Doctopic: Functional Analysis ARTICLE IN PRESS YJMAA:20152

J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Isometric isomorphisms of Beurling algebras

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ARTICLE INFO

Article history:
Received 1 November 2015
Available online xxxx
Submitted by T. Ransford

Keywords:
Locally compact groups
Beurling algebras
Isometric isomorphisms
Arens product

ABSTRACT

Let G and H be locally compact groups with continuous weights ω_1 and ω_2 respectively, such that $\omega_i(e_i) = 1$, $i = 1, 2$. In this article we show that if $M(G, \omega_1)$ (the weighted measure algebra on G) is isometrically algebra isomorphic to $M(H, \omega_2)$, then the underlying weighted groups are isomorphic, i.e. there exists an isomorphism of topological groups $\phi : G \to H$ such that $\dfrac{\omega_1}{\omega_2 \circ \phi}$ is multiplicative. Similarly, we show that any weighted locally compact group (G, ω) is completely determined by its Beurling group algebra $L^1(G, \omega)$, $LUC(G, \omega^{-1})^*$ and $L^1(G, \omega)^{**}$, when the two last algebras are equipped with an Arens product.
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1. Introduction and preliminaries

One of the most fundamental objectives in all areas of mathematics is to describe the general form of the maps that preserve the basic structure of the objects being investigated. In abstract harmonic analysis, the first paper along this line is due to J.G. Wendel [16]. Wendel proved that if G and H are locally compact topological groups and T is an isometric algebra isomorphism mapping $L^1(G)$ onto $L^1(H)$, then there exists an isomorphism $\phi : G \to H$ of the topological groups G and H, and a continuous character $\alpha : G \to \mathbb{T}$, such that

$$T(\psi) = c\left(\alpha \circ \phi^{-1}\right) \cdot \left(\psi \circ \phi^{-1}\right) \quad \left(\psi \in L^1(G)\right),$$

where c is the measure adjustment constant. Therefore, the group algebra $L^1(G)$ determines G as a topological group. B.E. Johnson in [11] showed that a similar result is true if we replace group algebras with measure algebras. In [12], A.T. Lau and K. McKennon generalized Johnson's result by giving a direct proof showing that as a Banach algebra with left Arens product, the dual of a left introverted subspace of $C_b(G)$ that contains $C_0(G)$, determines G. In particular, it can be concluded that $LUC(G)^*$, the dual of the space of left uniformly continuous functions on a locally compact group G, determines G. In [7], F. Ghahramani

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http://dx.doi.org/10.1016/j.jmaa.2016.01.060
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Please cite this article in press as: S. Zadeh, Isometric isomorphisms of Beurling algebras, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.01.060

and A.T. Lau showed that the bidual algebra $L^1(G)^{**}$ determines the topological group G. In [4], H. Farhadi used [7] to prove that the order structure of $L^1(G)^{**}$ combined with the algebra structure also determines the locally compact group G.

In this paper we will extend these results to the context of weighted locally compact groups and their associated Banach algebras, also known as Beurling algebras. These Beurling algebras have been studied by many authors over the years. The memoir [3] contains a wealth of information about Beurling algebras.

Throughout, G denotes a locally compact topological group with a fixed left Haar measure dx. A weight on G is a strictly positive continuous function ω such that $\omega(xy) \le \omega(x)\omega(y)$, for all $x, y \in G$ and $\omega(e_G) = 1$, where e_G denotes the identity element of G. By a weighted locally compact group, we mean a pair (G, ω), where G is a locally compact group and ω is a weight function on G.

We recall that given a weight ω on G, $L^1(G, \omega)$ is the Banach space of all measurable functions satisfying $\|f\|_{1,\omega} = \int_G |f(x)|\omega(x)dx < \infty$. With the convolution multiplication

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy,$$

$L^1(G, \omega)$ is a Banach algebra called the Beurling group algebra on G associated with the weight ω. It is easy to check that every compactly supported function in $L^1(G)$ belongs to $L^1(G, \omega)$. Like $L^1(G)$, the Banach algebra $L^1(G, \omega)$ has a bounded approximate identity (see [6, Lemma 2.1]).

Since $L^1(G, \omega)$ is, as a Banach space, isometrically isomorphic to $L^1(G)$ via $f \mapsto f\omega$, we can see that the dual of $L^1(G, \omega)$ is

$$L^\infty(G, \omega^{-1}) := \{g : \ g/\omega \in L^\infty(G)\},$$

equipped with the norm $\|f\|_{\infty,\omega^{-1}} = \sup_{x\in G} |f(x)/\omega(x)|$, and the duality is given by $f \mapsto \int_G f(x)g(x)dx$, where $f \in L^1(G, \omega)$ and $g \in L^\infty(G, \omega^{-1})$.

Let $C_0(G, \omega^{-1})$ denote the linear space of all (continuous) functions f on G such that $f/\omega \in C_0(G)$. It can be easily checked that $C_0(G, \omega^{-1})$ forms a Banach space with respect to the norm $\|f\|_{\infty,\omega^{-1}} = \sup_{x\in G} |f(x)/\omega(x)|$.

The set of all complex regular Borel measures on G such that $\int_G \omega(s)d|\mu|(s)$ is finite forms a Banach algebra, denoted by $M(G, \omega)$, with respect to the norm $\|\mu\|_\omega = \int_G \omega(s)d|\mu|(s)$. It can be seen that as a Banach space $M(G, \omega)$ is isometrically isomorphic to $C_0(G, \omega^{-1})^*$ with the duality given by $f \mapsto \int_G f(x)\, d\mu(x)$.

The Banach space $M(G, \omega)$ is a Banach algebra if it is equipped with the following multiplication defined by duality:

$$\langle \mu * \nu, f\rangle = \int_G \int_G f(st)d\mu(s)\, d\nu(t) \quad \left(f \in C_0(G, \omega^{-1})\right).$$

It is not difficult to see that multiplication in $M(G, \omega)$ is separately weak-star continuous i.e., $M(G, \omega)$ is a dual Banach algebra [14, Exercise 4.4.1]. We can identify each $f \in L^1(G, \omega)$, with the measure $h \mapsto \int_G h(x)f(x)dx$ in $C_0(G, \omega^{-1})^*$ and it can be seen that $L^1(G, \omega)$ is a closed ideal of $M(G, \omega)$.

The above definitions and identifications can be found in various places, for example [3] or [6].

The following remark is used several times in our work throughout Section 2.

Remark 1.1. Let $\omega\mu \in M(G)$ be defined by $d(\omega\mu)(x) = \omega(x)d\mu(x)$. Then the mapping $\mu \mapsto \omega\mu$ is a weak-star continuous isometric linear isomorphism from $M(G, \omega)$ onto $M(G)$.

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Let $LUC(G,\omega^{-1})$ denote the linear space of all (continuous) functions f on G such that $f/\omega \in LUC(G)$. Then it can be seen that $LUC(G,\omega^{-1})$ with the norm defined by $\|f\|_{\infty,\omega^{-1}} := \|\frac{f}{\omega}\|_{\infty}$, is a Banach space isometrically linear isomorphic to $LUC(G)$ via $\Phi : LUC(G,\omega^{-1}) \to LUC(G)$; $f \mapsto f\omega^{-1}$. Let

$$G^{luc}_{\omega} := \Phi^*(G^{luc}),$$

where G^{luc} denotes the Gelfand spectrum of the commutative C^*-algebra $LUC(G)$.

The second dual algebra A^{**} of a Banach algebra A can be equipped with two Banach algebra products, known as Arens products. Of these we will equip $L^1(G)^{**}$ with the first (left) product. Given f in $L^\infty(G,\omega^{-1})$ and ψ_1 in $L^1(G,\omega)$, we have a dual action specified by

$$\langle f\Box\psi_1, \psi_2\rangle = \langle f, \psi_1 * \psi_2\rangle \quad \left(\psi_2 \in L^1(G,\omega)\right).$$

By [3, Prop. 7.15], $L^\infty(G,\omega^{-1})\Box L^1(G,\omega) = LUC(G,\omega^{-1})$. Hence

$$LUC(G,\omega^{-1}) = LUC(G,\omega^{-1})\Box L^1(G,\omega),$$

since $L^1(G,\omega)$ factors, so that we have the following module action of $LUC(G,\omega^{-1})^*$ on $LUC(G,\omega^{-1})$:

$$\langle m\Box f, \psi\rangle = \langle m, f\Box\psi\rangle,$$

where $m \in LUC(G,\omega^{-1})^*$, $f \in LUC(G,\omega^{-1})$ and $\psi \in L^1(G,\omega)$. This module operation gives rise to an Arens product on $LUC(G,\omega^{-1})^*$ and turns it into a Banach algebra via

$$\langle m\Box n, f\rangle = \langle m, n\Box f\rangle,$$

where $m, n \in LUC(G,\omega^{-1})^*$, $f \in LUC(G,\omega^{-1})$. Note that the Banach algebras $LUC(G)^*$ and $LUC(G,\omega^{-1})^*$ are isomorphic if and only if ω is multiplicative. We can embed $M(G,\omega)$ isometrically as a Banach algebra into $LUC(G,\omega^{-1})^*$ via the natural embedding $\mu \mapsto \int_G f d\mu$, $f \in LUC(G,\omega^{-1})$ and $\mu \in M(G,\omega)$.

The above definitions can be found for example in [3].

Let (μ_α) be a net in $M(G,\omega)$. We say that (μ_α) converges to some μ in $M(G,\omega)$ in the strong operator topology, SO-topology, if $\|\mu_\alpha * \psi - \mu * \psi\| \to 0$, for all $\psi \in L^1(G,\omega)$.

Lemma 1.2. *The map $x \mapsto \frac{1}{\omega(x)}\delta_x$ from G into $M(G,\omega)$ is strong operator continuous.*

Proof. See [5, Lemma 1]. $\square$

The following lemma is an analogue of [9, Lemma 1.1.3], for Beurling algebras. The proof is an easy modification of the proof of [9, Lemma 1.1.3] and is therefore omitted. Full details of the proof of Lemma 1.3 are found in [17, Lemma 3.1.9].

Lemma 1.3. *Let (G,ω) be a weighted locally compact group. Then the strong operator closed convex hull of the set $\{\gamma\frac{\delta_x}{\omega(x)} : \ x \in G, \ \gamma \in \mathbb{T}\}$ is the unit ball of $M(G,\omega)$.*

Throughout this article, unless otherwise stated, all the isometric isomorphisms are surjective isometric algebra isomorphisms.

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2. Isometric isomorphisms of Beurling measure algebras

Let (G, ω) be a weighted locally compact group. A simple observation is that $M(G, \omega)$ always contains an algebraic copy of G, namely $\{\delta_g : g \in G\}$.

Let τ denote the given topology on the locally compact group G. Then by complete regularity of $C_0(G)$, the topology on G agrees with the relative w^*-topology induced by $M(G)$ and so we have that

$$g_\alpha \to g \ \text{ in } \tau\text{-topology} \quad \iff \quad f(g_\alpha) \to f(g) \quad \forall\, f \in C_0(G).$$

Let τ' denote the w^*-topology on G as a subset of $M(G, \omega)$; thus

$$g_\alpha \to g \quad \text{in } \tau'\text{-topology} \quad \iff \quad f(g_\alpha) \to f(g) \quad \forall\, f \in C_0(G, \omega^{-1}).$$

The proof of the following lemma is straightforward and is therefore omitted.

Lemma 2.1. *Let (G, τ) be a locally compact group, ω a weight on G. Then the topologies τ and τ' are the same.*

Lemma 2.2. *Let (G, ω) be a weighted locally compact group. Then the set of extreme points of the unit ball of the weighted measure algebra $M(G, \omega)$ is $\{\gamma \frac{\delta_g}{\omega(g)} : \ g \in G, \ \gamma \in \mathbb{T}\}$, where $\mathbb{T}$ is the circle group.*

Proof. As noted in the introduction, the map $\mu \mapsto \omega\mu$ is an isometric linear isomorphism of $M(G, \omega)$ onto $M(G)$. The inverse map, $\nu \mapsto \frac{1}{\omega}\nu$, maps $\{\gamma\delta_g : \gamma \in \mathbb{T}, \ g \in G\}$, the set of extreme points of the unit ball of $M(G)$ (see [2, Thm. V.8.4]) to $\{\gamma \frac{\delta_g}{\omega(g)} \ : \ g \in G, \ \gamma \in \mathbb{T}\}$, the set of extreme points of the unit ball of $M(G, \omega)$. $\square$

Definition 2.3. We say that two weighted locally compact groups (G, ω_1) and (H, ω_2) are isomorphic if there is a topological group isomorphism $\phi : G \to H$ such that $\frac{\omega_1}{\omega_2 \circ \phi}$ is multiplicative on G; we call such map ϕ an isomorphism of weighted locally compact groups.

Let $\phi : G \to H$ be an isomorphism of the weighted locally compact groups (G, ω_1) and (H, ω_2) and let $\gamma : G \to \mathbb{T}$ be a continuous character on G. Define the mapping

$$j_{\gamma,\phi} : C_0(H, \omega_2^{-1}) \to C_0(G, \omega_1^{-1}) \ \text{ where } \ j_{\gamma,\phi}(f) = \gamma \frac{\omega_1}{\omega_2 \circ \phi} f \circ \phi.$$

Then it is not difficult to see that $j_{\gamma,\phi}$ is an isometric linear isomorphism mapping $C_0(H, \omega_2^{-1})$ onto $C_0(G, \omega_1^{-1})$. Hence, the dual mapping $T_{\gamma,\phi} := j^*_{\gamma,\phi} : M(G, \omega_1) \to M(H, \omega_2)$ is also an isometric linear isomorphism. We observe that $T_{\gamma,\phi}$ also preserves the convolution product. To see this, first note that

$$\langle j^*_{\gamma,\phi}(\delta_x), f\rangle = \langle \delta_x, j_{\gamma,\phi}(f)\rangle = \langle \gamma(x) \frac{\omega_1(x)}{\omega_2 \circ \phi(x)} \delta_{\phi(x)}, f\rangle \quad (f \in C_0(H, \omega_2^{-1}).$$

Since γ, ϕ and $\frac{\omega_1}{\omega_2 \circ \phi}$ are multiplicative, it can be readily seen that $T_{\gamma,\phi}$ is multiplicative on point masses. Now, to see that $T_{\gamma,\phi}$ is also multiplicative on $M(G, \omega_1)$, note that the linear span of point masses is weak-star dense in $M(G, \omega_1)$, the convolution product is separately weak-star continuous and $T_{\gamma,\phi} = j^*_{\gamma,\phi}$ is weak-star continuous.

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Theorem 2.4. *Let (G,ω_1) and (H,ω_2) be weighted locally compact groups. Then the Banach algebras $M(G,\omega_1)$ and $M(H,\omega_2)$ are isometrically isomorphic if and only if the weighted locally compact groups (G,ω_1) and (H,ω_2) are isomorphic. Moreover, if T is any isometric algebra isomorphism from $M(G,\omega_1)$ onto $M(H,\omega_2)$, then there exists a continuous character $\gamma : G \to \mathbb{T}$ and an isomorphism $\phi : G \to H$ of the weighted locally compact groups (G,ω_1) and (H,ω_2) such that for each $g \in G$, we have*

$$\frac{T(\delta_g)}{\omega_1(g)} = \frac{\gamma(g)}{\omega_2(\phi(g))}\delta_{\phi(g)}.$$

Proof. In light of the preceding argument, if the weighted locally compact groups (G,ω_1) and (H,ω_2) are isomorphic, then the weighted measure algebras $M(G,\omega_1)$ and $M(H,\omega_2)$ are isomorphic. We now establish the converse. Suppose that $T : M(G,\omega_1) \to M(H,\omega_2)$ is an isometric algebra isomorphism. Since $T : M(G,\omega_1) \to M(H,\omega_2)$ is an isometric linear isomorphism, it preserves the extreme points of the unit ball. So by Lemma 2.2, for each $g \in G$, there exists $\phi(g) \in H$ and $\gamma(g) \in \mathbb{T}$ such that

$$T\left(\frac{\delta_g}{\omega_1(g)}\right) = \gamma(g)\frac{\delta_{\phi(g)}}{\omega_2(\phi(g))}. \tag{1}$$

It is not hard to see that $\phi : G \to H$ and $\gamma : G \to \mathbb{T}$ defined via the equation (1) are multiplicative, and that $\dfrac{\omega_1}{\omega_2 \circ \phi}$ is multiplicative. (See [17, Theorem 3.2.4] if more details needed.) We now show that both $\phi : G \to H$ and $\gamma : G \to \mathbb{T}$ are continuous. Let $x_\alpha \to x$ in G. Then by Lemma 1.2, for every $\psi \in L^1(G,\omega_1)$, we have that $\dfrac{1}{\omega_1(x_\alpha)}\delta_{x_\alpha} * \psi \xrightarrow{\|\cdot\|_{1,\omega_1}} \dfrac{1}{\omega_1(x)}\delta_x * \psi$. Since the net $(T(\dfrac{1}{\omega_1(x_\alpha)}\delta_{x_\alpha}))$ is bounded in $M(H,\omega_2)$, it has a subnet $(T(\dfrac{1}{\omega_1(x_{\alpha(i)})}\delta_{x_{\alpha(i)}}))$ converging weak-star to some $\mu \in M(H,\omega_2)$. For $\psi \in L^1(G,\omega_1)$, we have $T(\dfrac{1}{\omega_1(x_{\alpha(i)})}\delta_{x_{\alpha(i)}}) * T(\psi) \xrightarrow{w^*} \mu * T(\psi)$, or equivalently $T(\dfrac{1}{\omega_1(x_{\alpha(i)})}\delta_{x_{\alpha(i)}} * \psi) \xrightarrow{w^*} \mu * T(\psi)$. But, $T(\dfrac{1}{\omega_1(x_{\alpha(i)})}\delta_{x_{\alpha(i)}} * \psi) \xrightarrow{\|\cdot\|_{\omega_2}} T(\dfrac{1}{\omega(x)}\delta_x * \psi)$. Hence $T(\dfrac{1}{\omega_1(x)}\delta_x * \psi) = \mu * T(\psi)$, for all $\psi \in L^1(G,\omega_1)$. Thus,

$$\frac{1}{\omega_1(x)}\delta_x * \psi = T^{-1}(\mu) * \psi \quad \left(\psi \in L^1(G,\omega_1)\right). \tag{2}$$

Let (ψ_i) be an approximate identity as in [6, Lemma 2.1]. Then by equation (2), we have that

$$\frac{1}{\omega_1(x)}\delta_x * \psi_i = T^{-1}(\mu) * \psi_i. \tag{3}$$

Since by [6, Lemma 2.2], $\psi_i \to \delta_{e_G}$ in the w^*-topology of $M(G,\omega_1)$, taking the w^*-limit in (3) gives $\dfrac{1}{\omega_1(x)}\delta_x = T^{-1}(\mu)$, or equivalently $\mu = T(\frac{1}{\omega_1(x)}\delta_x)$. This argument also shows that every subnet of $\left(T(\frac{1}{\omega_1(x_\alpha)}\delta_{x_\alpha})\right)$ has a subnet that is weak-star convergent to $T(\dfrac{1}{\omega_1(x)}\delta_x)$ and hence $T\left(\dfrac{1}{\omega_1(x_\alpha)}\delta_{x_\alpha}\right) \xrightarrow{w^*} T\left(\dfrac{1}{\omega_1(x)}\delta_x\right)$. That is,

$$\gamma(x_\alpha)\frac{1}{\omega_2(\phi(x_\alpha))}\delta_{\phi(x_\alpha)} \xrightarrow{w^*} \gamma(x)\frac{1}{\omega_2(\phi(x))}\delta_{\phi(x)}. \tag{4}$$

It now follows from (4), using an standard argument, that ϕ and γ are continuous.

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Finally, using the isometric isomorphism $T^{-1} : M(H,\omega_2) \to M(G,\omega_1)$, it is not hard to see that $\phi : G \to H$ is a homeomorphism. □

Next we show that every isometric isomorphism $T : M(G,\omega_1) \to M(H,\omega_2)$ is weak-star continuous. Some of the ideas in proof are based on the proofs of Lemmas 1.4 and 1.5 of [8].

Theorem 2.5. *Let $T : M(G,\omega_1) \to M(H,\omega_2)$ be an isometric isomorphism. Then there exists an isomorphism of weighted locally compact groups $\phi : G \to H$ and a continuous character $\gamma : G \to \mathbb{T}$ such that $T = j^*_{\gamma,\phi}$. In particular, T is weak-star continuous.*

Proof. By Theorem 2.4 there exists an isomorphism of weighted locally compact groups $\phi : G \to H$ and a continuous character $\gamma : G \to \mathbb{T}$ such that

$$T(\delta_x) = \gamma(x)\frac{\omega_1(x)}{\omega_2 \circ \phi(x)}\delta_{\phi(x)} = j^*_{\gamma,\phi}(\delta_x), \tag{5}$$

for each $x \in G$. Letting $\mu \in M(G,\omega_1)$ be such that $\mu \geq 0$ and $\|\mu\|_{\omega_1} = 1$, it suffices to show that $T(\mu) = j^*_{\gamma,\phi}(\mu)$.

Taking μ in $M(G,\omega_1)$ with norm 1, by Lemma 1.3, we can find a net (μ_β) in the convex hull of the set $\{\gamma\frac{\delta_x}{\omega(x)} : x \in G, \gamma \in \mathbb{T}\}$ such that

$$\lim_\beta \|\mu_\beta * \psi - \mu * \psi\|_{\omega_1} = 0, \quad (\psi \in L^1(G,\omega)). \tag{6}$$

By equation (5),

$$T(\mu_\beta) = j^*_{\gamma,\phi}(\mu_\beta) \tag{7}$$

for each β.

We claim that $T(\mu_\beta) \xrightarrow{w^*} T(\mu)$ in $M(H,\omega_2)$. To see this, let ν be a weak-star limit point of $(T(\mu_\beta))$ in $M(H,\omega_1)$ and let $\left(\mu_{\beta(i)}\right)$ be a subnet of (μ_β) such that $T(\mu_{\beta(i)}) \xrightarrow{w^*} \nu$. Observe that it suffices now to show that $\nu = T(\mu)$. To simplify notation, we can assume that $T(\mu_\beta) \xrightarrow{w^*} \nu$. Let $\psi \in L^1(G,\omega_1)$ be fixed. Then

$$\|T(\mu_\beta) * T(\psi) - T(\mu) * T(\psi)\|_{\omega_2} \to 0,$$

by equation (6). Note that because $M(H,\omega_2) = C_0(G,\omega_2^{-1})^*$ is a dual Banach algebra (multiplication in $M(H,\omega_2)$ is separately weak-star continuous), $C_0(H,\omega_2^{-1})$ is a submodule of the dual Banach $M(H,\omega_2)$-module $M(H,\omega_2)^* = C_0(H,\omega_2^{-1})^{**}$ (see [14, Exercise 4.4.1]). Therefore, given $k \in C_0(H,\omega_2^{-1})$, $T(\psi) \cdot k \in M(H,\omega_2) \cdot C_0(H,\omega_2^{-1}) \subseteq C_0(H,\omega_2^{-1})$, and hence

$$\begin{aligned}\langle T(\mu) * T(\psi), k\rangle &= \lim_\beta \langle T(\mu_\beta) * T(\psi), k\rangle \\ &= \lim_\beta \langle T(\mu_\beta), T(\psi) \cdot k\rangle \\ &= \langle \nu, T(\psi) \cdot k\rangle = \langle \nu * T(\psi), k\rangle.\end{aligned}$$

Thus $T(\mu * \psi) = T(\mu) * T(\psi) = \nu * T(\psi)$, and so

$$\mu * \psi = T^{-1}(\nu) * \psi, \quad \left(\psi \in L^1(G,\omega_1)\right). \tag{8}$$

Let (ψ_i) be an approximate identity as in [6, Lemma 2.1]. Then by equation (8), we have that

$$\mu * \psi_i = T^{-1}(\nu) * \psi_i. \tag{9}$$

Since by [6, Lemma 2.2], $\psi_i \to \delta_{e_G}$ in the w^*-topology of $M(G, \omega_1)$, by taking the weak-star limit in (9) we have that $\mu = T^{-1}(\nu)$ and therefore $T(\mu) = \nu$. This gives the claim.

Finally, because $j^*_{\gamma,\phi}$ is weak-star continuous and $T(\mu_\beta) \xrightarrow{w^*} T(\mu)$, equation (7) yields $T(\mu) = w^* - \lim_\beta T(\mu_\beta) = w^* - \lim_\beta j^*_{\gamma,\phi}(\mu_\beta) = j^*_{\gamma,\phi}(\mu)$. □

3. Isometric isomorphisms of Beurling group algebras

We recall that a linear operator L on the Banach algebra $\mathcal{A}$ is a left multiplier if

$$L(ab) = L(a)b \quad (a, b \in \mathcal{A}).$$

By [10], if the Banach algebra $\mathcal{A}$ has a bounded approximate identity then every left multiplier on $\mathcal{A}$ is continuous. The reader is referred to [13, Sections 1.2.1–1.2.7] for definitions and basic theorems regarding the Banach algebra of left multipliers. We denote the Banach algebra of left multipliers of $L^1(G, \omega)$ by $\mathcal{M}(L^1(G, \omega))$.

Lemma 3.1. *Let (G, ω) be a weighted locally compact group. Suppose that L is a left multiplier on $L^1(G, \omega)$. Then, there exists $\mu \in M(G, \omega)$ such that $L(\psi) = \mu * \psi$, $\psi \in L^1(G, \omega)$.*

Proof. See [6, Lemma 2.3]. □

The following result is also observed in [3, Thm. 7.14] when the weight ω satisfies $\omega(x) \geq 1$ for all $x \in G$.

Theorem 3.2. *Let (G, ω) be a weighted locally compact group. The left multiplier algebra of $L^1(G, \omega)$ is isometrically isomorphic to $M(G, \omega)$.*

Proof. Let $\theta : M(G, \omega) \to \mathcal{M}\left(L^1(G, \omega)\right)$, $\theta(\mu) := L_\mu$. Then by Lemma 3.1, θ is a contractive surjective homomorphism. Letting $\mu \in M(G, \omega)$, we complete the proof by showing that $\|L_\mu\| \geq \|\mu\|$. Let (U_i) be a shrinking neighbourhood system of the identity all contained in a compact neighbourhood of the identity element. Then the proof of [6, Lemma 2.3] shows that $\psi_i := \dfrac{\chi_{U_i}}{\lambda(U_i)}$ forms a bounded approximate identity for $L^1(G, \omega)$, where λ denotes the Haar measure on G. Then $L_\mu(\psi_i) \xrightarrow{w^*} \mu$ by [6, Lemma 2.2]. Given $\epsilon > 0$, let $h \in C_0(G, \omega^{-1})$ be such that $\|h\|_{\infty,\omega} = 1$ and $|\mu(h)| \geq \|\mu\|_\omega - \epsilon$. Then by the weak-star convergence, we have that $L_\mu(\psi_i)(h) \to \mu(h)$ and therefore, $\lim_i |L_\mu(\psi_i)(h)| = |\mu(h)| \geq \|\mu\| - \epsilon$. Now

$$|L_\mu(\psi_i)(h)| \leq \|L_\mu(\psi_i)\|_{1,\omega} \leq \|L_\mu\| \|\psi_i\|_{1,\omega} \leq \|L_\mu\| \frac{\|\chi_{U_i}\|_{1,\omega}}{\lambda(U_i)},$$

and continuity of ω at e gives $\lim_i \dfrac{\|\chi_{U_i}\|_{1,\omega}}{\lambda(U_i)} = \omega(e) = 1$. Therefore, $\lim_i |L_\mu(\psi_i)(h)| \leq \|L_\mu\|$. Thus, for any $\epsilon > 0$, $\|\mu\| - \epsilon \leq |\mu(h)| = \lim_i |L_\mu(\psi_i)(h)| \leq \|L_\mu\|$. Since we have already observed that $\|L_\mu\| \leq \|\mu\|$, we have that $\|L_\mu\| = \|\mu\|$. □

The following proposition is similar to [13, Prop. 1.2.7]. The proof is straightforward.

Proposition 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Then, for every Banach algebra isometric isomorphism T from $\mathcal{A}$ onto $\mathcal{B}$, there is a Banach algebra isometric isomorphism $\tilde{T}$ defined by $\tilde{T}(L) = T \circ L \circ T^{-1}$, mapping the left multiplier algebra $\mathcal{M}(\mathcal{A})$ onto $\mathcal{M}(\mathcal{B})$. Moreover, $\tilde{T}(L_a) = L_{T(a)}$ for all $a \in \mathcal{A}$.*

Theorem 3.4. *Let (G, ω_1) and (H, ω_2) be weighted locally compact groups. Suppose that $T : L^1(G, \omega_1) \to L^1(H, \omega_2)$ is an isometric isomorphism. Then the weighted locally compact groups (G, ω_1) and (H, ω_2) are isomorphic. Conversely, if the weighted locally compact groups (G, ω_1) and (H, ω_2) are isomorphic, then $L^1(G, \omega_1)$ is isometrically isomorphic to $L^1(H, \omega_2)$.*

Proof. The first statement follows from Proposition 3.3 and Theorems 2.4 and 3.2. For the converse, let $\phi : G \to H$ be a topological isomorphism such that $\dfrac{\omega_1}{\omega_2 \circ \phi}$ is multiplicative on G. Then the isometric isomorphisms

$$j_\phi : C_0(H, \omega_2^{-1}) \to C_0(G, \omega_1^{-1}); f \mapsto \frac{\omega_1}{\omega_2 \circ \phi} f \circ \phi,$$
$$J_\phi : C_0(H) \to C_0(G); f \mapsto f \circ \phi,$$
$$\theta_G : C_0(G) \to C_0(G, \omega_1^{-1}); f \mapsto \omega_1 f,$$

and

$$\theta_H : C_0(H) \to C_0(H, \omega_2^{-1}); f \mapsto \omega_2 f,$$

satisfy

$$j_\phi \circ \theta_H = \theta_G \circ J_\phi.$$

It is well-known that J_ϕ^* maps $L^1(G)$ onto $L^1(H)$ – e.g. see [15, Prop. 5.1 (iv)] for a general result – and we have already noted that $\theta_G^* : M(G, \omega_1) \to M(G)$; $\mu \mapsto \omega_1 \mu$ maps $L^1(G, \omega_1)$ onto $L^1(G)$. Hence, $j_\phi^* \left(L^1(G, \omega_1)\right) = \left(\theta_H^{-1}\right)^* \circ J_\phi^* \circ \theta_G^*(L^1(G, \omega_1)) = L^1(H, \omega_2)$. Thus j_ϕ^* maps $L^1(G, \omega_1)$ onto $L^1(H, \omega_2)$ which, by the remarks preceding Theorem 2.4, is an isometric isomorphism of Banach algebras. □

Corollary 3.5. *Let (G, ω_1) and (H, ω_2) be weighted locally compact groups and suppose that $T : L^1(G, \omega_1) \to L^1(H, \omega_2)$ is an isometric isomorphism. Then there exists a continuous character $\gamma : G \to \mathbb{T}$ and an isomorphism of weighted locally compact groups $\phi : G \to H$ such that for all $\psi \in L^1(G, \omega_1)$*

$$T(\psi) = j_{\gamma,\phi}^*(\psi) = c\gamma \circ \phi^{-1} \left(\frac{\omega_1 \circ \phi^{-1}}{\omega_2} \right) \cdot \psi \circ \phi^{-1}$$

where c is a measure adjustment constant.

Proof. Let $T : L^1(G, \omega_1) \to L^1(H, \omega_2)$ be an isometric isomorphism, $\tilde{T} : M(G, \omega_1) \to M(H, \omega_2)$ the isometrically isomorphic extension that exists by Theorem 3.2 and Proposition 3.3. By Theorem 2.5, $\tilde{T} = j_{\gamma,\phi}^*$ where $\phi : G \to H$ is an isomorphism of weighted locally compact groups and $\gamma : G \to \mathbb{T}$ is a continuous character on G. As $\psi \mapsto \int_G \psi(\phi(x))dx$ defines a Haar integral on $L^1(H)$, there exists $c > 0$ such that for every $\psi \in L^1(H)$, the equation $\int_G \psi(\phi(x))dx = c \int_H \psi(y)dy$ holds. Therefore, for $\psi \in L^1(G, \omega_1)$ and $f \in C_0(H, \omega_2^{-1})$ we have

$$\langle \tilde{T}(\psi), f \rangle = \langle j_{\gamma,\phi}^*(\psi), f \rangle = \int_G \gamma(x) \left(\frac{\omega_1(x)}{\omega_2 \circ \phi(x)} \right) f \circ \phi(x) \psi(x) dx$$

$$= \int_H c\gamma \circ \phi^{-1}(y) \left(\frac{\omega_1 \circ \phi^{-1}(y)}{\omega_2(y)} \right) f(y) \psi \circ \phi^{-1}(y) dy$$

$$= \langle c\gamma \circ \phi^{-1} \left(\frac{\omega_1 \circ \phi^{-1}}{\omega_2} \right) \psi \circ \phi^{-1}, f \rangle.$$

Hence $T(\psi) = \tilde{T}(\psi) = c\gamma \circ \phi^{-1} \left(\frac{\omega_1 \circ \phi^{-1}}{\omega_2} \right) \psi \circ \phi^{-1}$. $\square$

4. Isometric isomorphisms of the dual of weighted LUC-functions and the bidual of weighted group algebras

In this section, we will first show that $LUC(G, \omega^{-1})^*$ determines the weighted locally compact group (G, ω). We will then use this result to prove that the Banach algebraic structure of $L^1(G, \omega)^{**}$, the bidual of the Beurling group algebra, also determines the weighted locally compact group (G, ω).

Let $\phi : G \to H$ be an isomorphism of the weighted locally compact groups (G, ω_1) and (H, ω_2) and let $\gamma : G \to \mathbb{T}$ be a continuous character on G. Define the mapping

$$J_{\gamma,\phi} : LUC(H, \omega_2^{-1}) \to LUC(G, \omega_1^{-1}) \quad \text{where} \quad J_{\gamma,\phi}(f) = \gamma \frac{\omega_1}{\omega_2 \circ \phi} f \circ \phi.$$

Then it is not difficult to see that $J_{\gamma,\phi}$ is an isometric linear isomorphism mapping $LUC(H, \omega_2^{-1})$ onto $LUC(G, \omega_1^{-1})$. Hence, the dual mapping $T_{\gamma,\phi} := J_{\gamma,\phi}^* : LUC(G, \omega_1^{-1})^* \to LUC(H, \omega_2^{-1})^*$ is also an isometric linear isomorphism such that

$$T_{\gamma,\phi}(\delta_x) = \gamma(x) \frac{\omega_1(x)}{\omega_2 \circ \phi(x)} \delta_{\phi(x)}, \; x \in G. \tag{10}$$

We observe that $T_{\gamma,\phi}$ also preserves the product. To see this, first note that since γ, ϕ and $\frac{\omega_1}{\omega_2 \circ \phi}$ are multiplicative, it can be readily seen from the equation (10) that $T_{\gamma,\phi}$ is multiplicative on point masses. Now, to see that $T_{\gamma,\phi}$ is multiplicative on $LUC(G, \omega_1^{-1})^*$, note that the linear span of point masses is weak-star dense in $LUC(G, \omega_1^{-1})^*$; for each $n \in LUC(G, \omega_1^{-1})^*$, $m \mapsto m\Box n$ is weak-star continuous on $LUC(G, \omega_1^{-1})^*$; for each $\mu \in M(G, \omega_1)$, $n \mapsto \mu\Box n$ is weak-star continuous; and from equation (10), $T_{\gamma,\phi}$ maps the linear span of point masses in $LUC(G, \omega_1^{-1})^*$ into $M(H, \omega_2)$. In particular, this shows that $LUC(G, \omega_1^{-1})^*$ and $LUC(H, \omega_2^{-1})^*$ are isometrically isomorphic Banach algebras whenever the weighted locally compact groups (G, ω_1) and (H, ω_2) are isomorphic.

Let

$$C_0(G, \omega^{-1})^\perp := \{ m \in LUC(G, \omega^{-1})^*; \; m(f) = 0, \; \forall f \in C_0(G, \omega^{-1}) \}.$$

The next lemma is a Beurling algebra version of [8, Lemma 1.1].

Lemma 4.1. *Suppose that (G, ω) is a weighted locally compact group. Then the ℓ^1-direct sum decomposition $LUC(G, \omega^{-1})^* = M(G, \omega) \oplus_1 C_0(G, \omega^{-1})^\perp$ holds, and $C_0(G, \omega^{-1})^\perp$ is an ideal in $LUC(G, \omega^{-1})^*$.*

Proof. The map $\Phi : LUC(G, \omega^{-1}) \to LUC(G)$, $f \mapsto \omega^{-1} f$ is an isometric linear isomorphism mapping $C_0(G, \omega^{-1})$ onto $C_0(G)$. As $LUC(G)^* = M(G) \oplus_1 C_0(G)^\perp$ by [8, Lemma 1.1], it can be readily seen that $LUC(G, \omega^{-1})^* = M(G, \omega) \oplus_1 C_0(G, \omega^{-1})^\perp$.

To show that $C_0(G, \omega^{-1})^\perp$ is an ideal in $LUC(G, \omega^{-1})^*$, we first show that for $\psi \in L^1(G, \omega)$, $\mu \in M(G, \omega) \subseteq LUC(G, \omega^{-1})^*$ and $h \in C_0(G, \omega^{-1})$,

Please cite this article in press as: S. Zadeh, Isometric isomorphisms of Beurling algebras, J. Math. Anal. Appl. (2016), http://dx.doi.org/10.1016/j.jmaa.2016.01.060

$$h\Box\psi \in C_0(G,\omega^{-1}) \text{ and } \mu\Box h \in C_0(G,\omega^{-1}). \tag{11}$$

We first note that because multiplication is separately weak-star continuous in $M(G,\omega)$, i.e., $M(G,\omega)$ is a dual Banach algebra, $\mu \cdot h, h \cdot \mu \in C_0(G,\omega^{-1}) = (M(G,\omega), w^*)^*$, where

$$\langle \mu \cdot h, \nu \rangle_{M^*-M} = \langle \nu * \mu, h \rangle_{M-C_0}, \text{ and } \langle h \cdot \mu, \nu \rangle_{M^*-M} = \langle \mu * \nu, h \rangle_{M-C_0}.$$

For $\phi \in L^1(G,\omega)$,

$$\begin{aligned}\langle h\Box\psi, \phi\rangle_{L^\infty-L^1} &= \langle h, \psi * \phi\rangle_{L^\infty-L^1}\\ &= \langle \psi * \phi, h\rangle_{M-C_0}\\ &= \langle h \cdot \psi, \phi\rangle_{M^*-M}\\ &= \langle \phi, h \cdot \psi\rangle_{M-C_0} = \langle h \cdot \psi, \phi\rangle_{L^\infty-L^1}\end{aligned}$$

so $h\Box\psi = h \cdot \psi$ in $L^\infty(G,\omega^{-1})$. By continuity, $h\Box\psi = h \cdot \psi \in C_0(G,\omega^{-1})$. Also from the above argument, $h\Box\phi = h \cdot \phi \in C_0(G,\omega^{-1})$, so

$$\begin{aligned}\langle \mu\Box h, \phi\rangle_{L^\infty-L^1} &= \langle \mu, h\Box\phi\rangle_{LUC^*-LUC}\\ &= \langle \mu, h \cdot \phi\rangle_{M-C_0}\\ &= \langle \mu \cdot h, \phi\rangle_{M^*-M}\\ &= \langle \phi, \mu \cdot h\rangle_{M-C_0} = \langle \mu \cdot h, \phi\rangle_{L^\infty-L^1}.\end{aligned}$$

Hence, $\mu\Box h = \mu \cdot h \in C_0(G,\omega^{-1})$.

Now suppose that $m \in LUC(G,\omega^{-1})^*$, $n \in C_0(G,\omega^{-1})^\perp$, $h \in C_0(G,\omega^{-1})$. Then for $\psi \in L^1(G,\omega)$, equation (11) gives $\langle n\Box h, \psi\rangle = \langle n, h\Box\psi\rangle = 0$ and hence $\langle m\Box n, h\rangle = \langle m, n\Box h\rangle = 0$. Thus, $C_0(G,\omega^{-1})^\perp$ is a left ideal in $LUC(G,\omega^{-1})^*$. Writing m as $m = \mu + m_1$ where $\mu \in M(G,\omega)$ and $m_1 \in C_0(G,\omega^{-1})^\perp$, $n\Box m = n\Box\mu + n\Box m_1$, with $n\Box m_1 \in C_0(G,\omega^{-1})^\perp$ from the above. Also, equation (11) gives $\langle n\Box\mu, h\rangle = \langle n, \mu\Box h\rangle = 0$, so $n\Box\mu \in C_0(G,\omega^{-1})^\perp$ as well. Thus, $C_0(G,\omega^{-1})^\perp$ is also a right ideal in $LUC(G,\omega^{-1})^*$. □

Let (m_α) be a net in $LUC(G,\omega^{-1})^*$. We say that (m_α) converges strictly to some m in $LUC(G,\omega^{-1})^*$ if $\|m_\alpha\Box\phi - m\Box\phi\| \to 0$, for each $\phi \in L^1(G,\omega)$.

The following lemma is an analogue of [8, Theorem 1.4], for Beurling algebras. The proof is an easy modification of the proof of [8, Theorem 1.4] and is therefore omitted. Full details of the proof of Lemma 4.2 are found in [17, Lemma 3.4.2].

Lemma 4.2. *Suppose that (G,ω_1) and (H,ω_2) are weighted locally compact groups and $T : LUC(G,\omega_1^{-1})^* \to LUC(H,\omega_2^{-1})^*$ is an isometric isomorphism. Let (m_α) be a net in $M(G,\omega_1)$ converging strictly to $m \in M(G,\omega_1)$ with $\|m_\alpha\|_{\omega_1} = \|m\|_{\omega_1} = 1$. Then $T(m_\alpha)$ converges to $T(m)$ in the w^*-topology of $LUC(H,\omega_2^{-1})^*$.*

Lemma 4.3. *Let (G,ω) be a weighted locally compact group. Then $G_\omega^{luc} = G_\omega \dot\cup G_\omega^*$, where $G_\omega = \{\omega(x)^{-1}\delta_x : x \in G\} = G_\omega^{luc} \cap M(G,\omega)$ and $G_\omega^* = G_\omega^{luc} \setminus G_\omega = G_\omega^{luc} \cap C_0(G,\omega^{-1})^\perp$. Moreover, $\{\gamma p : \gamma \in \mathbb{T}, p \in G_\omega^{luc}\}$ is the set of extreme points of the unit ball of $LUC(G,\omega^{-1})^*$.*

Proof. Since $LUC(G)$ is a commutative unital C^*-algebra, the Gelfand representation theorem for commutative unital C^*-algebras implies that $LUC(G) = C(G^{luc})$, where G^{LUC} denotes the Gelfand spectrum of $LUC(G)$. It follows from [2, Thm. V.8.4] that the set of extreme points of the unit ball of $LUC(G)^* = M(G^{LUC})$ is $\{\gamma\delta_x : \gamma \in \mathbb{T}, x \in G^{LUC}\}$. As noted in the introduction $\Phi : LUC(G,\omega^{-1}) \to LUC(G) : f \mapsto$

$\omega^{-1}f$ is an isometric linear isomorphism of Banach spaces. Therefore, $\Phi^* : LUC(G)^* \to LUC(G,\omega^{-1})^*$ is also an isometric linear isomorphism and so Φ^* maps the extreme points of the unit ball of $LUC(G)^*$ onto the extreme points of the unit ball of $LUC(G,\omega^{-1})^*$. Now, by the definition of G_ω^{LUC}, we have that $\Phi^*(G^{LUC}) = G_\omega^{LUC}$. Thus, the set of the extreme points of the unit ball of $LUC(G,\omega^{-1})^*$ is $\{\gamma p : \gamma \in \mathbb{T},\ p \in G_\omega^{luc}\}$. Moreover, as noted in the proof of Lemma 4.1, $\Phi^*(M(G)) = M(G,\omega)$ and $\Phi^*(C_0(G)^\perp) = C_0(G,\omega^{-1})^\perp$. Hence,

$$\begin{aligned} G_\omega^{luc} &= \Phi^*(G^{luc}) = \Phi^*(G)\dot\cup\Phi^*(G^{luc}\setminus G) \\ &= \left\{\frac{\delta_x}{\omega(x)} : x \in G\right\}\dot\cup\left(G_\omega^{luc}\setminus\Phi^*(G)\right) = G_\omega\dot\cup\left(G_\omega^{luc}\setminus G_\omega\right), \\ G_\omega &= \Phi^*(G) = \Phi^*\left(G^{luc}\cap M(G)\right) = G_\omega^{luc}\cap M(G,\omega) \end{aligned}$$

and

$$G_\omega^{luc}\setminus G_\omega = \Phi^*\left(G^{luc}\setminus G\right) = \Phi^*\left(G^{luc}\cap C_0(G)^\perp\right) = G_\omega^{luc}\cap C_0(G,\omega^{-1})^\perp. \qquad \square$$

Before proceeding to the next theorem, note that δ_{e_G} is the identity for $LUC(G,\omega^{-1})^*$ where e_G is the identity in G. To see this, first note that

$$\delta_{e_G}\Box\frac{\delta_x}{\omega(x)} = \frac{\delta_x}{\omega(x)} \quad\text{and}\quad \frac{\delta_x}{\omega(x)}\Box\delta_{e_G} = \frac{\delta(x)}{\omega(x)},$$

for each x in G. The linear span of $\{\frac{\delta_x}{\omega(x)} : x \in G\}$ is weak-star dense in $LUC(G,\omega^{-1})^*$ so, by the weak-star continuity of the product by δ_{e_G} on both left and right we obtain $\delta_{e_G}\Box m = m$ and $m\Box\delta_{e_G} = m$, for each m in $LUC(G,\omega^{-1})^*$. Thus δ_{e_G} is the identity.

Theorem 4.4. *Suppose that (G,ω_1) and (H,ω_2) are weighted locally compact groups and $T : LUC(G,\omega_1^{-1})^* \to LUC(H,\omega_2^{-1})^*$ is an isometric isomorphism. Then T maps $M(G,\omega_1)$ onto $M(H,\omega_2)$ and hence (G,ω_1) and (H,ω_2) are isomorphic weighted locally compact groups.*

Proof. As T is an isometric isomorphism, it maps an extreme point of the unit ball of $LUC(G,\omega_1^{-1})^*$ onto an extreme point of the unit ball of $LUC(H,\omega_2^{-1})^*$. Hence, by Lemma 4.3, for each $x \in G$ there exists $\gamma(x) \in \mathbb{T}$ and $p_x \in H_{\omega_2}^{luc}$ such that

$$T\left(\frac{\delta_x}{\omega_1(x)}\right) = \gamma(x)p_x.$$

Moreover, as a surjective algebra homomorphism, T maps δ_{e_G}, the identity of $LUC(G,\omega_1^{-1})^*$ to the identity, δ_{e_H}, of $LUC(H,\omega_2^{-1})^*$. Therefore, for $x \in G$,

$$\begin{aligned} \delta_{e_H} &= T(\delta_{e_G}) = T(\delta_x)\Box T(\delta_{x^{-1}}) \\ &= \omega_1(x)\omega_1(x^{-1})\gamma(x)\gamma(x^{-1})p_x\Box p_{x^{-1}}. \end{aligned}$$

If p_x belongs to $H_{\omega_2}^* = H_{\omega_2}^{luc}\cap C_0(H,\omega_2^{-1})^\perp$, then because $C_0(H,\omega_2^{-1})^\perp$ is an ideal in $LUC(H,\omega_2^{-1})^*$, we would have $\delta_{e_H} \in C_0(H,\omega_2^{-1})^\perp$ as well. This is not possible since $\delta_{e_H} \in M(H,\omega_2)$ and $M(H,\omega_2)\cap C_0(H,\omega_2^{-1})^\perp = \{0\}$. Hence, p_x belongs to $H_{\omega_2} = \{\omega_2(y)^{-1}\delta_y : y \in H\}$. Therefore there exists $\phi(x)$ in H such that $p_x = \omega_2(\phi(x))^{-1}\delta_{\phi(x)}$. Thus we have mappings $\gamma : G \to \mathbb{T}$ and $\phi : G \to H$ such that

$$T(\delta_x) = \gamma(x)\frac{\omega_1(x)}{\omega_2(\phi(x))}\delta_{\phi(x)} \quad (x \in G).$$

Moreover, the arguments found in the proof of Theorem 2.4 show that γ, $\frac{\omega_1}{\omega_2 \circ \phi}$ and ϕ are multiplicative on G. To see that γ and ϕ are continuous, let $x_\alpha \to x$ in G. Then by Lemmas 1.2 and 4.2, $T(\frac{1}{\omega_1(x_\alpha)}\delta_{x_\alpha}) \xrightarrow{w^*} T(\frac{1}{\omega_1(x)}\delta_x)$ in $LUC(H, \omega_2^{-1})^*$. It follows that $\gamma(x_\alpha)\frac{1}{\omega_2(\phi(x_\alpha))}\delta_{\phi(x_\alpha)} \xrightarrow{w^*}$ $\gamma(x)\frac{1}{\omega_2(\phi(x))}\delta_{\phi(x)}$ in $M(H, \omega_2^{-1})$, which is exactly equation (4) in the proof of Theorem 2.4. The proof of Theorem 2.4 hence shows that γ and ϕ are continuous.

Hence,

$$J_{\gamma,\phi} : LUC(H, \omega_2^{-1}) \to LUC(G, \omega_1^{-1}) : f \mapsto \gamma \frac{\omega_1}{\omega_2 \circ \phi} f \circ \phi$$

is a well-defined isometric isomorphism and $J^*_{\gamma,\phi} : LUC(G, \omega_1^{-1})^* \to LUC(H, \omega_2^{-1})^*$ satisfies

$$J^*_{\gamma,\phi}(\delta_x) = T(\delta_x) \quad (x \in G). \tag{12}$$

Moreover, for any $\mu \in M(G, \omega_1)$, $J^*_{\gamma,\phi}(\mu) \in M(H, \omega_2)$. To see this note that by Lemma 4.1, $J^*_{\gamma,\phi}(\mu) = \nu + m$ where $\nu = J^*_{\gamma,\phi}(\mu)|_{C_0(H,\omega_2^{-1})}$ and $m \in C_0(H, \omega_2^{-1})^\perp$. Letting $j^*_{\gamma,\phi} : M(G, \omega_1) \to M(H, \omega_2)$ be the isometric isomorphism defined in Section 2, it is clear $\nu = J^*_{\gamma,\phi}(\mu)|_{C_0(H,\omega_2^{-1})} = j^*_{\gamma,\phi}(\mu)$, so, $\|\mu\| = \|J^*_{\gamma,\phi}(\mu)\| = \|j^*_{\gamma,\phi}(\mu)\| + \|m\| = \|\mu\| + \|m\|$. Hence, $m = 0$, as needed.

Let $\mu \in M(G, \omega_1)$ with $\|\mu\| = 1$. As with the proof of Theorem 2.5, we can choose a net (μ_β) in the convex hull of $\{\gamma\frac{\delta_x}{\omega_1(x)},\ \gamma \in \mathbb{T}\ , x \in G\}$, such that $\mu_\beta \to \mu$ in $M(G, \omega_1)$ strictly. Therefore $T(\mu_\beta) \xrightarrow{w^*} T(\mu)$ in $LUC(H, \omega_2^{-1})$, by Lemma 4.2. But equation (12) and weak-star continuity of $J^*_{\gamma,\phi}$ give $T(\mu_\beta) = J^*_{\gamma,\phi}(\mu_\beta) \xrightarrow{w^*} J^*_{\gamma,\phi}(\mu)$ in $LUC(H, \omega_2^{-1})^*$, so $T(\mu) = J^*_{\gamma,\phi}(\mu)$ belongs to $M(H, \omega_2)$, as needed. $\square$

We conclude by showing that the bidual of the weighted group algebra $L^1(G, \omega)^{**}$ determines the weighted topological group (G, ω). The following lemma is needed for the proof of Proposition 4.6.

Lemma 4.5. *Let (G, ω) be a weighted locally compact group. Then $L^1(G, \omega)^{**}$ has a right identity of norm one.*

Proof. First we show that $L^1(G, \omega)$ has a bounded approximate identity whose terms are of norm one. To see this let U be a compact neighbourhood of the identity element e_G of G and let $(U_i)_{i\in I}$ be a neighbourhood system of e_G directed downwards by inclusion such that $U_i \subseteq U$, for each $i \in I$. Then the proof of [6, Lemma 2.3] shows that the net $(f_i)_{i\in I}$ where $f_i := \frac{\chi_{U_i}}{\lambda(U_i)}$, for each $i \in I$ is a bounded approximate identity of $L^1(G, \omega)$. Let $\psi_i := \frac{1}{\omega} f_i$, for each $i \in I$. Then $\|\psi_i\|_{1,\omega} = 1$, for each $i \in I$. Routine calculations using continuity of ω at e_G show that $\|\psi_i - f_i\|_{1,\omega} \to 0$. Hence, for any $\phi \in L^1(G, \omega)$,

$$\begin{aligned} \|\psi_i * \phi - \phi\|_{1,\omega} &\le \|\psi_i * \phi - f_i * \phi\|_{1,\omega} + \|f_i * \phi - f\|_{1,\omega} \\ &\le \|\psi_i - f_i\|_{1,\omega}\|\phi\|_{1,\omega} + \|f_i * \phi - \phi\|_{1,\omega} \to 0. \end{aligned}$$

Therefore, (ψ_i) is a bounded approximate identity for $L^1(G, \omega)$, with each term having $\|\psi_i\|_{1,\omega} = 1$.

Let E denote a weak-star cluster point of the canonical image of the bounded approximate identity (ψ_i) of $L^1(G, \omega)$ in $L^1(G, \omega)^{**}$. Then by [1, Thm. 7], E is a right identity for $L^1(G, \omega)^{**}$. Since $\|\psi_i\|_{1,\omega} = 1$ for each $i \in I$, it follows that $\|E\| \le 1$. Moreover, $\|E\| = \|E \square E\| \le \|E\|\|E\|$, so $\|E\| \ge 1$ as well. $\square$

Proposition 4.6. *Suppose that (G, ω_1) and (H, ω_2) are weighted locally compact groups, and suppose that $T : L^1(G, \omega_1)^{**} \to L^1(H, \omega_2)^{**}$ is an isometric isomorphism. Then $LUC(G, \omega_1^{-1})^*$ is isometrically isomorphic to $LUC(H, \omega_2^{-1})^*$, and hence (G, ω_1) and (H, ω_2) are isomorphic weighted locally compact groups.*

Proof. Replacing $L^1(G)^{**}$, $L^1(H)^{**}$, $LUC(G)^*$ and $LUC(H)^*$ with their weighted counterparts, the proof of the main theorem of [7] works in the same way. $\square$

Acknowledgments

I would like to thank my PhD supervisor, Professor Ross Stokke, for suggesting this problem as well as his constant interest. Also, I would like to express my gratitude to my PhD supervisors, Professor Fereidoun Ghahramani and Professor Ross Stokke, for many helpful conversations and for their encouragement.

References

[1] F.F. Bonsall, J. Duncan, Complete Normed Algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 80, Springer-Verlag, New York–Heidelberg, 1973.
[2] J.B. Conway, A Course in Functional Analysis, second edition, Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990.
[3] H.G. Dales, A.T. Lau, The second duals of Beurling algebras, Mem. Amer. Math. Soc. 177 (836) (2005), vi+191.
[4] H. Farhadi, Isomorphisms between the second duals of group algebras of locally compact groups, Math. Proc. Cambridge Philos. Soc. 119 (4) (1996).
[5] F. Ghahramani, Compact elements of weighted group algebras, Pacific J. Math. 113 (1) (1984) 77–84.
[6] F. Ghahramani, Weighted group algebra as an ideal in its second dual space, Proc. Amer. Math. Soc. 90 (1) (1984) 71–76.
[7] F. Ghahramani, A.T. Lau, Isometric isomorphisms between the second conjugate algebras of group algebras, Bull. Lond. Math. Soc. 20 (4) (1988) 342–344.
[8] F. Ghahramani, A.T. Lau, V. Losert, Isometric isomorphisms between Banach algebras related to locally compact groups, Trans. Amer. Math. Soc. 321 (1) (1990) 273–283.
[9] F.P. Greenleaf, Norm decreasing homomorphisms of group algebras, Pacific J. Math. 15 (1965).
[10] B.E. Johnson, An introduction to the theory of centralizers, Proc. Lond. Math. Soc. (3) 14 (1964) 299–320.
[11] B.E. Johnson, Isometric isomorphisms of measure algebras, Proc. Amer. Math. Soc. 15 (1964) 186–188.
[12] A.T. Lau, K. McKennon, Isomorphisms of locally compact groups and Banach algebras, Proc. Amer. Math. Soc. 79 (1) (1980) 55–58.
[13] T.W. Palmer, Banach Algebras and the General Theory of *-Algebras. Vol. I, Encyclopedia of Mathematics and Its Applications, vol. 49, Cambridge University Press, 1994.
[14] V. Runde, Lectures on Amenability, Lecture Notes in Mathematics, vol. 1774, Springer-Verlag, Berlin, 2002.
[15] R. Stokke, Homomorphisms of convolution algebras, J. Funct. Anal. 261 (12) (2011) 3665–3695.
[16] J.G. Wendel, On isometric isomorphism of group algebras, Pacific J. Math. 1 (1951) 305–311.
[17] S. Zadeh, Isomorphisms of Banach algebras associated with locally compact groups, PhD thesis, University of Manitoba, Canada, 2015.