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On the conditions for entire functions related to the partial theta function to belong to the Laguerre–Pólya class

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A B S T R A C T

In this paper, we study entire functions of order zero with positive Taylor coefficients and investigate sufficient or necessary conditions for such functions to have only real zeros. We answer the following question: for which values of $a > 1$ and $m \geq 1$ an entire function $\varphi_{a,m}(z) = \sum_{k=0}^{\infty} \frac{z^k}{a^{k^2}}(k!)^m$ and its Taylor sections belong to the Laguerre–Pólya class?

1. Introduction

There are many interesting papers concerning the zero distribution of entire functions, its sections and tails, see, for example, the remarkable survey of the topic in [13]. In this paper, we consider entire functions with positive Taylor coefficients and investigate the question whether or not they (and their Taylor sections) belong to the famous Laguerre–Pólya class.

Definition 1. A real entire function f is said to be in the *Laguerre–Pólya class*, written $f \in \mathcal{L} - \mathcal{P}$, if it can be expressed in the form

$$f(x) = cx^n e^{-\alpha x^2 + \beta x} \prod_{k=1}^{\infty} \left(1 - \frac{x}{x_k}\right) e^{\frac{x}{x_k}}, \tag{1}$$

where $c, \beta, x_k \in \mathbb{R}$, $x_k \neq 0$, $\alpha \geq 0$, n is a non-negative integer and $\sum_{k=1}^{\infty} \frac{1}{x_k^2} < \infty$. As usual the product on the right-hand side can be finite or empty (in the latter case the product equals 1).

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This class is essential in the theory of entire functions due to the fact that these and only these functions are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros. The following prominent theorem states even stronger fact.

Theorem A *(E. Laguerre and G. Pólya). (See, for example, [4, pp. 42–46].)*

(i) Let $(P_n)_{n=1}^{\infty}$ be a sequence of complex polynomials having only real zeros which converges uniformly in the circle $|z| \le A$, $A > 0$. Then this sequence converges uniformly on compact sets in $\mathbb{C}$ to an entire function f, and f is from $\mathcal{L}-\mathcal{P}$ class.

(ii) And conversely, for any $f \in \mathcal{L}-\mathcal{P}$ there is a sequence of complex polynomials with only real zeros which converges uniformly on compact sets of $\mathbb{C}$ to f.

Note that although every entire function from the Laguerre–Pólya class is the uniform limit of the polynomials with only real zeros, Taylor sections of this function may have non-real zeros. For example, $f(z) = e^z \in \mathcal{L}-\mathcal{P}$ and the sequence of polynomials $P_n(z) = \left(1+\frac{z}{n}\right)^n$ having only real zeros converges uniformly, on compact subsets of $\mathbb{C}$, to $f(z)$. But for every $n \in \mathbb{N}$ the n-th Taylor section of f has not more than one real zero counting multiplicity (see, for example, [18, Chapter 5, Problem 74]).

For various properties and characterizations of the Laguerre–Pólya class see [16, p. 100], [17] or [12, Kapitel II].

Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be an entire function with positive coefficients. We use two notations:

$$p_n = p_n(f) := \frac{a_{n-1}}{a_n},\ n \ge 1; \quad q_n = q_n(f) := \frac{p_n}{p_{n-1}} = \frac{a_{n-1}^2}{a_{n-2}a_n},\ n \ge 2. \tag{2}$$

Note that

$$a_n = \frac{a_0}{p_1 p_2 \dots p_n},\ n \ge 1; \quad a_n = \frac{a_1}{q_2^{n-1} q_3^{n-2} \dots q_{n-1}^2 q_n}\left(\frac{a_1}{a_0}\right)^{n-1},\ n \ge 2. \tag{3}$$

In this paper, we study entire functions with positive Taylor coefficients such that $q_n(f)$ are increasing in n and $q_2(f) > 1$. By (3) every entire function having these properties is of order zero. Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ be an entire function with positive coefficients and order less than 2. It is well known and often used that such function has only real zeros if and only if the sequence $(k!a_k)_{k=0}^{\infty}$ is the multiplier sequence.

Definition 2. A sequence $(\gamma_k)_{k=0}^{\infty}$ of real numbers is called a multiplier sequence if, whenever the real polynomial $P(x) = \sum_{k=0}^{n} a_k z^k$ has only real zeros, the polynomial $\sum_{k=0}^{n} \gamma_k a_k z^k$ has only real zeros. The class of multiplier sequences is denoted $\mathcal{MS}$.

The following famous theorem by G. Pólya and I. Schur provided both algebraic and transcendental characterizations of multiplier sequences.

Theorem B. *(See [17,16] or [11, Chapter VIII, Sec. 3].) Let $(\gamma_k)_{k=0}^{\infty}$ be a given real sequence. The following three statements are equivalent.*

1. *$(\gamma_k)_{k=0}^{\infty}$ is a multiplier sequence.*
2. *For every $n \in \mathbb{N}$ the polynomial $P_n(z) = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^k$ has only real zeros of the same sign.*
3. *The power series $\Phi(z) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} z^k$ converges absolutely in the whole complex plane and the entire function $\Phi(z)$ or the entire function $\Phi(-z)$ admits the representation*

$$Ce^{\sigma z} z^m \prod_{k=1}^{\infty}(1+\frac{z}{x_k}), \tag{4}$$

where $C \in \mathbb{R}$, $\sigma \geq 0$, $m \in \mathbb{N}\cup\{0\}$, $0 < x_k \leq \infty$, $\sum_{k=1}^{\infty} \frac{1}{x_k} < \infty$.

The simple consequence of Theorem B is the fact that the sequence $(\gamma_0, \gamma_1, \dots, \gamma_l, 0, 0, \dots)$ is a multiplier sequence if and only if the polynomial $P(z) = \sum_{k=0}^{l} \frac{\gamma_k}{k!} z^k$ has only real zeros of the same sign.

We need also the notion of a complex zero decreasing sequence. For a real polynomial P we will denote by $Z_c(P)$ the number of non-real zeros of P counting multiplicities.

Definition 3. A sequence $(\gamma_k)_{k=0}^{\infty}$ of real numbers is said to be a complex zero decreasing sequence if

$$Z_c(\sum_{k=0}^{n} \gamma_k a_k z^k) \leq Z_c(\sum_{k=0}^{n} a_k z^k), \tag{5}$$

for any real polynomial $\sum_{k=0}^{n} a_k z^k$. The class of complex zero decreasing sequences we will denote by $\mathcal{CZDS}$.

Obviously, $\mathcal{CZDS} \subset \mathcal{MS}$. The existence of nontrivial $\mathcal{CZDS}$ sequences is a consequence of the following remarkable theorem proved by Laguerre and extended by Pólya.

Theorem C. *(See [15] or [16, pp. 314–321].) Suppose f is an entire function from the Laguerre–Pólya class having only negative zeros. Then the sequence $(f(k))_{k=0}^{\infty}$ is a complex zero decreasing sequence.*

As it follows from the above theorem,

$$\left(a^{-k^2}\right)_{k=0}^{\infty} \in \mathcal{CZDS},\ a \geq 1, \quad \left(\frac{1}{k!}\right)_{k=0}^{\infty} \in \mathcal{CZDS}. \tag{6}$$

Note that the problem of finding whether or not a given polynomial has only real zeros is rather difficult and subtle. In 1926, Hutchinson [5, p. 327] extended the work of Petrovitch [14] and Hardy ([2] or [3, pp. 95–100]) and found the following sufficient condition for a polynomial (entire function) with positive coefficients to have only real zeros.

Theorem D. *(See [5, p. 327].) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with positive coefficients. Inequalities $q_n(f) \geq 4$, $\forall n \geq 2$, are valid if and only if the following two properties hold:*

(i) The zeros of f are all real, simple and negative and
(ii) the zeros of any polynomial $\sum_{k=m}^{n} a_k z^k$, $m < n$, formed by taking any number of consecutive terms of f, are all real and non-positive.

For some extensions of Hutchinson's results see, for example, [1, §4]. In Theorem 1.1(2) we will present a slight improvement of Hutchinson's theorem.

In [6] the following entire function $g_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{a^{j^2}}$, $a > 1$, so called "partial theta-function", was investigated. This paper gives the exhaustive answer to the question: for which $a > 1$ the entire function g_a belongs to the Laguerre–Pólya class?

Theorem E. *(See [6].) There exists a constant q_∞ ($q_\infty \approx 3.23363666$) such that:*

(1) *$S_{2k+1}(z, g_a) := \sum_{j=0}^{2k+1} \frac{z^j}{a^{j^2}} \in \mathcal{L}-\mathcal{P}$ for every $k \in \mathbb{N} \Leftrightarrow a^2 \geq q_\infty$;*

(2) $\exists N_0 \in \mathbb{N}\ \forall k \geq N_0\ S_{2k}(z, g_a) := \sum_{j=0}^{2k} \frac{z^j}{a^{j^2}} \in \mathcal{L} - \mathcal{P} \Leftrightarrow a^2 > q_\infty$;
(3) $g_a(z) \in \mathcal{L} - \mathcal{P} \Leftrightarrow a^2 \geq q_\infty$.

A wonderful paper [10] among the other results explains the role which the constant q_∞ plays in the study of the set of entire functions with positive coefficients having all Taylor sections with only real zeros. About interesting properties of zeros of the partial theta-function see [8] and [9].

Let $f(z) = \sum_{k=0}^\infty a_k z^k = 1 + z + \sum\limits_{k=2}^{+\infty} \frac{z^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$ be a given entire function with positive Taylor coefficients and of order less than 1. We will denote by $S_n(z)$ the n-th Taylor section of f: $S_n(z) = 1 + z + \sum\limits_{k=2}^{n} \frac{z^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$. We will investigate necessary or sufficient conditions for the function f to belong to the Laguerre–Pólya class.

We note a simple necessary condition. Let $f(z) = \sum_{k=0}^\infty a_k z^k \in \mathcal{L} - \mathcal{P}$ be a function of order less than 1, $f(0) \neq 0$, and $(x_k)_{k=1}^\infty \subset \mathbb{R}$ be a sequence of all its zeros. We have

$$\sum_{k=1}^\infty \frac{1}{x_k^2} = a_1^2 - 2a_0 a_2 > 0.$$

Thus for such functions

$$f \in \mathcal{L} - \mathcal{P} \Rightarrow q_2(f) > 2. \tag{7}$$

Our first result is the following theorem.

Theorem 1.1. *Let* $f(z) = \sum_{k=0}^\infty a_k z^k = 1 + z + \sum\limits_{k=2}^{+\infty} \frac{z^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k}$ *be an entire function with positive Taylor coefficients.*

(1) If $q_2 \geq 3$, $q_3 > \frac{q_2^3}{q_2^2 - 1}$ and $q_j \geq q_2$ for $j \geq 4$, then the function f and its sections S_n for all $n \geq 2$ have exactly two roots in the disk $\{z : |z| \leq q_2\}$.
(2) If $q_2 > 1$ and $q_{j+1} \geq 4$ for all $j = 2, 3, \ldots, n-1$, then S_n has at least $(n-2)$ real roots on $[q_2; +\infty)$. Furthermore, there exists a point $y \in [q_2; q_3]$ such that $S_n(-y) > 0$.
(3) If $f \in \mathcal{L} - \mathcal{P}$, $q_2 \geq 3$, $q_3 > \frac{q_2^3}{q_2^2 - 1}$ and $q_j \geq 4$ for all $j = 3, 4, \ldots$, then for every $n \in \mathbb{N}$ we have $S_{2n+1} \in \mathcal{L} - \mathcal{P}$.
(4) If $2 \leq q_2 < 3$ and $q_k \geq \frac{4q_2}{3}$ for all $k \geq 3$, then $f \notin \mathcal{L} - \mathcal{P}$.

Later on, we will investigate a family of entire functions

$$f^{(m,a)}(z) = \sum_{k=0}^{+\infty} \frac{z^k (k!)^m}{a^{k^2}}, \quad a > 1,\ m \geq 1,$$

and their Taylor sections

$$S_n^{(m,a)}(z) = \sum_{k=0}^{n} \frac{z^k (k!)^m}{a^{k^2}}.$$

By (6) we obtain $\left(a^{-k^2}\right)_{k=0}^\infty \in \mathcal{CZDS} \subset \mathcal{MS}$ for every $a \geq 1$. Thus we conclude that if for some $a_0 > 1$ we have $f^{(m,a_0)} \in \mathcal{L} - \mathcal{P}$ then for all $a \geq a_0$ we have $f^{(m,a)} \in \mathcal{L} - \mathcal{P}$. Analogously if for some $a_0 > 1$ we have $S_n^{(m,a_0)} \in \mathcal{L}-\mathcal{P}$ then for all $a \geq a_0$ we have $S_n^{(m,a)} \in \mathcal{L}-\mathcal{P}$. Thus we obtain that for every $n \in \mathbb{N}$, $n \geq 2$, there exists a constant $d_{(n,m)} \geq 1$ such that

$$S_n^{(m,a)} \in \mathcal{L}-\mathcal{P} \Leftrightarrow a^2 \geq d_{(n,m)}. \tag{8}$$

Analogously we obtain that there exists a constant $d_{(\infty,m)} > 1$ such that

$$f^{(m,a)} \in \mathcal{L}-\mathcal{P} \Leftrightarrow a^2 \geq d_{(\infty,m)}. \tag{9}$$

The main result of this paper is the following theorem.

Theorem 1.2. *In the notations introduced above we have:*

(1) For every fixed $m \geq 1$ the function $f^{(m,a)}$ belongs to the class $\mathcal{L}-\mathcal{P}$ if and only if there exists $x_0 = x_0(m) \in [-q_2(f^{(m,a)}); -1]$ such that $f^{(m,a)}(x_0) \leq 0$. For every fixed $m \geq 1$ and $n \in \mathbb{N}, n \geq 2$, the polynomial $S_n^{(m,a)}$ has only real zeros if and only if there exists $x_0 = x_0(m,n) \in [-q_2(f^{(m,a)}); -1]$ such that $S_n^{(m,a)}(x_0) \leq 0$.
(2) $3 \cdot 2^m < d_{(3,m)} < d_{(5,m)} < d_{(7,m)} < \ldots < d_{(\infty,m)}$.
(3) $\lim_{n\to\infty} d_{(2n+1,m)} = d_{(\infty,m)}$.
(4) $4 \cdot 2^m = d_{(2,m)} > d_{(4,m)} > d_{(6,m)} > \ldots > d_{(\infty,m)}$.
(5) $\lim_{n\to\infty} d_{(2n,m)} = d_{(\infty,m)}$.
(6) For every $n \in \mathbb{N}$, $n \geq 2$, the function $d_{(n,m)}$ is the continuous increasing function as a function of m. The function $d_{(\infty,m)}$ is also the continuous increasing function of m.

2. Proof of Theorem 1.1

By a small abuse of notation we will investigate the following function $f(z) = \sum_{k=0}^{\infty}(-1)^k a_k z^k = 1 - z + \sum_{k=2}^{+\infty}(-1)^k \frac{z^k}{q_2^{k-1}q_3^{k-2}\cdots q_{k-1}^2 q_k}$. The following lemma will provide some information about the behavior of minimal values of the second section of f.

Lemma 2.1. $\min_{\varphi\in[0;2\pi]} |S_2(q_2e^{i\varphi})| = \begin{cases} 1, & \text{if } q_2 \geq 3, \\ \sqrt{1-\frac{(3-q_2)^2}{4}q_2}, & \text{if } 2 \leq q_2 \leq 3. \end{cases}$

Proof. For $S_2(z) = 1 - z + \frac{z^2}{q_2}$ we get by direct calculation

$$|S_2(q_2e^{i\varphi})|^2 = 1 + 4q_2^2 \sin^2\frac{\varphi}{2} - 4q_2 \sin\frac{\varphi}{2}\sin\frac{3\varphi}{2}. \tag{10}$$

Substituting $v = \sin^2\frac{\phi}{2} \in [0;1]$ we have $\sin\frac{\phi}{2}\sin\frac{3\phi}{2} = v(3-4v)$ and so we get

$$|S_2(q_2e^{i\varphi})|^2 = 16q_2v^2 + 4q_2v(q_2-3) + 1 =: h(v).$$

Note that $h(v)$ represents a quadratic parabola opening upward with the vertex $v_0 = \frac{3-q_2}{8}$. If $2 \leq q_2 \leq 3$ then $\min_{v\in[0;1]} h(v) = h(v_0) = 1 - \frac{(3-q_2)^2q_2}{4}$. If $q_2 > 3$ then $v_0 < 0$ and $\min_{v\in[0;1]} h(v) = h(0) = 1$. □

The next lemma gives the estimate of the 3-rd remainder of our series.

Lemma 2.2. *Let $q_2 > 1$ and $q_j \geq q_2$ for $j \geq 3$. Then*

$$|R_3(q_2 e^{i\varphi})| \leq \frac{q_2^3}{q_3(q_2^2 - 1)}.$$

Proof.

$$\begin{aligned}|R_3(q_2 e^{i\varphi})| &\leq \sum_{k=3}^{+\infty} \frac{q_2^k}{q_2^{k-1} q_3^{k-2} \cdots q_{k-1}^2 q_k} \\ &= \frac{q_2}{q_3}\left(1 + \frac{1}{q_3 q_4} + \frac{1}{q_3^2 q_4^2 q_5} + \cdots\right) \leq \frac{q_2}{q_3}\left(1 + \frac{1}{q_2^2} + \frac{1}{q_2^5} + \frac{1}{q_2^9} \cdots\right) \\ &\leq \frac{q_2}{q_3}\left(1 + \frac{1}{q_2^2} + \frac{1}{q_2^4} + \frac{1}{q_2^6} \cdots\right) = \frac{q_2^3}{q_3(q_2^2 - 1)}. \qquad \square\end{aligned}$$

Now we will prove Theorem 1.1(1).

Lemma 2.3. *If $q_2 \geq 3$, $q_3 > \frac{q_2^3}{q_2^2 - 1}$ and $q_j \geq q_2$ for $j \geq 4$, then the function f and its sections S_n for all $n \geq 2$ have exactly two roots in the disk $\{z : |z| < q_2\}$.*

Proof. For $q_2 \geq 3$ using Lemma 2.1 we get $\min\limits_{\varphi \in [0;2\pi]} |S_2(q_2 e^{i\varphi})| = 1$. Then from Lemma 2.2 we have $|R_3(q_2 e^{i\phi})| \leq \frac{q_2^3}{q_3(q_2^2-1)} < 1$. Next, consider the roots of $S_2(z) = 1 - z + \frac{z^2}{q_2}$. If $D = 1 - \frac{4}{q_2} \geq 0 \Leftrightarrow q_2 \geq 4$, then both roots of S_2 are real and positive, and for the biggest root we have $\frac{q_2\left(1+\sqrt{1-\frac{4}{q_2}}\right)}{2} < q_2$. If $D < 0$, then S_2 has two non-real roots z_0 and $\bar{z}_0$ such that $z_0 \bar{z}_0 = q_2$. It means that $|z_0| = \sqrt{q_2} < q_2$ and $S_2(z)$ has two roots in $|z| < q_2$. Now, applying Rouché's theorem to $S_2(z)$ and $R_3(z)$ (or to $S_2(z)$ and $S_n(z) - S_2(z)$, where the same proof as in Lemma 2.2 shows that $|S_n(z) - S_2(z)| \leq \frac{q_2^3}{q_3(q_2^2-1)}$), we obtain the statement required. $\square$

Now, we will describe the behavior of $f(x)$, $S_{2n+1}(x)$ and $S_{2n}(x)$ on $[0; q_2]$.

Lemma 2.4. *Suppose that $q_j > 1$ for $j \geq 2$.*

1) If $x \in [0; q_2]$ then $S_3(x) < S_5(x) < \ldots < f(x)$.
2) If $x \in [0; 1]$ then $0 < S_1(x) < S_3(x) < S_5(x) < \ldots < f(x)$.
3) If $x \in [0; q_2]$ then $S_2(x) > S_4(x) > \ldots > f(x)$.

Proof. We rewrite $S_{2n+1}(x) = (1-x) + (a_2x^2 - a_3x^3) + (a_4x^4 - a_5x^5) + \ldots + (a_{2n}x^{2n} - a_{2n+1}x^{2n+1})$. We have $a_k x^k > a_{k+1}x^{k+1} \Leftrightarrow x < \frac{a_k}{a_{k+1}} \Leftrightarrow x < \frac{q_2^k q_3^{k-1} \ldots q_k^2 q_{k+1}}{q_2^{k-1} q_3^{k-2} \ldots q_{k-1}^2 q_k} = q_2 q_3 \ldots q_{k+1}$, $k = 2, 4, 6, \ldots$. The last inequality holds because $x \leq q_2$ and $q_j > 1$. This means that $S_{2n+1}(x) > S_{2n-1}(x)$. The same argument is applied to prove that $f(x) > S_{2k+1}(x)$. Moreover, if $x \in [0; 1]$ then one may apply the same logic starting with $S_1(x)$ (not $S_3(x)$).

To prove 3), we rewrite $S_{2n}(x)$ as $(1 - x + a_2x^2) + (-a_3x^3 + a_4x^4) + \ldots + (-a_{2n-1}x^{2n-1} + a_{2n}x^{2n})$. Thus, $S_{2n} < S_{2n-2k}$ follows from $x < q_2 q_3 \ldots q_{2n-2k}$. The last inequality holds because $x \leq q_2$ and $q_j > 1$. The same argument is applied to prove that $f(x) < S_{2k}(x)$ for all $k \in \mathbb{N}$. $\square$

The following lemma gives the necessary and sufficient condition for S_3 to have only real zeros.

Lemma 2.5. *If $f \in \mathcal{L}-\mathcal{P}$, $q_2 \geq 3$, $q_3 > \frac{q_2^3}{q_2^2-1}$, then S_3 has only real zeros if and only if there exists $x_0 \in [1; q_2]$ such that $S_3(x_0) < 0$.*

Proof. Since $q_2 \geq 3$, $q_3 > \frac{q_2^3}{q_2^2-1}$ and $f \in \mathcal{L}-\mathcal{P}$, we derive that $f(x)$ has two real roots on $[0; q_2]$ (by Lemma 2.3). Using statement 2) of Lemma 2.4 $f(x) > 0$ on $[0; 1]$. So $f(x)$ has two real roots on $[1; q_2]$. It means that there exists $x_0 \in [1; q_2]$ such that $f(x_0) \leq 0$. Now, $S_3(x) < f(x)$ on $[1; q_2]$ implies that $S_3(x_0) < 0$. Therefore, as $S_3(0) > 0, S_3(x_0) < 0$, under our assumptions $S_3(q_2) > 0$ and $S_3(+\infty) = -\infty$ it follows that S_3 has only real zeros. □

Remark 2.6. It is well-known that S_3 has only real zeros if and only if its discriminant is non-negative. In our notations we can state that S_3 has only real zeros if and only if $\delta(q_2, q_3) \geq 0$, where $\delta(u, v) = -4uv^2 + u^2v^2 - 4u^2v + 18uv - 27$.

Remark 2.7. $\delta(3, v) \geq 0 \Leftrightarrow v = 3$.

Proof. We observe that $\delta(3, v) = -3(v-3)^2 \geq 0$. This means that $v = 3$. □

The following lemma extends Hutchinson's ideas about isolation of three consecutive terms of the series. So we will prove Theorem 1.1(2).

Lemma 2.8. *If $q_2 > 1$ and $q_{j+1} \geq 4$ for all $j = 2, 3, \ldots, n-1$ then $S_n(x)$ has at least $(n-2)$ real roots on $[q_2; +\infty)$. Furthermore, there exists a point $y \in [q_2; q_3]$ such that $S_n(y) > 0$.*

Proof. Obviously there exists $x > q_2q_3 \ldots q_n$ such that $\operatorname{sign} S_n(x) = (-1)^n$. For every $j = 2, 3, \ldots, n-1$ we consider $x_0(j) = q_2q_3 \ldots q_j\sqrt{q_{j+1}} = \sqrt{q_2q_3 \ldots q_j}\sqrt{q_2q_3 \ldots q_{j+1}}$. Clearly, $x_0(j) \in [q_2q_3 \ldots q_j; q_2q_3 \ldots q_{j+1}]$.

Now we obtain

$$\begin{aligned}(-1)^j S_n(x_0(j)) &= (-1)^j\left(1 - x_0 + \frac{x_0^2}{q_2} + \ldots + \frac{(-1)^{j-2}x_0^{j-2}}{q_2^{j-3}q_3^{j-4}\ldots q_{j-2}}\right) \\ &\quad + \left(\frac{x_0^j}{q_2^{j-1}\ldots q_j} - \frac{x_0^{j-1}}{q_2^{j-2}\ldots q_{j-1}} - \frac{x_0^{j+1}}{q_2^{j}\ldots q_{j+1}}\right) \\ &\quad + (-1)^j\left(\frac{(-1)^{j+2}x_0^{j+2}}{q_2^{j+1}\ldots q_{j+2}} + \ldots + \frac{(-1)^n x_0^n}{q_2^{n-1}\ldots q_n}\right) =: A_1 + A_2 + A_3.\end{aligned}$$

Firstly, consider A_3. Analogously to computations made in Lemma 2.4, $a_kx^k > a_{k+1}x^{k+1} \Leftrightarrow x < q_2q_3 \ldots q_{k+1}$, $k \geq j+2$. This is true because $x_0(j) \leq q_2q_3 \ldots q_{j+1}$. So, $A_3 > 0$.

Similarly, $a_kx^k < a_{k+1}x^{k+1} \Leftrightarrow x > q_2q_3 \ldots q_{k+1}$, $k \leq j-3$. This is true because $x_0(j) \geq q_2q_3 \ldots q_j$. So, $A_1 > 0$.

At last, since $q_{j+1} \geq 4$ we have $A_2 = \frac{x_0^{j-1}}{q_2^{j-2}\ldots q_{j-1}}\left(-1 + \frac{q_2q_3\ldots q_j\sqrt{q_{j+1}}}{q_2q_3\ldots q_j} - \frac{q_2^2q_3^2\ldots q_j^2 q_{j+1}}{q_2^2q_3^2\ldots q_j^2 q_{j+1}}\right) = \frac{x_0^{j-1}}{q_2^{j-2}\ldots q_{j-1}}\left(\sqrt{q_{j+1}} - 2\right) \geq 0$. Thus, we found $n-1$ different points of sign changes for $S_n(x)$. It means that we found $n-2$ real roots of $S_n(x)$. □

Remark 2.9. We obtain that if $q_2 > 1$ and $q_{j+1} \geq 4$ for all $j = 2, 3, \ldots, n-1$, then f has not more than 2 non-real roots.

The next lemma summarizes all the facts that were stated earlier and proves Theorem 1.1(3).

Lemma 2.10. *If $f \in \mathcal{L}-\mathcal{P}$, $q_2 \geq 3$, $q_3 > \frac{q_2^3}{q_2^2-1}$, $q_j \geq 4$ for $j = 3, 4, \ldots$, then for every $n \in \mathbb{N}$ we get $S_{2n+1}(x) \in \mathcal{L}-\mathcal{P}$.*

Proof. By Lemma 2.5 there exists $x_0 \in [1; q_2]$ such that $S_3(x_0) < 0$, whence S_3 has only real zeros. From Lemma 2.4 we get $S_{2k+1}(x_0) < 0$ for all $k \in \mathbb{N}$. Since $S_{2k+1}(0) = 1$, we obtain that $\exists r_k \in [1; q_2) : S_{2k+1}(r_k) = 0$. From Lemma 2.8 we see that $S_{2k+1}(x)$ has $2k-1$ real roots on $[q_2; +\infty)$. So, $S_{2k+1}(x)$ has $2k$ real roots. The latter means that S_{2k+1} has only real zeros. □

The following lemma states that if coefficients q_2 and q_3 differ a lot, then the function f cannot belong to the Laguerre–Pólya class that proves Theorem 1.1(4).

Lemma 2.11. *If $2 \le q_2 < 3$ and $\forall k \ge 3$ $q_k \ge 4$, then $f \notin \mathcal{L} - \mathcal{P}$.*

Proof. Suppose that $f \in \mathcal{L} - \mathcal{P}$. Consider the following entire function: $g_a(x) = \sum\limits_{k=0}^{\infty} \frac{(-1)^k a_k x^k}{a^{k^2}}$, where $a > 1$. Since $\left(\frac{1}{a^{k^2}}\right)_{k=0}^{\infty} \in \mathcal{CZDS}$ we have $g_a(x) \in \mathcal{L} - \mathcal{P}$ for all $a > 1$. It is easy to verify that $q_k(g_a) = q_k(f) \cdot a^2$. So there exists $a > 1$ such that $q_2(g_a) = 3$. Obviously $q_k(g_a) > 4$ for all $k > 2$. Using Lemma 2.5, we get $\sum\limits_{k=0}^{3} \frac{(-1)^k a_k x^k}{a^{k^2}} \in \mathcal{L} - \mathcal{P}$ (we can use this lemma because $q_2(g_a) = 3$, $\frac{q_2^3(g_a)}{q_2^2(g_a)-1} = \frac{27}{8}$, $q_3(g_a) > \frac{27}{8}$). By Remark 2.6 $\delta(q_2(g_a), q_3(g_a)) = \delta(3, q_3(g_a)) \ge 0$. Then, by Remark 2.7 $q_3(g_a) = 3$, but $q_3(g_a) > 4$. We obtained a contradiction. □

Remark 2.12. As it is seen from the proof, theorem is correct in the following form: if $2 \le q_2 < 3$ and for all $k \ge 3$ we have $q_k \ge \frac{4q_2}{3}$, then $f \notin \mathcal{L} - \mathcal{P}$.

We have proved Theorem 1.1(4), thus the proof of Theorem 1.1 is completed.

3. Proof of Theorem 1.2

As in the proof of Theorem 1.1 by a small abuse of notation we will investigate the function

$$f^{(m,a)}(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k z^k (k!)^m}{a^{k^2}}, \quad a > 1,\ m \ge 1,$$

and its Taylor sections

$$S_n^{(m,a)}(z) = \sum_{k=0}^{n} \frac{(-1)^k z^k (k!)^m}{a^{k^2}}.$$

The question we are interested in remains the same: which necessary and sufficient conditions are there for the function $f^{(m,a)}$ and its Taylor sections to belong to the Laguerre–Pólya class? We will use some reasonings close to those from [6] (see also [7]).

Let us investigate the behavior of $d_{(\infty,m)}$ with different values of m.

Lemma 3.1. $d_{(\infty,m)} \ge 2^{m+1}$.

Proof. According to the G. Pólya and J. Schur Theorem B,

$$f^{(m,a)} \in \mathcal{L} - \mathcal{P} \ \Leftrightarrow\ T_n^{(m,a)} := \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k z^k (k!)^m}{a^{k^2}} \in \mathcal{L} - \mathcal{P}$$

for all $n \in \mathbb{N}$. Let us consider $T_2^{(m,a)}(z) = 1 - 2\frac{z}{a} + 2^{m+1}\frac{z^2}{a^4}$. We have $T_2^{(m,a)} \in \mathcal{L} - \mathcal{P}$ if and only if $a^2 \ge 2^{m+1}$. □

We observe that $q_k(f^{(m,a)}) = \left(1 - \frac{1}{k}\right)^m a^2$. Thus, $q_k(f^{(m,a)}) \leq q_{k+1}(f^{(m,a)})$ for all $k \geq 2$. It is important to note that $q_2(f^{(m,a)}) = \frac{1}{2^m} a^2$, $q_3(f^{(m,a)}) = \frac{2^m}{3^m} a^2$.

From now on, we will use the results obtained earlier in this work. To do that, we should assume first that $q_2(f^{(m,a)}) \geq 3$ and ascertain that $q_j(f^{(m,a)}) \geq 4$ for all $j \geq 3$.

Lemma 3.2. *If $q_2(f^{(m,a)}) \geq 3$, then $q_j(f^{(m,a)}) \geq 4$ for all $j \geq 3$.*

Proof. Plainly, $q_2(f^{(m,a)}) \geq 3 \Leftrightarrow a^2 \geq 3 \cdot 2^m$. The inequality we want to prove now turns into the next one: $a^2 \geq 4\left(1 - \frac{1}{j}\right)^{-m}$. One may see that $\max\limits_{j \geq 3} \frac{4}{\left(1-\frac{1}{j}\right)^m} = \frac{4 \cdot 3^m}{2^m}$. Obviously, $3 \cdot 2^m \geq 4 \cdot \left(\frac{3}{2}\right)^m \Leftrightarrow \left(\frac{4}{3}\right)^m \geq \frac{4}{3}$. This is true for all $m \in \mathbb{N}$. $\square$

So, using Lemma 2.8, one may obtain the following statement.

Corollary 3.3. *If $a^2 \geq 3 \cdot 2^m$ then $S_n^{(m,a)}$ has at least $n - 2$ real roots on $[\frac{a^2}{2^m}; +\infty)$ for all $n > 2$.*

Thus, the behavior of $d_{(n,m)}$ is determined by two roots of $S_n^{(m,a)}$ near the origin (namely, in the disk of radius $\frac{a^2}{2^m}$).

Next, we will find out the connection between $d_{(2n+1,m)}$ for different values of n and $d_{(\infty,m)}$. The following lemmas will assume that $a^2 \geq 3 \cdot 2^m$. Later we will show that this assertion is essential.

We will denote $\widetilde{d}_{(n,m)} = \max(d_{(n,m)}, 3 \cdot 2^m)$ and $\widetilde{d}_{(\infty,m)} = \max(d_{(\infty,m)}, 3 \cdot 2^m)$.

Now we will check that for $a^2 \geq 3 \cdot 2^m$ the inequality $q_3(f^{(m,a)}) > \frac{q_2^3(f^{(m,a)})}{q_2^2(f^{(m,a)})-1}$ is valid. This inequality is equivalent to $\left(\frac{2}{3}\right)^m > \frac{a^4}{2^m a^4 - 2^{3m}}$, or after transformations $a^4 > \frac{2^{4m}}{4^m - 3^m}$. Since $a^2 \geq 3 \cdot 2^m$ it is sufficient to prove that $9 \cdot 2^{2m} > \frac{2^{4m}}{4^m - 3^m}$. Thus we get the inequality $\left(\frac{4}{3}\right)^m > \frac{9}{8}$ which is true for all $m \geq 1$.

Lemma 3.4. $\widetilde{d}_{(3,m)} \leq \widetilde{d}_{(5,m)} \leq \widetilde{d}_{(7,m)} \leq \ldots \leq \widetilde{d}_{(\infty,m)}$.

Proof. Let us prove that $\widetilde{d}_{(2n-1,m)} \leq \widetilde{d}_{(2n+1,m)}$. If $a^2 \geq \widetilde{d}_{(2n+1,m)}$, then $S_{2n+1}^{(m,a)} \in \mathcal{L} - \mathcal{P}$. Since as $a^2 \geq 3 \cdot 2^m$ we know that $S_{2n-1}^{(m,a)}(x)$ has $2n - 3$ real roots on $[\frac{a^2}{2^m}; +\infty)$ (this follows from Corollary 3.3) and $S_{2n-1}^{(m,a)}(x) < S_{2n+1}^{(m,a)}(x)$ on $[1; \frac{a^2}{2^m}]$ (which follows from Lemma 2.4). So, $S_{2n+1}^{(m,a)}$ has 2 roots in $[1; \frac{a^2}{2^m}]$ (this follows from Lemma 2.3) and $S_{2n+1}^{(m,a)}(0) > 0$. So we derive that there exists $x_0 \in [1; \frac{a^2}{2^m}] : S_{2n+1}^{(m,a)}(x_0) \leq 0$. Then $S_{2n-1}^{(m,a)}(x_0) < 0$. Combined with $S_{2n-1}^{(m,a)}(0) > 0$ one obtains that $S_{2n-1}^{(m,a)}(x)$ has at least 1 real root on $[q; \frac{a^2}{2^m}]$. Thus $S_{2n-1}^{(m,a)}$ has 2 roots in $[1; \frac{a^2}{2^m}]$, these roots are real and so $S_{2n-1}^{(m,a)} \in \mathcal{L} - \mathcal{P}$.

Proof of the statement concerning $\widetilde{d}_{(\infty,m)}$ is exactly the same (up to changing $S_{2n+1}^{(m,a)}$ by $f^{(m,a)}$). $\square$

Next lemma will provide an information concerning $d_{(3,m)}$.

Lemma 3.5. $d_{(3,m)} > 3 \cdot 2^m$.

Proof. $S_3^{(m,a)}(x) = 1 - x + \frac{1}{q_2}x^2 - \frac{1}{q_2^2 q_3}x^3$. If $S_3^{(m,a)}(x) \in \mathcal{L} - \mathcal{P}$, then $P(x) = x^3 S_3^{(m,a)}(\frac{1}{x}) = x^3 - x^2 + \frac{1}{q_2}x - \frac{1}{q_2^2 q_3} \in \mathcal{L} - \mathcal{P}$. Then $\frac{d}{dx}P(x) = 3x^2 - 2x + \frac{1}{q_2} \in \mathcal{L} - \mathcal{P}$. The latter is equivalent to $D = 4 - 4\frac{3}{q_2} \geq 0 \Leftrightarrow q_2 \geq 3$. Thus, $d_{(3,m)} \geq 3 \cdot 2^m$.

Let us assume that $a^2 = d_{(3,m)} = 3 \cdot 2^m \Leftrightarrow q_2(S_3^{(m,a)}) = 3$. Then we have that $S_3^{(m,a)}(x) \in \mathcal{L} - \mathcal{P}$. But from the Remark 2.6 and Remark 2.7 one obtains that $q_3(S_3^{(m,a)}) = 3$, whilst from Lemma 3.2 $q_3(S_3^{(m,a)}) \geq 4$. The contradiction we have obtained proves this lemma. $\square$

Thus we obtained that $\widetilde{d}_{(2k+1,m)} = d_{(2k+1,m)}$, $k \in \mathbb{N}$, and $\widetilde{d}_{(\infty,m)} = d_{(\infty,m)}$.

Our next step is to prove that $(d_{(2n+1,m)})_{n=1}^{\infty}$ is a strictly monotonously increasing sequence.

Lemma 3.6. $d_{(2n-1,m)} \neq d_{(2n+1,m)}$.

Proof. Let $a^2 = d_{(2n+1,m)}$. Then $S_{2n+1}^{(m,a)}$ has exactly one double root on $[1; \frac{a^2}{2^m}]$. If $d_{(2n+1,m)} = d_{(2n-1,m)}$, then $S_{2n-1}^{(m,a)}$ also has exactly one double root. But $S_{2n-1}^{(m,a)}(0) > 0$, and so $S_{2n-1}^{(m,a)}(x) \geq 0$ on $[1; \frac{a^2}{2^m}]$. From Lemma 2.4 we get that $0 \leq S_{2n-1}^{(m,a)}(x) < S_{2n+1}^{(m,a)}(x)$, so $S_{2n+1}^{(m,a)}(x) > 0$ for every $x \in [1; \frac{a^2}{2^m}]$. But it contradicts to the statement that $S_{2n+1}^{(m,a)} \in \mathcal{L} - \mathcal{P}$. $\square$

Remark 3.7. If $a^2 = d_{(n,m)}$, then $S_n^{(m,a)}$ has exactly one double root and no other roots on $[1; \frac{a^2}{2^m}]$. If $a^2 = d_{(\infty,m)}$, then $f^{(m,a)}$ has exactly one double root and no other roots on $[1; \frac{a^2}{2^m}]$.

Our next goal is to find out what is the limit of $d_{(2n+1,m)}$ as $n \to \infty$.

Lemma 3.8. $\lim\limits_{n\to\infty} d_{(2n+1,m)} = d_{(\infty,m)}$.

Proof. Lemma 3.6 implies that the sequence $\left(d_{(2n+1,m)}\right)_{n=1}^{\infty}$ is monotonous and this sequence is bounded from above with the upper bound $d_{(\infty,m)}$. So we obtain that there exists the limit $\lim\limits_{n\to\infty} d_{(2n+1,m)}$ which we denote by L_0. Let us prove that $L_0 = d_{(\infty,m)}$. Obviously, $L_0 \leq d_{(\infty,m)}$. Let us assume that $L_0 < d_{(\infty,m)}$. Let us choose a_0 such that $a_0^2 \in (L_0; d_{(\infty,m)})$. Since $a_0^2 > L_0 = \sup_{n\in\mathbb{N}} d_{(2n+1,m)}$ we get $S_{2n+1}^{(m,a_0)} \in \mathcal{L} - \mathcal{P}$ for all $n \in \mathbb{N}$. But $f^{(m,a_0)}(z) = \lim\limits_{n\to\infty} S_{2n+1}^{(m,a_0)}(z)$, and this limit is uniform on compact subsets of $\mathbb{C}$, so from the Hurwitz's theorem $f^{(m,a_0)} \in \mathcal{L} - \mathcal{P}$. But $a_0^2 < d_{(\infty,m)}$ implies that $f^{(m,a_0)} \notin \mathcal{L} - \mathcal{P}$. This contradiction proves the required statement. $\square$

Now we will investigate the behavior of $d_{(2n,m)}$. Obviously $d_{(2,m)} = 4 \cdot 2^m$.

The next statement provides us with a lower bound on $d_{(2n,m)}$.

Lemma 3.9. $d_{(\infty,m)} < d_{(2n,m)}$ *for all* $n \in \mathbb{N}$.

Proof. We have proved that $d_{(\infty,m)} > 3\cdot 2^m$. Then by Remark 3.7 $f^{(m,a)}$ has exactly one double root and no other roots on $[1; \frac{a^2}{2^m}]$. From $f^{(m,a)}(0) > 0$ we obtain $f^{(m,a)}(x) \geq 0$ for all $x \in [0; \frac{a^2}{2^m}]$. So, from paragraph 3) of Lemma 2.4 $S_{2n}^{(m,a)}(x) > f^{(m,a)}(x) \geq 0 \Rightarrow S_{2n}^{(m,a)}(x) > 0$ for all $x \in [0; \frac{a^2}{2^m}]$. The latter implies that $S_{2n}^{(m,a)}$ has no real roots on that interval. But Lemma 2.3 grants that $S_{2n}^{(m,a)}$ has exactly two roots in the disk $|z| \leq \frac{a^2}{2^m}$. So, these are non-real roots and $S_{2n}^{(m,a)} \notin \mathcal{L} - \mathcal{P}$. Thus $d_{(2n,m)} > d_{(\infty,m)}$. $\square$

Corollary 3.10. $d_{(2n,m)} > 3 \cdot 2^m$ *for all* $n \in \mathbb{N}$.

The next statement describes the monotonicity of the sequence $\left(d_{(2,m)}\right)_{n=1}^{\infty}$.

Lemma 3.11. $d_{(2,m)} > d_{(4,m)} > d_{(6,m)} > \cdots$.

Proof. Let $a^2 = d_{(2n,m)}$. Then $S_{2n}^{(m,a)} \in \mathcal{L} - \mathcal{P}$. From Corollary 3.10 we obtain that $a^2 > 3\cdot 2^m$. It means that $S_{2n+2}^{(m,a)}$ has at least $2n$ real roots on $(\frac{a^2}{2^m}; \infty)$ (Corollary 3.3). Now, there exists $x_0 \in [0; \frac{a^2}{2^m}] : S_{2n}^{(m,a)}(x_0) = 0$ and x_0 is the unique double root on that interval (Remark 3.7). Now, $S_{2n+2}^{(m,a)}(x_0) < S_{2n}^{(m,a)}(x_0) = 0$ (Lemma 2.4). So, because $S_{2n+2}^{(m,a)}(0) > 0$ and $S_{2n+2}^{(m,a)}(x_0) < 0$, we get that $S_{2n+2}^{(m,a)}(x_0)$ has at least one root on $[0; \frac{a^2}{2^m}]$. Then $S_{2n+2}^{(m,a)}$ has at least $2n+1$ roots overall and thus $S_{2n+2}^{(m,a)} \in \mathcal{L} - \mathcal{P}$.

Moreover, it is easy to see that $S_{2n+2}^{(m,a)}$ has two distinct roots on $[0; \frac{a^2}{2^m}]$ and thus $a^2 \neq d_{(2n+2,m)}$. $\square$

In the following lemma we prove that a double root of $f^{(m,a)}$ cannot 'slide' along the axis.

Lemma 3.12. *Suppose that $3 \cdot 2^m \le a_1^2 \le a_2^2$. Then for all $m \ge 1$ and $n \ge 2$ it follows that: 1) if $S_n^{(m,a_1)}(x_1) \le 0$ for some $x_1 \in (0; \frac{a_1^2}{2^m})$, then $S_n^{(m,a_2)}(x_2) < 0$, where $x_2 = \frac{x_1 a_2}{a_1} \in (0; \frac{a_2^2}{2^m})$, and 2) if $f^{(m,a_1)}(x_1) \le 0$ for some $x_1 \in (0; \frac{a_1^2}{2^m})$, then $f^{(m,a_2)}(x_2) < 0$, where $x_2 = \frac{x_1 a_2}{a_1} \in (0; \frac{a_2^2}{2^m})$.*

Proof. Let us consider $y_1 = \frac{x_1}{a_1}$, $C_n^{(m,a)}(y) = S_n^{(m,a)}(ay)$ and $g^{(m,a)}(y) = f^{(m,a)}(ay)$.

Since $x_1 \in \left(0; \frac{a_1^2}{2^m}\right)$, we have $y_1 \in \left(0; \frac{a_1}{2^m}\right)$. We obtain for $a \in (a_1; a_2)$ that

$$\frac{\partial}{\partial a} C_n^{(m,a)}(y_1) = -\frac{y_1}{a^3}\left(2 \cdot 2^m - \frac{6y_1 \cdot 6^m}{a^4} + \sum_{p=2}^{n-2} \frac{(-1)^p y_1^p (p+2)(p+1)((p+1)!)^m}{a^{p^2+3p}}\right) \text{ and}$$

$$\frac{\partial}{\partial a} g^{(m,a)}(y_1) = -\frac{y_1}{a^3}\left(2 \cdot 2^m - \frac{6y_1 \cdot 6^m}{a^4} + \sum_{p=2}^{\infty} \frac{(-1)^p y_1^p (p+2)(p+1)((p+1)!)^m}{a^{p^2+3p}}\right).$$

For $a^2 \ge a_1^2 \ge 3 \cdot 2^m$ and $y_1 \in (0; \frac{a_1}{2^m})$ we have $2 \cdot 2^m \ge \frac{6y_1 \cdot 6^m}{a^4} \ge \frac{y_1^p (p+2)(p+1)((p+1)!)^m}{a^{p^2+3p}} \ge \frac{y_1^{p+1}(p+3)(p+2)((p+2)!)^m}{a^{(p+1)^2+3(p+1)}}$.

It means that $\frac{\partial}{\partial a} C_n^{(m,a)}(y_1) < 0$ and $\frac{\partial}{\partial a} g^{(m,a)}(y_1) < 0$ for $a \in (a_1; a_2)$ and thus $C_n^{(m,a_1)}(y_1) > C_n^{(m,a_2)}(y_1)$ and $g^{(m,a_1)}(y_1) > g^{(m,a_2)}(y_1)$. □

Corollary 3.13.

a) If $a^2 = d_{(n,m)}$ then $S_n^{(m,a)}$ has one double root on $[0; \frac{a^2}{2^m}]$,
b) if $a^2 > d_{(n,m)}$ then $S_n^{(m,a)}$ has two distinct roots on $[0; \frac{a^2}{2^m}]$,
c) if $a^2 = d_{(\infty,m)}$ then $f^{(m,a)}$ has one double root on $[0; \frac{a^2}{2^m}]$,
d) if $a^2 > d_{(\infty,m)}$ then $f^{(m,a)}$ has two distinct roots on $[0; \frac{a^2}{2^m}]$.

Proof. Statements a) and c) are true due to Remark 3.7.

To prove b), let us consider $a_1 = d_{(n,m)}$ and $a_2 :\ a_2^2 > d_{(n,m)}$. Plainly, there exists a unique double root x_1 of $S_n^{(m,a_1)}$. Now from Lemma 3.12 it follows that there exists $x_2 \in (0; \frac{a_2^2}{2^m})$ such that $S_n^{(m,a_2)}(x_2) < 0$. Plainly, $S_n^{(m,a_2)}$ cannot have a double root on $(0; \frac{a_2^2}{2^m})$, thus $S_n^{(m,a_2)}$ has two distinct roots on $(0; \frac{a_2^2}{2^m})$. The same logic is used to prove d). □

Now we describe the limit of the sequence $\left(d_{(2,m)}\right)_{n=1}^{\infty}$.

Lemma 3.14. $\lim\limits_{n\to\infty} d_{(2n,m)} = d_{(\infty,m)}$.

Proof. It follows from Lemma 3.11 that the sequence $\left(d_{(2n,m)}\right)_{n=1}^{\infty}$ is monotonous and from Lemma 3.9 that this sequence is bounded from below with the lower bound $d_{(\infty,m)}$. Then we get that this sequence has the limit and we will denote by $L_1 := \lim\limits_{n\to\infty} d_{(2n,m)}$. Let us prove that $L_1 = d_{(\infty,m)}$. Plainly, $L_1 \ge d_{(\infty,m)}$. Let us assume that $L_1 > d_{(\infty,m)}$. We will then choose a_0 such that $a_0^2 \in (d_{(\infty,m)}; L_1)$. Since $a_0^2 > d_{(\infty,m)}$ we have $f^{(m,a_0)}$ has two distinct roots on $[1; \frac{a_0^2}{2^m})$. Thus there exists $x_0 \in [1; \frac{a_0^2}{2^m})$ such that $f^{(m,a_0)}(x_0) < 0$. But $S_{2n}^{(m,a_0)}(x_0) \xrightarrow[n\to\infty]{} f^{(m,a_0)}(x_0)$. Thus, there exists $N_0 \in \mathbb{N}$ such that for every $n > N_0$ we have $S_{2n}^{(m,a_0)}(x_0) < 0$. It means that $S_{2n}^{(m,a_0)}$ has at least one root on $[1; \frac{a_0^2}{2^m})$. Using Corollary 3.3, we obtain that

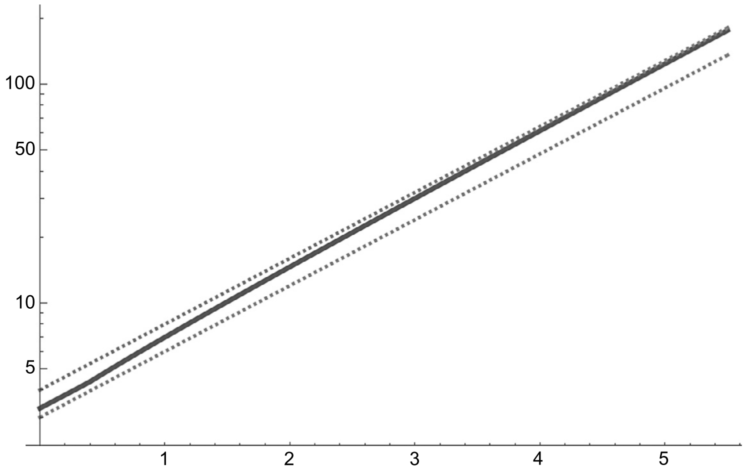

Fig. 1. Behavior of $d_{(3,m)}$ (on y-axis with a logarithmic scale) with different values of m (on x-axis, m ranges from 0 to 5.6) shown as a thick line. Dashed ones are the bounds given by theoretical estimates.

$S_{2n}^{(m,a_0)}$ has at least $2n-2$ roots on $[\frac{a_0^2}{2^m};\infty)$. So, $S_{2n}^{(m,a_0)}$ has at least $2n-1$ real roots and thus $S_{2n}^{(m,a_0)} \in \mathcal{L}-\mathcal{P}$ for all $n > N_0$. It means that $d_{(2n,m)} \leq a_0^2$ for every $n > N_0$. Then $\lim\limits_{n\to\infty} d_{(2n,m)} \leq a_0^2$. But then $a_0^2 \geq L_1$, whilst we picked $a_0^2 \in (d_{(\infty,m)}; L_1)$. This contradiction proves the required statement. □

Our final step is to prove that functions $d_{(n,m)}$ and $d_{(\infty,m)}$ are continuous and monotonous functions of m.

Lemma 3.15. *$d_{(n,m)} \in C([1;+\infty))$ and $d_{(\infty,m)} \in C([1;+\infty))$.*

Proof. 1) If $a^2 = d_{(n,m)}+\delta, \delta > 0$, then $S_n^{(m,a)}$ has n real distinct roots (Corollary 3.13). Now from Hurwitz's theorem there exists $\varepsilon_0^{(\delta)}$ such that $S_n^{(m+\varepsilon,a)} \in \mathcal{HP}$ for all $|\varepsilon| < \varepsilon_0^{(\delta)}$. Thus, $d_{(n,m+\varepsilon)} \leq a^2 = d_{(n,m)} + \delta$.

2) If $a^2 = d_{(n,m)} - \delta, \delta > 0$, then $S_n^{(m,a)}$ has at least two non-real roots. Now from Hurwitz's theorem there exists $\varepsilon_0^{(\delta)}$ such that $S_n^{(m+\varepsilon,a)}$ also has at least two non-real roots for all $|\varepsilon| < \varepsilon_0^{(\delta)}$. Thus, $d_{(n,m+\varepsilon)} \geq a^2 = d_{(n,m)} - \delta$.

Combining 1) and 2) brings us to the required conclusion. This proof remains valid for $d_{(\infty,m)}$. □

Lemma 3.16. *$d_{(n,m_1)} \leq d_{(n,m_2)}$ for all $m_1 < m_2$.*

Proof. Let us consider $\frac{\partial}{\partial m} f^{(m,a)}(x) = \sum\limits_{k=2}^{\infty} \frac{(-1)^k (k!)^m \log(k!) x^k}{a^{k^2}}$. It is easy to verify that for all $a^2 \geq 3\cdot 2^m$ and $x \in (0; \frac{a^2}{2^m})$ it follows that $\frac{(k!)^m \log(k!) x^k}{a^{k^2}} > \frac{((k+1)!)^m \log((k+1)!) x^{k+1}}{a^{(k+1)^2}}$, $k = 2, 3, \ldots$.

Thus, $\frac{\partial}{\partial m} f^{(m,a)}(x) > 0$.

Now, let us pick m_2 and $a^2 = d_{(n,m_2)}$. From Corollary 3.13 we get that $S_n^{(m_2,a)}$ has one double root on $(0; \frac{a^2}{2^{m_2}})$, namely x_0. For $\frac{\partial}{\partial m} S_n^{(m_2,a)}(x) > 0 \ \forall x \in (0; x_0)$ and $S_n^{(m_2,a)}(x_0) = 0$, then $S_n^{(m_1,a)}(x_0) < 0$ for all $m_1 < m_2$ ($x_0 \in (0; \frac{a^2}{2^{m_2}}) \subset (0; \frac{a^2}{2^{m_1}})$), which means that $S_n^{(m_1,a)} \in \mathcal{L}-\mathcal{P}$ for all $m_1 < m_2$ (by Corollary 3.3 it has $n-2$ real roots on $[\frac{a^2}{2^{m_1}}; +\infty)$, and using that $S_n^{(m_1,a)}(0) = 1 > 0$ and $S_n^{(m_1,a)}(x_0) < 0$, we get the required statement). □

To illustrate the statements given above, we present a few graphics (see Figs. 1 and 2) concerning the values of $d_{(n,m)}$.

Theorem 1.2 is proved.

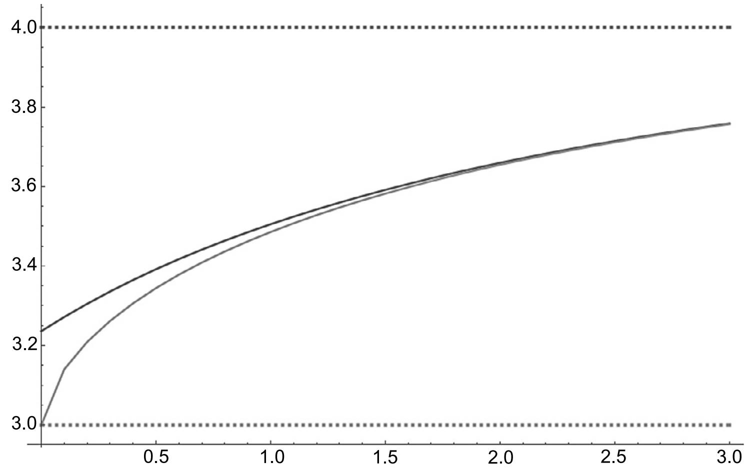

Fig. 2. Behavior of $d_{(3,m)}$ (the lower regular line) and $d_{(4,m)}$ (the upper one) with different values of m. Dashed lines are the bounds given by theoretical estimates.

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