J. Math. Anal. Appl. 451 (2017) 623–628

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Spectral property of certain fractal measures ☆

Dao-Xin Ding

Department of Mathematics, Hubei University of Education, Wuhan, 430205, PR China

ARTICLE INFO

Article history:
Received 24 October 2016
Available online 24 February 2017
Submitted by B.S. Thomson

Keywords:
Infinite convolution
Compatible pair
Spectral measure
Spectra

ABSTRACT

Let $\{0, a_j, b_j\} = \{0, 1, 2\}(mod\ 3)$ be a sequence of digit sets in $\mathbb{Z}$, and let $\{N_j = 3r_j\}$ be a sequence of integers bigger than 1. We call $\{f_{j,d}(x) = N_j^{-1}(x + d) : d \in \{0, a_j, b_j\}\}_{j=0}^{\infty}$ a Moran iterated function system, which is a generalization of an IFS. We prove that the associated Moran measure is spectral.

1. Introduction

We say that a compactly supported probability measure μ is a spectral measure if there exists a set of complex exponentials $E(\Lambda) := \{e^{2\pi i\langle\lambda,x\rangle}\}_{\lambda\in\Lambda}$ such that it is an orthonormal basis of $L^2(\mu)$. If such Λ exists, it is called a spectrum of μ. We also say a set Ω is a spectral set if $\chi_\Omega dx$ is a spectral measure. The study of spectral sets was first initiated from B. Fuglede in 1974 [10]. He proposed a reasonable conjecture on spectral sets:

Fuglede's Conjecture. $\Omega \subset \mathbb{R}^s$ is a spectral set if and only if Ω is a translational tile.

The problem of spectral measures is exciting when we consider fractal measures. Jorgensen and Pedersen [13] showed that the standard Cantor measure is a spectral measure if the contraction is $\frac{1}{2n}$, while there are at most two orthogonal exponentials when the contraction is $\frac{1}{2n+1}$. Following this discovery, more spectral fractal measures were found [1–6,8,7,9,11,12,14–16]. In particular, An and He [1] constructed a class of Moran spectral measures. Motivated by their ideas, we will focus on certain Moran measures.

Two finite sets $\mathcal{A} = \{a_j\}$ and $\mathcal{S} = \{s_j\}$ of cardinality q in $\mathbb{R}$ form a compatible pair, following the terminology of [16], if the matrix $M = [\frac{1}{\sqrt{q}}e(a_js_k)]$ is a unitary matrix. In other words $(\delta_{\mathcal{A}}, \mathcal{S})$ is a spectral pair, where

☆ The research is supported by the National Natural Science Foundation of China (No. 11401189) and the NSF of Hubei Province of China (No. 2014CFC1128, 2015CFC883).

E-mail address: daoxinding@yeah.net.

http://dx.doi.org/10.1016/j.jmaa.2017.02.040
0022-247X/

$$\delta_{\mathcal{A}} := \sum_{a \in \mathcal{A}} \frac{1}{q} \delta_a.$$

A compatible tower is a sequence (finite or infinite) of compatible pairs

$$\{B_0, L_0\}, \{B_1, L_1\}, \{B_2, L_2\}, \cdots$$

with $B_j \subset M_j^{-1}\mathbb{Z}^n$ and $L_j \subset \mathbb{Z}^n$, and matrices $R_j \in GL(n, M_{j-1}\mathbb{Z}^n)$ for $j \geq 1$.

Let $\{N_j\}_{j=0}^{\infty}$ be a sequence of integers with all $N_j \geq 2$ and let $\{D_j\}_{j=0}^{\infty}$ be a sequence of digit sets with $0 \in D_j \subset \mathbb{N}$ for each $j \geq 0$. We say $\{f_{j,d}(x) = N_j^{-1}(x + d) : d \in D_j\}_{j=0}^{\infty}$ is a Moran iterated function system, which is a generalization of an IFS. If $\sup\{d : d \in D_j, j \geq 0\} < \infty$, Strichartz [17] proved that there exists a compact set T and a Borel probability measure μ_T supported on T. Moreover,

$$T = \sum_{j=0}^{\infty} (N_0 N_1 \cdots N_j)^{-1} D_j = \left\{ \sum_{j=0}^{\infty} (N_0 N_1 \cdots N_j)^{-1} d_j : d_j \in D_j, j \geq 0 \right\}$$

and

$$\mu_T = \delta_{N_0^{-1} D_0} * \delta_{(N_0 N_1)^{-1} D_1} * \cdots * \delta_{(N_0 N_1 \cdots N_j)^{-1} D_j} * \cdots ,$$

where $*$ is the convolution sign.

Let $\mathcal{N} = \{N_j : N_j = 3r_j, r_j \in \mathbb{Z}^+, j = 0, 1, 2, \cdots\}$ and $\mathcal{D} = \{D_j : D_j = \{0, a_j, b_j\} \subset \mathbb{N}\}$ where $\sup\{d : d \in D_j, j \geq 0\} < \infty$ and $a_j \in 3\mathbb{Z} + 1, b_j \in 3\mathbb{Z} + 2, j = 0, 1, 2, \cdots$. We use $\mu_{\mathcal{N},\mathcal{D}}$ to denote the corresponding Moran measure.

Theorem 1.1. *$\mu_{\mathcal{N},\mathcal{D}}$ is a spectral measure with a spectrum*

$$\Lambda = r_0 L + r_0 N_1 L + \cdots + r_0 N_1 \cdots N_k L + \cdots$$

where $L = \{-1, 0, 1\}$ and each element of Λ is a finite sum.

Remark 1.2.

(1) Theorem 2.8 in [16] indicates that, to obtain uniform control in the use of Dominated Convergence Theorem, expanding matrices $\{R_j\}$ must be chosen from a finite set of expanding matrices. However, $\{N_j\}$ in Theorem 1.1 can be chosen from an infinite set of integers.

(2) If $N_j = 3$, $\mathcal{D}_j = \{0, 1, 2\}$ for all $j \in \mathbb{N}$, then $\mu_{\mathcal{N},\mathcal{D}} = \chi_{[0,1]} dx$. In this case, $\Lambda = \mathbb{Z}$ is a spectrum for $\chi_{[0,1]} dx$. In addition, given a $\mathcal{N}$, we see that, for any $\mathcal{D}$, the corresponding Moran measure has the same spectrum.

2. Proof of Theorem

The mask function of a finite set D in $\mathbb{R}$ is defined by

$$m_D(\xi) = \frac{1}{\#D} \sum_{d \in D} e^{-2\pi i d \xi}.$$

As usual, the Fourier transform of a probability measure μ in $\mathbb{R}$ is defined by

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi x} d\mu(x).$$

Then $m_D(\xi) = \widehat{\delta_D}(\xi)$. Let $\mathcal{Z}(\hat{\mu}) := \{\xi : \hat{\mu}(\xi) = 0\}$ denote the zero set of $\hat{\mu}$. A set of complex exponentials $\{e_\lambda = e^{2\pi i \lambda x} : \lambda \in \Lambda\}$ is said to be orthogonal if $\hat{\mu}(\lambda_i - \lambda_j) = 0$ for any $\lambda_i, \lambda_j \in \Lambda$. We say that Λ is a bi-zero set of $\hat{\mu}$ if $(\Lambda - \Lambda)\backslash\{0\} \subset \mathcal{Z}(\hat{\mu})$ and $0 \in \Lambda$, and call it a maximal bi-zero set if it is maximal in $\mathcal{Z}(\hat{\mu})$ to have the set difference property. Clearly Λ is a bi-zero set if and only if $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal subset with respect to μ.

We first give the following lemma, which plays an important role in observing the zero set of $\widehat{\mu_{\mathcal{N},\mathcal{D}}}$.

Lemma 2.1. *Let a, b be two integers and $\gcd(a, b) = 1$. Then $1 + z^a + z^b$ has zero point on the unit circle if and only if $\{a, b\} = \{1, 2\}(mod\ 3)$. In particular, when $\{a, b\} = \{1, 2\}(mod\ 3)$, $1 + z^a + z^b$ and $1 + z + z^2$ have the same zero set on the unit circle.*

Proof. The sufficiency is obvious. As for necessity, let $z = e^{2\pi i \xi}$ be a zero point of $1 + z^a + z^b$. We have

$$\begin{cases} 2\pi a\xi = 2k_1\pi + \frac{2\pi}{3} \\ 2\pi b\xi = 2k_2\pi + \frac{4\pi}{3} \end{cases} \text{or} \begin{cases} 2\pi a\xi = 2k_3\pi + \frac{4\pi}{3} \\ 2\pi b\xi = 2k_4\pi + \frac{2\pi}{3} \end{cases}$$

where k_1, k_2, k_3, k_4 are integers. Thus $3ak_2 + 2a = 3bk_1 + b$ or $3ak_4 + a = 3bk_3 + 2b$. Since $\gcd(a, b) = 1$, we get $\{a, b\} = \{1, 2\}(\text{mod } 3)$.

The set of zero points on the unit circle of $1 + z + z^2$ is $\{\omega, \omega^2\}$ where $\omega = e^{\frac{2\pi i}{3}}$. Evidently, when $\{a, b\} = \{1, 2\}(mod\ 3)$, ω and ω^2 are also zero points of $1 + z^a + z^b$. Conversely, if z is a zero point of $1 + z^a + z^b$, we have

$$\begin{cases} z^a = \omega \\ z^b = \omega^2 \end{cases} \text{or} \begin{cases} z^a = \omega^2 \\ z^b = \omega. \end{cases}$$

Since $\gcd(a, b) = 1$, there exist two integers p, q such that $pa + qb = 1$. Therefore,

$$z = z^{pa+qb} = \omega^{p+2q} \text{ or } z = z^{pa+qb} = \omega^{2p+q}.$$

Obviously, $z \neq 1$, so $z = \omega$ or $z = \omega^2$. We complete the proof. □

By $\sup\{d : d \in D_j, j \geq 0\} < \infty$, we can take two integers a, b such that $|a| \geq \sup\{|a_j|\}$, $|b| \geq \sup\{|b_j|\}$, $|a - b| \geq \sup\{|a_j - b_j|\}$ and $\gcd(a, b) = 1$.

Lemma 2.2. *There exists a positive constant δ such that*

$$|m_{D_j}(\xi)| \geq |m_D(\xi)|, \forall \xi \in (0, \delta), \forall j \in \mathbb{N},$$

where $D = \{0, a, b\}$. In particular, $|m_D(\xi)|$ is decreasing on $(0, \delta)$.

Proof. It is easy to calculate that

$$9|m_D(\xi)|^2 = 3 + 2cos2\pi a\xi + 2cos2\pi b\xi + 2cos(a - b)\xi.$$

For $\delta > 0$ small enough, the results hold. □

Given any $n \in \mathbb{N}$, we can expand it into the unique finite triadic expansion,

$$n = \sum_{j=1}^{k} \sigma_j 3^{j-1}, \ \sigma_j \in \{0, 1, 2\}, \ \sigma_k \neq 0.$$

In this way n is uniquely corresponding to one word $\sigma = \sigma_1 \cdots \sigma_k$ which is called the triadic expansion of n. Define a mapping τ from $\{0, 1, 2\}$ to $\{-1, 0, 1\}$ as $\tau(0) = 0, \ \tau(1) = 1$ and $\tau(2) = -1$. It is easy to check that $\Lambda = \{r_0\lambda_n\}_{n=0}^{\infty}$ holds, where

$$\lambda_n = \tau(\sigma_1) + \sum_{k=2}^{\infty} \tau(\sigma_k) \left(\prod_{j=1}^{k-1} N_j \right).$$

We have the following lemma.

Lemma 2.3. *Let $\{N_{n_j}\}$ be a subsequence of $\{N_j\}$ with $N_{n_j} \to \infty$ as $j \to \infty$, and let $0 \leq k \leq 3^{n_{j+1}-1}$ for all $j \in \mathbb{Z}$. For an arbitrary but fixed $\xi \in \mathbb{R}$, we have*

$$\nu_j := \prod_{l=0}^{\infty} \left| m_{D_{n_{j+1}+l}} \left(\frac{\xi + r_0\lambda_k}{\prod_{i=0}^{n_{j+1}+l} N_i} \right) \right| \to 1$$

as $j \to \infty$.

Proof. It is easy to check that there exists a positive constant c such that

$$\left| \frac{\xi + r_0\lambda_k}{\prod_{i=0}^{n_{j+1}-1} N_i} \right| \leq c.$$

Since $n_j \to \infty$ as $j \to \infty$, for all j large enough, we have

$$\left| \frac{\xi + r_0\lambda_k}{\prod_{i=0}^{n_{j+1}+l} N_i} \right| \leq \delta, \ \forall l \geq 0.$$

Notice $|m_D(\xi)| = |m_D(-\xi)|$. By Lemma 2.2 and $N_{n_j} \to \infty$ as $j \to \infty$, we have

$$\begin{aligned}
\nu_j(\xi) &\geq \prod_{l=0}^{\infty} \left| m_D \left(\frac{\xi + r_0\lambda_k}{\prod_{i=0}^{n_{j+1}+l} N_i} \right) \right| \\
&\geq \prod_{l=0}^{\infty} \left| m_D \left(\frac{1}{N^l} \cdot \frac{1}{N_{n_{j+1}}} \cdot \frac{\xi + r_0\lambda_k}{\prod_{i=0}^{n_{j+1}-1} N_i} \right) \right| \\
&= \left| m_D \left(\frac{1}{N_{n_{j+1}}} \cdot \frac{\xi + r_0\lambda_k}{\prod_{i=0}^{n_{j+1}-1} N_i} \right) \right| \cdot \left| \widehat{\mu_{N,D}} \left(\frac{1}{N_{n_{j+1}}} \cdot \frac{\xi + r_0\lambda_k}{\prod_{i=0}^{n_{j+1}-1} N_i} \right) \right| \to 1
\end{aligned}$$

as $j \to \infty$, where $N = \min\{N_i\}$. $\square$

We recall also the Jorgensen–Pedersen Lemma about checking when Λ is a spectrum for μ.

Lemma 2.4. *[13] Λ is a spectrum for a probability measure μ on $\mathbb{R}^d$ if and only if*

$$Q(\xi) := \sum_{\lambda\in\Lambda} |\widehat{\mu}(\xi+\lambda)|^2 \equiv 1.$$

Moreover, if Λ is an orthogonal set, then Q is an entire function on $\mathbb{C}^d$ with $0 \le Q(x) \le 1$ for $x \in \mathbb{R}^d$.

For any $n \in \mathbb{N}$, set

$$\mu_n = \delta_{N_0^{-1}D_0} * \delta_{(N_0N_1)^{-1}D_1} * \cdots * \delta_{(N_0N_1\cdots N_{n-1})^{-1}D_{n-1}}.$$

We have the following lemma.

Lemma 2.5.

$$\sum_{k=0}^{3^{n_{j+1}}-1} |\widehat{\mu_{n_{j+1}}}(\xi + r_0\lambda_k)|^2 \equiv 1, \quad \forall \xi \in \mathbb{R}.$$

Proof. It is easy to check that $\{r_0\lambda_k\}_{k=0}^{3^{n_{j+1}}-1}$ is a bi-zero set of $\mu_{n_{j+1}}$. Then $E(\{r_0\lambda_k\}_{k=0}^{3^{n_{j+1}}-1})$ is an orthogonal subset with respect to $\mu_{n_{j+1}}$. The dimension of the space $L^2(\mu_{n_{j+1}})$ is just $3^{n_{j+1}}$, hence $E(\{r_0\lambda_k\}_{k=0}^{3^{n_{j+1}}-1})$ is an orthonormal basis of $L^2(\mu_{n_{j+1}})$, that is, $\{r_0\lambda_k\}_{k=0}^{3^{n_{j+1}}-1}$ is a spectrum of $\mu_{n_{j+1}}$. According to Lemma 2.4, we get the desired result. $\square$

Proof of Theorem 1.1. Case I: When $\{N_j\}$ are chosen from a finite set of integers. Let $B_j = (N_j)^{-1}D_j$ and $L_j = r_jL$ for $j \ge 0$. Then $\{B_j, L_j\}$ forms a compatible power with $R_j = N_{j-1}$ for $j \ge 1$. We have by Lemma 2.1 that the zero set of μ_n is

$$\mathcal{Z}(\widehat{\mu_n}) = \{\xi \in \mathbb{R} : \widehat{\mu_n}(\xi) = 0\} = \cup_{j=0}^{n-1} r_0r_1\cdots r_j\{a : 3 \nmid a, a \in \mathbb{Z}\},$$

and

$$\begin{aligned} T_n &= (R_n^*)^{-1}\cdots(R_1^*)^{-1}L_0 + (R_n^*)^{-1}\cdots(R_2^*)^{-1}L_1 + \cdots + (R_n^*)^{-1}L_{n-1} \\ &= (N_{n-1})^{-1}\cdots(N_0)^{-1}L_0 + (N_{n-1})^{-1}\cdots(N_1)^{-1}L_1 + \cdots + (N_{n-1})^{-1}L_{n-1}. \end{aligned}$$

It is easy to calculate that $T_n \subset (-\frac{1}{2}, \frac{1}{2})$ for any $n \in \mathbb{N}$. Hence the zero set $\mathcal{Z}(\widehat{\mu_n})$ is separated from the set T_n by $\frac{1}{2}$, uniformly in n. According to Theorem 2.8 in [16], we have that $\mu_{\mathcal{N},\mathcal{D}}$ is a spectral measure with a spectrum Λ.

Case II: When $\{N_j\}$ are chosen from an infinite set of integers, there exists a subsequence $\{N_{n_j}\}$ of $\{N_j\}$ with $N_{n_j} \to \infty$ as $j \to \infty$. We argue by contradiction. Notice that $\Lambda = \{r_0\lambda_n\}_{n=0}^{\infty}$. Suppose that $\{r_0\lambda_n\}_{n=0}^{\infty}$ is not a spectrum of $\mu_{\mathcal{N},\mathcal{D}}$, then we can pick a $\xi_0 \in \mathbb{R}$ such that $0 \le Q(\xi_0) < 1$. By Lemma 2.3, for j large enough, we have $\nu_j(\xi_0) \ge \frac{Q(\xi_0)+1}{2}$. Therefore,

$$|\widehat{\mu_{\mathcal{N},\mathcal{D}}}(\xi_0 + r_0\lambda_k)| = |\widehat{\mu_{n_{j+1}}}(\xi_0 + r_0\lambda_k)| \cdot \nu_j(\xi_0) \ge |\widehat{\mu_{n_{j+1}}}(\xi_0 + r_0\lambda_k)| \cdot \frac{Q(\xi_0)+1}{2}.$$

We then get from Lemma 2.5 that the following inequalities hold.

$$1 \ge Q_{n_{j+1}}(\xi_0) = Q_{n_j}(\xi_0) + \sum_{k=3^{n_j}}^{3^{n_{j+1}}-1} |\widehat{\mu_{\mathcal{N},\mathcal{D}}}(\xi_0 + r_0\lambda_k)|^2$$

$$
\begin{aligned}
&= Q_{n_j}(\xi_0) + \sum_{k=3^{n_j}}^{3^{n_{j+1}}-1} |\widehat{\mu_{n_{j+1}}}(\xi_0 + r_0\lambda_k)|^2 \cdot \nu_j(\xi_0) \\
&\geq Q_{n_j}(\xi_0) + \left(1 - \sum_{k=0}^{3^{n_j}-1} |\widehat{\mu_{n_{j+1}}}(\xi_0 + r_0\lambda_k)|^2\right) \cdot \frac{Q(\xi_0)+1}{2} \\
&\geq Q_{n_j}(\xi_0) + \left(1 - \frac{2}{Q(\xi_0)+1} \sum_{k=0}^{3^{n_j}-1} |\widehat{\mu_{\mathcal{N},\mathcal{D}}}(\xi_0 + r_0\lambda_k)|^2\right) \cdot \frac{Q(\xi_0)+1}{2} \\
&\geq Q_{n_j}(\xi_0) + \frac{1-Q(\xi_0)}{2}.
\end{aligned}
$$

Iterating the above inequalities, we have $1 \geq \infty$. This is impossible. Hence, $\{r_0\lambda_n\}_{n=0}^{\infty}$ must be a spectrum of $\mu_{\mathcal{N},\mathcal{D}}$.

References

[1] L.X. An, X.G. He, A class of spectral Moran measures, J. Funct. Anal. 266 (2014) 343–354.
[2] L.X. An, X.G. He, K.S. Lau, Spectrality of a class of infinite convolutions, Adv. Math. 283 (2015) 362–376.
[3] L.X. An, X.G. He, H.X. Li, Spectrality of infinite Bernoulli convolutions, J. Funct. Anal. 269 (2015) 1571–1590.
[4] X.R. Dai, When does a Bernoulli convolution admit a spectrum?, Adv. Math. 231 (2012) 1681–1693.
[5] X.R. Dai, X.G. He, C.K. Lai, Spectral property of Cantor measures with consecutive digits, Adv. Math. 242 (2013) 187–208.
[6] X.R. Dai, X.G. He, K.S. Lau, On spectral N-Bernoulli measures, Adv. Math. 259 (2014) 511–531.
[7] D. Dutkay, D.G. Han, Q.Y. Sun, On spectra of Cantor measure, Adv. Math. 221 (2009) 251–276.
[8] D. Dutkay, P. Jorgensen, Fourier frequencies in affine iterated function systems, J. Funct. Anal. 247 (2007) 110–137.
[9] Y.S. Fu, Z.X. Wen, Spectral property of a class of Moran measures on $\mathbb{R}$, J. Math. Anal. Appl. 430 (2015) 572–584.
[10] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, J. Funct. Anal. 16 (1974) 101–121.
[11] X.G. He, C.K. Lai, K.S. Lau, Exponential spectra in $L^2(\mu)$, Appl. Comput. Harmon. Anal. 34 (2013) 327–338.
[12] T.Y. Hu, K.S. Lau, Spectral property of the Bernoulli convolution, Adv. Math. 219 (2008) 554–567.
[13] P. Jorgensen, S. Pedersen, Dense analytic subspaces in fractal L^2 spaces, J. Anal. Math. 75 (1998) 185–228.
[14] I. Łaba, Y. Wang, On spectral Cantor measures, J. Funct. Anal. 193 (2002) 409–420.
[15] J.L. Li, Spectra of a class of self-affine measures, J. Funct. Anal. 260 (2011) 1086–1095.
[16] R. Strichartz, Mock Fourier series and transforms associated with certain Cantor measures, J. Anal. Math. 81 (2000) 209–238.
[17] R. Strichartz, Convergence of Mock Fourier series, J. Anal. Math. 99 (2006) 333–353.