



Homogenization of degenerate coupled fluid flows and heat transport through porous media

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Abstract

We establish a homogenization result for a fully nonlinear degenerate parabolic system with critical growth arising from the heat and moisture flow through a partially saturated porous media. Existence of a global weak solution of the mesoscale problem is proven by means of a semidiscretization in time, a priori estimates and passing to the limit from discrete approximations. After that, porous material exhibiting periodic spatial oscillations is considered and the two-scale convergence (as the oscillation period vanishes) to a corresponding homogenized problem is rigorously proven.

Keywords: Nonlinear degenerate parabolic system, Homogenization, Asymptotic analysis, Two-scale convergence, Coupled transport processes in porous media

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1. Introduction

This paper deals with the two-scale homogenization of a class of doubly nonlinear degenerate parabolic problems arising from coupled transport processes in porous media.

Model problem. Let Ω be a bounded domain in \mathbb{R}^2 , $\Omega \in C^{0,1}$ and let Γ_D and Γ_N be open disjoint subsets of $\partial\Omega$ (not necessarily connected) such that $\Gamma_D \neq \emptyset$ and the 1-dimensional measure of $\partial\Omega \setminus (\Gamma_D \cup \Gamma_N)$ equals zero. Let $T \in (0, \infty)$ be fixed throughout the paper. Let us abbreviate $\Omega_T = \Omega \times (0, T)$, $\Gamma_{DT} = \Gamma_D \times (0, T)$ and $\Gamma_{NT} = \Gamma_N \times (0, T)$ and let $\mathcal{Y} = (0, 1)^2$ be a periodicity cell. Vectors and vector functions are denoted by boldface letters. Let $\varepsilon > 0$ be a *small* scalar parameter. From the geometrical point of view, ε is the characteristic length representing the small scale variability of the porous media. We study the homogenization of the doubly nonlinear degenerate system indexed by the scale parameter ε , namely,

$$\partial_t b(x/\varepsilon, u^\varepsilon) + \nabla \cdot \mathbf{q}^\varepsilon = 0 \quad \text{in } \Omega_T, \quad (1.1)$$

$$\partial_t [b(x/\varepsilon, u^\varepsilon) \theta^\varepsilon + \rho(x/\varepsilon) \theta^\varepsilon] = \nabla \cdot [\chi_2(x/\varepsilon) \lambda(\theta^\varepsilon, u^\varepsilon) \nabla \theta^\varepsilon - \theta^\varepsilon \mathbf{q}^\varepsilon] \quad \text{in } \Omega_T, \quad (1.2)$$

$$u^\varepsilon = 0 \quad \text{in } \Gamma_{DT}, \quad (1.3)$$

$$\theta^\varepsilon = 0 \quad \text{in } \Gamma_{DT}, \quad (1.4)$$

$$\mathbf{q}^\varepsilon \cdot \mathbf{n} = 0 \quad \text{in } \Gamma_{NT}, \quad (1.5)$$

$$\nabla \theta^\varepsilon \cdot \mathbf{n} = 0 \quad \text{in } \Gamma_{NT}, \quad (1.6)$$

$$u^\varepsilon(0) = u_0 \quad \text{in } \Omega, \quad (1.7)$$

$$\theta^\varepsilon(0) = \theta_0 \quad \text{in } \Omega. \quad (1.8)$$

Here

$$\mathbf{q}^\varepsilon = -[\chi_1(x/\varepsilon) a(\theta^\varepsilon) \nabla u^\varepsilon + \mathbf{g}(x/\varepsilon, \theta^\varepsilon, u^\varepsilon)]. \quad (1.9)$$

From the physical point of view, equations (1.1) and (1.2), respectively, represent the mass balance of moisture and the heat equation for the porous system after the so-called Kirchhoff transformation. Functions $u^\varepsilon : \Omega_T \rightarrow \mathbb{R}$ and $\theta^\varepsilon : \Omega_T \rightarrow \mathbb{R}$ are the unknowns. \mathbf{n} is the outer unit normal vector to $\partial\Omega$, $a : \mathbb{R} \rightarrow \mathbb{R}$, $b : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$, $\lambda : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $\mathbf{g} : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2$, $\chi_1, \chi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$, $\rho : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $u_0 : \Omega \rightarrow \mathbb{R}$ and $\theta_0 : \Omega \rightarrow \mathbb{R}$ are given functions. Roughly speaking, the coefficient functions χ_1 , χ_2 and ρ are \mathcal{Y} -periodic functions on \mathbb{R}^2 and similarly, b and \mathbf{g} are \mathcal{Y} -periodic with respect

to the first variable (see the next section for precise assumptions). As ε gets smaller, the coefficient functions χ_1 , χ_2 , ρ , b and \mathbf{g} in (1.1)–(1.2) oscillate more rapidly. We solve the classical homogenization problem, namely investigate the behavior of the solution $[u^\varepsilon, \theta^\varepsilon]$ in the limit (as $\varepsilon \rightarrow 0$).

Remark 1.1. *The analysis presented here can be straightforwardly extended to a setting with nonhomogeneous boundary conditions (see [10] for details). Here we work with homogeneous boundary conditions just to simplify and shorten the presentation and avoid unnecessary technicalities in the existence result.*

System (1.1)–(1.2) can be seen as the special case of more general problem, the so-called doubly nonlinear problem, $\partial_t \mathbf{B}(x, \mathbf{u}) - \nabla \cdot \mathbf{A}(x, \mathbf{u}, \nabla \mathbf{u}) = \mathbf{0}$, with nonlinearities in both, parabolic as well as elliptic parts. From theoretical point of view, as far as we know, no general existence, uniqueness and regularity theory is developed for such kinds of problems. Partial results assuming special structure of operators \mathbf{B} and \mathbf{A} can be found e.g. in [4, 19]. However, these results are not applicable if \mathbf{B} does not take the subgradient structure, which is the case of the coupled system (1.1)–(1.2) due to a non-symmetry in the parabolic term. Moreover, under rather general assumptions on b , the equation (1.1) degenerates in the parabolic part. Finally, it is worth noting that further difficulty lies in the convective term in the heat equation, which represents strong nonlinearity in the model. Hence, we deal with the nonlinear degenerate system under critical growth assumptions. In the present paper we prove the existence of the weak solution to (1.1)–(1.9) and rigorously derive the corresponding homogenized problem letting $\varepsilon \rightarrow 0$ and using the two-scale convergence theory (in the sense of [40], see also [3]). As far as we know, this is the first attempt to carry out such an analysis for (1.1)–(1.9).

A brief bibliographical survey. Qualitative properties of particular problems like (1.1), (1.3), (1.5) and (1.7), such as existence, uniqueness and regularity, have been studied by several authors, see e.g. [4, 19]. More recently, in [12], the authors studied homogenization of the decoupled nonlinear degenerate parabolic problem (1.1), (1.3) and (1.7) with $b(y, s) = s$, however, including more general nonlinear operator $\mathbf{q}^\varepsilon = \mathbf{q}^\varepsilon(y, s, \nabla s)$. The operator \mathbf{q}^ε was assumed to be periodic in $y = x/\varepsilon$ and degenerated in ∇s . The authors derived the limit equation letting $\varepsilon \rightarrow 0$ using the two-scale convergence theory and proven a strong convergence (corrector results). Similar

results can be found in [13], where the same authors considered the homogenization of multi-scale degenerate problem like (1.1), (1.3) and (1.7) with $b(x/\varepsilon, s) = s$ and $\mathbf{q}^\varepsilon = -[A(x/\varepsilon)\nabla u^\varepsilon + \mathbf{g}(x/\varepsilon, u^\varepsilon)]$. The authors derived the homogenized equation and present results on the first order corrector. In [37], the authors studied the homogenization of nonlinear parabolic equations like (1.1) with mixed boundary conditions under restrictive assumptions on b and with non-degenerate and monotone elliptic part excluding the degenerate van Genuchten model. A similar problem has been considered in [25].

Homogenization of a coupled system of diffusion-convection equations in a domain with periodic microstructure, modeling the flow of isothermal immiscible compressible fluids through porous media, was theoretically studied e.g. in [5, 6]. Homogenization of complex chemical processes in porous media has been recently studied e.g. in [18, 28, 29, 30]. Particularly, in [18], the authors rigorously derived upscaled reaction-diffusion models based on a system of coupled semi-linear microstructural reaction-diffusion equations modeling concrete corrosion in sewer pipes. In [29], the authors applied two scale convergence techniques to obtain the macro model of thermal-diffusion reaction problems (including the Dufour and Soret effects) modeling e.g. behavior of concrete at high temperatures, drug delivery in biology tissues etc. Just recently, in [28], the authors proved results on the weak solvability and homogenization of a microscopic semi-linear elliptic systems modeling diffusion processes in porous media taking into account surface and chemical reactions.

In the present paper, following the results in [13] concerning the isothermal degenerate problem, we intend to exploit the specific physical structure of the transport model to obtain homogenized equations based on the two-scale convergence theory [3, 40]. To the best of our knowledge, mathematical homogenization of degenerate coupled-transport problems in partially saturated porous media remains largely unexplored and remains open due to the nonlinear coupling and degeneracies of the system.

Applications. The use of mathematical models in the simulation of heat and moisture flow through heterogeneous porous media has received substantial attention in recent years. Research effort has concentrated on the simulation of geothermal reservoirs, the effect of degradation due to the presence of moisture in building and engineering structures, heat and moisture transport in fresh concrete, heat resistance of concrete in fire or under nuclear accidents and many other common or extreme scenarios. Therefore advanced models

allowing to accurately analyze and forecast the coupled heat and mass transfer in porous media are of great importance. The main difficulty lies in the determination of several parameters used to define relevant efficient transport coefficients of heterogeneous materials. In realistic situations, their experimental determination is rather complicated due to the complexity and lack of enough reliable and accurate data. In virtue of the multi-scale character of porous media, design of advanced engineering applications require the use of materials with complex microstructures.

Several engineering models of coupled moisture flow and heat transport in porous media possess a common structure, derived from two balance equations of mass of moisture and heat energy [7, 8, 35, 41, 42, 44] (neglecting heat and moisture sources)

$$\partial_t(\Theta^\varepsilon \varrho_w) + \nabla \cdot (\varrho_w \Theta^\varepsilon \mathbf{v}_w^\varepsilon) = 0 \quad (1.10)$$

and

$$\partial_t(c_w \varrho_w \Theta^\varepsilon \theta + c_S^\varepsilon \varrho_S^\varepsilon \theta) + \nabla \cdot (\mathbf{q}_h^\varepsilon + \theta c_w \varrho_w \Theta^\varepsilon \mathbf{v}_w^\varepsilon) = 0. \quad (1.11)$$

In (1.10)–(1.11), ε is a small scale parameter (the characteristic scale of heterogeneities) that express explicitly the multi-scale structure of the functions Θ^ε , \mathbf{v}_w^ε , \mathbf{q}_h^ε , c_S^ε and ϱ_S^ε . Here, Θ^ε is the water content of the porous material, θ is the temperature, \mathbf{v}_w^ε represents the velocity of the moisture in the porous material, \mathbf{q}_h^ε stands for the conductive heat flux vector. Further, ϱ_w is the density of liquid water, c_w represents the isobaric heat capacity of water, ϱ_S^ε and c_S^ε , respectively, are the mass density and the isobaric heat capacity of the solid. The amount of water present at a certain matric potential of the porous medium is characterized by the water retention curve $\Theta^\varepsilon = \Theta^\varepsilon(x, h)$ (moisture sorption isotherms), where h is the so called matric potential.

Many natural and man-made porous media typically exhibit a hierarchical structure on many length scales. Moreover, the structural elements themselves have structure. We assume that in each of these structures the moisture flow is governed by Darcy's law and the heat flow is described by Fourier's law. Here the original moisture flow model is already a homogenized model based on Stokes equations at the pore level (see e.g. [24, Chapter 3] and the references therein). Our goal is to obtain the homogenized model, valid at the macro level starting from the mesoscopic one. Hence, starting at the mesoscale level, we apply the Darcy's constitutive law for the mass flux

$$\Theta^\varepsilon \mathbf{v}_w^\varepsilon = -k^\varepsilon(x, \theta, h) (\nabla h + \mathbf{g}_z), \quad (1.12)$$

where \mathbf{g}_z stands for the vertical unit vector and k^ε represents the hydraulic conductivity of porous medium. Similarly, we assume the conductive heat flux \mathbf{q}_h^ε to be given by Fourier's law

$$\mathbf{q}_h^\varepsilon = -\lambda^\varepsilon(x, \theta, h) \nabla \theta \quad (1.13)$$

with the thermal conductivity function λ^ε . Combining Darcy's law (1.12) and the conservation of mass (1.10) yields the following equation

$$\partial_t \Theta^\varepsilon(x, h) = \nabla \cdot (k^\varepsilon(x, \theta, h) \nabla h + \mathbf{g}_z k^\varepsilon(x, \theta, h)). \quad (1.14)$$

Similarly, combining Darcy's law (1.12), Fourier's law (1.13) and the conservation of energy (1.11) yields

$$\begin{aligned} \partial_t (c_w \varrho_w \Theta^\varepsilon(x, h) \theta + c_s^\varepsilon(x) \varrho_s^\varepsilon(x) \theta) &= \nabla \cdot (\lambda^\varepsilon(x, \theta, h) \nabla \theta) \\ &+ \nabla \cdot [\theta c_w \varrho_w (k^\varepsilon(x, \theta, h) \nabla h + \mathbf{g}_z k^\varepsilon(x, \theta, h))]. \end{aligned} \quad (1.15)$$

The primary unknowns in the model are the absolute temperature $\theta = \theta(x, t)$ and the matric potential (the capillary pressure head) $h = h(x, t)$ (single-valued functions of the time t and the spatial position x).

Typically, neglecting residual water content, Θ^ε can be expressed as

$$\Theta^\varepsilon(x, h) = \Theta_s^\varepsilon(x) S_E(h),$$

where Θ_s^ε is the saturated water content of the porous material and S_E is the effective saturation. Many parametric models have been proposed. The models of Brooks and Corey [11] and of van Genuchten [21] are the most popular for applications to air-water systems. The Brooks-Corey relationship can be written as

$$S_E(h) = |\vartheta h|^{-\beta}, \quad h < 0,$$

where ϑ and β are fitting parameters related to the inverse of the entry pressure head and the pore-size distribution. The model proposed by van Genuchten [21] is given by the relationship

$$S_E(h) = [1 + |\alpha h|^n]^{-m}, \quad h < 0,$$

where α , m and n are parameters. Retention curves in the saturated zone can be assumed as [8]

$$S_E(h) = s_E h + 1, \quad h \geq 0,$$

where $s_E > 0$ is the specific (elasticity) storativity coefficient.

The moisture and temperature dependence of hydraulic conductivity is given by [16, 17, 44, 47]

$$k^\varepsilon(x, \theta, h) = \frac{k_s^\varepsilon(x)k_r(h)}{\mu_w(\theta)},$$

where k_s^ε is the saturated hydraulic conductivity, k_r stands for the h -dependent relative hydraulic conductivity, which is commonly based on the power law form [16, 44]

$$k_r(h) = \begin{cases} S_E(h)^{A_w} & \text{for } h < 0, \\ 1 & \text{for } h \geq 0. \end{cases}$$

Here the exponent A_w is obtained from fitting the data. Finally, μ_w represents the temperature dependent kinematic viscosity [51].

The effective thermal conductivity λ^ε of partially saturated porous media depends upon the degree of saturation with water and temperature. We consider that the dependence may be approximated by a relationship

$$\lambda^\varepsilon(x, \theta, h) = \lambda_{d0}^\varepsilon(x)\lambda_d(\theta)\lambda_s(S_E(h)),$$

where λ_{d0}^ε , λ_d and λ_s are given functions obtained from fitting the data.

Let us stress that, due to the structural complexity of highly porous media, the coefficient functions Θ_s^ε , k_s^ε and λ_{d0}^ε are assumed to be rapidly oscillating with large contrast. In our simplified heuristic approach we consider a periodic composite $\mathcal{Y} = \bigcup_i \mathcal{Y}_i$ at the mesoscale level, in which the moisture flow equation (1.14) as well as the heat equation (1.15) are valid in each component. Then the coefficient functions Θ_s^ε , k_s^ε and λ_{d0}^ε become fast oscillating in \mathcal{Y} and jump on the interfaces of \mathcal{Y}_i having the form $\Theta_s^\varepsilon(y) = \sum_i \nu_i(y)\bar{\Theta}_{si}^\varepsilon$, $k_s^\varepsilon(y) = \sum_i \nu_i(y)\bar{k}_{si}^\varepsilon$, $\lambda_{d0}^\varepsilon(y) = \sum_i \nu_i(y)\bar{\lambda}_{d0i}^\varepsilon$. Here, ν_i is the characteristic function of \mathcal{Y}_i and $\bar{\Theta}_{si}^\varepsilon$, \bar{k}_{si}^ε and $\bar{\lambda}_{d0i}^\varepsilon$ are the individual characteristics of each component. Our goal is to obtain the upscaled equations, valid at the macro level based on (1.14)–(1.15), starting from the mesoscopic geometry described above. The present asymptotic analysis is of topical interest in mathematical analysis [12, 13, 14, 23, 25, 36, 43], numerical homogenization, as well as geotechnical and civil engineering (see e.g. [1, 34, 39, 45, 48, 49, 50]).

Kirchhoff transformation. It is worth noting that the nonlinear parabolic equation (1.14) is degenerate in both elliptic and parabolic parts. A powerful tool for handling the degenerate mass balance equation (1.14) is provided by

the so called Kirchhoff transformation, which employs the primitive function $\beta : \mathbb{R} \rightarrow \mathbb{R}$, $\zeta = \beta(\xi)$, defined by

$$\beta(\xi) = \int_0^\xi k_r(s) ds$$

and transforms degeneracies only to the parabolic term (see e.g. [4, 27]). This formally leads to the system (1.1)–(1.2), where

$$\begin{aligned} b(y, u) &= \Theta_s^\varepsilon(y) S_E(\beta^{-1}(u)), \\ a(\theta) &= \frac{1}{\mu_w(\theta)}, \\ \rho(y) &= \frac{c_S^\varepsilon(y) \varrho_S^\varepsilon(y)}{c_w \varrho_w}, \\ \lambda(\theta, u) &= \frac{\lambda_d(\theta) \lambda_s(S_E(\beta^{-1}(u)))}{c_w \varrho_w}, \\ \mathbf{g}(y, \theta, u) &= \frac{\mathbf{g}_z k^\varepsilon(y, \theta, \beta^{-1}(u))}{c_w \varrho_w}. \end{aligned}$$

General assumptions on coefficient functions b , a , λ and \mathbf{g} are specified in Section 2.

Remark 1.2. *Let us note that, assuming a third type boundary conditions (Robin conditions) for the temperature and the capillary pressure head, the application of the Kirchhoff transformation leads to the more complicated problem due to the coupled nonlinear boundary integral in the variational formulation. Our existence and regularity result for system (1.1)–(1.8) can not be extended to such kinds of problems straightforwardly. More rigorous qualitative study is required in this way.*

Plan of the paper. In Section 2, we introduce necessary notation and definition of some function spaces, present some auxiliary results and specify our assumptions on data and structure conditions on coefficient functions under which the main result of the paper is proven. In Section 3, we provide the weak formulation of the initial-boundary value problem for the system (1.1)–(1.9). It is worth noting that, to make the text more readable, technical details of the existence proof for the multi-scale problem (1.1)–(1.9) are

collected in Appendix A following the ideas coming from [10]. Theorem 3.5 represents the main result of the paper, derivation of the homogenized model based on upscaling the problem (1.1)–(1.9). The proof of the main result is based on several preliminary results presented in Sections 4 and 5. In particular, in Section 4 we prove some a-priori estimates with respect to the scale parameter ε , which are sufficient for passing to the limit $\varepsilon \rightarrow 0$ in (1.1)–(1.9) using the two-scale convergence theory [40, 3], see Section 5. The proof of the main result is completed in Section 6.

2. Preliminaries

Notation and definition of some function spaces. Throughout the paper, we will always use positive constants C, c, c_1, c_2, \dots , which are not specified and which may differ from line to line. \mathbf{I}_d denotes the identity matrix of size 2×2 . We suppose $p, q, p' \in [1, \infty]$, p' denotes the conjugate exponent to p , $p > 1$, $1/p + 1/p' = 1$. $L^p(\Omega)$ denotes the usual Lebesgue space equipped with the norm $\|\cdot\|_{L^p(\Omega)}$ and $W^{k,p}(\Omega)$, $k \geq 0$ (k need not to be an integer, see [31]), denotes the usual Sobolev-Slobodecki space with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$. Recall that $W^{0,p}(\Omega) = L^p(\Omega)$. Further, we define $V := \left\{ \phi \in W^{1,2}(\Omega); \phi|_{\Gamma_D} = 0 \right\}$. We denote by V^* the dual space of V , (\cdot, \cdot) stands for the inner product on $L^2(\Omega)$ and $\langle \cdot, \cdot \rangle$ represents the duality pairing between V and V^* . Let X be a Banach space. By $L^p(0, T; X)$ we denote the usual Bochner space (see [2]). In the paper we shall use the following embedding theorems (see [2, 31]):

$$\begin{cases} W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), & \|\varphi\|_{L^q(\Omega)} \leq c \|\varphi\|_{W^{k,p}(\Omega)}, & 1 \leq q < +\infty, kp = 2, \\ W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), & \|\varphi\|_{L^q(\Omega)} \leq c \|\varphi\|_{W^{k,p}(\Omega)}, & 1 \leq q \leq 2p/(2 - kp), \\ & & kp < 2, \\ W^{k,p}(\Omega) \hookrightarrow L^\infty(\Omega), & \|\varphi\|_{L^\infty(\Omega)} \leq c \|\varphi\|_{W^{k,p}(\Omega)}, & kp > 2 \end{cases} \quad (2.1)$$

for every $\varphi \in W^{k,p}(\Omega)$. Let $W_{per}^{1,2}(\mathcal{Y})$ be the space of elements of $W^{1,2}(\mathcal{Y})$ having the same trace on opposite face of \mathcal{Y} . $L_{per}^p(\mathcal{Y})$ is the subspace of $L^p(\mathcal{Y})$ of \mathcal{Y} -periodic functions φ , i.e. $\varphi(x + k\mathbf{e}_i) = \varphi(x)$ a.e. on \mathbb{R}^2 , for all $k \in \mathbb{Z}$ and $i \in \{1, 2\}$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the canonical basis of \mathbb{R}^2 . By $C_{per}(\mathcal{Y})$ we denote the Banach space of continuous and \mathcal{Y} -periodic functions. $L^p(\Omega; X)$ stands for the set of measurable functions $\varphi : x \in \Omega \rightarrow \varphi(x) \in X$ such that $\|\varphi(x)\|_X \in L^p(\Omega)$. $L_{per}^2(\mathcal{Y}; C(\overline{\Omega}))$ represents the space of measurable functions $\varphi : y \in \mathcal{Y} \rightarrow \varphi(y) \in C(\overline{\Omega})$ such that $\|\varphi(y)\|_{C(\overline{\Omega})} \in L_{per}^2(\mathcal{Y})$.

Structure and data properties. We begin by introducing our assumptions on coefficient functions in (1.1)–(1.9).

- (A1) $b(y, \xi)$ is a positive and continuous function, \mathcal{Y} -periodic with respect to the first variable (with a basic unit cell $(0, 1)^2$) and strictly monotone and Lipschitz continuous in ξ such that

$$\begin{aligned} \nabla_y b(y, \cdot) &\in C(\mathbb{R}^2 \times \mathbb{R})^2, \\ 0 < \partial_\xi b(y, \xi) &\leq C_b, \quad 0 < b(y, \xi) \leq C_b(1 + |\xi|) \quad (C_b = \text{const}), \\ (b(y, \xi_1) - b(y, \xi_2)) &(\xi_1 - \xi_2) > 0 \end{aligned} \quad (2.2)$$

for all $y \in \mathbb{R}^2$ and $\xi, \xi_1, \xi_2 \in \mathbb{R}$, $\xi_1 \neq \xi_2$. Moreover, there exists $\tilde{y} \in \mathbb{R}^2$ such that

$$(b(y, \xi_1) - b(y, \xi_2))(\xi_1 - \xi_2) \geq (b(\tilde{y}, \xi_1) - b(\tilde{y}, \xi_2))(\xi_1 - \xi_2) \quad (2.3)$$

for all $y \in \mathbb{R}^2$ and $\xi_1, \xi_2 \in \mathbb{R}$.

- (A2) Functions ρ and χ_k , $k = 1, 2$, are smooth \mathcal{Y} -periodic functions on \mathbb{R}^2 , such that $\rho(y) \geq \rho_0 > 0$ and $\chi_k(y) \geq \chi_0 > 0$ for all $y \in \mathbb{R}^2$, $\rho_0, \chi_0 = \text{const}$.

- (A3) a and λ are continuous functions satisfying

$$\begin{aligned} 0 < a_1 &\leq a(\xi) \leq a_2 < +\infty & (a_1, a_2 = \text{const}), \\ 0 < \lambda_1 &\leq \lambda(\xi, \zeta) \leq \lambda_2 < +\infty & (\lambda_1, \lambda_2 = \text{const}) \end{aligned} \quad (2.4)$$

for all $\xi, \zeta \in \mathbb{R}$.

$\mathbf{g} : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^2$ is a smooth vector function, \mathcal{Y} -periodic with respect to the first variable with the basic unit cell $(0, 1)^2$ such that

$$|\mathbf{g}(y, \xi, \zeta)| \leq c < +\infty \quad (c = \text{const})$$

for all $y \in \mathbb{R}^2$ and $\xi, \zeta \in \mathbb{R}$.

- (A4) (Initial data) We assume $u_0, \theta_0 \in V \cap L^\infty(\Omega)$.

Auxiliary results.

Remark 2.1 ([4], Section 1.1). *Let us note that (A1) implies that there is a C^1 -function $\Phi : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$, $\Phi(y, 0) = 0$, $\partial_\xi \Phi(y, \xi)|_{\xi=0} = 0$, (strictly) convex in $\xi \in \mathbb{R}$ such that $b(\cdot, \xi) - b(\cdot, 0) = \partial_\xi \Phi(\cdot, \xi)$ for all $\xi \in \mathbb{R}$. Introduce the Legendre transform*

$$B(y, \xi) = \int_0^1 (b(y, \xi) - b(y, s\xi))\xi ds = \int_0^\xi (b(y, \xi) - b(y, s))ds, \quad y \in \mathbb{R}^2.$$

Let us present some properties of B (see [4] for more details):

$$\begin{aligned} B(y, z) &= \int_0^1 (b(y, z) - b(y, sz))z ds \geq 0 & \forall z \in \mathbb{R}, y \in \mathbb{R}^2, \\ B(y, z) - B(y, r) &\geq (b(y, z) - b(y, r))r & \forall r, z \in \mathbb{R}, y \in \mathbb{R}^2, \\ [b(y, z) - b(y, 0)]z - \Phi(y, z) &= B(y, z) \leq [b(y, z) - b(y, 0)]z & \forall z \in \mathbb{R}, y \in \mathbb{R}^2. \end{aligned}$$

Further, note that (A1) implies

$$B(y, z) \geq B(\tilde{y}, z) \tag{2.5}$$

for all $y \in \mathbb{R}^2$ and $z \in \mathbb{R}$, where \tilde{y} is taken from (2.3).

Proposition 2.2 (see [4], Lemma 1.5.). *Suppose (A4). Let $\varepsilon > 0$ and $u \in L^2(0, T; V)$, such that*

$$b(x/\varepsilon, u) \in L^\infty(0, T; L^1(\Omega)), \quad \partial_t b(x/\varepsilon, u) \in L^2(0, T; V^*)$$

and

$$\int_0^T \langle \partial_t b(x/\varepsilon, u), \phi \rangle dt + \int_{\Omega_T} (b(x/\varepsilon, u) - b(x/\varepsilon, u_0)) \partial_t \phi dx dt = 0$$

for every test function $\phi \in L^2(0, T; V) \cap W^{1,1}(0, T; L^\infty(\Omega))$ with $\phi(T) = 0$. Then $B(x/\varepsilon, u) \in L^\infty(0, T; L^1(\Omega))$ and for almost all t the following formula holds

$$\int_\Omega B(x/\varepsilon, u(t)) dx - \int_\Omega B(x/\varepsilon, u_0) dx = \int_0^t \langle \partial_s b(x/\varepsilon, u), u \rangle ds. \tag{2.6}$$

3. Main result

We first formulate the problem (1.1)–(1.8) in a weak sense.

Definition 3.1. *Let $\varepsilon > 0$ be given. We say that a pair $[u^\varepsilon, \theta^\varepsilon]$, such that*

$$u^\varepsilon \in L^2(0, T; V) \quad \text{and} \quad \theta^\varepsilon \in L^2(0, T; V) \cap L^\infty(\Omega_T),$$

is a weak solution of the system (1.1)–(1.8) iff

- (i) $b(\cdot, u^\varepsilon) \in L^\infty(0, T; L^1(\Omega))$, $\partial_t b(\cdot, u^\varepsilon) \in L^2(0, T; V^*)$, $\partial_t [(b(\cdot, u^\varepsilon) + \rho) \theta^\varepsilon] \in L^2(0, T; V^*)$ and

$$\int_0^T \langle \partial_t b(x/\varepsilon, u^\varepsilon), \phi \rangle dt + \int_0^T \int_\Omega [b(x/\varepsilon, u^\varepsilon) - b(x/\varepsilon, u_0)] \partial_t \phi dx dt = 0$$

for every test function $\phi \in L^2(0, T; V) \cap W^{1,1}(0, T; L^\infty(\Omega))$ with $\phi(T) = 0$ and

$$\begin{aligned} & \int_0^T \langle \partial_t [(b(x/\varepsilon, u^\varepsilon) + \rho(x/\varepsilon)) \theta^\varepsilon], \psi \rangle dt \\ & + \int_{\Omega_T} [(b(x/\varepsilon, u^\varepsilon) + \rho(x/\varepsilon)) \theta^\varepsilon - (b(x/\varepsilon, u_0) + \rho(x/\varepsilon)) \theta_0] \partial_t \psi dx dt \\ & = 0 \end{aligned}$$

for every test function $\psi \in L^2(0, T; V) \cap W^{1,1}(0, T; L^\infty(\Omega))$ with $\psi(T) = 0$.

- (ii) the couple $[u^\varepsilon, \theta^\varepsilon]$ satisfies the following system

$$\begin{aligned} & \int_0^T \langle \partial_t b(x/\varepsilon, u^\varepsilon), \phi \rangle dt \\ & + \int_{\Omega_T} [\chi_1(x/\varepsilon) a(\theta^\varepsilon) \nabla u^\varepsilon + \mathbf{g}(x/\varepsilon, \theta^\varepsilon, u^\varepsilon)] \cdot \nabla \phi dx dt \\ & = 0 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
 & \int_0^T \langle \partial_t [(b(x/\varepsilon, u^\varepsilon) + \rho(x/\varepsilon)) \theta^\varepsilon], \psi \rangle dt \\
 & + \int_{\Omega_T} \chi_2(x/\varepsilon) \lambda(\theta^\varepsilon, u^\varepsilon) \nabla \theta^\varepsilon \cdot \nabla \psi \, dx dt \\
 & + \int_{\Omega_T} \theta^\varepsilon [\chi_1(x/\varepsilon) a(\theta^\varepsilon) \nabla u^\varepsilon + \mathbf{g}(x/\varepsilon, \theta^\varepsilon, u^\varepsilon)] \cdot \nabla \psi \, dx dt \\
 & = 0
 \end{aligned} \tag{3.2}$$

for all test functions $\phi \in L^2(0, T; V)$ and $\psi \in L^2(0, T; V)$.

Theorem 3.2. *Let the assumptions (A1)–(A4) be satisfied. Then there exists at least one weak solution of the system (1.1)–(1.8). Moreover,*

$$\operatorname{ess\,sup}_{\Omega_T} |\theta^\varepsilon| \leq C, \tag{3.3}$$

where the constant C is independent of ε .

Proof. The existence of the weak solution to the problem like (1.1)–(1.8) with $b = b(u)$, $\mathbf{g} = \mathbf{g}(\theta, u)$, $\chi_1 \equiv \chi_2 = 1$ and $\rho = \text{const}$ is studied e.g. in [10, 41]. The proof of Theorem 3.2 follows similar arguments, see Appendix A for more details. \square

Before we state the main result of our paper, we introduce the following definition of the two-scale convergence, see [3].

Definition 3.3. *Let $1 < q < +\infty$. A sequence $v^\varepsilon \in L^q(\Omega)$ is said to two-scale converge to a function $v \in L^q(\Omega \times \mathcal{Y})$ if $(\varepsilon \rightarrow 0)$*

$$\int_{\Omega} v^\varepsilon \psi(x, x/\varepsilon) dx \rightarrow \int_{\Omega} \int_{\mathcal{Y}} v \psi(x, y) dy dx \tag{3.4}$$

for all $\psi \in L^{q'}(\Omega; C_{\text{per}}(\mathcal{Y}))$, $q' = q/(q-1)$.

Remark 3.4. [15, Remark 9.4] *Due to density properties, it is easily seen that if v^ε two-scale converges to v , convergence (3.4) (with $q = 2$) holds also for any $\psi \in L^2_{\text{per}}(\mathcal{Y}; C(\overline{\Omega}))$ as well as for any ψ of the form $\psi(x, y) = \psi_1(y)\psi_2(x, y)$ with $\psi_1 \in L^\infty(\mathcal{Y})$ and $\psi_2 \in L^2_{\text{per}}(\mathcal{Y}; C(\overline{\Omega}))$.*

The main result of this paper is summarized in the following theorem.

Theorem 3.5 (Main result). *Let $\varepsilon > 0$. Suppose the data satisfy (A1)–(A4) and $[u^\varepsilon, \theta^\varepsilon]$ is the couple according to Definition 3.1, i.e. the weak solution of (1.1)–(1.8). Then there exist the pairs $[u, u_1]$ and $[\theta, \theta_1]$,*

$$\begin{aligned} u &\in L^2(0, T; V), \quad u_1 \in L^2(\Omega_T; W_{per}^{1,2}(\mathcal{Y})), \\ \theta &\in L^2(0, T; V) \cap L^\infty(\Omega_T), \quad \theta_1 \in L^2(\Omega_T; W_{per}^{1,2}(\mathcal{Y})), \end{aligned}$$

and a sequence $[u^{\varepsilon_j}, \theta^{\varepsilon_j}]$ such that $\lim_{j \rightarrow +\infty} \varepsilon_j = 0^+$ and

$$\begin{aligned} u^{\varepsilon_j} &\rightharpoonup u && \text{weakly in } L^2(0, T; W^{1,2}(\Omega)), \\ u^{\varepsilon_j} &\rightarrow u && \text{almost everywhere in } \Omega_T, \\ \nabla u^{\varepsilon_j} &\rightarrow \nabla_x u + \nabla_y u_1 && \text{in the two-scale sense} \end{aligned}$$

and

$$\begin{aligned} \theta^{\varepsilon_j} &\rightharpoonup \theta && \text{weakly in } L^2(0, T; W^{1,2}(\Omega)), \\ \theta^{\varepsilon_j} &\rightharpoonup \theta && \text{weakly star in } L^\infty(\Omega_T), \\ \theta^{\varepsilon_j} &\rightarrow \theta && \text{almost everywhere in } \Omega_T, \\ \nabla \theta^{\varepsilon_j} &\rightarrow \nabla_x \theta + \nabla_y \theta_1 && \text{in the two-scale sense.} \end{aligned}$$

Further, u and θ satisfy (in a weak sense) the following coupled homogenized problem

$$\partial_t b^*(u) + \nabla \cdot \mathbf{q}^* = 0 \quad \text{in } \Omega_T, \quad (3.5)$$

$$\partial_t (b^*(u)\theta + \rho^*\theta) = \nabla \cdot (\mathbf{\Lambda}^*(\theta, u, x, t)\nabla\theta - \theta\mathbf{q}^*) \quad \text{in } \Omega_T, \quad (3.6)$$

$$u = 0 \quad \text{in } \Gamma_{DT}, \quad (3.7)$$

$$\theta = 0 \quad \text{in } \Gamma_{DT}, \quad (3.8)$$

$$\mathbf{q}^* \cdot \mathbf{n} = 0 \quad \text{in } \Gamma_{NT}, \quad (3.9)$$

$$\nabla\theta \cdot \mathbf{n} = 0 \quad \text{in } \Gamma_{NT}, \quad (3.10)$$

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \quad (3.11)$$

$$\theta(x, 0) = \theta_0(x) \quad \text{in } \Omega, \quad (3.12)$$

where

$$\mathbf{q}^* = -(\mathbf{A}^*(\theta, x, t)\nabla u + \mathbf{g}^*(\theta, u, x, t)). \quad (3.13)$$

In (3.5), (3.6) and (3.13), the homogenized coefficient functions are defined as

$$b^*(s) = \int_{\mathcal{Y}} b(y, s) \, dy, \quad (3.14)$$

$$\rho^* = \int_{\mathcal{Y}} \rho(y) \, dy, \quad (3.15)$$

$$\mathbf{A}^*(r, x, t) = a(r) \int_{\mathcal{Y}} \chi_1(y) (\mathbf{I}_d + \nabla_y \mathbf{w}(x, y, t)) \, dy, \quad (3.16)$$

$$\mathbf{g}^*(r, s, x, t) = \int_{\mathcal{Y}} (a(r) \chi_1(y) \nabla_y \eta(x, y, t) + \mathbf{g}(y, r, s)) \, dy, \quad (3.17)$$

$$\mathbf{\Lambda}^*(r, s, x, t) = \int_{\mathcal{Y}} \chi_2(y) \lambda(r, s) (\mathbf{I}_d + \nabla_y \mathbf{\Lambda}(x, y, t)) \, dy, \quad (3.18)$$

where $\mathbf{w} = (w_1, w_2)$, $\nabla_y \mathbf{w}$ is the matrix $(\nabla_y \mathbf{w})_{ij} = \partial w_i / \partial y_j$, $i, j = 1, 2$. Similarly, $\mathbf{\Lambda} = (\Lambda_1, \Lambda_2)$ and $\nabla_y \mathbf{\Lambda}$ is the matrix $(\nabla_y \mathbf{\Lambda})_{ij} = \partial \Lambda_i / \partial y_j$, $i, j = 1, 2$. Further, w_i , η and $\Lambda_i \in W_{per}^{1,2}(\mathcal{Y})$ are periodic solutions of the following cell problems:

$$-\nabla_y \cdot (\chi_1(y) (\mathbf{e}_i + \nabla_y w_i)) = 0 \quad \text{in } \mathcal{Y}, \quad i = 1, 2, \quad (3.19)$$

$$\int_{\mathcal{Y}} w_i \, dy = 0, \quad (3.20)$$

further

$$-\nabla_y \cdot (a(r) \chi_1(y) \nabla_y \eta) = \nabla_y \cdot \mathbf{g}(y, r, s) \quad \text{in } \mathcal{Y}, \quad (3.21)$$

$$\int_{\mathcal{Y}} \eta \, dy = 0, \quad (3.22)$$

and, finally,

$$-\nabla_y \cdot (\chi_2(y) (\mathbf{e}_i + \nabla_y \Lambda_i)) = 0 \quad \text{in } \mathcal{Y}, \quad i = 1, 2, \quad (3.23)$$

$$\int_{\mathcal{Y}} \Lambda_i \, dy = 0. \quad (3.24)$$

Schedule of the proof. In the proof of the main result we use the single- and two-scale convergence theory. First we derive suitable a priori estimates for the solution $[u^\varepsilon, \theta^\varepsilon]$ of the two-scale problem (1.1)–(1.8), see Section 4, key estimates (4.2), (4.5), (4.6), (4.12), (4.13) and (4.17). Based on these

estimates together with (3.3) we can pass to the limit as the space period ε vanishes, and prove the convergence to a solution of a purely coarse-scale formulation, recall (3.5) and (3.6). The two-scale convergence is carefully studied in Section 5 and the proof of the main result is completed in Section 6 proving (3.19)–(3.24).

4. A-priori estimates.

Here we present some a-priori bounds for the solution of (3.1)–(3.2). First we derive some estimates for u^ε . Setting $\chi_{(0,t)}u^\varepsilon$ as a test function in (3.1) ($\chi_{(0,t)}$ is the characteristic function of $(0, t)$) and using the integration by parts (2.6) we arrive at

$$\begin{aligned} \int_{\Omega} B(x/\varepsilon, u^\varepsilon(t)) dx + \int_{\Omega_t} \chi_1(x/\varepsilon) a(\theta^\varepsilon) |\nabla u^\varepsilon|^2 dx ds \\ = \int_{\Omega} B(x/\varepsilon, u_0) dx - \int_{\Omega_t} \mathbf{g}(x/\varepsilon, \theta^\varepsilon, u^\varepsilon) \cdot \nabla u^\varepsilon dx ds \end{aligned}$$

for almost all $t \in (0, T)$. Applying the Young's inequality to the second integral on the right hand side and using (A3) we get

$$\int_{\Omega} B(x/\varepsilon, u^\varepsilon(t)) dx + \int_0^t \|u^\varepsilon(s)\|_{W^{1,2}(\Omega)}^2 ds \leq C$$

for almost all $t \in (0, T)$. Hence

$$\sup_{0 \leq t \leq T} \int_{\Omega} B(x/\varepsilon, u^\varepsilon(t)) dx + \int_0^T \|u^\varepsilon(t)\|_{W^{1,2}(\Omega)}^2 dt \leq C \quad (4.1)$$

and thus

$$\|u^\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C. \quad (4.2)$$

Next we proceed analogously as in [19]. Using $\chi_{(t,t+h)}w$, $w \in V$, as a test function in (3.1) we obtain

$$\begin{aligned} \langle b(x/\varepsilon, u^\varepsilon(t+h)) - b(x/\varepsilon, u^\varepsilon(t)), w \rangle + \int_t^{t+h} \int_{\Omega} \chi_1(x/\varepsilon) a(\theta^\varepsilon) \nabla u^\varepsilon \cdot \nabla w dx ds \\ = - \int_t^{t+h} \int_{\Omega} \mathbf{g}(x/\varepsilon, \theta^\varepsilon, u^\varepsilon) \cdot \nabla w dx ds. \end{aligned} \quad (4.3)$$

Now we set $w = u^\varepsilon(t+h) - u^\varepsilon(t)$ and integrate (4.3) with respect to t over $(0, T-h)$ to obtain

$$\begin{aligned} \int_0^{T-h} (b(x/\varepsilon, u^\varepsilon(t+h)) - b(x/\varepsilon, u^\varepsilon(t)), u^\varepsilon(t+h) - u^\varepsilon(t)) dt \\ \leq Ch \int_0^T (\|u^\varepsilon\|_{W^{1,2}(\Omega)}^2 + 1) dt \end{aligned}$$

and owing to (4.1) we have

$$\int_0^{T-h} (b(x/\varepsilon, u^\varepsilon(t+h)) - b(x/\varepsilon, u^\varepsilon(t)), u^\varepsilon(t+h) - u^\varepsilon(t)) dt \leq Ch. \quad (4.4)$$

Now using (A1) there exists $\tilde{y} \in \mathbb{R}^2$ such that

$$\int_0^{T-h} (b(\tilde{y}, u^\varepsilon(t+h)) - b(\tilde{y}, u^\varepsilon(t)), u^\varepsilon(t+h) - u^\varepsilon(t)) dt \leq Ch. \quad (4.5)$$

Further, combining (4.1) and (2.5) we deduce

$$\sup_{0 \leq t \leq T} \int_{\Omega} B(\tilde{y}, u^\varepsilon(t)) dx \leq C. \quad (4.6)$$

Note that C is independent of ε . From (4.2) it follows that

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)) \quad (4.7)$$

(through a subsequence which we shall again denote by “ $\{\varepsilon\}$ ”). Using the compactness arguments of [4, Lemma 1.9] (see also [19, Eqs. (2.10)–(2.12)]) and the estimates (4.5) and (4.6) we get

$$b(\tilde{y}, u^\varepsilon) \rightarrow b(\tilde{y}, u) \quad \text{in } L^1(\Omega_T) \quad (4.8)$$

and almost everywhere on Ω_T . Since b is strictly monotone (recall (2.2)), it follows from (4.8) that, see [26, Proposition 3.35],

$$u^\varepsilon \rightarrow u \quad \text{almost everywhere on } \Omega_T. \quad (4.9)$$

Now let us derive some uniform estimates for θ^ε . First, let $\tau > 0$ and define $u^\varepsilon(t) = u_0$ and $\theta^\varepsilon(t) = \theta_0$ for $-\tau < t < 0$. Letting $\tau \rightarrow 0$ in the

estimate

$$\begin{aligned}
 & -\frac{1}{\tau} \int_0^t \int_{\Omega} [b(x/\varepsilon, u^\varepsilon(s)) - b(x/\varepsilon, u^\varepsilon(s-\tau))] \theta^\varepsilon(s)^2 dx ds \\
 & + \frac{2}{\tau} \int_0^t \int_{\Omega} [b(x/\varepsilon, u^\varepsilon(s)) \theta^\varepsilon(s) - b(x/\varepsilon, u^\varepsilon(s-\tau)) \theta^\varepsilon(s-\tau)] \theta^\varepsilon(s) dx ds \\
 & + \frac{2}{\tau} \int_0^t \int_{\Omega} [\rho(x/\varepsilon) \theta^\varepsilon(s) - \rho(x/\varepsilon) \theta^\varepsilon(s-\tau)] \theta^\varepsilon(s) dx ds \\
 = & -\frac{1}{\tau} \int_0^t \langle b(x/\varepsilon, u^\varepsilon(s)) - b(x/\varepsilon, u^\varepsilon(s-\tau)), \theta^\varepsilon(s)^2 \rangle ds \\
 & + \frac{2}{\tau} \int_0^t \langle b(x/\varepsilon, u^\varepsilon(s)) \theta^\varepsilon(s) - b(x/\varepsilon, u^\varepsilon(s-\tau)) \theta^\varepsilon(s-\tau), \theta^\varepsilon(s) \rangle ds \\
 & + \frac{2}{\tau} \int_0^t \langle \rho(x/\varepsilon) \theta^\varepsilon(s) - \rho(x/\varepsilon) \theta^\varepsilon(s-\tau), \theta^\varepsilon(s) \rangle ds \\
 \geq & \frac{1}{\tau} \int_{t-\tau}^t \theta^\varepsilon(s)^2 [b(x/\varepsilon, u^\varepsilon(s)) + \rho(x/\varepsilon)] ds \\
 & - \frac{1}{\tau} \int_{-\tau}^0 \theta^\varepsilon(s)^2 [b(x/\varepsilon, u^\varepsilon(s)) + \rho(x/\varepsilon)] ds
 \end{aligned}$$

we obtain

$$\begin{aligned}
 & \int_{\Omega} \theta^\varepsilon(t)^2 [b(x/\varepsilon, u^\varepsilon(t)) + \rho(x/\varepsilon)] dx - \int_{\Omega} \theta_0^2 [b(x/\varepsilon, u_0) + \rho(x/\varepsilon)] dx \\
 \leq & - \int_0^t \langle \partial_s b(x/\varepsilon, u^\varepsilon), \theta^\varepsilon(s)^2 \rangle ds \\
 & + \int_0^t \langle \partial_s ([b(x/\varepsilon, u^\varepsilon(s)) + \rho(x/\varepsilon)] \theta^\varepsilon(s)), 2\theta^\varepsilon(s) \rangle ds
 \end{aligned} \tag{4.10}$$

for almost all $t \in (0, T)$. Now, using $\phi = [\theta^\varepsilon]^2$ as a test function in (3.1) and $\psi = 2\theta^\varepsilon$ in (3.2) and combining both equations together with (4.10) we arrive at

$$\begin{aligned}
 & \int_{\Omega} \theta^\varepsilon(t)^2 [b(x/\varepsilon, u^\varepsilon(t)) + \rho(x/\varepsilon)] dx + \int_{\Omega_t} 2\chi_2(x/\varepsilon) \lambda(\theta^\varepsilon, u^\varepsilon) |\nabla \theta^\varepsilon|^2 dx ds \\
 & \leq \int_{\Omega} \theta_0^2 (b(x/\varepsilon, u_0) + \rho(x/\varepsilon)) dx
 \end{aligned}$$

for almost all $t \in (0, T)$. Hence, using (A3) we arrive at the inequality

$$\begin{aligned} \int_{\Omega} \theta^\varepsilon(t)^2 (b(x/\varepsilon, u^\varepsilon(t)) + \rho(x/\varepsilon)) dx \\ + 2\lambda_1 \int_{\Omega_t} |\nabla \theta^\varepsilon|^2 dx ds \leq \int_{\Omega} \theta_0^2 (b(x/\varepsilon, u_0) + \rho(x/\varepsilon)) dx. \end{aligned}$$

Therefore we conclude that

$$\sup_{0 \leq t \leq T} \|\theta^\varepsilon(t)\|_{L^2(\Omega)}^2 + \int_0^T \|\theta^\varepsilon\|_{W^{1,2}(\Omega)}^2 dt \leq C, \quad (4.11)$$

which immediately yields

$$\|\theta^\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega))} \leq C. \quad (4.12)$$

It follows that θ^ε converges weakly in $L^2(0, T; W^{1,2}(\Omega))$ to θ (along a selected subsequence). To get *almost everywhere convergence* of θ^ε we prove the following estimate

$$\int_0^{T-h} \int_{\Omega} |\theta^\varepsilon(t+h) - \theta^\varepsilon(t)|^2 dx dt \leq Ch, \quad (4.13)$$

which can be obtained similarly as (4.4). Namely, using $\chi_{(t,t+h)} w$, $w \in V$, as a test function in (3.2) we obtain

$$\begin{aligned} & \langle [b(x/\varepsilon, u^\varepsilon(t+h)) + \rho(x/\varepsilon)] \theta^\varepsilon(t+h) - [b(x/\varepsilon, u^\varepsilon(t)) + \rho(x/\varepsilon)] \theta^\varepsilon(t), w \rangle \\ & + \int_t^{t+h} \int_{\Omega} \chi_2(x/\varepsilon) \lambda(\theta^\varepsilon, u^\varepsilon) \nabla \theta^\varepsilon \cdot \nabla w dx ds \\ & + \int_t^{t+h} \int_{\Omega} \theta^\varepsilon [a(x/\varepsilon, \theta^\varepsilon) \nabla u^\varepsilon + \mathbf{g}(x/\varepsilon, \theta^\varepsilon, u^\varepsilon)] \cdot \nabla w dx ds \\ & = 0. \end{aligned} \quad (4.14)$$

Now we set $w = \theta^\varepsilon(t+h) - \theta^\varepsilon(t)$ and integrate (4.14) with respect to t over

$(0, T - h)$ to obtain

$$\begin{aligned}
 & \int_0^{T-h} (b(x/\varepsilon, u^\varepsilon(t+h)) [\theta^\varepsilon(t+h) - \theta^\varepsilon(t)], \theta^\varepsilon(t+h) - \theta^\varepsilon(t)) \, dt \\
 & + \int_0^{T-h} ([b(x/\varepsilon, u^\varepsilon(t+h)) - b(x/\varepsilon, u^\varepsilon(t))] \theta^\varepsilon(t), \theta^\varepsilon(t+h) - \theta^\varepsilon(t)) \, dt \\
 & + \int_0^{T-h} \int_\Omega \rho(x/\varepsilon) |\theta^\varepsilon(t+h) - \theta^\varepsilon(t)|^2 \, dx \, dt \\
 & + \int_0^{T-h} \int_t^{t+h} \int_\Omega \chi_2(x/\varepsilon) \lambda(\theta^\varepsilon(s), u^\varepsilon(s)) \nabla \theta^\varepsilon(s) \cdot \nabla (\theta^\varepsilon(t+h) - \theta^\varepsilon(t)) \, dx \, ds \, dt \\
 & + \int_0^{T-h} \int_t^{t+h} \int_\Omega \theta^\varepsilon(s) [a(x/\varepsilon, \theta^\varepsilon(s)) \nabla u^\varepsilon(s) + \mathbf{g}(x/\varepsilon, \theta^\varepsilon(s), u^\varepsilon(s))] \\
 & \quad \cdot \nabla (\theta^\varepsilon(t+h) - \theta^\varepsilon(t)) \, dx \, ds \, dt \\
 & = 0.
 \end{aligned}$$

Hence

$$\begin{aligned}
 & c_1 \int_0^{T-h} \int_\Omega |\theta^\varepsilon(t+h) - \theta^\varepsilon(t)|^2 \, dx \, dt \\
 & \leq \int_0^{T-h} ([b(x/\varepsilon, u^\varepsilon(t+h)) - b(x/\varepsilon, u^\varepsilon(t))] \theta^\varepsilon(t), \theta^\varepsilon(t+h) - \theta^\varepsilon(t)) \, dt \\
 & \quad + Ch \int_0^T \left(\|\theta^\varepsilon\|_{W^{1,2}(\Omega)}^2 + \|\theta^\varepsilon\|_{L^\infty(\Omega)} (\|u^\varepsilon\|_{W^{1,2}(\Omega)} + 1) \|\theta^\varepsilon\|_{W^{1,2}(\Omega)} \right) \, dt.
 \end{aligned} \tag{4.15}$$

The first integral on the right-hand side in (4.15) can be further estimated using (A1) (Lipschitz continuity of b in the second variable), (3.3) and the

Young's inequality to get

$$\begin{aligned}
 & \int_0^{T-h} ([b(x/\varepsilon, u^\varepsilon(t+h)) - b(x/\varepsilon, u^\varepsilon(t))] \theta^\varepsilon(t), \theta^\varepsilon(t+h) - \theta^\varepsilon(t)) \, dt \\
 & \leq C(\delta) \|\theta^\varepsilon\|_{L^\infty(\Omega_T)}^2 \int_0^{T-h} \int_\Omega |b(x/\varepsilon, u^\varepsilon(t+h)) - b(x/\varepsilon, u^\varepsilon(t))|^2 \, dx \, dt \\
 & \quad + \delta \int_0^{T-h} \int_\Omega |\theta^\varepsilon(t+h) - \theta^\varepsilon(t)|^2 \, dx \, dt \\
 & \leq C(\delta) \int_0^{T-h} \int_\Omega (b(x/\varepsilon, u^\varepsilon(t+h)) - b(x/\varepsilon, u^\varepsilon(t)), u^\varepsilon(t+h) - u^\varepsilon(t)) \, dx \, dt \\
 & \quad + \delta \int_0^{T-h} \int_\Omega |\theta^\varepsilon(t+h) - \theta^\varepsilon(t)|^2 \, dx \, dt. \tag{4.16}
 \end{aligned}$$

Now, choosing δ sufficiently small, combining (4.15), (4.16) and using (3.3), (4.2), (4.4) and (4.12) we get (4.13).

Finally, from (3.2), using (A2), (A3), (4.2), (4.12) and (3.3), we can write

$$\|\partial_t (b(x/\varepsilon, u^\varepsilon) \theta^\varepsilon + \rho(x/\varepsilon) \theta^\varepsilon)\|_{L^2(0,T;V^*)} \leq C. \tag{4.17}$$

As a consequence of the preceding a priori estimates, in particular, (3.3), (4.2), (4.5), (4.6), (4.11), (4.12), (4.13) and (4.17), we see that there exist functions $u, \theta \in L^2(0, T; W^{1,2}(\Omega))$, $\beta, \omega \in L^2(\Omega_T)$ and the functionals $\delta, \gamma \in L^2(0, T; V^*)$, such that, along a selected subsequence denoted again by $\{\varepsilon\}$, we have

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)), \tag{4.18}$$

$$u^\varepsilon \rightarrow u \quad \text{almost everywhere in } \Omega_T, \tag{4.19}$$

$$b(x/\varepsilon, u^\varepsilon) \rightharpoonup \beta \quad \text{weakly in } L^2(\Omega_T), \tag{4.20}$$

$$\partial_t b(x/\varepsilon, u^\varepsilon) \rightharpoonup \delta \quad \text{weakly in } L^2(0, T; V^*) \tag{4.21}$$

and

$$\theta^\varepsilon \rightharpoonup \theta \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega)), \tag{4.22}$$

$$\theta^\varepsilon \rightarrow \theta \quad \text{almost everywhere in } \Omega_T, \tag{4.23}$$

$$\theta^\varepsilon \rightharpoonup \theta \quad \text{weakly in } L^p(\Omega_T), \, 1 < p < +\infty, \tag{4.24}$$

$$\theta^\varepsilon \rightharpoonup \theta \quad \text{weakly star in } L^\infty(\Omega_T), \tag{4.25}$$

$$b(x/\varepsilon, u^\varepsilon) \theta^\varepsilon + \rho(x/\varepsilon) \theta^\varepsilon \rightharpoonup \omega \quad \text{weakly in } L^2(\Omega_T), \tag{4.26}$$

$$\partial_t [b(x/\varepsilon, u^\varepsilon) \theta^\varepsilon + \rho(x/\varepsilon) \theta^\varepsilon] \rightharpoonup \gamma \quad \text{weakly in } L^2(0, T; V^*). \tag{4.27}$$

5. Two-scale limit

First we review basic properties of the two-scale (in the sense of [3, 40]) and single-scale convergence, see e.g. [24, Appendix A] and [15, Chapter 9]. We then state some results concerning the weak limits of the nonlinear parabolic parts in (1.1) and (1.2) which will be used in Subsection 5.2. The main result of this section is Theorem 5.5.

5.1. Auxiliary results

Theorem 5.1. [3, Chapter 2] *Let v^ε be a bounded sequence in $W^{1,p}(\Omega)$ with $1 < p < +\infty$ that converges weakly to a limit v in $W^{1,p}(\Omega)$. Then, v^ε two-scale converges to $v(x)$, and there exists a function $v_1 \in L^p(\Omega; W_{per}^{1,p}(\mathcal{Y}))$ such that, up to a subsequence,*

$$\nabla v^\varepsilon \rightarrow \nabla_x v + \nabla_y v_1 \quad \text{in the two-scale sense.}$$

Lemma 5.2. [15, Lemma 9.1] *Let $\phi \in L^p(\Omega; C_{per}(\mathcal{Y}))$ with $1 \leq p < +\infty$. Then the function $\phi(\cdot/\varepsilon, \cdot) \in L^p(\Omega)$ with*

$$\|\phi(\cdot/\varepsilon, \cdot)\|_{L^p(\Omega)} \leq \|\phi(\cdot, \cdot)\|_{L^p(\Omega; C_{per}(\mathcal{Y}))}$$

and $\phi(\cdot/\varepsilon, \cdot)$ converges weakly in $L^p(\Omega)$ to

$$\int_{\mathcal{Y}} \phi(y, x) dy.$$

Lemma 5.3. [33, Lemma 4.2] *Let (\mathcal{M}, μ) be a σ -finite measure space and f_n and $g_n \in L^1(\mathcal{M})$ be two sequences of functions, and let $f, g, h \in L^1(\mathcal{M})$ such that (as $n \rightarrow +\infty$)*

$$\begin{aligned} f_n &\rightarrow f && \text{almost everywhere in } \mathcal{M}, \\ g_n &\rightharpoonup g && \text{weakly in } L^1(\mathcal{M}), \\ f_n g_n &\rightharpoonup h && \text{weakly in } L^1(\mathcal{M}). \end{aligned}$$

Then $h = fg$ almost everywhere in \mathcal{M} .

Lemma 5.4. *There exists a subsequence of $\{\varepsilon\}$, still denoted by $\{\varepsilon\}$, such that*

$$b(x/\varepsilon, u^\varepsilon) \rightharpoonup b^*(u) \quad \text{weakly in } L^2(\Omega_T), \quad (5.1)$$

$$\partial_t b(x/\varepsilon, u^\varepsilon) \rightharpoonup \partial_t b^*(u) \quad \text{weakly in } L^2(0, T; V^*), \quad (5.2)$$

$$b(x/\varepsilon, u^\varepsilon) \theta^\varepsilon + \rho(x/\varepsilon) \theta^\varepsilon \rightharpoonup b^*(u) \theta + \rho^* \theta \quad \text{weakly in } L^2(\Omega_T) \quad (5.3)$$

and

$$\partial_t [b(x/\varepsilon, u^\varepsilon) \theta^\varepsilon + \rho(x/\varepsilon) \theta^\varepsilon] \rightharpoonup \partial_t [b^*(u) \theta + \rho^* \theta] \quad \text{weakly in } L^2(0, T; V^*), \quad (5.4)$$

Proof. Limits (5.1) and (5.2) are proven in [37, Corollary 4.3]. The limit (5.3) follows from Lemma 5.3, (5.1) and the estimate

$$\left\| b(x/\varepsilon, u^\varepsilon) \theta^\varepsilon + \rho(x/\varepsilon) \theta^\varepsilon \right\|_{L^2(\Omega_T)} \leq c,$$

where c is independent of ε . Now, (5.4) follows from (5.3) and (4.27). \square

5.2. Passage to the limit for $\varepsilon \rightarrow 0$

Theorem 5.5. *There exist a function $u_1 \in L^2(\Omega_T; W_{per}^{1,2}(\mathcal{Y}))$ and a subsequence u^ε (still denoted by u^ε) such that*

$$\nabla u^\varepsilon \rightarrow \nabla_x u + \nabla_y u_1(x, y, t) \quad \text{in the two-scale sense.}$$

Similarly, there exist a function $\theta_1 \in L^2(\Omega_T; W_{per}^{1,2}(\mathcal{Y}))$ and a subsequence θ^ε (still denoted by θ^ε) such that

$$\nabla \theta^\varepsilon \rightarrow \nabla_x \theta + \nabla_y \theta_1(x, y, t) \quad \text{in the two-scale sense.}$$

Further, the pairs (u, u_1) and (θ, θ_1) satisfy the following two-scale homogenized coupled problem

$$\begin{aligned} & \int_0^T \left\langle \int_{\mathcal{Y}} \partial_t b(y, u) dy, \phi \right\rangle dt \\ & + \int_{\Omega_T} \int_{\mathcal{Y}} \chi_1(y) a(\theta) (\nabla_x u + \nabla_y u_1(x, y, t)) \cdot (\nabla_x \phi + \nabla_y \phi_1(x, y, t)) dy dx dt \\ & + \int_{\Omega_T} \int_{\mathcal{Y}} \mathbf{g}(y, \theta, u) \cdot (\nabla_x \phi + \nabla_y \phi_1(x, y, t)) dy dx dt \\ & = 0 \end{aligned} \quad (5.5)$$

and

$$\begin{aligned}
 & \int_0^T \langle \int_{\mathcal{Y}} \partial_t [b(y, u) \theta + \rho(y) \theta] dy, \psi \rangle dt \\
 & + \int_{\Omega_T} \int_{\mathcal{Y}} \chi_2(y) \lambda(\theta, u) (\nabla_x \theta + \nabla_y \theta_1(x, y, t)) \cdot (\nabla_x \psi + \nabla_y \psi_1(x, y, t)) dy dx dt \\
 & + \int_{\Omega_T} \int_{\mathcal{Y}} \theta \chi_1(y) a(\theta) (\nabla_x u + \nabla_y u_1(x, y, t)) \cdot (\nabla_x \psi + \nabla_y \psi_1(x, y, t)) dy dx dt \\
 & + \int_{\Omega_T} \int_{\mathcal{Y}} \theta g(y, \theta, u) \cdot (\nabla_x \psi + \nabla_y \psi_1(x, y, t)) dy dx dt \\
 & = 0
 \end{aligned} \tag{5.6}$$

for all test functions $\phi, \psi \in C_0^\infty(\Omega_T)$ and $\phi_1, \psi_1 \in C_0^\infty(\Omega_T; C_{per}^\infty(\mathcal{Y}))$.

Proof. Let $\phi, \psi \in C_0^\infty(\Omega_T)$ and $\phi_1, \psi_1 \in C_0^\infty(\Omega_T; C_{per}^\infty(\mathcal{Y}))$. We take test functions as

$$\phi^\varepsilon = \phi(x, t) + \varepsilon \phi_1(x, x/\varepsilon, t) \tag{5.7}$$

and

$$\psi^\varepsilon = \psi(x, t) + \varepsilon \psi_1(x, x/\varepsilon, t), \tag{5.8}$$

respectively, in (3.1) and (3.2). Note that, by (5.2) and the strong convergence of ϕ^ε to ϕ in $L^p(\Omega_T)$ we deduce

$$\int_0^T \langle \partial_t b(x/\varepsilon, u^\varepsilon), \phi^\varepsilon \rangle dt \rightarrow \int_0^T \langle \partial_t b^*(u), \phi \rangle dt. \tag{5.9}$$

Similarly, by (5.4) we have

$$\begin{aligned}
 & \int_0^T \langle \partial_t [(b(x/\varepsilon, u^\varepsilon) + \rho(x/\varepsilon)) \theta^\varepsilon], \psi^\varepsilon \rangle dt \\
 & \rightarrow \int_0^T \langle \partial_t [(b^*(u) + \rho^*) \theta], \psi \rangle dt.
 \end{aligned} \tag{5.10}$$

Further, using (2.4), (4.18) and (4.23) we deduce

$$a(\theta^\varepsilon) \nabla u^\varepsilon \rightharpoonup a(\theta) \nabla u \quad \text{weakly in } L^2(\Omega_T)^2.$$

Indeed, $\chi_1 \in L^\infty(\mathcal{Y})$ and $\nabla \phi^\varepsilon \in L^2_{per}(\mathcal{Y}; C(\overline{\Omega}))$ so that $\chi_1(x/\varepsilon)\nabla \phi^\varepsilon$ can be used as a test function in the two-scale convergence of $a(\theta^\varepsilon)\nabla u^\varepsilon$. Hence

$$\begin{aligned} \int_{\Omega_T} \chi_1(x/\varepsilon) a(\theta^\varepsilon) \nabla u^\varepsilon \cdot \nabla \phi^\varepsilon \, dx dt &= \int_{\Omega_T} a(\theta^\varepsilon) \nabla u^\varepsilon \cdot [\chi_1(x/\varepsilon) \nabla \phi^\varepsilon] \, dx dt \\ &\rightarrow \int_{\Omega_T} \int_{\mathcal{Y}} \chi_1(y) a(\theta) (\nabla_x u + \nabla_y u_1(x, y, t)) \cdot (\nabla_x \phi + \nabla_y \phi_1(x, y, t)) \, dy dx dt. \end{aligned} \quad (5.11)$$

We deduce similarly that

$$\begin{aligned} \int_{\Omega_T} \mathbf{g}(x/\varepsilon, \theta^\varepsilon, u^\varepsilon) \cdot \nabla \phi^\varepsilon \, dx dt \\ \rightarrow \int_{\Omega_T} \int_{\mathcal{Y}} \mathbf{g}(y, \theta, u) \cdot (\nabla_x \phi + \nabla_y \phi_1(x, y, t)) \, dy dx dt. \end{aligned} \quad (5.12)$$

In the same manner we conclude that

$$\begin{aligned} \int_{\Omega_T} \chi_2(x/\varepsilon) \lambda(\theta^\varepsilon, u^\varepsilon) \nabla \theta^\varepsilon \cdot \nabla \psi^\varepsilon \, dx dt \\ \rightarrow \int_{\Omega_T} \int_{\mathcal{Y}} \chi_2(y) \lambda(\theta, u) (\nabla_x \theta + \nabla_y \theta_1(x, y, t)) \cdot (\nabla_x \psi + \nabla_y \psi_1(x, y, t)) \, dy dx dt \end{aligned} \quad (5.13)$$

and

$$\begin{aligned} \int_{\Omega_T} \theta^\varepsilon \chi_1(x/\varepsilon) a(\theta^\varepsilon) \nabla u^\varepsilon \cdot \nabla \psi^\varepsilon \, dx dt \\ \rightarrow \int_{\Omega_T} \int_{\mathcal{Y}} \theta \chi_1(y) a(\theta) (\nabla_x u + \nabla_y u_1(x, y, t)) \cdot (\nabla_x \psi + \nabla_y \psi_1(x, y, t)) \, dy dx dt \end{aligned} \quad (5.14)$$

and finally

$$\begin{aligned} \int_{\Omega_T} \theta^\varepsilon \mathbf{g}(x/\varepsilon, \theta^\varepsilon, u^\varepsilon) \cdot \nabla \psi^\varepsilon \, dx dt \\ \rightarrow \int_{\Omega_T} \int_{\mathcal{Y}} \theta \mathbf{g}(y, \theta, u) \cdot (\nabla_x \psi + \nabla_y \psi_1(x, y, t)) \, dy dx dt. \end{aligned} \quad (5.15)$$

Thus, choosing (5.7) and (5.8), respectively, as test functions in (3.1) and (3.2) the above convergences (5.9)–(5.10) and (5.11)–(5.15) are sufficient for taking the limit $\varepsilon_j \rightarrow 0$ as $j \rightarrow +\infty$ (along a selected subsequence) to get (5.5)–(5.6). This completes the proof of Theorem 5.5. \square

6. Proof of the main result

In this section we complete the proof of Theorem 3.5, the main result of our paper. In particular, we identify the homogenized problem corresponding to (1.1)–(1.9), which can be obtained in the following way. Setting $\phi = 0$ in (5.5) we arrive at

$$\begin{aligned} \int_{\Omega_T} \int_{\mathcal{Y}} \chi_1(y) a(\theta) (\nabla_x u + \nabla_y u_1(x, y, t)) \cdot \nabla_y \phi_1(x, y, t) dy dx dt \\ = - \int_{\Omega_T} \int_{\mathcal{Y}} \mathbf{g}(y, \theta, u) \cdot \nabla_y \phi_1(x, y, t) dy dx dt \end{aligned} \quad (6.1)$$

for all $\phi_1 \in C_0^\infty(\Omega_T; C_{per}^\infty(\mathcal{Y}))$. Here we determine u_1 (up to a constant) as

$$u_1(x, y, t) = \nabla_x u(x, t) \cdot \mathbf{w}(x, y, t) + \eta(x, y, t), \quad (6.2)$$

where $\mathbf{w}(x, y, t)$, $\mathbf{w} = (w_1, \dots, w_d)$, and $\eta(x, y, t)$ can be obtained in the following way. Indeed, combining (6.1) and (6.2) we arrive at

$$\begin{aligned} \int_{\Omega_T} \int_{\mathcal{Y}} \chi_1(y) a(\theta) (\nabla_x u + \nabla_y [\nabla_x u(x, t) \cdot \mathbf{w}(x, y, t) \\ + \eta(x, y, t)]) \cdot \nabla_y \phi_1(x, y, t) dy dx dt \\ = - \int_{\Omega_T} \int_{\mathcal{Y}} \mathbf{g}(y, \theta, u) \cdot \nabla_y \phi_1(x, y, t) dy dx dt \end{aligned}$$

and thus

$$\begin{aligned} \int_{\Omega_T} a(\theta) \int_{\mathcal{Y}} [\chi_1(y) \nabla_x u (\mathbf{I}_d + \nabla_y \mathbf{w}(x, y, t))] \cdot \nabla_y \phi_1(x, y, t) dy dx dt \\ + \int_{\Omega_T} \int_{\mathcal{Y}} [\chi_1(y) a(\theta) \nabla_y \eta(x, y, t) + \mathbf{g}(y, \theta, u)] \cdot \nabla_y \phi_1(x, y, t) dy dx dt \\ = 0 \end{aligned} \quad (6.3)$$

for all $\phi_1 \in C_0^\infty(\Omega_T; C_{per}^\infty(\mathcal{Y}))$. Here and in what follows, $\nabla_y \mathbf{w}$ is the matrix $(\nabla_y \mathbf{w})_{ij} = \partial w_i / \partial y_j$. Integrating by parts in (6.3) we deduce

$$\begin{aligned} - \int_{\Omega_T} a(\theta) \int_{\mathcal{Y}} \nabla_y \cdot [\chi_1(y) \nabla_x u (\mathbf{I}_d + \nabla_y \mathbf{w}(x, y, t))] \phi_1(x, y, t) dy dx dt \\ - \int_{\Omega_T} \int_{\mathcal{Y}} \nabla_y \cdot [\chi_1(y) a(\theta) \nabla_y \eta(x, y, t) + \mathbf{g}(y, \theta, u)] \phi_1(x, y, t) dy dx dt \\ = 0. \end{aligned}$$

From this, it is easy to show that \mathbf{w} and η can be obtained as solutions of periodic cell problems (3.19)–(3.20) and (3.21)–(3.22).

Now let $\phi_1 = 0$ in (5.5). Then we have

$$\begin{aligned} & \int_0^T \left\langle \int_{\mathcal{Y}} \partial_t b(y, u) dy, \phi \right\rangle dt \\ & + \int_{\Omega_T} a(\theta) \int_{\mathcal{Y}} \chi_1(y) [\nabla_x u + \nabla_y u_1(x, y, t)] \cdot \nabla_x \phi dy dx dt \\ & + \int_{\Omega_T} \int_{\mathcal{Y}} \mathbf{g}(y, \theta, u) \cdot \nabla_x \phi dy dx dt \\ & = 0 \end{aligned}$$

and taking into account (6.2) we can write

$$\begin{aligned} & \int_0^T \left\langle \partial_t \int_{\mathcal{Y}} b(y, u) dy, \phi \right\rangle dt \\ & + \int_{\Omega_T} \left[\nabla_x u(x, t) a(\theta) \left(\int_{\mathcal{Y}} \chi_1(y) (\mathbf{I}_d + \nabla_y \mathbf{w}(x, y, t)) dy \right) \right] \cdot \nabla_x \phi dx dt \\ & + \int_{\Omega_T} \left(\int_{\mathcal{Y}} (a(\theta) \chi_1(y) \nabla_y \eta(x, y, t) + \mathbf{g}(y, \theta, u)) dy \right) \cdot \nabla_x \phi dx dt \\ & = 0 \end{aligned} \tag{6.4}$$

for all $\phi \in C_0^\infty(\Omega_T)$. Note that (6.4) represents the weak form of the equation (3.5).

Similar procedure applies for the energy equation (5.6). Setting $\psi = 0$ in (5.6) leads to

$$\begin{aligned} & \int_{\Omega_T} \int_{\mathcal{Y}} \chi_2(y) \lambda(\theta, u) (\nabla_x \theta + \nabla_y \theta_1(x, y, t)) \cdot \nabla_y \psi_1(x, y, t) dy dx dt \\ & + \int_{\Omega_T} \theta \int_{\mathcal{Y}} a(\theta) \chi_1(y) (\nabla_x u + \nabla_y u_1(x, y, t)) \cdot \nabla_y \psi_1(x, y, t) dy dx dt \\ & + \int_{\Omega_T} \theta \int_{\mathcal{Y}} \mathbf{g}(y, \theta, u) \cdot \nabla_y \psi_1(x, y, t) dy dx dt \\ & = 0 \end{aligned} \tag{6.5}$$

for all $\psi_1 \in C_0^\infty(\Omega_T; C_{per}^\infty(\mathcal{Y}))$. Here, θ_1 can be computed (up to a constant) through the relationship

$$\theta_1(x, y, t) = \nabla_x \theta(x, t) \cdot \mathbf{\Lambda}(x, y, t), \quad \mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_d), \tag{6.6}$$

which yields, substituting (6.2) and (6.6) into (6.5),

$$\begin{aligned}
 & \int_{\Omega_T} \int_{\mathcal{Y}} \chi_2(y) \lambda(\theta, u) \nabla_x \theta [\mathbf{I}_d + \nabla_y \mathbf{\Lambda}(x, y, t)] \cdot \nabla_y \psi_1(x, y, t) dy dx dt \\
 & + \int_{\Omega_T} a(\theta) \theta \underbrace{\int_{\mathcal{Y}} \chi_1(y) \nabla_x u [\mathbf{I}_d + \nabla_y \mathbf{w}(x, y, t)] \cdot \nabla_y \psi_1(x, y, t) dy}_{0} dx dt \\
 & + \int_{\Omega_T} \theta \underbrace{\int_{\mathcal{Y}} [a(\theta) \chi_1(y) \nabla_y \eta(x, y, t) + \mathbf{g}(y, \theta, u)] \cdot \nabla_y \psi_1(x, y, t) dy}_{0} dx dt \\
 & = 0
 \end{aligned} \tag{6.7}$$

for all $\psi_1 \in C_0^\infty(\Omega_T; C_{per}^\infty(\mathcal{Y}))$. Here and subsequently, $\nabla_y \mathbf{\Lambda}$ is the matrix $(\nabla_y \mathbf{\Lambda})_{ij} = \partial \Lambda_i / \partial y_j$. Taking into account (3.19)–(3.22), the second and third integral on the left hand side in (6.7) vanish. Hence, Λ_i are obtained as the unique solutions of the periodic cell problems (3.23)–(3.24).

Now let $\psi_1 = 0$ in (5.6). This leads to

$$\begin{aligned}
 & \int_0^T \left\langle \int_{\mathcal{Y}} \partial_t [b(y, u) \theta + \rho(y) \theta] dy, \psi \right\rangle dt \\
 & + \int_{\Omega_T} \int_{\mathcal{Y}} \chi_2(y) \lambda(\theta, u) (\nabla_x \theta + \nabla_y \theta_1(x, y, t)) \cdot \nabla_x \psi dy dx dt \\
 & + \int_{\Omega_T} \int_{\mathcal{Y}} \theta a(\theta) \chi_1(y) (\nabla_x u + \nabla_y u_1(x, y, t)) \cdot \nabla_x \psi dy dx dt \\
 & + \int_{\Omega_T} \int_{\mathcal{Y}} \theta \mathbf{g}(y, \theta, u) \cdot \nabla_x \psi dy dx dt \\
 & = 0
 \end{aligned}$$

for all $\psi \in C_0^\infty(\Omega_T)$ and, using (6.2) and (6.6), we can write

$$\begin{aligned}
 & \int_0^T \langle \partial_t \left[\theta \int_{\mathcal{Y}} b(y, u) dy + \theta \int_{\mathcal{Y}} \rho(y) dy \right], \psi \rangle dt \\
 & + \int_{\Omega_T} \left(\int_{\mathcal{Y}} \chi_2(y) \lambda(\theta, u) \nabla_x \theta (\mathbf{I}_d + \nabla_y \mathbf{\Lambda}(x, y, t)) dy \right) \cdot \nabla_x \psi dx dt \\
 & + \int_{\Omega_T} \theta \left(\int_{\mathcal{Y}} \chi_1(y) a(\theta) \nabla_x u (\mathbf{I}_d + \nabla_y \mathbf{w}(x, y, t)) dy \right) \cdot \nabla_x \psi dx dt \\
 & + \int_{\Omega_T} \theta \left(\int_{\mathcal{Y}} (a(\theta) \chi_1(y) \nabla_y \eta(x, y, t) + \mathbf{g}(y, \theta, u)) dy \right) \cdot \nabla_x \psi dx dt \\
 & = 0.
 \end{aligned} \tag{6.8}$$

We conclude that (6.8) is the weak form of the equation (3.6). The proof of Theorem 3.5 is complete.

Appendix A. The existence of the weak solution to (1.1)–(1.8)

Step 1. Approximations. Let us fix $N \in \mathbb{N}$ and set $\tau = T/N$ be a time step. We approximate our evolution problem by a semi-implicit time discretization scheme. Let $u_N^0 \equiv u_0 \in V \cap L^\infty(\Omega)$ and $\theta_N^0 \equiv \theta_0 \in V \cap L^\infty(\Omega)$ be given. Then we define, in each time step, $[u_N^n, \theta_N^n]$ as a solution of the following recurrence problem:

Find a pair $[u_N^n, \theta_N^n] \in V^2$, $n = 1, \dots, N$, such that

$$\begin{aligned}
 & \int_{\Omega} \frac{b(x/\varepsilon, u_N^n) - b(x/\varepsilon, u_N^{n-1})}{\tau} \phi dx \\
 & + \int_{\Omega} (\chi_1(x/\varepsilon) a(\theta_N^{n-1}) \nabla u_N^n + \mathbf{g}(x/\varepsilon, \theta_N^{n-1}, u_N^n)) \cdot \nabla \phi dx \\
 & = 0
 \end{aligned} \tag{A.1}$$

for all $\phi \in V$;

$$\begin{aligned}
 & \int_{\Omega} \frac{b(x/\varepsilon, u_N^n) \theta_N^n - b(x/\varepsilon, u_N^{n-1}) \theta_N^{n-1}}{\tau} \psi dx + \int_{\Omega} \rho(x/\varepsilon) \frac{\theta_N^n - \theta_N^{n-1}}{\tau} \psi dx \\
 & + \int_{\Omega} \chi_2(x/\varepsilon) \lambda(\theta_N^{n-1}, u_N^{n-1}) \nabla \theta_N^n \cdot \nabla \psi dx \\
 & + \int_{\Omega} \theta_N^n [\chi_1(x/\varepsilon) a(\theta_N^{n-1}) \nabla u_N^n + \mathbf{g}(x/\varepsilon, \theta_N^{n-1}, u_N^n)] \cdot \nabla \psi dx \\
 & = 0
 \end{aligned} \tag{A.2}$$

for all $\psi \in V$.

Step 2. Existence of the solution to (A.1)–(A.2). Let $[u_N^{n-1}, \theta_N^{n-1}] \in V^2 \cap L^\infty(\Omega)^2$ be given and τ be “small enough”. Combining the standard existence theory for elliptic problems, see e.g. [46, Chapter 2], [38, Chapter 3], [22, 32], together with the regularity results in [20], the problem (A.1)–(A.2) possesses the solution $[u_N^n, \theta_N^n] \in V^2 \cap L^\infty(\Omega)^2$.

Step 2. A-priori estimates. Here we prove some uniform estimates (with respect to N) for the time interpolants of the solution. We define the piecewise constant interpolants ($n = 1, 2, \dots, N$) $\bar{\phi}_N(t) = \phi_N^n$ for $t \in ((n-1)\tau, n\tau]$ and, in addition, we extend $\bar{\phi}_N$ for $t \leq 0$ by $\bar{\phi}_N(t) = \phi_0$ for $t \in (-\tau, 0]$. For a function φ we often use the simplified notation $\varphi = \varphi(t)$, $\varphi_\tau(t) = \varphi(t - \tau)$, $\partial_t^{-\tau} \varphi(t) = \frac{\varphi(t) - \varphi(t-\tau)}{\tau}$, $\partial_t^\tau \varphi(t) = \frac{\varphi(t+\tau) - \varphi(t)}{\tau}$. Then, following (A.1) and (A.2), the piecewise constant interpolants $\bar{u}_N \in L^\infty(0, T; W^{1,2}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$, and $\bar{\theta}_N \in L^\infty(0, T; W^{1,2}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$ satisfy the equations

$$\begin{aligned} & \int_{\Omega} \partial_t^{-\tau} b(x/\varepsilon, \bar{u}_N(t)) \phi \, dx \\ & + \int_{\Omega} [\chi_1(x/\varepsilon) a(\bar{\theta}_N(t-\tau)) \nabla \bar{u}_N(t) + \mathbf{g}(x/\varepsilon, \bar{\theta}_N(t-\tau), \bar{u}_N(t))] \cdot \nabla \phi \, dx \\ & = 0 \end{aligned} \tag{A.3}$$

for all $\phi \in V$ and $t \in (0, T)$ and

$$\begin{aligned} & \int_{\Omega} \partial_t^{-\tau} [b(x/\varepsilon, \bar{u}_N(t)) \bar{\theta}_N(t) + \varrho(x/\varepsilon) \bar{\theta}_N(t)] \psi \, dx \\ & + \int_{\Omega} \chi_2(x/\varepsilon) \lambda(\bar{\theta}_N(t-\tau), \bar{u}_N(t-\tau)) \nabla \bar{\theta}_N(t) \cdot \nabla \psi \, dx \\ & + \int_{\Omega} \bar{\theta}_N(t) [\chi_1(x/\varepsilon) a(\bar{\theta}_N(t-\tau)) \nabla \bar{u}_N(t) + \mathbf{g}(x/\varepsilon, \bar{\theta}_N(t-\tau), \bar{u}_N(t))] \cdot \nabla \psi \, dx \\ & = 0 \end{aligned} \tag{A.4}$$

for all $\psi \in V$ and $t \in (0, T)$.

We test (A.3) with $\phi = \bar{u}_N(t)$ and integrate over t from 0 to s . For the

parabolic term we can write

$$\begin{aligned} & \int_0^s \int_{\Omega} \partial_t^{-\tau} b(x/\varepsilon, \bar{u}_N(t)) \bar{u}_N(t) \, dx dt \\ & \geq \frac{1}{\tau} \int_{s-\tau}^s \int_{\Omega} B(x/\varepsilon, \bar{u}_N(t)) - B(x/\varepsilon, u_0) \, dx dt. \end{aligned} \quad (\text{A.5})$$

Now, using (A.5) and standard estimates for the elliptic part and applying the Gronwall's lemma we deduce, cf. [4],

$$\sup_{0 \leq t \leq T} \int_{\Omega} B(x/\varepsilon, \bar{u}_N(t)) \, dx + \int_0^T \int_{\Omega} |\nabla \bar{u}_N(t)|^2 \, dx dt \leq c. \quad (\text{A.6})$$

Hence it follows that

$$\bar{u}_N \rightharpoonup u \quad \text{weakly in } L^2(0, T; V) \quad (\text{A.7})$$

along a selected subsequence (letting $N \rightarrow +\infty$). Let $k \in \mathbb{N}$ and use

$$\phi(t) = \partial_t^{k\tau} \bar{u}_N(s)$$

for $j\tau \leq t \leq (j+k)\tau$ with $(j-1)\tau \leq s \leq j\tau$ and $1 \leq j \leq \frac{T}{\tau} - k$, as a test function in (A.3). Integrating over s and using (A.6) we obtain

$$\int_0^{T-k\tau} (b(x/\varepsilon, \bar{u}_N(s+k\tau)) - b(x/\varepsilon, \bar{u}_N(s))) (\bar{u}_N(s+k\tau) - \bar{u}_N(s)) \, ds \leq ck\tau. \quad (\text{A.8})$$

Using the compactness arguments as in [4, Lemma 1.9] and [19, Eqs. (2.10)–(2.12)] we conclude

$$b(x/\varepsilon, \bar{u}_N) \rightarrow b(x/\varepsilon, u) \text{ in } L^1(\Omega_T) \quad (\text{A.9})$$

(for a subsequence) and almost everywhere on Ω_T . Since b is strictly monotone in the second variable, it follows from (A.9) that [26, Proposition 3.35]

$$\bar{u}_N \rightarrow u \quad \text{almost everywhere.} \quad (\text{A.10})$$

The above a priori estimates furthermore imply

$$\begin{aligned} \int_0^T \langle \omega_N, \phi \rangle \, dt &= \int_0^T \int_{\Omega} \partial_t^{-\tau} b(x/\varepsilon, \bar{u}_N(t)) \phi(t) \, dx dt \\ &= - \int_0^{T-h} \int_{\Omega} [b(x/\varepsilon, \bar{u}_N(t)) - b(x/\varepsilon, u_0)] \partial_t^{\tau} \phi(t) \, dx dt \end{aligned}$$

for all $\phi \in L^2(0, T; V)$ with $\phi(t) = 0$ for (a.a.) $t > T - h$. From this we have

$$\omega_N \rightharpoonup \omega \quad \text{weakly in } L^2(0, T; V^*) \quad (\text{A.11})$$

and $\omega = \partial_t b(x/\varepsilon, u)$.

Now we use $\psi = 2\bar{\theta}_N$ as a test function in (A.4) to obtain

$$\begin{aligned} & 2 \int_{\Omega} \partial_t^{-\tau} b(x/\varepsilon, \bar{u}_N(t)) \bar{\theta}_N(t)^2 dx \\ & + 2 \int_{\Omega} \chi_2(x/\varepsilon) \lambda(\bar{\theta}_N(t - \tau), \bar{u}_N(t - \tau)) \nabla \bar{\theta}_N(t) \cdot \nabla \bar{\theta}_N(t) dx \\ & + 2 \int_{\Omega} [\chi_1(x/\varepsilon) a(\bar{\theta}_N(t - \tau)) \nabla \bar{u}_N(t) + \mathbf{g}(x/\varepsilon, \bar{\theta}_N(t - \tau), \bar{u}_N(t))] \cdot \bar{\theta}_N(t) \nabla \bar{\theta}_N(t) dx \\ & = 0. \end{aligned} \quad (\text{A.12})$$

We are allowed to use $\phi(t) = \bar{\theta}_N(t)^2$ as a test function in (A.3) to obtain

$$\begin{aligned} & \int_{\Omega} \partial_t^{-\tau} b(x/\varepsilon, \bar{u}_N(t)) \bar{\theta}_N(t)^2 dx \\ & + \int_{\Omega} [\chi_1(x/\varepsilon) a(\bar{\theta}_N(t - \tau)) \nabla \bar{u}_N(t) + \mathbf{g}(x/\varepsilon, \bar{\theta}_N(t - \tau), \bar{u}_N(t))] \cdot \nabla \bar{\theta}_N(t)^2 dx \\ & = 0. \end{aligned} \quad (\text{A.13})$$

Combining (A.12) and (A.13) we deduce

$$\begin{aligned} & \int_{\Omega} \partial_t^{-\tau} [\bar{\theta}_N(t)^2 (b(x/\varepsilon, \bar{u}_N(t)) + \varrho(x/\varepsilon))] dx \\ & + \int_{\Omega} \frac{1}{\tau} (\bar{\theta}_N(t) - \bar{\theta}_N(t - \tau))^2 (b(x/\varepsilon, \bar{u}_N(t - \tau)) + \varrho(x/\varepsilon)) dx \\ & + 2 \int_{\Omega} \chi_2(x/\varepsilon) \lambda(\bar{\theta}_N(t - \tau), \bar{u}_N(t - \tau)) |\nabla \bar{\theta}_N(t)|^2 dx \\ & = 0. \end{aligned} \quad (\text{A.14})$$

Integrating (A.14) with respect to time t we obtain the a-priori estimate

$$\sup_{0 \leq t \leq T} \int_{\Omega} |\bar{\theta}_N(t)|^2 dx + \int_0^T \|\bar{\theta}_N(t)\|_{W^{1,2}(\Omega)}^2 dt \leq c. \quad (\text{A.15})$$

This allows us to conclude that there exists $\theta \in L^2(0, T; V)$ such that, letting $N \rightarrow +\infty$ (along a selected subsequence),

$$\bar{\theta}_N \rightharpoonup \theta \quad \text{weakly in } L^2(0, T; V). \quad (\text{A.16})$$

Further, in much the same way as in (A.8), we arrive at

$$\int_0^{T-k\tau} |w(x/\varepsilon, \bar{u}_N(s+k\tau), \bar{\theta}_N(s+k\tau)) - w(x/\varepsilon, \bar{u}_N(s), \bar{\theta}_N(s))|^2 ds \leq ck\tau,$$

where $w(x/\varepsilon, \bar{u}_N, \bar{\theta}_N) = [b(x/\varepsilon, \bar{u}_N) + \varrho(x/\varepsilon)] \bar{\theta}_N$. From this we conclude that

$$\int_0^{T-k\tau} |\bar{\theta}_N(s+k\tau) - \bar{\theta}_N(s)|^2 ds \leq ck\tau. \quad (\text{A.17})$$

Hence

$$\bar{\theta}_N \rightarrow \theta \quad \text{almost everywhere.} \quad (\text{A.18})$$

Let ℓ be an odd integer. Using $\phi = \bar{\theta}_N^\ell$ as a test function in (A.3) and $\psi = [\ell/(\ell+1)]\bar{\theta}_N^{\ell+1}$ in (A.4) and combining both equations we have

$$\begin{aligned} & \int_{\Omega} \bar{\theta}_N(t)^{\ell+1} [b(x/\varepsilon, \bar{u}_N(t)) + \rho(x/\varepsilon)] dx \\ & + \int_{\Omega_t} \ell \bar{\theta}_N(s)^{\ell-1} \chi_2(x/\varepsilon) \lambda(\bar{\theta}_N(s), \bar{u}_N(s)) |\nabla \bar{\theta}_N(s)|^2 dx ds \\ & = \int_{\Omega} \theta_0^{\ell+1} [b(x/\varepsilon, u_0) + \rho(x/\varepsilon)] dx. \end{aligned} \quad (\text{A.19})$$

Recall that ℓ is the odd integer. Hence the second integral in (A.19) is nonnegative. Moreover, from (A.19) it follows that

$$\|\bar{\theta}_N\|_{L^\infty(0,T;L^{\ell+1}(\Omega))} \leq C, \quad (\text{A.20})$$

where the constant C is independent of ℓ , τ and ε . Now, let $\ell \rightarrow +\infty$ in (A.20), we get

$$\|\bar{\theta}_N\|_{L^\infty(\Omega_T)} \leq C, \quad (\text{A.21})$$

where the constant C is independent of τ and ε . Hence, letting $N \rightarrow +\infty$ (along a selected subsequence), we have

$$\bar{\theta}_N \rightharpoonup \theta \quad \text{weakly star in } L^\infty(\Omega_T). \quad (\text{A.22})$$

Step 3. Passing to the limit. The above established convergences (A.7), (A.10) and (A.16), (A.18) and (A.22) are sufficient for taking the limit $N \rightarrow +\infty$ in (A.1) and (A.2) (along a selected subsequence) to get the weak solution of the system (1.1)–(1.8) in the sense of Definition 3.1 and satisfying (3.3). This completes the proof of Theorem 3.2.

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