



Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa



Compactness and the fixed point property in ℓ_1 [☆]

T. Domínguez-Benavides, M. Japón ^{*}

Facultad de Matemáticas, Universidad de Sevilla, c/Tarfia s/n, 41012-Sevilla, Spain

ARTICLE INFO

Article history:

Received 11 February 2016

Available online xxxx

Submitted by Richard M. Aron

Keywords:

Nonexpansive mappings

Uniformly Lipschitzian mappings

Cascading nonexpansive mappings

Fixed point

Compact domain

ABSTRACT

In this paper we prove that compactness can be characterized by means of the existence of a fixed point for some classes of mappings defined on convex closed subsets of the space ℓ_1 . Nominally, our result involves nonexpansive mappings, uniformly Lipschitzian mappings and cascading nonexpansive mappings. We also extend the results to some more general classes of Banach spaces.

© 2016 Elsevier Inc. All rights reserved.

1. Introduction

The main goal of this paper is to characterize compactness of a convex closed subset of ℓ_1 by means of the existence of a fixed point for several classes of mappings. Many different topological, metric or geometrical properties are often used in Fixed Point Theory to prove the existence of a fixed point. In some rare cases, these properties are not only sufficient but also necessary and so, they are characterized by fixed point results. For instance, while Schauder's Theorem assures the existence of a fixed point for continuous mappings defined on a convex compact subset C of a linear normed space X , V. Klee [18] proved that compactness is also a necessary assumption, and so, he stated: A convex closed subset C of a linear normed space is compact if and only if every continuous mapping defined from C into C has a fixed point. P.K. Lin and Y. Sternfeld [24] improved Klee's result in 1985 proving that for any convex closed noncompact subset C of a linear normed space there exists a Lipschitzian mapping f which is fixed point free. (In fact, they proved the following much stronger result: $\inf\{\|x - fx\| : x \in C\} > 0$.) Since the mapping $\lambda f + (1 - \lambda)I$ ($\lambda \in (0, 1)$) has the same fixed point set as f and it is $\lambda L + (1 - \lambda)$ -Lipschitzian whenever f is L -Lipschitzian, letting $\lambda \rightarrow 0^+$, we can easily check that the fixed point free mapping f can be chosen with a Lipschitz constant as close to 1^+ as wanted. (For more aspects concerning the failure of Schauder's Theorem in noncompact

[☆] The authors are partially supported by MCIN, Grant MTM2015-65242-C2-1-P and Andalusian Regional Government Grant FQM-127.

^{*} Corresponding author.

E-mail addresses: tomasd@us.es (T. Domínguez-Benavides), japon@us.es (M. Japón).

setting, see [14].) To fix the notation, we will say that C satisfies the Fixed Point Property (FPP) for a class of mappings \mathcal{A} if every mapping $f \in \mathcal{A}$ defined from C into C has a fixed point. Thus, the result in [24] can be stated as follows:

Theorem 1.1. *Let L be any number greater than 1. A convex closed subset C of a Banach space satisfies the FPP for L -Lipschitzian mappings if and only if it is a compact set.*

One could wonder if the same is true for $L = 1$, but it was already known that the behavior of 1-Lipschitzian mappings (i.e. nonexpansive mappings) with respect to the FPP is quite different. Indeed, F. Browder [4] had proved in 1965 that any closed convex bounded subset of a Hilbert space satisfies the FPP for nonexpansive mappings. In fact, since W. Ray [28] proved in 1985 that every closed convex unbounded set of a Hilbert space fails the FPP for nonexpansive mappings, we have the following:

Theorem 1.2. *A convex closed subset C of a Hilbert space satisfies the FPP for nonexpansive mappings if and only if it is bounded.*

The reflexivity of Hilbert spaces let us state Theorem 1.2 in the following equivalent form:

Theorem 1.3. *A convex closed subset C of a Hilbert space satisfies the FPP for nonexpansive mappings if and only if it is weakly compact.*

No similar characterization is known for any other reflexive space. Furthermore, Theorem 1.2 does not hold for the classic nonreflexive spaces c_0 and ℓ_1 because it is well known that in these spaces there are some closed convex and bounded subsets which fail the FPP for nonexpansive mappings (for instance, the closed unit ball of c_0 or the positive face of the unit sphere of ℓ_1). However, Theorem 1.3 does also hold for the space c_0 as proved in [7], extending [11] and [26]. Thus, it is natural to consider the possibility of stating a similar result for the space ℓ_1 . Since weak compactness is equivalent to compactness for subsets of ℓ_1 , it is clear that every weakly compact convex subset of ℓ_1 satisfies the FPP for nonexpansive mappings. However, it is well known that there are many convex noncompact subsets of ℓ_1 (for instance, weak* compact sets) which satisfy the FPP for nonexpansive mappings (more detailed information about subsets of ℓ_1 satisfying the FPP for nonexpansive mappings can be found in [8]). In fact, K. Goebel and T. Kuczumow [15] constructed a nested sequence of convex closed subsets of the space ℓ_1 which alternatively satisfy or fail the FPP for nonexpansive mappings. This sharp example seemed to point to that compact sets are the only convex closed subsets of ℓ_1 that satisfy the hereditary FPP for nonexpansive mappings (i.e. every closed convex subset satisfies the FPP for nonexpansive mappings). This assertion was, in fact, proved by P. Dowling et al. [10] in case that the set is norm-bounded.

In this paper we will give a characterization of norm compactness in ℓ_1 where, firstly the boundedness condition is not longer required in the hypothesis and secondly, we can also drop the hereditary assumption because this characterization can be achieved by means of the existence of fixed points for certain families of self-mappings defined over the set C itself, in contrast to the results in [10]. In particular, in case of nonexpansive mappings, we prove that compactness for a closed convex subset of ℓ_1 is equivalent to satisfy the FPP for Lipschitzian mappings which are nonexpansive on their ranges.

Furthermore, we could wonder if Theorem 1.1 is also true for uniformly Lipschitzian mappings, i.e. mappings such that all iterates are L -Lipschitzian. Looking at the literature on fixed points for uniformly Lipschitzian mappings (see, for instance, [5,6,9,16,22,23]), it is very clear that this is not, in general, the case, because for small values of $L > 1$ it is possible to obtain the existence of a fixed point in convex bounded closed subsets of several classes of Banach spaces. For instance, for $X = \ell_2$, uniformly L -Lipschitzian mappings defined from a convex bounded closed subset C into C have a fixed point if $L < \sqrt{2}$ [22]. In spite of these existence results for noncompact sets and small values of L , we can still obtain a characterization of

compactness for a closed convex subset of ℓ_1 by means of the FPP for uniformly L -Lipschitzian mappings if $L > 2$.

Moreover, C. Lennard and Nezir [21] defined the notion of cascading non-expansive mapping. Let C be a closed convex subset of a Banach space X and $T : C \rightarrow C$ a mapping. Let $C_0 = C$ and $C_n = \overline{\text{co}} T(C_{n-1})$.

Definition 1.4. Let X be a Banach space and C be a closed convex subset of X . Let $T : C \rightarrow C$ be a mapping and (C_n) be defined as above. The mapping T is said to be cascading nonexpansive if there exists a sequence λ_n in $[1, \infty)$ such that $\lambda_n \rightarrow 1$ and for all $n \geq 0$, for all $x, y \in C_n$ we have $\|Tx - Ty\| \leq \lambda_n \|x - y\|$.

They used this notion to obtain a characterization of the reflexivity for Banach lattices or Banach spaces with unconditional basis. In particular, they have proved that any Banach space containing isomorphically ℓ_1 , also contains a set K which fails the FPP for cascading non-expansive mappings. In this paper we prove, that, in the case of the space ℓ_1 , any closed convex noncompact set fails the FPP for cascading non-expansive mappings and so, we give another characterization of compactness by means of the existence of fixed points for a class of mappings.

In fact, our results on the space ℓ_1 are based upon some properties of this space which are shared by some other more general Banach spaces. Thus, in the last section we extend our characterization results to some other spaces, including, in particular the space of trace class operators endowed with the trace norm.

2. Preliminaries

A subset C of a Banach space is said to have the approximate fixed point property (AFPP) for nonexpansive mappings if $\inf\{\|x - Tx\| : x \in C\} = 0$ for all nonexpansive self-mappings defined from C into C . It is easy to check that every closed convex and bounded subset C has this property. However, there exist some closed convex and unbounded subsets also verifying the AFPP. In fact, the convex subsets with the AFPP are characterized by the following condition: A convex unbounded subset $C \subset X$ is said to be directionally bounded if for every $(x_n) \subset C$ with $\lim_n \|x_n\| = +\infty$ and for every $f \in X^*$ with $\|f\| = 1$,

$$\limsup_n f\left(\frac{x_n}{\|x_n\|}\right) < 1.$$

Theorem 2.1. [29, Theorem 3.2] *A convex unbounded subset C of a Banach space X has the AFPP if and only if it is directionally bounded.*

In fact, every infinite dimensional Banach space contains an unbounded closed convex set which is directionally bounded and therefore an unbounded set which has the AFPP [25].

The following characterization of norm-compactness for closed convex bounded subsets of ℓ_1 was given in [10]:

Theorem 2.2. *Let C be a closed convex bounded subset of ℓ_1 . The following assertions are all equivalent:*

- a) C is compact,
- b) every closed convex bounded subset of C has the FPP for nonexpansive mappings.

As mentioned in the introduction, the above equivalence is not longer true if we replace condition b) by the assertion that the set C has itself the FPP for nonexpansive mappings.

The proof of Theorem 2.2 is based on the existence of asymptotically isometric copies of ℓ_1 :

Theorem 2.3. [10] *Let X be a Banach space with a norm $\|\cdot\|$ and let C be a convex closed subset of X . Let (ϵ_n) be a null sequence in $(0, 1)$. If C contains a sequence (x_n) such that*

$$\sum_{n=1}^{\infty} (1 - \epsilon_n) |t_n| \leq \left\| \sum_{n=1}^{\infty} t_n x_n \right\| \leq \sum_{n=1}^{\infty} (1 + \epsilon_n) |t_n|$$

for all $(t_n) \in \ell_1$, then C contains a non-empty closed convex subset K such that there is a nonexpansive mapping $T : K \rightarrow K$ which fails to have a fixed point.

If a sequence (x_n) verifies the previous inequalities for some null sequence $(\epsilon_n)_n$ it is said that it spans an asymptotically isometric copy (a.i.c.) of ℓ_1 . We can assume that (ϵ_n) is a decreasing sequence.

Proof. We will give a brief outline of the proof, because we will use the same arguments later.

Let (x_n) be a sequence spanning an a.i.c. of ℓ_1 . By passing to a particular subsequence we can assume that

$$\begin{aligned} (1 + \epsilon_{2n+1})(3^{-2n-2} + 2^{-1}) + (1 + \epsilon_{2n+2})(3^{-2n-2}) \\ < (1 - \epsilon_{2n-1})(3^{-2n} + 2^{-1}) + (1 - \epsilon_{2n})(3^{-2n}) \end{aligned} \quad (\diamond)$$

for every $n \in \mathbb{N}$. We define

$$q = \sum_{j=1}^{\infty} \frac{1}{3^j} x_j, \quad u_n = \left(\frac{1}{3^{2n}} + \frac{1}{2} \right) x_{2n-1} - \frac{1}{3^{2n}} x_{2n}.$$

From the convexity of C we have that $q + u_n \in C$ for every $n \in \mathbb{N}$. Let $T : K_U \rightarrow K_U$, where $K_U = \{\sum_{i=1}^{\infty} t_n u_n : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1\}$, be the right shift, that is,

$$T \left(\sum_{i=1}^{\infty} t_n u_n \right) = \left(\sum_{i=1}^{\infty} t_n u_{n+1} \right)$$

It is proved in [10] that T is nonexpansive. Finally, the set $K := q + K_U$ is contained in C and $H : K \rightarrow K$ given by $H(q + x) = q + T(x)$ is a nonexpansive fixed point free mapping from a closed convex subset K of C into K . \square

By using a translation and a dilatation argument, the following result easily follows:

Corollary 2.4. *Let (x_n) be a bounded sequence and assume that there exist some x_0 and $L > 0$ such that $((x_n - x_0)/L)$ spans an a.i.c. of ℓ_1 . Then there exists some closed convex subset K in $\overline{\text{co}}(x_n)$ and a nonexpansive mapping $T : K \rightarrow K$ which is fixed point free.*

We also recall the following result [13, Proposition 1]:

Proposition 2.5. *Let (x_n) be a normalized sequence in a Banach space such that*

$$\limsup_n \|x_n + x\| = \limsup_n \|x_n\| + \|x\| \quad (*)$$

for every $x \in [x_n]$, the closed span generated by the sequence (x_n) . If $[x_n]$ is infinite dimensional, then there exists a subsequence (x_{n_k}) which spans an a.i.c. of ℓ_1 .

Notice that if a sequence (x_n) spans an a.i.c. of ℓ_1 , then so does $(x_n + z_n)$ for every sequence (z_n) with $\lim_n \|z_n\| = 0$. Therefore, it is clear that we can remove the normalization in the previous result and consider that the sequence (x_n) verifies $\lim_n \|x_n\| = 1$.

Note that every weak*-null sequence in ℓ_1 verifies the previous condition $(*)$ for all $x \in \ell_1$. In particular:

Lemma 2.6. *Let C be a closed convex subset of ℓ_1 . If C contains a weak*-convergent sequence (x_n) which is not norm convergent, there exists some closed convex bounded subset $K \subset C$ and a nonexpansive mapping $T : K \rightarrow K$ which is fixed point free.*

Proof. Let (x_n) be such sequence and let x_0 be the weak*-limit of x_n and $L := \limsup_n \|x_n - x_0\| > 0$. Taking a subsequence we can assume that $\lim_n \|(x_n - x_0)/L\| = 1$ which implies that $((x_n - x_0)/L)$ spans an a.i.c. of ℓ_1 . Applying Corollary 2.4 we obtain a subset $K \subset \overline{\text{co}}(x_n) \subset C$ and a fixed point free nonexpansive mapping $T : K \rightarrow K$. \square

3. Equivalence between compactness and the FPP

In this section we will achieve a characterization of compactness by means of the FPP for classes of mappings defined over the set C itself. In order to do that, we need the following technical assertions:

Lemma 3.1. *Let (e_n) be the canonical basis in ℓ_1 and let*

$$K := \left\{ \sum_{n=1}^{\infty} t_n e_n : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

Then there exists a 2-uniform Lipschitzian retraction

$$r : (\ell_1, \|\cdot\|) \rightarrow (K, \|\cdot\|).$$

Proof. Denote $K^+ = \{\sum_{n=1}^{\infty} t_n e_n : t_n \geq 0; \sum_{n=1}^{\infty} t_n < \infty\}$.

For $x = \sum_{n=1}^{\infty} t_n e_n$ define $a : \ell_1 \rightarrow K^+$ by

$$a(x) = \sum_{n=1}^{\infty} |t_n| e_n.$$

It is not difficult to check that a is a 1-Lipschitzian retraction. Define $b : K^+ \rightarrow K$ by

$$b(x) = \begin{cases} (1 - \sum_{n=2}^{\infty} t_n) e_1 + \sum_{n=2}^{\infty} t_n e_n & \text{if } \sum_{n=1}^{\infty} t_n \leq 1 \\ \frac{x}{\sum_{n=1}^{\infty} t_n} & \text{if } \sum_{n=1}^{\infty} t_n \geq 1 \end{cases}$$

Let $x = \sum_{n=1}^{\infty} t_n e_n, y = \sum_{n=1}^{\infty} s_n e_n \in K^+$. Denote $A = \sum_{n=1}^{\infty} t_n$ and $B = \sum_{n=1}^{\infty} s_n$. If $A \geq 1, B \geq 1$ we have

$$\begin{aligned} \|b(x) - b(y)\| &= \left\| \sum_{n=1}^{\infty} \frac{Bt_n - As_n}{AB} e_n \right\| \leq \sum_{n=1}^{\infty} |t_n - s_n| + |B - A| \\ &\leq 2 \sum_{n=1}^{\infty} |t_n - s_n| = 2\|x - y\|. \end{aligned}$$

If $A \leq 1, B \leq 1$ we have

$$\begin{aligned} \|b(x) - b(y)\| &= \left\| \sum_{n=2}^{\infty} (s_n - t_n) e_1 + \sum_{n=2}^{\infty} (t_n - s_n) e_n \right\| \\ &\leq 2 \sum_{n=1}^{\infty} |t_n - s_n| = 2\|x - y\|. \end{aligned}$$

Finally, if $A > 1$, $B < 1$, choose $z = \lambda x + (1 - \lambda)y$, $\lambda \in (0, 1)$ such that $\lambda A + (1 - \lambda)B = 1$. We have

$$\begin{aligned} \|b(x) - b(y)\| &\leq \|b(x) - b(z)\| + \|b(z) - b(y)\| \\ &\leq 2(\|x - z\| + \|z - y\|) = 2\|x - y\|. \end{aligned}$$

Thus, the mapping $r = b \circ a$ satisfies the required conditions. \square

The above retraction can be extended if we consider basic sequences equivalent to the standard basis in ℓ_1 .

Lemma 3.2. *Let (u_n) be a basic sequence in a Banach space X equivalent to the standard basis of ℓ_1 . Let $U = [u_n]$ be the closed subspace generated by the vectors (u_n) and let $G : \ell_1 \rightarrow [u_n]$ be the mapping $G(\sum_{n=1}^{\infty} t_n e_n) = \sum_{n=1}^{\infty} t_n u_n$. Then there is a retraction R_U from the total space U onto K_U , where*

$$K_U := \left\{ \sum_{n=1}^{\infty} t_n u_n : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

Moreover this retraction is $2\|G\|\|G^{-1}\|$ -uniform Lipschitzian.

Proof. It is enough to consider the composition $G \circ r \circ G^{-1}$, where r is the retraction given in Lemma 3.1. \square

We will use the following result for basic sequences. It is usually known as the Principle of small perturbations (see for instance the statement and the proof of [1, Theorem 1.3.9]):

Lemma 3.3. *Let $(x_n)_n$ be a basic sequence in a Banach space X with basis constant M . Assume that $(y_n)_n$ is another sequence in X such that*

$$2M \sum_{n=1}^{\infty} \frac{\|x_n - y_n\|}{\|x_n\|} := \theta < 1.$$

- The sequence (y_n) is a basic sequence equivalent to (x_n) . Moreover the basis constant of (y_n) is at most $M(1 + \theta)(1 - \theta)^{-1}$.*
- Let $(\hat{x}_n^*)_n$ denote the Hahn-Banach extensions of the biorthogonal functionals of $(x_n)_n$ to the whole space X . Then,*

$$A(x) = x + \sum_{n=1}^{\infty} \hat{x}_n^*(x)(y_n - x_n),$$

is an invertible isomorphism from X onto X with $A(x_n) = y_n$ for every $n \in \mathbb{N}$, $\|A\| \leq 1 + \theta$ and $\|A^{-1}\| \leq (1 - \theta)^{-1}$.

- If there exists a projection $P : X \rightarrow [x_n]$ then $\tilde{P} = A \circ P \circ A^{-1} : X \rightarrow [y_n]$ is a projection with $\|\tilde{P}\| \leq \|P\|(1 + \theta)(1 - \theta)^{-1}$.*

Finally we deduce the following characterization of compactness for closed convex sets of ℓ_1 . Notice that the boundedness condition is not longer necessary in the hypotheses:

Theorem 3.4. Let C be a convex closed subset of ℓ_1 . Then, the following conditions are all equivalent:

- (1) C is a compact set.
- (2) C satisfies the FPP for Lipschitzian mappings $T : C \rightarrow C$ which are nonexpansive on $\overline{\text{co}}T(C)$.
- (3) C verifies the FPP for cascading nonexpansive mappings.
- (4) For every $L > 2$, C satisfies the FPP for uniformly L -Lipschitzian mappings.

Proof. Condition (1) implies (2), (3) and (4) from Schauder's Theorem. Assume that C fails to be norm-compact and fix some $L > 2$.

We first assume that C contains a weak*-convergent sequence (x_n) which fails to be norm-convergent.

Take $\epsilon \in (0, \frac{1}{2})$ such that

$$2 \frac{\frac{1}{2} + \epsilon}{\frac{1}{2} - \epsilon} \frac{1 + \epsilon}{1 - \epsilon} < L.$$

Fix a strictly decreasing sequence $(r_n)_n$ in $(0, \epsilon/2)$ such that $\sum_{n=1}^{\infty} r_n = \frac{1}{2}$.

Without loss of generality we assume that (x_n) is weak*-null and that $1 = \lim_n \|x_n\|$. By [Proposition 2.5](#) we can also assume that (x_n) spans an a.i.c. of ℓ_1 for a decreasing sequence $(\epsilon_n)_n$ in $(0, 1)$. Taking a further subsequence, we may also assume for every $n \in \mathbb{N}$:

$$\left(\left(r_{2n} + \frac{1}{2} \right) (1 + \epsilon_{2n-1}) + (1 + \epsilon_{2n}) r_{2n} \right) < \frac{1}{2} + \epsilon \quad (\dagger)$$

$$\left(\left(r_{2n} + \frac{1}{2} \right) (1 - \epsilon_{2n-1}) + (1 - \epsilon_{2n}) r_{2n} \right) > \frac{1}{2} - \epsilon, \quad (\ddagger)$$

and the following analogue of (\diamond) in the proof of [Theorem 2.3](#):

$$\begin{aligned} & (1 + \epsilon_{2n+1})(r_{2n+2} + 2^{-1}) + (1 + \epsilon_{2n+2})r_{2n+2} \\ & < (1 - \epsilon_{2n-1})(r_{2n} + 2^{-1}) + (1 - \epsilon_{2n})r_{2n} \end{aligned} \quad (\diamond\diamond)$$

Taking a subsequence, if necessary, we can assume that there exists a normalized block basic sequence (v_n) of the basis (e_n) in ℓ_1 such that $\sum_{n=1}^{\infty} \|x_n - v_n\| < \epsilon/8$.

For every $n \in \mathbb{N}$, define

$$\hat{v}_n := \left(r_{2n} + \frac{1}{2} \right) v_{2n-1} - r_{2n} v_{2n}$$

which is a block basic sequence of (e_n) in ℓ_1 with $1 > \|\hat{v}_n\| > \frac{1}{2}$ for all $n \in \mathbb{N}$. Moreover $(\hat{v}_n)_n$ has a basis constant equal to one.

Define $q = \sum_{n=1}^{\infty} r_n x_n$ and $u_n := (r_{2n} + \frac{1}{2})x_{2n-1} - r_{2n}x_{2n}$ as in the proof of [Theorem 2.3](#). Notice that

$$2 \sum_{n=1}^{\infty} \frac{\|\hat{v}_n - u_n\|}{\|\hat{v}_n\|} \leq 4 \sum_{n=1}^{\infty} \|\hat{v}_n - u_n\| \leq 8 \sum_{n=1}^{\infty} \|v_n - x_n\| < \epsilon.$$

On the other hand, since $(\hat{v}_n)_n$ is a block basic sequence of (e_n) , there exists a norm-one projection $P : \ell_1 \rightarrow [\hat{v}_n]$ (see, for instance, [\[1, Lemma 2.1.1. and Remark 2.1.2\]](#)). Applying [Lemma 3.3.c\)](#), there exists a projection $\tilde{P} : \ell_1 \rightarrow [u_n]$ with $\|\tilde{P}\| \leq \frac{1+\epsilon}{1-\epsilon}$.

As before we denote

$$K_U = \left\{ \sum_{n=1}^{\infty} t_n u_n : t_n \geq 0, \sum_{n=1}^{\infty} t_n = 1 \right\}.$$

The sequence (u_n) is equivalent to the canonical basis of ℓ_1 . In fact, if we define $G : \ell_1 \rightarrow [u_n]$ as $G(\sum_n t_n e_n) = \sum_n t_n u_n$, from (\dagger) we deduce that $\|G\| \leq \frac{1}{2} + \epsilon$ and from (\ddagger) that $\|G^{-1}\| \leq 1/(\frac{1}{2} - \epsilon)$. Using Lemma 3.2 there exists a retraction R_U from $[u_n]$ onto K_U , where R_U is $2\frac{\frac{1}{2}+\epsilon}{\frac{1}{2}-\epsilon}$ -uniform Lipschitzian.

Therefore $\tilde{R}_U = R_U \circ \tilde{P} : \ell_1 \rightarrow K_U$ is a retraction which is L -uniform Lipschitzian.

Let $T : K_U \rightarrow K_U$ be the shift operator which is a fixed point free nonexpansive mapping from the proof of Theorem 2.3 (here we use (\diamond) to prove that T is nonexpansive). Note that $q + K_U \subset C$. Now define $H : C \rightarrow q + K_U \subset C$ given by

$$H(x) = q + T(\tilde{R}_U(x - q))$$

Let us check that H is well defined. Take some $x \in C$. We have $\tilde{R}_U(x - q) \in K_U$, $T(\tilde{R}_U(x - q)) \in K_U$ and $q + T(\tilde{R}_U(x - q)) \in q + K_U \subset C$. Assume that there exists some $x \in C$ such that

$$H(x) = q + T(\tilde{R}_U(x - q)) = x.$$

The above implies that $x - q \in T(K_U) \subset K_U$ and hence $\tilde{R}_U(x - q) = x - q$ and $T(x - q) = x - q$ which contradicts the fact that T is fixed point free.

Let us prove that H is nonexpansive over $\overline{\text{co}}(H(C))$: Let $x, y \in \text{co}(H(C))$. Then $x = \sum_{i=1}^n t_i x_i$ and $y = \sum_{i=1}^n s_i y_i$ with $x_i, y_i \in H(C)$, $t_i, s_i \geq 0$ for $i = 1, \dots, n$ and $\sum_{i=1}^n t_i = \sum_{i=1}^n s_i = 1$. This implies that $x_i - q, y_i - q \in K_U$ for every $i = 1, \dots, n$ and, since K_U is convex, $x - q, y - q \in K_U$. It follows that $\|H(x) - H(y)\| \leq \|x - y\|$ and H is nonexpansive on $\overline{\text{co}}(H(C))$.

Finally we have constructed a fixed point free mapping $H : C \rightarrow C$ which is nonexpansive over $\overline{\text{co}}(H(C))$ (and therefore noncascading nonexpansive). Moreover it is not difficult to check that this mapping is also L -uniform Lipschitzian.

In case that C is bounded, then the equivalences follow from the above arguments. Assume that C is unbounded. If C is not weak*-closed, $C \cap RB_{\ell_1}$ is not w^* -closed for some $R > 0$, by a Theorem of Krein and Smulian [12, p. 429]. So there exists some sequence $(x_n) \subset C$ which is weak*-convergent to some $x_0 \notin C$. In particular, (x_n) does not converge in norm (since C is norm-closed) and we can follow the same arguments as above.

Assume that C is weak*-closed and take $(x_n) \subset C$ such that $\lim_n \|x_n\| = +\infty$. Define $z_n := x_n / \|x_n\|$ and we can assume that (z_n) is weak* convergent to some $a \in \ell_1$. If $a \neq 0$ we can find a ray contained in C . Indeed, choose $v_0 \in C$ arbitrary and for any $t \geq 0$ choose $n_0 \in \mathbb{N}$ such that $\|x_n\| > t$ for all $n \geq n_0$. Then the sequence

$$\frac{t}{\|x_n\|} x_n + \left(1 - \frac{t}{\|x_n\|}\right) v_0$$

lies in C for all $n \geq n_0$ and converges weak* to $v_0 + ta$. This in particular implies that C fails the FPP for nonexpansive maps (consider $T : C \rightarrow C$ given by $Tx = v_0 + (\|x - v_0\| + 1)\frac{a}{\|a\|}$), which is false whenever (2)–(4) are assumed.

If $a = 0$, in particular z_n does not converge to 0 in norm. Consider some $v_0 \in C$. Define

$$v_n := \left(1 - \frac{1}{\|x_n\|}\right) v_0 + \frac{x_n}{\|x_n\|} = v_0 - \frac{v_0}{\|x_n\|} + \frac{x_n}{\|x_n\|}$$

From the last equality we deduce that (v_n) is a sequence in C which is weak*-convergent to v_0 and without norm convergent subsequences because $\lim \|v_n - v_0\| = 1$. This let us finish the proof by the previous arguments. \square

We do not know if condition 4) could be relaxed by the FPP for uniform L -Lipschitzian mappings with some $L \leq 2$. If that were true, new arguments would be necessary. Indeed, one can wonder if there could exist a retraction from ℓ_1 onto the positive face K of the unit ball with Lipschitz constant strictly less than 2. The following Proposition shows that 2 is, in fact, the best possible choice.

Proposition 3.5. *Let R be a Lipschitz-retraction from ℓ_1 into K with Lipschitz constant L . Then, $L \geq 2$.*

Proof. We will prove this result following an indirect way which is based upon fixed point theory. Assume that there exists a Lipschitz retraction from ℓ_1 onto K with Lipschitz constant $L < 2$. Denote by T the right shift defined from K into K and $S = (I + T)/2$. By [17] the mapping S is asymptotically regular, i.e. $\lim_n S^{n+1}x - S^n x = 0$ for every $x \in K$. Since $(SR)^n = S^n R$ we have that the mapping SR is uniformly L -Lipschitzian and asymptotically regular. Theorem 3.1 and Corollary 2.2 in [9] imply that SR would have a fixed point in every closed convex weak*-compact subset containing K , and therefore a fixed point in K . But a fixed point of SR is a fixed point of S and so of T and this is a contradiction because the right shift on K is a fixed point free affine isometry. \square

4. Extensions to some other classes of Banach spaces

It is worthwhile to notice that the main tools in the proof of Theorem 3.4 are the existence of asymptotically isometric ℓ_1 copies and the existence of a projection onto a block basic sequence. We could consider more general Banach spaces which satisfy analogous conditions so that a similar equivalence between compactness and FPP can be stated.

Recall that if a Banach space X has a boundedly complete Schauder basis, then X is isomorphic to Z^* where Z is the closed subspace spanned by the orthogonal functionals of the vectors in the Schauder basis [1, Theorem 3.2.10]. Therefore, the weak*-topology in this case will refer to the $\sigma(X, Z)$ -topology and the weak*-convergence is just the coordinatewise convergence for bounded sequences. Recall that a block basic sequence (x_n) is said to be semi-normalized if there exist some $0 < a < b$ such that $a \leq \|x_n\| \leq b$ for every $n \in \mathbb{N}$.

Corollary 4.1. *Let X be a Banach space with a boundedly complete Schauder basis such that:*

- i) *Every normalized weak*-null sequence (x_n) contains a further subsequence which spans an asymptotically isometric copy of ℓ_1 .*
- ii) *For every semi-normalized block basic sequence $(u_n)_n$ there exists a further subsequence $(u_{n_k})_k$ which is 1-complemented in X .*

Let C be a closed convex subset of X . The following conditions are all equivalent:

- (1) *C is a compact set.*
- (2) *C satisfies the FPP for mappings $T : C \rightarrow C$ which are nonexpansive on $\overline{\text{co}}T(C)$.*
- (3) *C verifies the FPP for cascading nonexpansive mappings.*
- (4) *For every $L > 2$, C satisfies the FPP for uniformly L -Lipschitzian mappings.*

Examples of Banach spaces satisfying conditions i) and ii) in Corollary 4.1 are the following:

Example 4.2. Assume that X is a one-direct sum of finite dimensional Banach spaces $X = \oplus_1 \sum_n X_n$ with the usual $\|x\|_1 = \sum_n \|x_n\|_{X_n}$ if $x = (x_n)_n$ with $x = (x_n)$ with $x_n \in X_n$. In this class of Banach spaces we can include the Fourier–Stieltjes algebras $B(G)$ whenever G is a separable compact group (see the proof of [19, Lemma 3.1] and references therein). In the particular case that $G = \mathbb{T}$, we recover the sequence space ℓ_1 . For these classes of Banach spaces, condition $(*)$ stated in Proposition 2.5 holds for every weak*-null sequence. Here the predual is the c_0 -sum of the spaces X_n . Moreover, if (u_n) is a semi-normalized block basic sequence in X , we can extract a subsequence $(u_{n_k})_k$ for which two different vectors do not share coordinates in the same block X_n . Now it is easy to prove that the sequence $(u_{n_k})_k$ is 1-complemented in X .

Another class of Banach spaces satisfying conditions $i)$ and $ii)$ and which are not isomorphic to ℓ_1 are the following:

Example 4.3. [3] Let any $l > 1$ and denoted X as the Banach space defined by the completeness of c_{00} with the norm

$$|x| = \sup \sum_{i=1}^k |x_{n_i}|$$

where the supremum is taken over all admissible subsets S of \mathbb{N} , where $S = \{n_1, \dots, n_k\}$ is an admissible set whenever $n_{i+1} \geq ln_i$ for all $i = 1, \dots, k-1$. Then $(X, |\cdot|)$ is a Banach space with a boundedly complete Schauder basis satisfying the previous $(*)$ condition, and therefore, every normalized weak*-null sequence contains a subsequence spanning an asymptotically isometric copy of ℓ_1 (see [3, Section 6]). On the other hand, if (u_n) is a semi-normalized block basic sequence in X , we can extract a subsequence (u_{n_k}) such that $\min\{\text{supp}(u_{n_k})\} > l \max\{\text{supp}(u_{n_{k-1}})\}$. Then it is not difficult to check that there is a norm-one projection from X onto $[(u_{n_k})_k]$.

To finish, consider $C_1(H)$, the space of trace class operators endowed with the trace norm. $C_1(H)$ is a dual Banach space whose predual is the space of all compact operators defined on a Hilbert space H . In [10] it is proved that a closed convex bounded subset C of $C_1(H)$ is weakly compact if and only if every closed convex subset $K \subset C$ has the FPP for nonexpansive mappings (Corollary 7 in [10]). Using similar arguments as in the previous section, we can achieve a characterization or weakly compactness through the FPP for the set C .

Theorem 4.4. Let C be a closed convex bounded subset of $C_1(H)$. The following are equivalent:

- (1) C is weakly compact.
- (2) C satisfies the FPP for mappings which are nonexpansive over $\overline{co}T(C)$.
- (3) C satisfies the FPP for cascading nonexpansive mappings.

The assertion

- (4) For every $L > 2$, C satisfies the FPP for uniformly L -Lipschitzian mappings,

implies the remaining ones.

Proof. It is known that $C_1(H)$ verifies the weak*-FPP [20]. In particular every closed weakly compact subset has the FPP for nonexpansive mappings. From this (1) clearly implies (2). Furthermore, if T is a cascading nonexpansive mapping then T is nonexpansive on a minimal convex weakly T -invariant subset; and so (1) implies (3).

Assume now C fails to be weakly compact. We can find a sequence (x_n) failing to converge weakly and such that (x_n) is weak*-convergent in $X^{**} \setminus X$. Notice that $X = C_1(H)$ is an example of an L -embedded Banach space, which means that X is 1-complemented in its bidual. Applying [27, Theorem 2] there exists some $x_0 \in X$ and some $L > 0$ such that the sequence $\{(x_n - x_0)/L\}_n$ spans an asymptotically isometric copy of ℓ_1 . Using the main result in [2] and taking a subsequence if necessary, there is a projection $P : X \rightarrow [x_n]$ with $\|P\|$ as close to one as wanted. By using Lemma 3.2 and the same arguments as in the proof of Theorem 3.4 we can prove that each statement (2), (3) or (4) implies weakly compactness. \square

Acknowledgments

The authors would like to thank two anonymous reviewers for their constructive and valuable comments, which helped us to improve the quality of this manuscript.

References

- [1] F. Albiac, N.J. Kalton, Topics in Banach Space Theory, Grad. Texts in Math., Springer, 2006.
- [2] J. Arazy, Almost isometric embeddings of ℓ_1 in preduals of von Neumann algebras, Math. Scand. 54 (1984) 79–94.
- [3] A. Barrera-Cuevas, M. Japón, New families of non-reflexive Banach spaces with the fixed point property, J. Math. Anal. Appl. 425 (1) (2015) 349–363.
- [4] F.E. Browder, Fixed point theorems for noncompact mappings in Hilbert spaces, Proc. Natl. Acad. Sci. USA 43 (1965) 1272–1276.
- [5] E. Casini, E. Maluta, Fixed points of uniformly Lipschitzian mappings in spaces with uniformly normal structure, Nonlinear Anal. 9 (1985) 103–108.
- [6] T. Domínguez Benavides, Fixed point theorems for uniformly Lipschitzian mappings and asymptotically regular mappings, Nonlinear Anal. 32 (1) (1998) 15–27.
- [7] T. Domínguez Benavides, The failure of the Fixed Point Property for unbounded subsets in c_0 , Proc. Amer. Math. Soc. 140 (2) (2012) 645–650.
- [8] T. Domínguez Benavides, J. García-Falset, E. Llorens-Fuster, P. Lorenzo Ramírez, Fixed point properties and proximality in Banach spaces, Nonlinear Anal. 71 (2009) 1562–1571.
- [9] T. Domínguez Benavides, H.K. Xu, A new geometrical coefficient for Banach spaces and its applications in fixed point theory, Nonlinear Anal. 25 (1995) 311–325.
- [10] P.N. Dowling, C.J. Lennard, B. Turett, The fixed point property for subsets of some classical Banach spaces, Nonlinear Anal. 49 (2002) 141–145.
- [11] P.N. Dowling, C.J. Lennard, B. Turett, Weak compactness is equivalent to the fixed point property in c_0 , Proc. Amer. Math. Soc. 132 (6) (2004) 1659–1666.
- [12] N. Dunford, J. Schwartz, Linear Operators, Part I, Interscience Publishers, 1958.
- [13] H. Fetter, B. Gamboa de Buen, Banach spaces with a basis that are hereditarily asymptotically isometric to ℓ_1 and the fixed point property, Nonlinear Anal. 71 (2009) 4598–4608.
- [14] K. Goebel, W. Kaczor, Remarks on the failure of Schauder's theorem in noncompact setting, Ann. Univ. Mariae Curie-Skłodowska Sect. A 51 (2) (1997) 99–108.
- [15] K. Goebel, T. Kuczumow, Irregular convex sets with fixed-point property for nonexpansive mappings, Colloq. Math. 40 (2) (1978/79) 259–264.
- [16] J. Górnicki, M. Krüppel, Fixed points for uniformly Lipschitzian mappings, Bull. Pol. Acad. Sci. Math. 36 (1988) 57–62.
- [17] S. Ishikawa, Fixed points and iteration of a nonexpansive mapping in a Banach space, Proc. Amer. Math. Soc. 59 (1976) 65–71.
- [18] V. Klee, Some topological properties of convex sets, Trans. Amer. Math. Soc. 78 (1955) 30–45.
- [19] A.T.-M. Lau, P.F. Mah, Fixed point property for Banach algebras associated to locally compact groups, J. Funct. Anal. 258 (2) (2010) 357–372.
- [20] C. Lennard, C_1 is uniformly Kadec–Klee, Proc. Amer. Math. Soc. 109 (1) (1990) 71–77.
- [21] C. Lennard, V. Nezir, Reflexivity is equivalent to the perturbed fixed point property for cascading nonexpansive maps in Banach lattices, Nonlinear Anal. 95 (2014) 414–420.
- [22] E.A. Lifschitz, Fixed point theorems for operators in strongly convex spaces, Voronez. Gos. Univ. Trudy Mat. Fak. 16 (1975) 23–28 (in Russian).
- [23] T.C. Lim, Fixed point theorems for uniformly Lipschitzian mappings in L^p spaces, Nonlinear Anal. 7 (1983) 555–563.
- [24] P.K. Lin, Y. Sternfeld, Convex sets with the Lipschitz fixed point property are compact, Proc. Amer. Math. Soc. 93 (1985) 633–639.
- [25] E. Matouskova, S. Reich, Reflexivity and approximated fixed points, Studia Math. 159 (3) (2003) 403–415.
- [26] B. Maurey, Points fixes des contractions de certains faiblement compacts de L^1 , in: Seminaire d'Analyse Fonctionnelle, 1980–1981, Centre de Mathématiques, in: Exp., vol. VIII, École Polytech, Palaiseau, 1981, 19pp.
- [27] H. Pfitzner, A note on asymptotically isometric copies of ℓ_1 and c_0 , Proc. Amer. Math. Soc. 129 (5) (2000) 1367–1373.
- [28] W.O. Ray, The fixed point property and unbounded sets in Hilbert space, Trans. Amer. Math. Soc. 258 (1980) 531–537.
- [29] I. Shafir, The approximate fixed point property in Banach and hyperbolic spaces, Israel J. Math. 71 (2) (1990) 211–223.