda_2 + \lambda_2^2)}{(3\lambda_1-\lambda_2)^2}$, since $3\lambda_1^2 - 3\lambda_1\lambda_2 + \lambda_2^2 = 3\big(\lambda_1 - \frac{1}{2}\lambda_2\big)^2 + \frac{1}{4}\lambda_2^2 \neq 0$, and $\alpha_2 \neq 0$, we have $3\lambda_1^2 \neq \lambda_2^2$, so $a \neq 0$.

If $\frac{6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2}{3\lambda_1 - \lambda_2} < 0$, then $a = \frac{3\lambda_1\lambda_2 - \lambda_2^2}{3\lambda_1 - \lambda_2}$, $\alpha_2 = \frac{18\lambda_1^4 - \lambda_2^4 + 6\lambda_1\lambda_2^3 - 15\lambda_1^2\lambda_2^2}{(3\lambda_1-\lambda_2)^2} \neq 0$, so $18\lambda_1^4 - \lambda_2^4 + 6\lambda_1\lambda_2^3 - 15\lambda_1^2\lambda_2^2 \neq 0$.

Hence, we have

$$\underline{\lambda_2 \neq 3\lambda_1,\ \lambda_2^2 \neq 3\lambda_1^2,\ \frac{6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2}{3\lambda_1 - \lambda_2} > 0,\ p_1 \neq 0,\ p_2 = 0,\ \alpha_1 = 0,}$$

$$\underline{\alpha = \frac{18\lambda_1^4 - 6\lambda_1\lambda_2^3 + 24\lambda_1^2\lambda_2^2 - 36\lambda_1^3\lambda_2}{(3\lambda_1-\lambda_2)^2},\ \alpha_2 = \frac{18\lambda_1^4 - 2\lambda_2^4 + 6\lambda_1\lambda_2^3 - 18\lambda_1^3\lambda_2}{(3\lambda_1-\lambda_2)^2},\ \phi = ce^{\frac{3\lambda_1^2 - \lambda_2^2}{(3\lambda_1-\lambda_2)p_1}t},}\ [\mathbf{G}]$$

or

$$\underline{\lambda_2 \neq 3\lambda_1,\ 18\lambda_1^4 - \lambda_2^4 + 6\lambda_1\lambda_2^3 - 15\lambda_1^2\lambda_2^2 \neq 0,\ \frac{6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2}{3\lambda_1 - \lambda_2} < 0,\ p_1 \neq 0,\ p_2 = 0,}$$

$$\underline{\alpha_1 = 0,\ \alpha = \frac{18\lambda_1^4 - 6\lambda_1\lambda_2^3 + 24\lambda_1^2\lambda_2^2 - 36\lambda_1^3\lambda_2}{(3\lambda_1-\lambda_2)^2},\ \alpha_2 = \frac{18\lambda_1^4 - \lambda_2^4 + 6\lambda_1\lambda_2^3 - 15\lambda_1^2\lambda_2^2}{(3\lambda_1-\lambda_2)^2},\ \phi = ce^{\frac{3\lambda_1\lambda_2 - 3\lambda_1^2}{(3\lambda_1-\lambda_2)p_1}t}}.\ [\mathbf{H}]$$

In particular, when $\lambda_1 = \lambda_2 < 0$, we have $p_1 \neq 0$, $p_2 = 0$, $\alpha = \alpha_1 = 0$, $\alpha_2 = 2\lambda_1^2$, $\phi = c$.

$2'$. $\underline{p_1\alpha_2 = 0}$. $1°$. $\underline{p_1 = 0}$. Then $\zeta = 2p_2$, $\eta = 2p_2^2$, $\frac{\eta}{\zeta^2} = \frac{1}{2}$. By equation (52), we get $a = \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{2\lambda_2}$. On the other hand, $a = \frac{\lambda_2 + \sqrt{\lambda_2^2 - 2(\alpha + 3\lambda_1\lambda_2 - 3\lambda_1^2)}}{2}$, so we get

$$\alpha^2 + (6\lambda_1\lambda_2 - 2\lambda_2^2)\alpha + 3\lambda_1^2\lambda_2^2 - 6\lambda_1^3\lambda_2 + 3\lambda_1^4 = 0. \tag{53}$$

When $\lambda_1 = \lambda_2$, we get $\alpha^2 + 4\lambda_1^2\alpha = 0$, then $\alpha = 0$ or $\alpha = -4\lambda_1^2$. If $\alpha = 0$, we have $a = \lambda_1 = \lambda_2 > 0$, by equation (44c), we get $\alpha_2 = 0$, $\phi = \psi^{\frac{\zeta}{\eta}} = c_0e^{\frac{\lambda_1}{p_2}t}$. If $\alpha = -4\lambda_1^2$, then $\lambda_1 = \lambda_2 < 0$, $a = -\lambda_1$, and by equation (44c), we get $\lambda_1 = 0$, which is a contradiction.

When $\lambda_1 \neq \lambda_2$, $\Delta = 8\lambda_2^2(3\lambda_1^2 - \lambda_2^2)$,

if $\lambda_2^2 > 3\lambda_1^2$, we have no solution;

if $\lambda_2^2 = 3\lambda_1^2$, we have

$$\underline{a = 0,\ \alpha_1 = \alpha_2 = 0,\ \alpha = \lambda_2^2 - 3\lambda_1\lambda_2,\ \phi = c;}\ [\mathbf{I}]$$

if $\lambda_2^2 < 3\lambda_1^2$, we have $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 + \sqrt{2\lambda_2^2(3\lambda_1^2 - \lambda_2^2)}$ or $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 - \sqrt{2\lambda_2^2(3\lambda_1^2 - \lambda_2^2)}$. By equation (44c), we have $\alpha_2 = 0$ and

$$-2a^2 + (3\lambda_1 + \lambda_2)a = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2. \tag{54}$$

Then we have $\lambda_1 = \lambda_2 > 0$ or $\lambda_1 = \lambda_2 < 0$, but when $\lambda_1 = \lambda_2 < 0$, it is not satisfies equation (44c), so we get

$$\underline{\lambda_1 = \lambda_2 > 0,\ p_1 = 0,\ p_2 \neq 0,\ \alpha = \alpha_1 = \alpha_2 = 0,\ \phi = ce^{\frac{\lambda_1}{p_2}t}}. \ [\mathbf{J}]$$

2°. $\underline{\alpha_2 = 0}$. The case $p_1 = 0$ we have already discussed, so we assume that $p_1 \neq 0$. By equation (52), we have $a = \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}$, $\phi^\zeta = ce^{a\frac{\zeta^2}{\eta}t}$, then by equation (44b), we have $\alpha(\alpha - \lambda_2^2 + 3\lambda_1\lambda_2) = 0$. So when $\lambda_2 = 3\lambda_1$, we have $\alpha = 0$; when $\lambda_2 \neq 3\lambda_1$, we have $\alpha = 0$ or $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 \neq 0$.

When $\lambda_2 = 3\lambda_1$, $\alpha = 0$, then $a = 0$, $\eta = 0$, which is a contradiction.

When $\lambda_2 \neq 3\lambda_1$, $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 \neq 0$, then $a = 0$ and $(\lambda_2^2 - 3\lambda_1^2)\eta = 0$.

If $\lambda_2^2 \neq 3\lambda_1^2$, then $\eta = 0$, which is a contradiction;

If $\lambda_2^2 = 3\lambda_1^2$, which satisfies $\lambda_2 \neq 3\lambda_1$, then $\phi^\zeta = c$, which satisfies equations (44a)–(44c).

So we have

$$\underline{\lambda_2^2 = 3\lambda_1^2,\ p_1 \neq 0,\ p_2 \neq -\tfrac{1}{2}p_1,\ \alpha = \lambda_2^2 - 3\lambda_1\lambda_2,\ \alpha_1 = \alpha_2 = 0,\ \phi = c}. \ [\mathbf{K}]$$

When $\lambda_2 \neq 3\lambda_1$, $\alpha = 0$, then $a = (3\lambda_1 - \lambda_2)\frac{\eta}{\zeta^2}$, by equation (44a), we get $\frac{\eta}{\zeta^2} = \frac{3\lambda_1^2 - \lambda_2^2}{(3\lambda_1 - \lambda_2)^2} > 0$ and $\lambda_2^2 < 3\lambda_1^2$, then $a = \frac{3\lambda_1^2 - \lambda_2^2}{3\lambda_1 - \lambda_2}$. Since $a > \frac{\lambda_2}{2}$, we get $\frac{6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2}{3\lambda_1 - \lambda_2} > 0$. Considering that $\frac{p_1^2 + 2p_2^2}{(p_1 + 2p_2)^2} = \frac{\eta}{\zeta^2} = \frac{3\lambda_1^2 - \lambda_2^2}{(3\lambda_1 - \lambda_2)^2}$, we have $(3\lambda_1^2 - 3\lambda_1\lambda_2 + \lambda_2^2)p_1^2 - (6\lambda_1^2 - 2\lambda_2^2)p_1p_2 = (3\lambda_1^2 - 6\lambda_1\lambda_2 + 3\lambda_2^2)p_2^2$, no matter $\lambda_1 = \lambda_2$ or $\lambda_1 \neq \lambda_2$, we can get $p_2 \neq 0$.

So we get

$$\underline{\lambda_2 \neq 3\lambda_1,\ \lambda_2^2 < 3\lambda_1^2,\ \frac{6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2}{3\lambda_1 - \lambda_2} > 0,\ \frac{\eta}{\zeta^2} = \frac{3\lambda_1^2 - \lambda_2^2}{(3\lambda_1 - \lambda_2)^2},\ \alpha = \alpha_1 = \alpha_2 = 0,\ \phi = ce^{\frac{3\lambda_1^2 - \lambda_2^2}{3\lambda_1 - \lambda_2}\frac{\zeta}{\eta}t} = ce^{\frac{3\lambda_1 - \lambda_2}{\zeta}t}}. \ [\mathbf{L}]$$

3) $\underline{c_1 \neq 0, c_2 \neq 0, b \neq 0}$.

1′. $\underline{p_2 \neq 0}$. Then e^{at}, e^{bt} and $(c_1e^{at} + c_2e^{bt})^{1 - 2p_2\frac{\zeta}{\eta}}$ are linearly independent, by equation (46), we have $\left[a - \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}\right]c_1 = 0$, $\left[b - \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}\right]c_2 = 0$.

Considering that $c_1 \neq 0$, $c_2 \neq 0$, we have $a = b = \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}$, which is a contradiction.

2′. $\underline{p_2 = 0}$. Then by equation (46), we have

$$a - \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2} - \frac{p_1\alpha_2}{\lambda_2(p_2 - p_1)}\frac{\eta}{\zeta^2} = 0, \qquad b - \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2} - \frac{p_1\alpha_2}{\lambda_2(p_2 - p_1)}\frac{\eta}{\zeta^2} = 0.$$

Then $a = b$, which is a contradiction.

4) $\underline{c_1 \neq 0, c_2 \neq 0, b = 0}$. Then $a = \lambda_2 \neq 0$ and

$$ac_1e^{at} = \frac{p_1\alpha_2}{\lambda_2(p_2 - p_1)}\frac{\eta}{\zeta^2}(c_1e^{at} + c_2)^{1 - 2p_2\frac{\zeta}{\eta}} + \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}(c_1e^{at} + c_2). \tag{55}$$

1′. $\underline{1 - 2p_2\frac{\zeta}{\eta} \neq 0}$.

If $p_2 \neq 0$, then e^{at} and $(c_1e^{at} + c_2)^{1 - 2p_2\frac{\zeta}{\eta}}$ are linearly independent, by equation (55), we have $a = 0$, this is a contradiction to $a = \lambda_2 \neq 0$.

If $p_2 = 0$, using the same method we can get $a = 0$, this is a contradiction to $a = \lambda_2 \neq 0$.

$2'$. $\underline{1 - 2p_2 \frac{\zeta}{\eta} = 0}$.

Then $p_2 \neq 0$, $\eta = 2p_2\zeta$ and equation (55) becomes

$$\Big(\lambda_2 - \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}\Big)c_1 e^{at} - \frac{p_1\alpha_2}{\lambda_2(p_2 - p_1)}\frac{\eta}{\zeta^2} - \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2} = 0. \tag{56}$$

Then

$$\lambda_2 = \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}. \tag{57}$$

Since $b = 0$ which means $\lambda_2 = \sqrt{\lambda_2^2 - 4(\alpha + 3\lambda_1\lambda_2 - 3\lambda_1^2)\frac{\eta}{\zeta^2}}$, we get $\alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2$, then using equation (57), we have $\lambda_2 = \frac{3\lambda_1^2 - \lambda_2^2}{\lambda_2}\frac{\eta}{\zeta^2}$ and $\lambda_2^2 \neq 3\lambda_1^2$, considering that $\eta = 2p_2\zeta$, we get $p_1 = \frac{6\lambda_1^2 - 4\lambda_2^2}{\lambda_2^2}p_2$.

If $3\lambda_1^2 = 2\lambda_2^2$, then $p_1 = 0$, by $\eta = 2p_2\zeta$ we have $p_2 = 0$, which is a contradiction.

If $3\lambda_1^2 \neq 2\lambda_2^2$, then $p_1 \neq 0$, and by $\eta = 2p_2\zeta$ we have $18\lambda_1^4 - 30\lambda_1^2\lambda_2^2 + 11\lambda_2^4 = 0$, then $\lambda_1^2 = \frac{5\pm\sqrt{3}}{6}\lambda_2^2$, which satisfies $\lambda_2^2 \neq 3\lambda_1^2$ and $3\lambda_1^2 \neq 2\lambda_2^2$. Then $\alpha_2 = \frac{(p_2 - p_1)(\lambda_2^2 - 3\lambda_1\lambda_2)}{p_1}$, $p_1 = (1 \pm \sqrt{3})p_2 \neq 0$, $\zeta = (3 \pm \sqrt{3})p_2$, $\eta = (6 \pm 2\sqrt{3})p_2^2$, $\phi = ce^{\frac{\lambda_2}{2p_1}t}$.

So we get

$$\underline{\lambda_1^2 = \tfrac{5\pm\sqrt{3}}{6}\lambda_2^2,\ p_1 = (1 \pm \sqrt{3})p_2 \neq 0,\ \alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2,\ \alpha_1 = 0,\ \alpha_2 = \tfrac{(p_2 - p_1)(\lambda_2^2 - 3\lambda_1\lambda_2)}{p_1},\ \phi = ce^{\frac{\lambda_2}{2p_1}t}}. \ [\mathbf{M}]$$

4.3.3.2.2 $\alpha = \frac{\lambda_2^2\zeta^2}{4\eta} + 3\lambda_1^2 - 3\lambda_1\lambda_2$, $\psi = c_1 e^{\frac{\lambda_2}{2}t} + c_2 t e^{\frac{\lambda_2}{2}t}$. By equation (45), we have

$$\Big[\frac{\lambda_2}{2}c_1 + c_2 - a_0 c_1 + \Big(\frac{\lambda_2}{2}c_2 - a_0 c_2\Big)t\Big]e^{\frac{\lambda_2}{2}t} = \frac{p_1\alpha_2}{\lambda_2(p_2 - p_1)}\frac{\eta}{\zeta^2}(c_1 + c_2 t)^{1 - 2p_2\frac{\zeta}{\eta}}(e^{\frac{\lambda_2}{2}t})^{1 - 2p_2\frac{\zeta}{\eta}}, \tag{58}$$

where $a_0 = \frac{\lambda_2}{4} + \frac{3\lambda_1^2 - \lambda_2^2}{\lambda_2^2}\frac{\eta}{\zeta^2}$.

1) $\underline{c_2 \neq 0}$. Then by equation (58), we have $p_2 = 0$, by equation (44c), we get $\alpha_2 + \lambda_2\frac{(\phi^\zeta)'}{\phi^\zeta} = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2$, then $\phi^\zeta = c_0 e^{\frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2 - \alpha_2}{\lambda_2}t}$; on the other hand, $\phi^\zeta = \psi^{\frac{\zeta^2}{\eta}} = \Big(c_1 e^{\frac{\lambda_2}{2}t} + c_2 t e^{\frac{\lambda_2}{2}t}\Big)^{\frac{\zeta^2}{\eta}}$, then $c_0 e^{\frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2 - \alpha_2}{\lambda_2}t} = \Big(c_1 e^{\frac{\lambda_2}{2}t} + c_2 t e^{\frac{\lambda_2}{2}t}\Big)^{\frac{\zeta^2}{\eta}}$, this is a contradiction to $c_2 \neq 0$.

2) $\underline{c_2 = 0}$. Then equation (58) becomes

$$\Big(\frac{\lambda_2}{2}c_1 - a_0 c_1\Big)e^{\frac{\lambda_2}{2}t} = \frac{p_1\alpha_2}{\lambda_2(p_2 - p_1)}\frac{\eta}{\zeta^2}c_1^{1 - 2p_2\frac{\zeta}{\eta}}(e^{\frac{\lambda_2}{2}t})^{1 - 2p_2\frac{\zeta}{\eta}}. \tag{59}$$

$1'$. $\underline{a_0 = \frac{\lambda_2}{2}}$. By equation (59), we get $p_1\alpha_2 = 0$. Considering that $a_0 = \frac{\lambda_2}{4} + \frac{3\lambda_1^2 - \lambda_2^2}{\lambda_2^2}\frac{\eta}{\zeta^2}$, we have $\lambda_2^2 \neq 3\lambda_1^2$ and $\frac{\lambda_2^2}{4}\frac{\zeta^2}{\eta} = 3\lambda_1^2 - \lambda_2^2$, then $\alpha = 6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2$.

If $p_1 = 0$, then $\zeta = 2p_2$, $\eta = 2p_2^2$, $\frac{\eta}{\zeta^2} = \frac{1}{2}$. By $\frac{\lambda_2^2}{4}\frac{\zeta^2}{\eta} = 3\lambda_1^2 - \lambda_2^2$, we have $\lambda_2^2 = 2\lambda_1^2$ and $\alpha = 2\lambda_2^2 - 3\lambda_1\lambda_2$. By equation (44b), we get $\phi^\zeta = c_0 e^{\lambda_2 t}$, then by equation (44c), we get $\lambda_2 = \frac{3}{2}\lambda_1$, this is a contradiction to $\lambda_2^2 = 2\lambda_1^2$.

If $p_1 \neq 0$, $\alpha_2 = 0$, then $\psi = c_1 e^{\frac{\lambda_2}{2}t}$, $\phi^\zeta = c_0 e^{\frac{2(3\lambda_1^2 - \lambda_2^2)}{\lambda_2}t}$, then by equation (44b), we have $\alpha = 6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2 = 0$, which satisfies equation (44c). $\phi = c_0 e^{\frac{\lambda_2}{2}\frac{\zeta}{\eta}t}$.

So we get

$$\underline{p_1 \neq 0,\ \alpha = 6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2 = 0,\ \alpha_1 = \alpha_2 = 0,\ \phi = c_0 e^{\frac{\lambda_2}{2}\frac{\zeta}{\eta}t}}. \ [\mathbf{N}]$$

$2'$. $\underline{a_0 \neq \frac{\lambda_2}{2}}$. By equation (59), we have $p_2 = 0$, then $\zeta = p_1$, $\eta = p_1^2$, $\frac{\zeta^2}{\eta} = 1$, so $\alpha = \frac{\lambda_2^2}{4} + 3\lambda_1^2 - 3\lambda_1\lambda_2$, $\phi^\zeta = c_1 e^{\frac{\lambda_2}{2}t}$. By equation (44b), we have $6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2 = 0$, by equation (44c), we have $\alpha_2 = 3\lambda_1^2 - \frac{5}{4}\lambda_2^2$. So we get

$$\underline{6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2 = 0,\ p_1 \neq 0,\ p_2 = 0,\ \alpha = \frac{\lambda_2^2}{4} + 3\lambda_1^2 - 3\lambda_1\lambda_2,\ \alpha_2 = 3\lambda_1^2 - \frac{5}{4}\lambda_2^2,\ \phi = ce^{\frac{\lambda_2}{2\zeta}t}}. \ [\mathbf{O}]$$

4.3.3.2.3 $\alpha > \frac{\lambda_2^2\zeta^2}{4\eta} + 3\lambda_1^2 - 3\lambda_1\lambda_2$. Then $\psi = c_1 e^{\frac{\lambda_2}{2}t}\cos(at) + c_2 e^{\frac{\lambda_2}{2}t}\sin(at)$ and $a = \frac{\sqrt{4(\alpha + 3\lambda_1\lambda_2 - 3\lambda_1^2)\frac{\eta}{\zeta^2} - \lambda_2^2}}{2}$. By equation (45), we have

$$\begin{aligned}
&\Big(\frac{\lambda_2}{2}c_1 + ac_2\Big)\cos(at) + \Big(-ac_1 + \frac{\lambda_2}{2}c_2\Big)\sin(at) \\
&= \frac{p_1\alpha_2}{\lambda_2(p_2 - p_1)}\frac{\eta}{\zeta^2}\big(c_1\cos(at) + c_2\sin(at)\big)^{1-2p_2\frac{\zeta}{\eta}} e^{-\lambda_2 p_2 \frac{\zeta}{\eta}t} \\
&\quad + \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}\big(c_1\cos(at) + c_2\sin(at)\big).
\end{aligned} \tag{60}$$

1) $\underline{p_2 \neq 0}$. Then by equation (60), we get $p_1\alpha_2 = 0$ and

$$\frac{\lambda_2}{2}c_1 + ac_2 = \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}c_1; \tag{61a}$$

$$-ac_1 + \frac{\lambda_2}{2}c_2 = \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}c_2. \tag{61b}$$

By (61a) $\times\, c_2$ − (61b) $\times\, c_1$, we get $c_1^2 + c_2^2 = 0$, this is a contradiction.

2) $\underline{p_2 = 0}$. Then by equation (60), we have

$$\frac{\lambda_2}{2}c_1 + ac_2 = \frac{p_1\alpha_2}{\lambda_2(p_2 - p_1)}\frac{\eta}{\zeta^2}c_1 + \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}c_1; \tag{62a}$$

$$-ac_1 + \frac{\lambda_2}{2}c_2 = \frac{p_1\alpha_2}{\lambda_2(p_2 - p_1)}\frac{\eta}{\zeta^2}c_2 + \frac{\alpha - \lambda_2^2 + 3\lambda_1\lambda_2}{\lambda_2}\frac{\eta}{\zeta^2}c_2. \tag{62b}$$

By (62a) $\times\, c_2$ − (62b) $\times\, c_1$, we get $c_1^2 + c_2^2 = 0$, this is a contradiction.

So we have no solution in the case of $\alpha > \frac{\lambda_2^2\zeta^2}{4\eta} + 3\lambda_1^2 - 3\lambda_1\lambda_2$.

According to the above discussions, we get the following theorem:

Theorem 4.19. *Let $M = I \times_{\phi^{p_1}} F_1 \times_{\phi^{p_2}} F_2 \cdots \times_{\phi^{p_m}} F_m$ be a generalized Kasner space–time,* $\dim F_1 = 1$, $\dim F_2 = 2$ *and $P = \frac{\partial}{\partial t}$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if one of the following conditions is satisfied:*

(1) $\lambda_2^2 = 3\lambda_1^2$, $p_1 = p_2 = 0$, $\alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2 = \lambda_2^2 - 3\lambda_1\lambda_2$, $\alpha_1 = \alpha_2 = 0$;
(2) $\lambda_2^2 = 3\lambda_1^2$, $p_1^2 + p_2^2 \neq 0$, $\alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2 = \lambda_2^2 - 3\lambda_1\lambda_2$, $\alpha_1 = \alpha_2 = 0$, $\phi = c$;
(3) $\lambda_2^2 \neq 3\lambda_1^2$, $\lambda_2 \neq 3\lambda_1$, $\frac{6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2}{3\lambda_1 - \lambda_2} \neq 0$, $p_1 \neq 0$, $p_2 = 0$, $\alpha = \frac{18\lambda_1^4 - 6\lambda_1\lambda_2^3 + 24\lambda_1^2\lambda_2^2 - 36\lambda_1^3\lambda_2}{(3\lambda_1 - \lambda_2)^2}$, $\alpha_1 = 0$, $\alpha_2 = \frac{18\lambda_1^4 - 2\lambda_2^4 + 6\lambda_1\lambda_2^3 - 18\lambda_1^3\lambda_2}{(3\lambda_1 - \lambda_2)^2}$, $\phi = ce^{\frac{3\lambda_1^2 - \lambda_2^2}{(3\lambda_1 - \lambda_2)p_1}t}$;
(4) $\lambda_2^2 < 3\lambda_1^2$, $\lambda_2 \neq 3\lambda_1$, $\frac{6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2}{3\lambda_1 - \lambda_2} \neq 0$, $p_1 \neq 0$, $p_2 \neq -\frac{1}{2}p_1$, $\frac{\eta}{\zeta^2} = \frac{3\lambda_1^2 - \lambda_2^2}{(3\lambda_1 - \lambda_2)^2}$, $\alpha = \alpha_1 = \alpha_2 = 0$, $\phi = ce^{\frac{3\lambda_1 - \lambda_2}{\zeta}t}$;
(5) $\lambda_2 \neq 3\lambda_1$, $\frac{6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2}{3\lambda_1 - \lambda_2} < 0$, $18\lambda_1^4 - \lambda_2^4 + 6\lambda_1\lambda_2^3 - 15\lambda_1^2\lambda_2^2 \neq 0$, $p_1 \neq 0$, $p_2 = 0$, $\alpha_1 = 0$, $\alpha = \frac{18\lambda_1^4 - 6\lambda_1\lambda_2^3 + 24\lambda_1^2\lambda_2^2 - 36\lambda_1^3\lambda_2}{(3\lambda_1 - \lambda_2)^2}$, $\alpha_2 = \frac{18\lambda_1^4 - \lambda_2^4 + 6\lambda_1\lambda_2^3 - 15\lambda_1^2\lambda_2^2}{(3\lambda_1 - \lambda_2)^2}$, $\phi = ce^{\frac{3\lambda_1\lambda_2 - 3\lambda_1^2}{(3\lambda_1 - \lambda_2)p_1}t}$;

(6) $\lambda_1 = \lambda_2$, $p_1 = 0$, $p_2 \neq 0$, $\alpha = \alpha_1 = \alpha_2 = 0$, $\phi = c_0 e^{\frac{\lambda_1}{p_2}t}$;
(7) $\lambda_1^2 = \frac{5\pm\sqrt{3}}{6}\lambda_2^2$, $p_1 = (1 \pm \sqrt{3})p_2 \neq 0$, $\alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2$, $\alpha_1 = 0$, $\alpha_2 = \frac{(p_2-p_1)(\lambda_2^2-3\lambda_1\lambda_2)}{p_1}$, $\phi = ce^{\frac{\lambda_2}{2p_1}t}$;
(8) $6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2 = 0$, $p_1 \neq 0$, $\alpha = 6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2 = 0$, $\alpha_1 = \alpha_2 = 0$, $\phi = ce^{\frac{\lambda_2}{2}\frac{\zeta}{\eta}t}$;
(9) $6\lambda_1^2 - 3\lambda_1\lambda_2 - \lambda_2^2 = 0$, $p_1 \neq 0$, $p_2 = 0$, $\frac{\zeta^2}{\eta} = 1$, $\alpha = \frac{\lambda_2^2}{4} + 3\lambda_1^2 - 3\lambda_1\lambda_2$, $\alpha_2 = 3\lambda_1^2 - \frac{5}{4}\lambda_2^2$, $\phi = ce^{\frac{\lambda_2}{2}\frac{\zeta}{\eta}t}$.

Remark 13. By **[A]** we get Theorem 4.19(1), by **[B]**, **[F]**, **[I]**, **[K]** we get Theorem 4.19(2), by **[C]**, **[G]** we get Theorem 4.19(3), by **[E]**, **[L]** we get Theorem 4.19(4), by **[H]** we get Theorem 4.19(5), by **[D]**, **[J]** we get Theorem 4.19(6), by **[M]** we get Theorem 4.19(7), by **[N]** we get Theorem 4.19(8), by **[O]** we get Theorem 4.19(9).

4.3.4. Type (II) generalized Kasner space–times with a quarter-symmetric connection with constant scalar curvature

By Proposition 4.17, then (F_2, ∇^{F_2}) has constant scalar curvature S^{F_2} and

$$\overline{S} = \frac{S^{F_2}}{\phi^{2p_2}} - 2\zeta\frac{\phi''}{\phi} - (\eta + \zeta^2 - 2\zeta)\frac{(\phi')^2}{\phi^2} + 3(\lambda_1 + \lambda_2)\zeta\frac{\phi'}{\phi} + 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2). \tag{63}$$

(*a*) $\zeta = 0$. (1) $\eta = 0$. Then $p_1 = p_2 = 0$ and $\overline{S} = S^{F_2} + 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)$.
(2) $\eta \neq 0$. Then $\overline{S} = \frac{S^{F_2}}{\phi^{2p_2}} - \eta\frac{(\phi')^2}{\phi^2} + 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)$, which means

$$\eta\frac{(\phi')^2}{\phi^2} = \frac{S^{F_2}}{\phi^{2p_2}} - [\overline{S} - 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)]. \tag{64}$$

(*b*) $\zeta \neq 0$. Putting $\phi = \psi^{\frac{2\zeta}{\eta+\zeta^2}}$, we get

$$-\frac{4\zeta^2}{\eta + \zeta^2}\psi'' + \frac{6(\lambda_1 + \lambda_2)\zeta^2}{\eta + \zeta^2}\psi' + (3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2 - \overline{S})\psi + S^{F_2}\psi^{1-\frac{4p_2\zeta}{\eta+\zeta^2}} = 0. \tag{65}$$

4.3.5. Classification of Einstein type (III) generalized Kasner space–times with a quarter-symmetric connection

Considering $\dim F_1 = \dim F_2 = \dim F_3 = 1$, by Remark 4, we get $\alpha_i = 0$ and by Proposition 4.16, we have

$$\zeta\Big(\lambda_2\frac{\phi'}{\phi} - \frac{\phi''}{\phi}\Big) - (\eta - \zeta)\frac{(\phi')^2}{\phi^2} + 3(\lambda_1^2 - \lambda_1\lambda_2) = \alpha; \tag{66a}$$

$$-p_1\Big[\frac{\phi''}{\phi} + (\zeta - 1)\frac{(\phi')^2}{\phi^2} + (\lambda_2 - 3\lambda_1)\frac{\phi'}{\phi}\Big] + \lambda_2\zeta\frac{\phi'}{\phi} = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2; \tag{66b}$$

$$-p_2\Big[\frac{\phi''}{\phi} + (\zeta - 1)\frac{(\phi')^2}{\phi^2} + (\lambda_2 - 3\lambda_1)\frac{\phi'}{\phi}\Big] + \lambda_2\zeta\frac{\phi'}{\phi} = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2; \tag{66c}$$

$$-p_3\Big[\frac{\phi''}{\phi} + (\zeta - 1)\frac{(\phi')^2}{\phi^2} + (\lambda_2 - 3\lambda_1)\frac{\phi'}{\phi}\Big] + \lambda_2\zeta\frac{\phi'}{\phi} = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2. \tag{66d}$$

(*a*) $\zeta = \eta = 0$. By equation (66a), we have $\alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2$, and by equation (66b), we have $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2$, then we get $\lambda_2^2 = 3\lambda_1^2$.

So we obtain

$$\underline{\lambda_2^2 = 3\lambda_1^2,\ \alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2 = \lambda_2^2 - 3\lambda_1\lambda_2,\ \alpha_i = 0,\ \zeta = \eta = 0.}$$

(b) $\zeta = 0$, $\eta \neq 0$. By (66b) + (66c) + (66d), we get $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2$; by equation (66a), we have $\frac{(\phi')^2}{\phi^2} = \frac{3\lambda_1^2 - \lambda_2^2}{\eta}$.

1) $3\lambda_1^2 - \lambda_2^2 < 0$, we have no solution.

2) $3\lambda_1^2 - \lambda_2^2 = 0$, then $\phi = c$, which satisfies equation (66a).

3) $3\lambda_1^2 - \lambda_2^2 > 0$, then $\phi = c_0 e^{\pm\sqrt{\frac{3\lambda_1^2 - \lambda_2^2}{\eta}}t}$, since $\eta \neq 0$, so at least one $p_i \neq 0$, we assume $p_1 \neq 0$, by equation (66b), we get $\lambda_2 = 3\lambda_1$, but by $3\lambda_1^2 - \lambda_2^2 > 0$, we get $\lambda_1^2 < 0$, which is a contradiction.

So we have

$$\underline{\lambda_2^2 = 3\lambda_1^2,\ \alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2 = \lambda_2^2 - 3\lambda_1\lambda_2,\ \alpha_i = 0,\ \zeta = 0,\ \eta \neq 0,\ \phi = c}$$

in case (b).

(c) $\zeta \neq 0$, then $\eta \neq 0$. If $p_1 = p_2 = p_3$, we get type (I), if $p_1 = p_2$ or $p_2 = p_3$ or $p_1 = p_3$, we get type (II), so $p_1 \neq p_2 \neq p_3$. Let $\phi = \psi^{\frac{\zeta}{\eta}}$, then equations (66a)–(66d) become

$$\frac{\zeta^2}{\eta}\frac{\lambda_2\psi' - \psi''}{\psi} = \alpha + 3\lambda_1\lambda_2 - 3\lambda_2^2 \tag{67a}$$

$$\frac{p_1}{\zeta}\Big[-\frac{(\phi^\zeta)''}{\phi^\zeta} + (3\lambda_1 - \lambda_2)\frac{(\phi^\zeta)'}{\phi^\zeta}\Big] + \lambda_2\frac{(\phi^\zeta)'}{\phi^\zeta} = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2 \tag{67b}$$

$$\frac{p_2}{\zeta}\Big[-\frac{(\phi^\zeta)''}{\phi^\zeta} + (3\lambda_1 - \lambda_2)\frac{(\phi^\zeta)'}{\phi^\zeta}\Big] + \lambda_2\frac{(\phi^\zeta)'}{\phi^\zeta} = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2 \tag{67c}$$

$$\frac{p_3}{\zeta}\Big[-\frac{(\phi^\zeta)''}{\phi^\zeta} + (3\lambda_1 - \lambda_2)\frac{(\phi^\zeta)'}{\phi^\zeta}\Big] + \lambda_2\frac{(\phi^\zeta)'}{\phi^\zeta} = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2 \tag{67d}$$

By (67b) $\times p_2 -$ (67c) $\times p_1$ and considering that $p_1 \neq p_2$, we get

$$\lambda_2\frac{(\phi^\zeta)'}{\phi^\zeta} = \alpha - \lambda_2^2 + 3\lambda_1\lambda_2 \tag{68a}$$

$$\frac{(\phi^\zeta)'}{\phi^\zeta} = \frac{\alpha}{\lambda_2} - \lambda_2 + 3\lambda_1 \tag{68b}$$

By equations (67b) and (68a), we get $-\frac{(\phi^\zeta)''}{\phi^\zeta} + (3\lambda_1 - \lambda_2)\frac{(\phi^\zeta)'}{\phi^\zeta} = 0$, then by equation (68b), we have

$$\frac{(\phi^\zeta)''}{\phi^\zeta} = (\frac{3\lambda_1}{\lambda_2} - 1)\alpha + 9\lambda_1^2 - 6\lambda_1\lambda_2 + \lambda_2^2 \tag{69}$$

On the other hand, using equation (68b), we get

$$\phi^\zeta = c_0 e^{(\frac{\alpha}{\lambda_2} - \lambda_2 + 3\lambda_1)t} \tag{70}$$

By equations (69) and (70), we obtain $\alpha^2 + (3\lambda_1\lambda_2 - \lambda_2^2)\alpha = 0$, so when $\lambda_2 = 3\lambda_1$, we have $\alpha = 0$; when $\lambda_2 \neq 3\lambda_1$, we have $\alpha = 0$ or $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 \neq 0$.

1) $\lambda_2 = 3\lambda_1$. Then $\alpha = 0$, by equation (70), we get that ϕ^ζ is a constant, then ψ is a constant, so by equation (67a), we have $\lambda_1^2 = 0$, which is a contradiction to $\lambda_1 \neq 0$.

2) $\lambda_2 \neq 3\lambda_1$. 1′. $\alpha = \lambda_2^2 - 3\lambda_1\lambda_2 \neq 0$. By equation (70), we get that ϕ^ζ is a constant, then ψ is a constant, and $\phi = c$ is a constant. By equation (67a), we have $\alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2$ and $\lambda_2^2 = 3\lambda_1^2$.

So we get

$$\underline{\lambda_2^2 = 3\lambda_1^2,\ \alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2 = \lambda_2^2 - 3\lambda_1\lambda_2,\ \alpha_i = 0,\ \zeta \neq 0,\ \eta \neq 0,\ \phi = c.}$$

$2'$. $\alpha = 0$. By equation (67a), we have

$$\psi'' - \lambda_2\psi' + (3\lambda_1\lambda_2 - 3\lambda_1^2)\frac{\eta}{\zeta^2}\psi = 0. \tag{71}$$

By equation (70), we get $\phi^\zeta = c_0 e^{(3\lambda_1-\lambda_2)t}$, then $\phi = ce^{\frac{3\lambda_1-\lambda_2}{\zeta}t}$, $\psi = c_1 e^{(3\lambda_1-\lambda_2)\frac{\eta}{\zeta^2}t}$, by equation (71), we get $\frac{\eta}{\zeta^2} = \frac{3\lambda_1^2-\lambda_2^2}{(3\lambda_1-\lambda_2)^2}$.

So we have

$$\underline{\lambda_2 \neq 3\lambda_1,\ \alpha = 0,\ \alpha_i = 0,\ \frac{\eta}{\zeta^2} = \frac{3\lambda_1^2-\lambda_2^2}{(3\lambda_1-\lambda_2)^2},\ \phi = ce^{\frac{3\lambda_1-\lambda_2}{\zeta}t}.}$$

According to the above discussions, we get the following theorem:

Theorem 4.20. *Let $M = I \times_{\phi^{p_1}} F_1 \times_{\phi^{p_2}} F_2 \cdots \times_{\phi^{p_m}} F_m$ be a generalized Kasner space–time for $p_i \neq p_j$ for $i, j \in \{1, 2, 3\}$ and $\dim F_1 = \dim F_2 = \dim F_3 = 1$, and $P = \frac{\partial}{\partial t}$. Then $(M, \overline{\nabla})$ is Einstein with the Einstein constant α if and only if one of the following conditions is satisfied:*

(1) $\lambda_2^2 = 3\lambda_1^2$, $\alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2 = \lambda_2^2 - 3\lambda_1\lambda_2$, $\alpha_i = 0$, $\zeta = \eta = 0$;
(2) $\lambda_2^2 = 3\lambda_1^2$, $\alpha = 3\lambda_1^2 - 3\lambda_1\lambda_2 = \lambda_2^2 - 3\lambda_1\lambda_2$, $\alpha_i = 0$, $\eta \neq 0$, $\phi = c$;
(3) $\lambda_2 \neq 3\lambda_1$, $\alpha = 0$, $\alpha_i = 0$, $\frac{\eta}{\zeta^2} = \frac{3\lambda_1^2-\lambda_2^2}{(3\lambda_1-\lambda_2)^2}$, $\phi = ce^{\frac{3\lambda_1-\lambda_2}{\zeta}t}$.

4.3.6. Type (III) generalized Kasner space–times with a quarter-symmetric connection with constant scalar curvature

By Proposition 4.17, we get

$$\overline{S} = -2\zeta\frac{\phi''}{\phi} - (\eta + \zeta^2 - 2\zeta)\frac{(\phi')^2}{\phi^2} + 3(\lambda_1 + \lambda_2)\zeta\frac{\phi'}{\phi} + 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2). \tag{72}$$

(a) $\zeta = \eta = 0$. Then $p_1 = p_2 = p_3 = 0$, and $\overline{S} = 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)$.
(b) $\zeta = 0$, $\eta \neq 0$. Then $[(ln\phi)']^2 = \frac{3(\lambda_1^2+\lambda_2^2-4\lambda_1\lambda_2)-\overline{S}}{\eta}$, so we have:

1) $\overline{S} > 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)$, we have no solution.
2) $\overline{S} = 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)$, then $\phi = c$.
3) $\overline{S} < 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)$, then $\phi = c_0 e^{\pm\sqrt{\frac{3(\lambda_1^2+\lambda_2^2-4\lambda_1\lambda_2)-\overline{S}}{\eta}}t}$.

(c) $\zeta \neq 0$, then $\eta \neq 0$. Putting $\phi = \psi^{\frac{2\zeta}{\eta+\zeta^2}}$, we have

$$-\frac{4\zeta^2}{\eta+\zeta^2}\psi'' + \frac{6(\lambda_1+\lambda_2)\zeta^2}{\eta+\zeta^2}\psi' + (3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2 - \overline{S})\psi = 0. \tag{73}$$

So we get

1) $\overline{S} < \frac{9\zeta^2(\lambda_1+\lambda_2)^2}{4(\eta+\zeta^2)} + 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$,

$$\psi = c_1 e^{\frac{\frac{3}{2}(\lambda_1+\lambda_2)+\sqrt{\frac{9(\lambda_1+\lambda_2)^2}{4} - \frac{(\overline{S}-3\lambda_1^2-3\lambda_2^2+12\lambda_1\lambda_2)(\eta+\zeta^2)}{\zeta^2}}}{2}t} + c_2 e^{\frac{\frac{3}{2}(\lambda_1+\lambda_2)-\sqrt{\frac{9(\lambda_1+\lambda_2)^2}{4} - \frac{(\overline{S}-3\lambda_1^2-3\lambda_2^2+12\lambda_1\lambda_2)(\eta+\zeta^2)}{\zeta^2}}}{2}t};$$

2) $\overline{S} = \frac{9\zeta^2(\lambda_1+\lambda_2)^2}{4(\eta+\zeta^2)} + 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$, $\psi = c_1 e^{\frac{3(\lambda_1+\lambda_2)}{4}t} + c_2 t e^{\frac{3(\lambda_1+\lambda_2)}{4}t}$;

3) $\overline{S} > \frac{9\zeta^2(\lambda_1+\lambda_2)^2}{4(\eta+\zeta^2)} + 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$,

$$\psi = c_1 e^{\frac{3}{4}(\lambda_1+\lambda_2)t} \cos \frac{\sqrt{\frac{(\overline{S}-3\lambda_1^2-3\lambda_2^2+12\lambda_1\lambda_2)(\eta+\zeta^2)}{\zeta^2} - \frac{9(\lambda_1+\lambda_2)^2}{4}}}{2} t + c_2 e^{\frac{3}{4}(\lambda_1+\lambda_2)t} \sin \frac{\sqrt{\frac{(\overline{S}-3\lambda_1^2-3\lambda_2^2+12\lambda_1\lambda_2)(\eta+\zeta^2)}{\zeta^2} - \frac{9(\lambda_1+\lambda_2)^2}{4}}}{2} t$$

According to the above discussions, we get the following theorem:

Theorem 4.21. *Let $M = I \times_{\phi^{p_1}} F_1 \times_{\phi^{p_2}} F_2 \cdots \times_{\phi^{p_m}} F_m$ be a generalized Kasner space–time and $\dim F_1 = \dim F_2 = \dim F_3 = 1$, and $P = \frac{\partial}{\partial t}$. Then $(M, \overline{\nabla})$ has constant scalar curvature $\overline{S}$ if and only if one of the following conditions is satisfied:*

(1) $\zeta = \eta = 0$, $\overline{S} = 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)$.

(2) $\zeta = 0$, $\eta \neq 0$, *when* $\overline{S} > 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)$, *we have no solution; when* $\overline{S} = 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)$, *then* $\phi = c$; *when* $\overline{S} < 3(\lambda_1^2 + \lambda_2^2 - 4\lambda_1\lambda_2)$, *then* $\phi = c_0 e^{\pm\sqrt{\frac{3(\lambda_1^2+\lambda_2^2-4\lambda_1\lambda_2)-\overline{S}}{\eta}}t}$.

(3) $\zeta \neq 0$,

(a) $\overline{S} < \frac{9\zeta^2(\lambda_1+\lambda_2)^2}{4(\eta+\zeta^2)} + 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$,

$$\phi = \left[c_1 e^{\frac{\frac{3}{2}(\lambda_1+\lambda_2)+\sqrt{\frac{9(\lambda_1+\lambda_2)^2}{4} - \frac{(\overline{S}-3\lambda_1^2-3\lambda_2^2+12\lambda_1\lambda_2)(\eta+\zeta^2)}{\zeta^2}}}{2}t} + c_2 e^{\frac{\frac{3}{2}(\lambda_1+\lambda_2)-\sqrt{\frac{9(\lambda_1+\lambda_2)^2}{4} - \frac{(\overline{S}-3\lambda_1^2-3\lambda_2^2+12\lambda_1\lambda_2)(\eta+\zeta^2)}{\zeta^2}}}{2}t}\right]^{\frac{2\zeta}{\eta+\zeta^2}};$$

(b) $\overline{S} = \frac{9\zeta^2(\lambda_1+\lambda_2)^2}{4(\eta+\zeta^2)} + 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$, $\phi = \left[c_1 e^{\frac{3(\lambda_1+\lambda_2)}{4}t} + c_2 t e^{\frac{3(\lambda_1+\lambda_2)}{4}t}\right]^{\frac{2\zeta}{\eta+\zeta^2}}$;

(c) $\overline{S} > \frac{9\zeta^2(\lambda_1+\lambda_2)^2}{4(\eta+\zeta^2)} + 3\lambda_1^2 + 3\lambda_2^2 - 12\lambda_1\lambda_2$,

$$\phi = \left[c_1 e^{\frac{3}{4}(\lambda_1+\lambda_2)t} \cos \frac{\sqrt{\frac{(\overline{S}-3\lambda_1^2-3\lambda_2^2+12\lambda_1\lambda_2)(\eta+\zeta^2)}{\zeta^2} - \frac{9(\lambda_1+\lambda_2)^2}{4}}}{2} t + c_2 e^{\frac{3}{4}(\lambda_1+\lambda_2)t} \sin \frac{\sqrt{\frac{(\overline{S}-3\lambda_1^2-3\lambda_2^2+12\lambda_1\lambda_2)(\eta+\zeta^2)}{\zeta^2} - \frac{9(\lambda_1+\lambda_2)^2}{4}}}{2} t\right]^{\frac{2\zeta}{\eta+\zeta^2}}.$$

Remark 14. When $\lambda_1 = \lambda_2 = 1$, we get Propositions 32, 33 and Theorems 35, 37, 36 in [14] by Propositions 4.16, 4.17 and Theorems 4.19–4.21, respectively.

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