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The structure of solutions near a sonic line in gas dynamics via the pressure gradient equation

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A B S T R A C T

A key issue in gas dynamics in two space dimensions is the regularity of solutions near a sonic curve. We build a large class of regular solutions with given boundary conditions on the sonic line. The modeling equation is the pressure gradient equation, which is the same as Euler system when the parameters of the gas are pushed to certain extreme. We use a novel set of coordinates, involving both the space–time and state variables, to split regular terms from singular terms in the analysis.

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1. Introduction

The Euler system models the motion of compressible ideal fluids and has been discussed a lot theoretically, experimentally and numerically in the literature (see [5,7,8]). The pressure gradient (PG) system is derived out of the Euler system either through flux splitting [15] or asymptotic expansion [26,28]. The two dimensional PG system takes the form

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ p_t + pu_x + pv_y = 0, \end{cases}$$

where (u, v) is velocity and p is pressure. One feature of this system is that pressure can be decoupled from u, v to form its own second order quasilinear differential equation

$$(\frac{p_t}{p})_t - p_{xx} - p_{yy} = 0. \tag{1.1}$$

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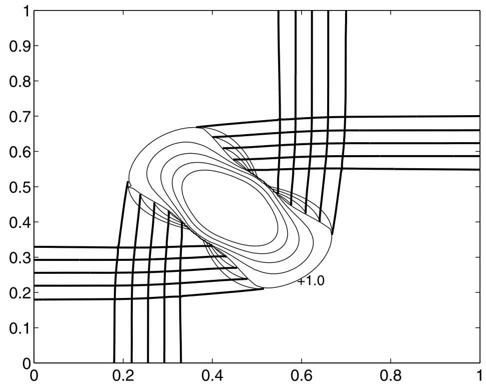

Fig. 1. Interaction of two forward and two backward rarefaction waves for Euler with $\gamma = 1.4$, pressure $p_1 = 0.444$, density $\rho_1 = 1.0$, $\rho_2 = 0.5197$, velocity $u_1 = v_1 = 0.00$ at time T = 0.25. The contour curves are pseudo-March lines where the outmost one marked with +1.0 is the sonic curve. The bold curves from the boundaries and light short ones are all characteristics. The four regions where the bold and the short light characteristics overlap are semi-hyperbolic regions. (Courtesy of Glimm et al. [10].)

The PG system is considered as a useful simplified model for the Euler equations, since interesting observations sometimes were first found in this model and later recovered for the Euler system. Furthermore, the PG system is easier to handle technically.

The well-posedness of the general Cauchy problem or the two dimensional Riemann problem, where the initial data is a constant along each ray through the origin on the physical (x, y) plane, remains a largely open question. There is work though in [6,3,2,27] for constructing the transonic shock solutions in particular situations for the Euler system and other models arising from gas dynamics. We refer to Numerical simulations of the Riemann problem in [11,12,16,4].

The four-wave Riemann problem refers to the initial data, where it is a constant in each quadrant and adjacent state is connected by a single wave. It is a special case of the 2-D Riemann problem. Self-similar solutions depending only on $(\xi = \frac{x}{t}, \eta = \frac{y}{t})$ are expected. And the equation (1.1) is turned into a new form

$$(p-\xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p-\eta^2)p_{\eta\eta} + \frac{1}{p}(\xi p_\xi + \eta p_\eta)^2 - 2(\xi p_\xi + \eta p_\eta) = 0. \tag{1.2}$$

The PG equation (1.2) changes type as the Euler equations in the self-similar plane. A *sonic curve* is where the equation changes type from hyperbolic to elliptic. A conjecture was proposed in [22] with supportive numerical results in [19,16,22] that the solutions of the 4-wave problem have 19 different configurations for polytropic gas modeled by Euler equations and 12 genuinely different configurations for the PG system.

The existence of solution in the elliptic region with data assigned on the sonic curve was proved in [25] for the PG system. Various wave interactions in the hyperbolic region were analyzed in [9,13,14,1,27]. A global classic solution was constructed in [17] to the interaction of four orthogonal planer rarefaction waves with two axes of symmetry for the Euler system. The strengths of the waves were chosen to be large to avoid the occurrence of sonic points and the solution was hyperbolic all the way to the vacuum. We remark that without the restriction on the wave strength, a subsonic domain would be involved in the self-similar plane and the solution will be transonic, which is more interesting yet much more challenging to study.

Another solution for the Euler system related with the transonic phenomenon was constructed in [18] with data assigned on two characteristics, which intersect the sonic curve in two distinguishable points. The solution was proved to exist in a region bounded by the two characteristics and part of the sonic curve. Such a solution is called a *semi-hyperbolic patch* since one family of characteristics emanating from the sonic boundary would form a transonic shock. It was also observed in the numerical simulation [10] in the region covered by overlapping bold and light curves (see Fig. 1). The characteristics are not tangential to the sonic curve when vanishing in the semi-hyperbolic patch (see Fig. 1), which behaves differently as in the

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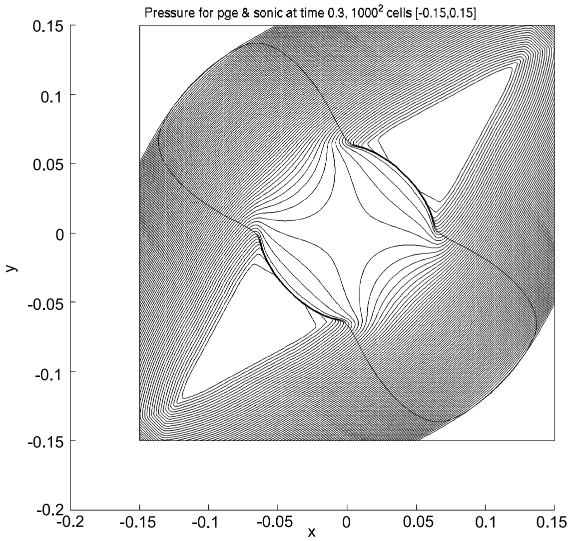

Fig. 2. Interaction for two forward and two backward rarefaction waves for pressure gradient system on square $[-0.15, 0.15] \times [-0.15, 0.15]$ at time T = 0.3. The solution has two axes of symmetry with pressure $p_1 = 1$, $c_2 = \sqrt{p_2} = 0.52$. The demarcation line between subsonic and supersonic regions is the closed contour curve in the center where the light sonic line is connected with the bold shock curve. Shown also are the pressure contour curves computed via Clawpack.

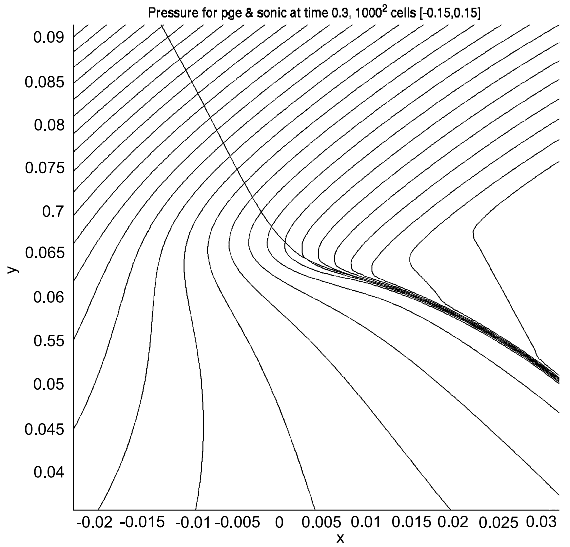

Fig. 3. Enlargement of the portion in Fig. 2 on $[-0.02, 0.03] \times [0.04, 0.09]$.

conjecture [22]. The regularity of the sonic curve in the semi-hyperbolic patch is discussed in the paper [20]. We remark that both observations in [17] and [18] were fist found in the PG system, see [21] for instance.

In our ongoing work, the authors are attempting to construct a class of smooth transonic solutions locally near the sonic curve for the PG system, as observed in the numerical computation [29] (see Fig. 2 and Fig. 3). In this paper we consider an intermediate situation. Namely, we construct solutions in the hyperbolic region with data assigned on the sonic curve.

Consider the polar coordinates (θ, r), where $\xi = r\cos\theta$ and $\eta = r\sin\theta$. Equation (1.2) is changed into

$$(p - r^2)p_{rr} + \frac{p}{r^2}p_{\theta\theta} + \frac{p}{r}p_r + \frac{1}{p}(rp_r)^2 - 2rp_r = 0. \tag{1.3}$$

The PG equation (1.3) is elliptic in the region $r^2 - p < 0$; and is hyperbolic in the region where $r^2 - p > 0$, with the two eigenvalues $\pm\lambda^{-1} = \pm\sqrt{\frac{r^2(r^2-p)}{p}}$. The two families of characteristics are defined by $\frac{dr}{d\theta} = \pm\lambda^{-1}$. The curve $r^2 - p = 0$ is called the *sonic curve* and the PG equation changes type across it.

In this paper, given a piece of *sonic curve*

$$\theta = \varphi(r) \in C^4([r_a, r_b]), \quad 0 < r_a < r < r_b, \tag{1.4}$$

denote it by Γ, we study a degenerate Cauchy problem for (1.3) with the initial value

$$p(\varphi(r), r) = r^2, \quad p_\theta(\varphi(r), r) = a_0(r)^2 \in C^3([r_a, r_b]). \tag{1.5}$$

We assume that

$$|\varphi'| \leq \gamma, \tag{1.6}$$

and

$$a_0^2 \geq \alpha > 0, \tag{1.7}$$

where γ and α are positive constants. We remark that (1.6) excludes that Γ is a circular arc. Furthermore at the sonic line

$$p_r = 2r - \varphi' \cdot p_\theta = 2r - \varphi' \cdot a_0^2 =: a_1(r). \tag{1.8}$$

Our theorem of existence is stated as follows.

Theorem 1. *Given a smooth sonic curve Γ as in (1.4), there is a classic solution to (1.3) in the hyperbolic region near Γ, and satisfies the boundary conditions (1.5) under the assumptions of (1.6) and (1.7).*

In section 2, new coordinates are introduced based on the geometry near the sonic curve. The main result of existence of solutions in the new coordinates is stated. The proof of the existence theorem is worked out in section 3 through the iteration method. Various *a priori* estimates are given, and we show that the iteration sequence converges under a new weighted metric. In the last section, we convert the solution in section 3 back to the polar coordinates and obtain Theorem 1.

2. Statement of the main result in new coordinates

In the hyperbolic region, there is a characteristic decomposition for (1.3)

$$\begin{aligned} \partial_+\partial_- p &= q(\partial_+ p - \partial_- p)\partial_- p, \\ \partial_-\partial_+ p &= q(\partial_- p - \partial_+ p)\partial_+ p, \end{aligned} \tag{2.1}$$

where $\partial_\pm = \partial_\theta \pm \lambda^{-1}\partial_r$ are differentiation along characteristics and $q = \frac{r^2}{4p(r^2-p)}$ blows up at the sonic line. We remark this decomposition was derived in [9].

After introducing $R = \partial_+ p$ and $S = \partial_- p$, for smooth solutions we rewrite the equations (2.1) as the following system for (p, R, S),

$$
\begin{aligned}
p_\theta &= \frac{R+S}{2}, \\
\partial_- R &= q(S-R)R, \\
\partial_+ S &= q(R-S)S.
\end{aligned} \tag{2.2}
$$

On the sonic curve there holds

$$
(p,R,S)\big|_\Gamma = (r^2, a_0^2, a_0^2). \tag{2.3}
$$

One difficulty in the above system for R and S is that the term q blows up in the order of $(r^2 - p)^{-1}$ when approaching the sonic curve, while $R-S$ reaches 0. We notice the two characteristics become tangential to a circular curve on the sonic line Γ, but the characteristics are non-tangential to the sonic curve. Observing this geometry, we therefore introduce the level curves $\sqrt{r^2-p} = constants$ and $r = constants$ as new grid curves. Indeed, to fix the idea, let

$$
\tilde{r} = r, \quad t = \sqrt{r^2 - p(r,\theta)}. \tag{2.4}
$$

Therefore $\partial_\theta = -\frac{p_\theta}{2t}\partial_t$, $\partial_r = \partial_{\tilde{r}} + \frac{2r-p_r}{2t}\partial_t$.

The first equation in system (2.2) becomes a decoupled one

$$
\partial_t p = -2t. \tag{2.5}
$$

Without abuse of notation, we will denote $\tilde{r}$ by r. The other equations can be simplified as the following system

$$
\begin{aligned}
R_t + \frac{2t\lambda^{-1}}{S+2r\lambda^{-1}} R_r &= -\frac{2t^2 q}{S+2r\lambda^{-1}} R\left(\frac{S-R}{t}\right), \\
S_t - \frac{2t\lambda^{-1}}{R-2r\lambda^{-1}} S_r &= -\frac{2t^2 q}{R-2r\lambda^{-1}} S\left(\frac{R-S}{t}\right),
\end{aligned} \tag{2.6}
$$

where

$$
\lambda^{-1}(t,r) = \frac{tr}{\sqrt{r^2-t^2}}, \quad q(t,r) = \frac{r^2}{4(r^2-t^2)t^2}. \tag{2.7}
$$

The system (2.6) is closed in the new coordinates and the sonic curve is flattened to be on $t=0$. We assign boundary conditions

$$
R(0,r) = S(0,r) = a_0^2(r), \quad R_t(0,r) = -S_t(0,r) = a_1(r), \tag{2.8}
$$

where a_1 is defined in (1.8). In fact, the first condition is from (2.3). For the second one, it is motivated by the following observation. When adding and subtracting the two equations in (2.6) and evaluating on the sonic curve, we have

$$
R_t - S_t = \frac{R-S}{t} = 2a_1, \quad R_t + S_t = 0.
$$

We will show the system (2.6) with boundary conditions (2.8) has a solution in the region $t>0$ near the sonic curve. To work out the proof more easily, we define the higher order terms as

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$$\begin{aligned} U(t,r) &= R(t,r) - a_0^2(r) - a_1(r)t, \\ V(t,r) &= S(t,r) - a_0^2(r) + a_1(r)t. \end{aligned} \tag{2.9}$$

The system (2.6) is transformed into

$$\begin{aligned} U_t + \frac{2t\lambda^{-1}}{S+2r\lambda^{-1}}U_r &= \frac{1}{2}\left(\frac{U-V}{t}\right) + \left(\frac{2t^2qR}{S+2r\lambda^{-1}} - \frac{1}{2}\right)\left(\frac{U-V}{t} + 2a_1\right) \\ &\quad - \frac{2t\lambda^{-1}}{S+2r\lambda^{-1}}\left(\partial_r(a_0^2 + ta_1)\right), \\ V_t - \frac{2t\lambda^{-1}}{R-2r\lambda^{-1}}V_r &= \frac{1}{2}\left(\frac{V-U}{t}\right) + \left(\frac{2t^2qS}{R-2r\lambda^{-1}} - \frac{1}{2}\right)\left(\frac{V-U}{t} - 2a_1\right) \\ &\quad + \frac{2t\lambda^{-1}}{R-2r\lambda^{-1}}\left(\partial_r(a_0^2 - ta_1)\right). \end{aligned} \tag{2.10}$$

And the boundary conditions (2.8) become homogeneous

$$U(0,r) = V(0,r) = U_t(0,r) = V_t(0,r) = 0, \quad r_a \le r \le r_b. \tag{2.11}$$

We indicate the two eigenvalues of (2.10) as

$$\Lambda_+(V) = \frac{2t\lambda^{-1}}{V + a_0^2 - a_1 t + 2r\lambda^{-1}}, \quad \Lambda_-(U) = -\frac{2t\lambda^{-1}}{U + a_0^2 + a_1 t - 2r\lambda^{-1}}.$$

Let

$$D(\delta_0) := \{(t,r) | 0 \le t \le \delta_0, \quad r_1(t) \le r \le r_2(t)\},$$

where $r_1(t), r_2(t)$ are continuously differentiable on $0 \le t \le \delta_0$, $r_1(0) = r_a, r_2(0) = r_b$ and $r_1 < r_2$ for $0 \le t \le \delta_0$.

Definition 1. The domain $D(\delta_0)$ is called a strong domain of determinacy to system (2.10) if for any (U,V) continuously differentiable, satisfying $U(0,\cdot) = V(0,\cdot) = U_t(0,\cdot) = V_t(0,\cdot) = 0$ and $(\xi,\eta) \in D(\delta_0)$, the plus and minus characteristics associated with it, i.e. the integral curves, denoted by $r_\pm(t;\xi,\eta)$, to $\frac{dr}{dt} = \Lambda_\pm(U,V)$ satisfying $r_\pm(\xi) = \eta$ are also inside $D(\delta_0)$ for $0 < t \le \xi$.

Next we consider a refined class of functions, because the higher order terms (U,V) have very small magnitude near the sonic line $t = 0$. Solutions of the Cauchy problem (2.10) with condition (2.11) will be obtained within this class. Let $\mathcal{S} = \mathcal{S}(M,\delta)$ consisting of all continuously differentiable functions $\mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : D(\delta) \to \mathbb{R}^2$ satisfying the following properties

(S1) $f_1(0,r) = f_2(0,r) = \partial_t f_1(0,r) = \partial_t f_2(0,r) = 0$,

(S2) $\left\|\frac{f_1}{t^2}\right\|_\infty + \left\|\frac{f_2}{t^2}\right\|_\infty \le M$,

(S3) $\left\|\frac{\partial_r f_1}{t^2}\right\|_\infty + \left\|\frac{\partial_r f_2}{t^2}\right\|_\infty \le M$, (S4) $\partial_r \mathbf{F}$ is Lipschitz continuous with respect to r with

$$\left\| \frac{\partial_{rr}^2 f_1}{t^2} \right\|_\infty + \left\| \frac{\partial_{rr}^2 f_2}{t^2} \right\|_\infty \le M,$$

where $||\cdot||_\infty$ is the supremum norm over domain $D(\delta)$. We denote $\mathcal{W}$ the larger class containing only continuous functions on $D(\delta)$ which satisfy the first two conditions (S1) and (S2). Both $\mathcal{S}$ and $\mathcal{W}$ are subsets of $C^0(D(\delta); \mathbb{R}^2)$. We use the following weighted metric for $\mathcal{S}$ and $\mathcal{W}$

$$d(\mathbf{F}, \mathbf{G}) := \left\| \frac{f_1 - g_1}{t^2} \right\|_\infty + \left\| \frac{f_2 - g_2}{t^2} \right\|_\infty . \tag{2.12}$$

Remark 1. $(\mathcal{W}, d)$ is a complete metric space. However, $(\mathcal{S}, d)$ is not a closed subset in $(\mathcal{W}, d)$.

Theorem 2. *Under the assumptions (1.4)–(1.7) for a_0 and φ, and that $D(\delta_0)$ is a strong domain of determinacy to the system (2.10), there exist constants $\delta \in (0, \delta_0)$ and M, such that the degenerate hyperbolic system (2.10) with boundary condition (2.11) has a classical solution in the function class $\mathcal{S}(M, \delta)$.*

In turn, Theorem 2 yields the existence of a classic solution to the system (2.6), locally near sonic curve, with data (2.8).

3. Proof of the existence theorem in the new coordinates

We will construct an integration iteration out of the system (2.10) and demonstrate the iteration mapping is a contraction in the function space $\mathcal{S}$. Hence the iteration sequence yields a limit which solves the equation in the classical sense.

Assume $u(t, r)$ and $v(t, r)$ are admissible functions on $D(\delta)$, i.e., they belong to set $\mathcal{S}$. Differentiation along the two associated characteristics is defined by

$$\frac{d}{d_+(v)} := \partial_t + \Lambda_+(v)\partial_r, \quad \frac{d}{d_-(u)} := \partial_t + \Lambda_-(u)\partial_r. \tag{3.1}$$

Let $b_1(u, v)$ and $b_2(u, v)$ be the expressions in the right hand sides of (2.10). The iteration we are going to use can be written briefly as

$$\frac{d}{d_+(v)} U = b_1(u, v), \quad \frac{d}{d_-(u)} V = b_2(u, v). \tag{3.2}$$

We rewrite (3.2) in integration form

$$U(\xi, \eta) = \int_0^\xi b_1(t, r_+(t; \xi, \eta), u, v)\, dt, \tag{3.3}$$

$$V(\xi, \eta) = \int_0^\xi b_2(t, r_-(t; \xi, \eta), u, v)\, dt, \tag{3.4}$$

where the arguments of u, v under the integral signs are $(t, r_+(t; \xi, \eta))$ and $(t, r_-(t; \xi, \eta))$ respectively.

A mapping is thus determined: $T\left(\begin{pmatrix} u \\ v \end{pmatrix}\right) = \begin{pmatrix} U \\ V \end{pmatrix}$.

The system (2.10) with boundary condition (2.11) has a solution is equivalent to that the mapping T has a fixed point. If (U, V) given by (3.3) and (3.4) is continuously differentiable, it is a solution to (3.2). Properties of mapping T are given in the following lemmas.

Lemma 1. *Under the assumptions of Theorem 2, there exist positive constants δ, M and $0 < \beta < 1$ such that*

i. T maps $\mathcal{S}$ into $\mathcal{S}$;
ii. for any pair $\mathbf{F}, \hat{\mathbf{F}}$ in $\mathcal{S}$, there holds

$$d\left(T(\mathbf{F}), T(\hat{\mathbf{F}})\right) \le \beta\, d\left(\mathbf{F}, \hat{\mathbf{F}}\right).$$

The constants M, δ, β depend only on the C^3 norms of a_0, a_1, α and $D(\delta_0)$.

Proof. We will use $k > 1$ to denote a constant depending only on C^3 norms of a_0, a_1, α and r_a, r_b. We use C to denote a constant depending on k and M; they may vary from one line to another.

Let $\mathbf{F} = (u, v)$, $\hat{\mathbf{F}} = (\hat{u}, \hat{v})$ be in set $\mathcal{S}$ and $\mathbf{G} = T(\mathbf{F}) = (U, V)$ and $\hat{\mathbf{G}} = T(\hat{\mathbf{F}}) = (\hat{U}, \hat{V})$. It is obvious $U(0, \eta) = V(0, \eta) = 0$.

Let us check that conditions (S2) and (S3) hold. We rewrite the first term in $b_1(u, v)$ as $\frac{t}{2}\left(\frac{u-v}{t^2}\right)$ and notice that

$$\begin{aligned} \frac{2t^2 qR}{S + 2r\lambda^{-1}} - \frac{1}{2} &= \frac{R - S + \dfrac{t^2 R}{r^2 - t^2} - 2r\lambda^{-1}}{2(S + 2r\lambda^{-1})} \\ &= t \cdot \frac{t \cdot \dfrac{u - v}{t^2} + 2a_1 + \dfrac{t(u + a_0^2 + a_1 t)}{r^2 - t^2} - \dfrac{2r^2}{\sqrt{r^2 - t^2}}}{2(v + a_0^2 - a_1 t + 2r\lambda^{-1})}. \end{aligned} \tag{3.5}$$

Hence

$$\left| \frac{2t^2 qR}{S + 2r\lambda^{-1}} - \frac{1}{2} \right| \le tk \cdot (k + Mt). \tag{3.6}$$

Furthermore if we differentiate (3.5) with respect to r, there hold

$$\left| \partial_r \left(\frac{2t^2 qR}{S + 2r\lambda^{-1}} - \frac{1}{2} \right) \right| \le tk \cdot (k + Mt), \tag{3.7}$$

$$\left| \partial_{rr} \left(\frac{2t^2 qR}{S + 2r\lambda^{-1}} - \frac{1}{2} \right) \right| \le tk \cdot (k + Mt). \tag{3.8}$$

For simplicity, denote the last term in $b_1(u, v)$ as

$$\Phi = \frac{2t\lambda^{-1}}{S + 2r\lambda^{-1}} \left(\partial_r (a_0^2 + t a_1) \right). \tag{3.9}$$

It is easy to see $|\Phi(a_0, a_1, v)| \le kt^2(1 + t)$, so b_1 can be estimated as

$$\begin{aligned} |b_1(u, v)| &= \left| \frac{1}{2} \left(\frac{U - V}{t} \right) + \left(\frac{2t^2 qR}{S + 2r\lambda^{-1}} - \frac{1}{2} \right) \left(\frac{U - V}{t} + 2a_1 \right) - \Phi(a_0, a_1, V) \right| \\ &\le \frac{tM}{2} + tk(k + Mt)^2 + kt^2(1 + t). \end{aligned} \tag{3.10}$$

Integrating b_1 along plus characteristic according to (3.3), we obtain

$$U(\xi,\eta) \le \frac{\xi^2}{4}M + k(k+M\delta)^2\frac{\xi^2}{2} + \xi^3 k.$$

Thus

$$\left|\frac{U(\xi,\eta)}{\xi^2}\right| \le \frac{M}{4} + k(1+M\delta)^2. \tag{3.11}$$

And from (3.4), the same bound can be obtained for $\frac{V}{\xi^2}$, i.e.,

$$\left|\frac{V(\xi,\eta)}{\xi^2}\right| \le \frac{M}{4} + k(1+M\delta)^2. \tag{3.12}$$

Adding the above two inequalities together we obtain

$$\left|\frac{U(\xi,\eta)}{\xi^2}\right| + \left|\frac{V(\xi,\eta)}{\xi^2}\right| \le \frac{M}{2} + k(1+M\delta)^2. \tag{3.13}$$

To see the bounds for $\frac{\partial_r U}{t^2}$ and $\frac{\partial_r V}{t^2}$, we differentiate (3.3) w.r.t. η and obtain

$$\frac{\partial U}{\partial \eta}(\xi,\eta) = \int_0^{\xi} \frac{\partial b_1(u,v)}{\partial r}\cdot\frac{\partial r_+}{\partial \eta}\,dt, \tag{3.14}$$

where

$$\frac{\partial r_+}{\partial \eta}(t;\xi,\eta) = \exp\int_{\xi}^{t} \frac{\partial \Lambda_+(v)}{\partial r}(\tau, r_+(\tau;\xi,\eta))\,d\tau. \tag{3.15}$$

The terms $\frac{\partial \Lambda_+(v)}{\partial r}$, $\frac{\partial b_1(u,v)}{\partial r}$ are the partial derivatives with respect to r of the composite function $\Lambda(t,r,v(t,r))$, $b_1(t,r,u(t,r),v(t,r))$. More precisely,

$$\begin{aligned}
&\frac{\partial \Lambda_+(v)}{\partial r}(t, r(t;\xi,\eta)) \\
&= \frac{2t^2\mu - 2rt^2[(2a_0a_0' - ta_1')\sqrt{r^2-t^2} + (a_0^2 - a_1 t)\frac{r}{\sqrt{r^2-t^2}} + 4tr]}{\mu^2} \\
&- \frac{2rt^2}{\mu^2}(\sqrt{r^2-t^2}\frac{\partial v}{\partial r} + \frac{rv}{\sqrt{r^2-t^2}}),
\end{aligned} \tag{3.16}$$

while

$$\mu(t,r,v) = (v + a_0^2(r) - a_1(r)t)\sqrt{r^2-t^2} + 2r^2 t. \tag{3.17}$$

We can see that there is a factor t^2 in (3.16), so

$$\int_{\xi}^{t} \frac{\partial \Lambda_+(v)}{\partial r}(\tau, r_+(\tau;\xi,\eta))\,d\tau \le (k + kM\delta^2)|\xi|^3.$$

We further notice $\frac{\partial r_+}{\partial \eta}$ is continuous and

$$\left|\frac{\partial r_+}{\partial \eta}\right| \leq e^{(k+kM\delta^2)\delta^3}. \tag{3.18}$$

Next we estimate the term

$$\begin{aligned}\frac{\partial b_1(u,v)}{\partial r} &= \frac{1}{2}\left(\frac{\partial_r u - \partial_r v}{t}\right) + \left(\frac{2t^2 qR}{S+2r\lambda^{-1}} - \frac{1}{2}\right)\left(\frac{\partial_r u - \partial_r v}{t} + 2a_1'(r)\right) \\ &+ \partial_r\left(\frac{2t^2 qR}{S+2r\lambda^{-1}}\right)\cdot\left(\frac{u-v}{t}+a_1\right) - \partial_r\Phi(a_0,a_1,v) \\ &= I + II + III + IV.\end{aligned} \tag{3.19}$$

For I, we rewrite it as $\frac{t}{2}\left(\frac{\partial_r u - \partial_r v}{t^2}\right)$, then

$$|I| \leq \frac{t}{2}\left(\left\|\frac{\partial_r u}{t^2}\right\|_\infty + \left\|\frac{\partial_r v}{t^2}\right\|_\infty\right) \leq \frac{t}{2}M.$$

For II, using (3.6) we obtain

$$|II| \leq tk\cdot(k+Mt)^2.$$

For III, by (3.7) we obtain

$$|III| \leq tk\cdot(k+Mt)^2.$$

For IV, there holds

$$|IV| \leq t^2 k.$$

Therefore from these estimates, (3.14) renders

$$\begin{aligned}\left|\frac{\partial U}{\partial \eta}(\xi,\eta)\right| &\leq e^{(k+kM\delta^2)\delta^3}\cdot\int_0^\xi |I+II+III+IV|\,dt \\ &\leq e^{(k+kM\delta^2)\delta^3}\cdot\left(\tfrac{\xi^2}{4}M + \tfrac{\xi^2}{2}k(k+M\delta)^2\right).\end{aligned} \tag{3.20}$$

Dividing the above estimate by ξ^2, we obtain

$$\left|\frac{1}{\xi^2}\frac{\partial U}{\partial \eta}(\xi,\eta)\right| \leq e^{(k+kM\delta^2)\delta^3}\left(\frac{1}{4}M + \frac{1}{2}k(k+M\delta)^2\right).$$

Same estimates can be applied to $\frac{\partial V}{\partial \eta}$, therefore

$$\left|\frac{1}{\xi^2}\frac{\partial V}{\partial \eta}(\xi,\eta)\right| \leq e^{(k+kM\delta^2)\delta^3}\left(\frac{1}{4}M + \frac{1}{2}k(k+M\delta)^2\right).$$

We add them to obtain

$$\left|\frac{1}{\xi^2}\frac{\partial U}{\partial \eta}(\xi,\eta)\right| + \left|\frac{1}{\xi^2}\frac{\partial V}{\partial \eta}(\xi,\eta)\right| \le e^{(k+kM\delta^2)\delta^3}\left(\frac{1}{2}M + k(1+M\delta)^2\right). \tag{3.21}$$

Let us check that (S4) is also preserved. We compute directly

$$\begin{aligned}\frac{\partial^2 U}{\partial \eta^2}(\xi,\eta) &= \int_0^\xi \frac{\partial}{\partial r}\left(\frac{\partial b_1}{\partial r}(t, r(t,\xi,\eta))\right)(\frac{\partial r}{\partial \eta})^2 + \frac{\partial b_1}{\partial r}\frac{\partial^2 r}{\partial \eta^2}\, dt \\ &:= I + II.\end{aligned} \tag{3.22}$$

For I, we notice

$$\begin{aligned}\frac{\partial^2 b_1}{\partial r^2} &= \frac{1}{2}\left(\frac{\partial^2_{rr}u - \partial^2_{rr}v}{t}\right) + \partial_r\left(\frac{2t^2qR}{S+2r\lambda^{-1}} - \frac{1}{2}\right)\cdot\left(\frac{\partial_r u - \partial_r v}{t} + 2a_1'(r)\right) \\ &+ \left(\frac{2t^2qR}{S+2r\lambda^{-1}} - \frac{1}{2}\right)\cdot\left(\frac{\partial^2_{rr}u - \partial^2_{rr}v}{t} + 2a_1''(r)\right) \\ &+ \partial_r\left(\frac{2t^2qR}{S+2r\lambda^{-1}}\right)\cdot\left(\frac{\partial_r u - \partial_r v}{t} + a_1'(r)\right) \\ &+ \partial^2_{rr}\left(\frac{2t^2qR}{S+2r\lambda^{-1}}\right)\cdot\left(\frac{u-v}{t} + a_1\right) - \partial^2_{rr}\Phi(a_0,a_1,v).\end{aligned} \tag{3.23}$$

According to (3.6), (3.7), (3.8) and (3.9), we have

$$\left|\frac{\partial^2 b_1}{\partial r^2}(t,r)\right| \le \frac{Mt}{2} + tk(k+Mt)^2 + t^2k(1+t). \tag{3.24}$$

Combining it with (3.18) and integrating from 0 to ξ with respect to t, we obtain

$$|I| \le e^{2(k+kM\delta^2)\delta^3}\left(\frac{\xi^2}{4}M + \xi^2 k(1+M\delta)^2\right).$$

For II, we differentiate (3.15) to obtain

$$\frac{\partial^2 r}{\partial \eta^2}(t;\xi,\eta) = \frac{\partial r}{\partial \eta}\cdot\int_\xi^t \frac{\partial^2\Lambda_+}{\partial r^2}\cdot\frac{\partial r}{\partial \eta}(\tau, r(\tau;\xi,\eta))\, d\tau. \tag{3.25}$$

From (3.16) we note $\frac{\partial^2\Lambda_+}{\partial r^2}(\tau, r(\tau;\xi,\eta))$ is bounded by $\tau^2(k+M\tau^2)$, so

$$\left|\frac{\partial^2 r}{\partial \eta^2}(t;\xi,\eta)\right| \le \left|\frac{\partial r}{\partial \eta}\right|^2 \int_\xi^t \frac{\partial^2\Lambda_+}{\partial r^2}\, d\tau \le e^{2(k+kM\delta^2)\delta^3}\xi^3(k+M\delta_0^2).$$

Due to (3.19) there holds

$$\int_0^\xi \left|\frac{\partial b_1}{\partial r}(t;\xi,\eta)\right| dt \le \xi^2\left(\frac{M}{4} + k(1+M\delta_0)^2\right).$$

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As a consequence we obtain

$$|II| \le \left|\frac{\partial^2 r}{\partial \eta^2}\right| \cdot \int_0^\xi \left|\frac{\partial b_1}{\partial r}\right| dt \le Ce^{2(k+kM\delta^2)\delta^3}\xi^5.$$

Dividing both sides of (3.22) by ξ^2 we obtain

$$\left|\frac{1}{\xi^2}\frac{\partial^2 U}{\partial \eta^2}\right| \le \frac{1}{\xi^2}(|I|+|II|) \le e^{2(k+kM\delta_0^2)\delta^3}(\frac{M}{4}+k+C(\xi+\xi^2+\xi^3)).$$

By the same reason we obtain

$$\left|\frac{1}{\xi^2}\frac{\partial^2 V}{\partial \eta^2}\right| \le e^{2(k+kM\delta_0^2)\delta^3}(\frac{M}{4}+k+C(\xi+\xi^2+\xi^3)).$$

Thus

$$\left|\frac{1}{\xi^2}\frac{\partial^2 U}{\partial \eta^2}\right| + \left|\frac{1}{\xi^2}\frac{\partial^2 V}{\partial \eta^2}\right| \le e^{k(1+M\delta_0^2)\delta^3}(\frac{M}{2}+k+C(\delta+\delta^2+\delta^3)). \tag{3.26}$$

We can choose M sufficiently large (for example $\frac{1}{2}M+k<\frac{3}{4}M$), and pick δ very small such that the right hand sides of (3.13), (3.21) and (3.26) do not exceed M. We have shown that $(S2-S4)$ are preserved by the mapping T.

We can also derive

$$\frac{\partial r}{\partial \xi}(t;\xi,\eta) = -\Lambda_+(\xi,\eta,v(\xi,\eta))\cdot\frac{\partial r}{\partial \eta}(t;\xi,\eta), \tag{3.27}$$

$$\frac{\partial U}{\partial \xi}(\xi,\eta) = \int_0^\xi \frac{\partial b_1(u,v)}{\partial r}\cdot\frac{\partial r}{\partial \xi}\,dt + b_1(\xi,\eta,u(\xi,\eta),v(\xi,\eta)). \tag{3.28}$$

Combining the previous estimates, we obtain that U,V are continuously differentiable as well and $\frac{\partial U}{\partial \xi}(0,\eta) = \frac{\partial V}{\partial \xi}(0,\eta)=0$. Hence the map T does map $\mathcal{S}$ into itself.

Next we show T is a contraction under the new metric, i.e.,

$$d(\mathbf{G},\hat{\mathbf{G}}) \le \beta\, d(\mathbf{F},\hat{\mathbf{F}}).$$

According to the definition of the mapping T, we have

$$\frac{d}{d_+(v)}U = b_1(u,v), \quad \frac{d}{d_+(\hat v)}\hat U = b_1(\hat u,\hat v). \tag{3.29}$$

Recalling $\frac{d}{d_+(v)}$ and $\frac{d}{d_-(u)}$ defined by (3.2), we obtain

$$\frac{d}{d_+(v)}(U-\hat U)(t,r) \;=\; b_1(u,v)-b_1(\hat u,\hat v)+(\Lambda_+(\hat v)-\Lambda_+(v))\,\partial_r\hat U. \tag{3.30}$$

We estimate the right-hand side term by term.

$$
\begin{aligned}
b_1(u,v)-b_1(\hat{u},\hat{v}) &= \frac{1}{2}\left(\frac{u-\hat{u}}{t}-\frac{v-\hat{v}}{t}\right)\\
&+\left(\frac{2t^2q(u+a_0^2+a_1t)}{v+a_0^2-a_1t+2r\lambda^{-1}}-\frac{1}{2}\right)\left(\frac{u-\hat{u}}{t}-\frac{v-\hat{v}}{t}\right)\\
&+2t^2q(\frac{\hat{u}-\hat{v}}{t}+a_1)\left(\frac{u+a_0^2+a_1t}{v+a_0^2-a_1t+2r\lambda^{-1}}-\frac{\hat{u}+a_0^2+a_1t}{\hat{v}+a_0^2-a_1t+2r\lambda^{-1}}\right)\\
&-(\Phi(v)-\Phi(\hat{v})) := I+II+III+IV.
\end{aligned}
\tag{3.31}
$$

For the first two terms

$$
|I|\le\frac{t}{2}\left(\left\|\frac{u-\hat{u}}{t^2}\right\|_\infty+\left\|\frac{v-\hat{v}}{t^2}\right\|_\infty\right)=\frac{t}{2}d(\mathbf{F},\hat{\mathbf{F}}),
$$

$$
|II|\le\frac{Ct(1+\delta+\delta^2)}{\alpha-C\delta(1+\delta)}\cdot t\cdot\left(\left\|\frac{u-\hat{u}}{t^2}\right\|_\infty+\left\|\frac{v-\hat{v}}{t^2}\right\|_\infty\right)\le Ct^2\cdot d(\mathbf{F},\hat{\mathbf{F}})
$$

due to (3.5).

For III, recall $q=\frac{r^2}{4(r^2-t^2)t^2}$ therefore

$$
\left|2t^2q(\frac{\hat{u}-\hat{v}}{t}-a_1)\right|\le\frac{2r_b^2}{r_a^2-\delta_0^2}(k+Mt)=k(k+M\delta_0).
$$

For the remaining terms, we have

$$
\begin{aligned}
&\frac{u+a_0^2+a_1t}{v+a_0^2-a_1t+2r\lambda^{-1}}-\frac{\hat{u}+a_0^2+a_1t}{\hat{v}+a_0^2-a_1t+2r\lambda^{-1}}\\
&=(\hat{u}+a_0^2+a_1t)\left(\frac{1}{v+a_0^2-a_1t+2r\lambda^{-1}}-\frac{1}{\hat{v}+a_0^2-a_1t+2r\lambda^{-1}}\right)\\
&+\frac{u-\hat{u}}{v+a_0^2-a_1t+2r\lambda^{-1}}\\
&\le(k+k\delta_0+M\delta_0^2)\frac{|v-\hat{v}|}{(\alpha-\delta_0(k+M\delta_0))^2}+\frac{|u-\hat{u}|}{\alpha-\delta_0(k+M\delta_0)}.
\end{aligned}
\tag{3.32}
$$

The consequent estimate for III is obtained as

$$
|III|\le C\left(\frac{|u-\hat{u}|}{\alpha-C\delta_0(1+\delta_0)}+\frac{|v-\hat{v}|}{(\alpha-C\delta_0(1+\delta_0))^2}\right)\le Ct^2\cdot d(\mathbf{F},\hat{\mathbf{F}}).
$$

For IV, from the definition of Φ in (3.9) we have

$$
\begin{aligned}
|IV| &= \left|2t\lambda^{-1}\cdot\partial_r(a_0^2+a_1t)\left(\frac{1}{v+a_0^2-a_1t}-\frac{1}{\hat{v}+a_0^2-a_1t}\right)\right|\\
&\le Ct^4d(\mathbf{F},\hat{\mathbf{F}}).
\end{aligned}
$$

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To estimate the remaining terms on the righthand side of (3.30), we notice that

$$\begin{aligned} &|(\Lambda_+(\hat{v})-\Lambda_+(v))| \\ = &\left|2t\lambda^{-1}\left(\frac{1}{v+a_0^2-a_1t+2r\lambda^{-1}}-\frac{1}{\hat{v}+a_0^2-a_1t+2r\lambda^{-1}}\right)\right| \\ \le &\ kt^2\frac{|v-\hat{v}|}{(\alpha-\delta_0(k+M\delta_0))^2}. \end{aligned} \tag{3.33}$$

Therefore

$$\left|(\Lambda_+(\hat{v})-\Lambda_+(v))\,\partial_r\hat{U}\right| \le Ct^2\cdot t^2\left\|\frac{v-\hat{v}}{t^2}\right\|_\infty\cdot t^2\left\|\frac{\partial_r\hat{U}}{t^2}\right\|_\infty \le Ct^6 d(\mathbf{F},\hat{\mathbf{F}}).$$

The last inequality holds because $\left\|\frac{\partial_r\hat{U}}{t^2}\right\|_\infty \le M$. Notice the above five estimates all contain factor t. If we integrate (3.30) along the characteristic, we will obtain the factor $\frac{t^2}{2}$ on the right-hand side, which leads to the following inequality

$$\left|\frac{U-\hat{U}}{t^2}\right| \le \left(\frac{1}{4}+Ct(1+t^2+t^4)\right)d(\mathbf{F},\hat{\mathbf{F}}).$$

Similarly, we have

$$\frac{d}{d_-(u)}V=b_2(u,v),\qquad \frac{d}{d_-(\hat{u})}\hat{V}=b_2(\hat{u},\hat{v}).$$

Following the same argument as above, we obtain

$$\left|\frac{V-\hat{V}}{t^2}\right| \le \left(\frac{1}{4}+Ct(1+t^2+t^4)\right)d(\mathbf{F},\hat{\mathbf{F}}).$$

All together we have

$$d(G,\hat{G})=\left\|\frac{U-\hat{U}}{t^2}\right\|_\infty+\left\|\frac{V-\hat{V}}{t^2}\right\|_\infty \le \left(\frac{1}{2}+C\delta(1+\delta^2+\delta^4)\right)d(\mathbf{F},\hat{\mathbf{F}}). \tag{3.34}$$

By probably making δ even smaller the number $\beta=\frac{1}{2}+C\delta(1+\delta^2+\delta^4)$ is strictly smaller than 1. Hence T is a contraction under the metric d. $\square$

For any $\mathbf{F}^{(1)}\in\mathcal{S}$, let $\mathbf{F}^{(\mathbf{n})}=T\mathbf{F}^{(\mathbf{n-1})}$, then the iteration sequence $\{\mathbf{F}^{(\mathbf{n})}\}$ is Cauchy in $(\mathcal{W},d)$ which is a complete metric space. So the limit is in $\mathcal{W}$.

Recalling Remark 1 that $(\mathcal{S},d)$ is not closed in $(\mathcal{W},d)$, the limit might not stay in $\mathcal{S}$. However the following lemmas guarantee that the limit is differentiable and does stay in $\mathcal{S}$.

Lemma 2. *Under the assumptions of Theorem 2, the iteration sequence $\{\mathbf{F}^{(n)}\}$ has the property that $\{\frac{\partial\mathbf{F}^{(\mathbf{n})}(\mathbf{t},\mathbf{r})}{\partial t}\}$ and $\{\frac{\partial\mathbf{F}^{(\mathbf{n})}(\mathbf{t},\mathbf{r})}{\partial r}\}$ are uniformly Lipschitz continuous on $D(\delta)$.*

We need several lemmas to see this point.

Lemma 3. *For the iteration sequence* $\mathbf{F}^{(n)} = (u^{(n)}, v^{(n)})$, *there holds*

$$\left\| \frac{\partial_t u^{(n)}}{t} \right\|_\infty + \left\| \frac{\partial_t v^{(n)}}{t} \right\|_\infty \leq 2M.$$

Proof. From (3.19), (3.27) and (3.28) and note the fact that the iteration sequence stays within set $\mathcal{S}$, we obtain

$$\left| \partial_\xi u^{(n)} \right| \leq C\xi^4 + \frac{\xi}{2}M + \xi k(k + M\xi)^2 + k\xi^2.$$

Thus

$$\left| \frac{\partial_\xi u^{(n)}}{\xi} \right| \leq \frac{M}{2} + k + C(\xi + \xi^2 + \xi^3).$$

Similarly we have same bound for $\left| \frac{\partial_\xi v^{(n)}}{\xi} \right|$. By choosing M large and δ small, the conclusion of the lemma is obtained. □

As a result, the sequence $\{\partial_t \mathbf{F}^{(n)}\}$ is uniformly bounded. We next prove another estimate on $\{\partial_{tr}^2 \mathbf{F}^{(n)}\}$.

Lemma 4. *For the iteration sequence* $\mathbf{F}^{(n)} = (u^{(n)}, v^{(n)})$, *there holds*

$$\left\| \frac{\partial_{tr}^2 u^{(n)}}{t} \right\|_\infty + \left\| \frac{\partial_{tr}^2 v^{(n)}}{t} \right\|_\infty \leq 2M.$$

Proof. Let $(U, V) = T(u, v)$. Differentiating (3.14) we obtain

$$\partial_\xi\left(\frac{\partial U}{\partial \eta}\right) = \frac{\partial b_1}{\partial r}\frac{\partial r_+}{\partial \eta}(\xi, \eta) + \int_0^\xi \frac{\partial}{\partial r}\left(\frac{\partial b_1}{\partial r}(t, r(t, \xi, \eta))\right)\frac{\partial r}{\partial \xi}\frac{\partial r}{\partial \eta} + \frac{\partial b_1}{\partial r}\frac{\partial^2 r}{\partial \xi \partial \eta}\,dt, \tag{3.35}$$

where

$$\frac{\partial^2 r}{\partial \xi \partial \eta}(t; \xi, \eta) = \frac{\partial r}{\partial \eta} \cdot \left(\int_\xi^t \frac{\partial^2 \Lambda_+}{\partial r^2} \cdot \frac{\partial r}{\partial \xi}\,dt - \frac{\partial \Lambda_+}{\partial r}(\xi, \eta) \right) \tag{3.36}$$

obtained by differentiating (3.15) with respect to ξ. According to (3.23), (3.27), (3.15), (3.19), (3.36) we can obtain

$$\left| \partial_\xi\left(\frac{\partial U}{\partial \eta}\right) \right| \leq e^{C\delta^3}\xi\left(\frac{1}{2}M + k + C(\xi + \xi^2)\right) + C\xi^4 + C\xi^3.$$

Same bound holds for $\partial_\xi(\frac{\partial V}{\partial \eta})$. Hence by probably making M larger and δ smaller we obtain

$$\left| \frac{1}{\xi}\partial_\xi\left(\frac{\partial U}{\partial \eta}\right) \right| + \left| \frac{1}{\xi}\partial_\xi\left(\frac{\partial V}{\partial \eta}\right) \right| \leq e^{C\delta^3}\left(M + k + C(\delta + \delta^2)\right) + C\delta^3 + C\delta^2 \leq 2M. \quad \square$$

By Lemmas 4 and 1, $\{\partial_r \mathbf{F}^{(n)}\}$ is uniformly Lipschitz continuous.

Lemma 5. *For the iteration sequence* $\mathbf{F}^{(n)} = (u^{(n)}, v^{(n)})$, *there holds*

$$\left\|\partial_{tt}^2 u^{(n)}\right\|_\infty + \left\|\partial_{tt}^2 v^{(n)}\right\|_\infty \le 7M.$$

Proof. Let $(U, V) = T(u, v)$. Differentiating (3.28) with respect to ξ we obtain

$$\frac{\partial^2 U}{\partial \xi^2} = \int_0^\xi \frac{\partial^2 b_1}{\partial r^2}\left(\frac{\partial r}{\partial \xi}\right)^2 + \frac{\partial b_1}{\partial r}\frac{\partial^2 r}{\partial \xi^2}\, dt + \frac{2\partial b_1}{\partial \xi}(\xi, \eta, u, v). \tag{3.37}$$

Similarly, by differentiating (3.27) with respect to ξ we have

$$\frac{\partial^2 r}{\partial \xi^2} = -\frac{\partial \Lambda_+(\xi, \eta, v(\xi, \eta))}{\partial \xi} \cdot \frac{\partial r}{\partial \eta}(t; \xi, \eta) - \Lambda_+(\xi, \eta, v(\xi, \eta))\frac{\partial^2 r}{\partial \xi \partial \eta}(t; \xi, \eta). \tag{3.38}$$

Via direct computation we also have

$$\frac{\partial \Lambda_+}{\partial \xi} = \frac{4\xi\eta - 2\xi^2\eta\frac{\partial \mu}{\partial \xi}}{\mu^2},$$

where μ is defined in (3.17) and its partial derivative is

$$\frac{\partial \mu(\xi, \eta, v(\xi, \eta))}{\partial \xi} = \left(\frac{\partial v}{\partial \xi} - a_1\right)\sqrt{\eta^2 - \xi^2} - \frac{\xi(v + a_0^2 - \xi a_1)}{\sqrt{\eta^2 - \xi^2}} + 2\eta^2.$$

Therefore we obtain the estimate $\left|\frac{\partial \Lambda_+}{\partial \xi}\right| \le C\xi(1+\xi)$ and consequently $\left|\frac{\partial^2 r}{\partial \xi^2}\right| \le C\xi(1+\xi)$. We notice that

$$\begin{aligned}\frac{\partial b_1(u, v)}{\partial t} &= \frac{1}{2}\left(\frac{u_t - v_t}{t} - \frac{u-v}{t^2}\right) + \partial_t\left(\frac{2t^2 qR}{S + 2r\lambda^{-1}} - \frac{1}{2}\right)\cdot\left(\frac{u-v}{t} + 2a_1\right) \\ &\quad + \left(\frac{2t^2 qR}{S + 2r\lambda^{-1}} - \frac{1}{2}\right)\cdot\left(\frac{u_t - v_t}{t} - \frac{u-v}{t^2}\right) - \partial_t f.\end{aligned} \tag{3.39}$$

According to Lemmas 1 and 3, we have

$$\left|\frac{\partial b_1}{\partial t}\right| \le \frac{3M}{2} + k(k + Mt)^2 + kMt(k + Mt) + kt(1+t).$$

Combining estimates on (3.19), (3.27), (3.23), (3.38) and (3.39) in (3.37), we obtain

$$\left|\frac{\partial^2 U}{\partial \xi^2}\right| \le C\delta^3(1+\delta^3) + 3M + k + C\delta(1+\delta).$$

For the same reason we can derive

$$\left|\frac{\partial^2 V}{\partial \xi^2}\right| \le C\delta^3(1+\delta^3) + 3M + k + C\delta(1+\delta).$$

Hence $\left|\frac{\partial^2 U}{\partial \xi^2}\right| + \left|\frac{\partial^2 U}{\partial \xi^2}\right| \le 6M + k + C\delta(1 + \delta + \delta^2 + \delta^5)$. The lemma is achieved by making M large and δ small. $\square$

In sum, $\{\partial_t \mathbf{F}^{(n)}\}$ is uniformly bounded and uniformly Lipschitz continuous by Lemmas 1, 4 and 5. Lemma 2 is proved as a result. Theorem 2 is then a natural consequence of Lemmas 1 and 2.

4. Convert solution back to (θ, r) plane

Notice the Jacobian for the coordinate change from (θ, r) to (t, r) is $\left|\frac{\partial(t,\tilde{r})}{\partial(\theta,r)}\right| = -\frac{p_\theta}{2t} < 0$, hence it is a one to one correspondence between $(t, \tilde{r})$ and (θ, r). More specifically, once $R(t, r), S(t, r)$ are found in the (t, r) plane as given in Theorem 2, we will get (θ, r) by integrating the following equation

$$\frac{\partial\theta}{\partial t} = -\frac{2t}{p_\theta} = \frac{-4t}{R+S}, \qquad \theta(0, r) = \varphi(r). \tag{4.1}$$

Straightforward calculation also gives

$$\frac{\partial\theta}{\partial r} = \frac{4r^2 t - (R-S)\sqrt{p}}{tr(R+S)}. \tag{4.2}$$

Therefore the pressure is known and $p(\theta(t, r), r) = r^2 - t^2$. Furthermore,

$$p_\theta(\theta(t,r), r) = \frac{R(t,r) + S(t,r)}{2}, \quad p_r(\theta(t,r), r) = \frac{(R(t,r) - S(t,r))\sqrt{r^2 - t^2}}{2tr}. \tag{4.3}$$

As a result, it can be verified by direct computation that the solution $p(\theta, r)$ satisfy the PGE (1.3) in the hyperbolic region $r^2 - p > 0$ up to but not include the sonic boundary. The boundary conditions (1.5) also hold on the sonic curve. Therefore we obtain the Theorem 1.

We remark that the regularity assumptions on the function class $\mathcal{S}$ for solutions can be weakened with some technical treatment in the *a priori* estimates. With this local solution at hand, we are motivated to consider two open problems. Namely, how to construct the sonic-supersonic solutions when the sonic curve becomes tangential to the characteristics at one point. Another future work is to construct smooth transonic solutions. This paper although was written before [23] and [24], it is still valuable as a paving stone. We believe it is more feasible to construct smooth transonic solutions for PGE than the 2D Euler system. This paper is one important step in the entire construction mechanism of global solutions for PGE.

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