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A characterization of connected self-affine fractals arising from collinear digits ☆

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ABSTRACT

Let A be an expanding integer matrix with characteristic polynomial $f(x) = x^2 + px + q$, and let $\mathcal{D} = \{0, 1, \dots, |q| - 2, |q| + m\}\mathbf{v}$ be a collinear digit set where $m \geqslant 0$, $\mathbf{v} \in \mathbb{Z}^2$. It is well known that there exists a unique self-affine fractal T satisfying $AT = T + \mathcal{D}$. In this paper, we give a complete characterization of connectedness of T. That generalizes the previous result for $|q| = 3$.

1. Introduction

Given an $n \times n$ integer matrix A, we assume it is expanding, i.e., its eigenvalues all have moduli strictly larger than 1. Let $\mathcal{D} = \{\mathbf{d}_1, \dots, \mathbf{d}_k\} \subset \mathbb{R}^n$ be a *digit set*. It is well known (see [14]) that there exists a unique attractor $T := T(A, \mathcal{D})$ satisfying:

$$T = A^{-1}(T + \mathcal{D}) = \left\{ \sum_{i=1}^{\infty} A^{-i}\mathbf{d}_{j_i} : \mathbf{d}_{j_i} \in \mathcal{D} \right\}. \tag{1.1}$$

We often call T a *self-affine fractal*. If moreover, $|\det(A)| = k$ and the interior of T is nonempty, then T can tile the whole space $\mathbb{R}^n$ by translations. We call such T a *self-affine tile*.

The fundamental theory and applications of self-affine fractals/tiles have been extensively studied in the literature ([3,6,7,10,14–16]). In the studies, people found that, given a matrix A, the structures of a digit

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set $\mathcal{D}$ strongly influence the topological properties of $T(A,\mathcal{D})$, such as connectedness and disk-likeness (see [1,2,4,5,8,11–13,18,19,21–23]). Among all the researches, the collinear digit sets perhaps attracted the most attentions. Say $\mathcal{D}$ is collinear if $\mathcal{D} = \{d_1,\dots,d_k\}\mathbf{v}$ for some vector $\mathbf{v} \in \mathbb{R}^n$ and $d_1 < d_2 < \cdots < d_k$. If moreover, $d_{i+1} - d_i = 1$ for all i, $\mathcal{D}$ is said to be *consecutive collinear (CC)*. If $d_{i+1} - d_i = 1$ for all i except one i_0 such that $d_{i_0+1} - d_{i_0} > 1$, then $\mathcal{D}$ is said to have a *jump*. The study on the connected self-affine fractals/tiles arising from CC digit sets has been an interesting topic. Hacon et al. [7] first proved that a self-affine tile T is always pathwise connected when $k = 2$. Lau and his coworkers ([9,12,13,18,20]) developed this direction and systematically studied the topology of self-affine tiles for any k. The connectedness of self-affine fractals with CC digit sets was also concerned in [19,23].

However, there are very limited results on the collinear digit set $\mathcal{D}$ with jumps. In [19], we made a first attempt in this area, especially we proved that

Theorem 1.1. *Let A be an expanding integer matrix with characteristic polynomial $f(x) = x^2 + px \pm 3$, and let $\mathcal{D} = \{0, 1, b\}\mathbf{v}$ where $2 \leqslant b \in \mathbb{Z}$ such that $\{\mathbf{v}, A\mathbf{v}\}$ is linearly independent. Then we have*

(i) when $b = 2$, T is always a connected self-affine tile;
(ii) when $b \geqslant 4$, T is always a disconnected self-affine fractal;
(iii) when $b = 3$, T is connected if and only if $(p, q) \in \{(\pm1, -3), (\pm2, 3), (\pm3, 3)\}$.

For an expanding 2×2 integer matrix A, it is known by [3] that the characteristic polynomial of A is given by

$$f(x) = x^2 + px + q, \text{ with } |p| \leqslant q, \text{ if } q \geqslant 2; \quad |p| \leqslant |q + 2|, \text{ if } q \leqslant -2. \tag{1.2}$$

In the paper, we will give a complete characterization on the connectedness of T arising from a collinear digit set with a jump. As for $|q| = 2$ and $\mathcal{D} = \{\mathbf{0}, \mathbf{v}\}$, T is always a connected self-affine tile ([7] or [12]). So we will exclude this trivial case.

Theorem 1.2. *Let A be an expanding integer matrix with characteristic polynomial $f(x) = x^2 + px + q$ where $|q| \geqslant 3$, let $\mathcal{D} = \{0, 1, \dots, |q| - 2, |q| + m\}\mathbf{v}$, where $0 \leqslant m \in \mathbb{Z}$ and $\mathbf{v} \in \mathbb{Z}^2$ such that $\{\mathbf{v}, A\mathbf{v}\}$ is linearly independent. Then*

(i) when $m \geqslant 1$, T is always disconnected;
(ii) when $m = 0$, T is connected if and only if

$$(p, q) \in \{(p, q) \in \mathbb{Z}^2 : 2|p| = |q + 2|\} \cup \{(\pm1, -3), (\pm2, 3), (\pm3, 3), (\pm4, 4)\}.$$

(See Fig. 1.)

The proof is elementary although it contains lots of calculations and multiple discussions. The main idea is to fully use the radix expansion like (1.1) and Cayley–Hamilton theorem (see basic tools in Section 2). Moreover, we remark that when $m = -1$, $\mathcal{D}$ becomes a CC digit set and T is always connected [12]. We also mention that the theorem still holds if the jump occurs elsewhere. Actually the proof is the same irrespective of the location of the jump occurs for the disconnected cases as it does not involve finding the exact radix expansion. For the connected cases when $m = 0$, the difference set $\mathcal{D} - \mathcal{D}$ is unchanged wherever the jump occurs.

For the organization of the paper, we provide some useful lemmas in Section 2 and prove Theorem 1.2 by five parts in Section 3. As Appendix A, we give a Matlab program to calculate all the upper bounds of α, β used in the proof.

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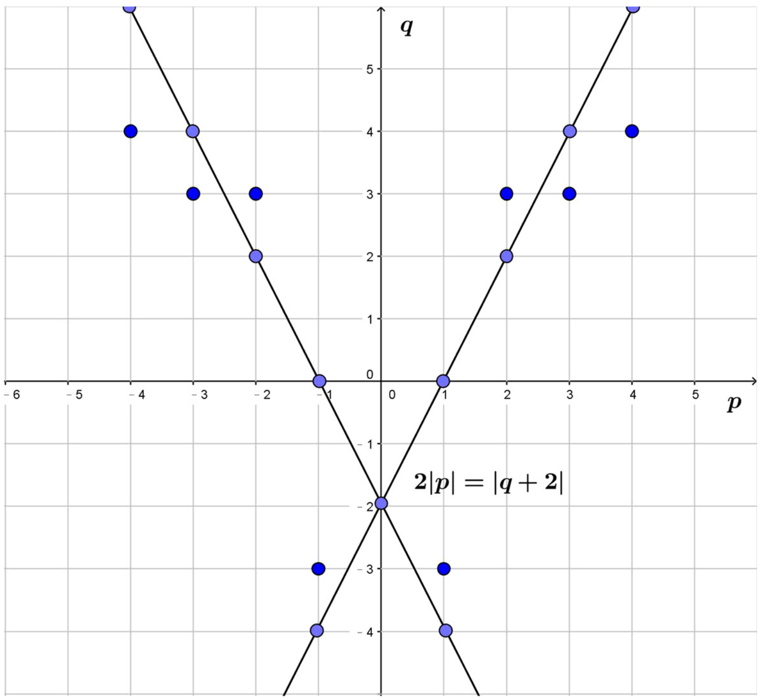

Fig. 1. The domain of (p, q) for connected self-affine fractals.

2. Basic lemmas

In this section, we prepare several basic results that will be used frequently in the next section. Let $(A, \mathcal{D})$ be given as in the assumption of Theorem 1.2. Denoted by $D = \{0, 1, \dots, |q| - 2, |q| + m\}$ and $\triangle D = D - D$, then $\mathcal{D} = D\mathbf{v}$ and $\triangle\mathcal{D} = \triangle D\mathbf{v}$.

First we provide a simple but useful lemma for connectedness of $T := T(A, \mathcal{D})$ (we refer to [8] or [12] for the general criterion of connectedness).

Lemma 2.1. *T is connected if $\mathbf{v} \in T - T$ and $(m + c)\mathbf{v} \in T - T$ for some $c \in \{2, 3, \dots, |q|\}$; T is disconnected if $(m + c)\mathbf{v} \notin T - T$ for all $c \in \{2, 3, \dots, |q|\}$.*

Proof. It follows immediately from Theorem 4.3 of [12] by using certain graph argument on $\mathcal{D}$. □

Let $\Delta = p^2 - 4q$ be the discriminant of the polynomial $f(x) = x^2 + px + q$, and define α_i, β_i by

$$A^{-i}\mathbf{v} = \alpha_i\mathbf{v} + \beta_i A\mathbf{v}, \quad i = 1, 2, \dots \tag{2.1}$$

From the Cayley–Hamilton theorem $f(A) = A^2 + pA + qI = 0$ where I is the identity matrix, we have

Lemma 2.2 *([17]). Let α_i, β_i be defined as the above. Then $q\alpha_{i+2} + p\alpha_{i+1} + \alpha_i = 0$ and $q\beta_{i+2} + p\beta_{i+1} + \beta_i = 0$, i.e.,*

$$\begin{bmatrix} \alpha_{i+1} \\ \alpha_{i+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/q & -p/q \end{bmatrix}^i \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}; \quad \begin{bmatrix} \beta_{i+1} \\ \beta_{i+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/q & -p/q \end{bmatrix}^i \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

and $\alpha_1 = -p/q, \alpha_2 = (p^2 - q)/q^2; \beta_1 = -1/q, \beta_2 = p/q^2$. Moreover for $\Delta \neq 0$, we have

$$\alpha_i = \frac{q(y_1^{i+1} - y_2^{i+1})}{\Delta^{1/2}} \quad \text{and} \quad \beta_i = \frac{-(y_1^i - y_2^i)}{\Delta^{1/2}},$$

where $y_1 = \frac{-p+\Delta^{1/2}}{2q}$ *and* $y_2 = \frac{-p-\Delta^{1/2}}{2q}$ *are the two roots of* $qx^2 + px + 1 = 0$.

Write

$$\alpha := \sum_{i=1}^{\infty} |\alpha_i|, \quad \beta := \sum_{i=1}^{\infty} |\beta_i|.$$

By Lemma 2.2, if $\Delta < 0$, then the value of β (similarly for α) can be bounded by

$$\beta \leqslant \sum_{i=1}^{n-1} |\beta_i| + \frac{2q^{-n/2}}{(1-q^{-1/2})(4q-p^2)^{1/2}}. \tag{2.2}$$

In Appendix A, we will provide all the upper bounds of α, β that are used frequently in Section 3.

Lemma 2.3 *([23]). If* $\Delta \geqslant 0$. *Then*

$$\alpha = \begin{cases} \frac{|p|-1}{q-|p|+1} & q > 0 \\ \frac{|p|+1}{|q|-|p|-1} & q < 0; \end{cases} \qquad \beta = \begin{cases} \frac{1}{q-|p|+1} & q > 0 \\ \frac{1}{|q|-|p|-1} & q < 0. \end{cases}$$

Let $L := \{\gamma\mathbf{v} + \delta A\mathbf{v} : \gamma, \delta \in \mathbb{Z}\}$ be the *lattice* generated by $\{\mathbf{v}, A\mathbf{v}\}$. For $l \in L \setminus \{\mathbf{0}\}$, $T + l$ is called a *neighbor* of T if $T \cap (T + l) \neq \emptyset$. It is easy to see that $T + l$ is a neighbor of T if and only if l can be expressed as

$$l = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T, \text{ where } b_i \in \triangle D.$$

By using (2.1), the above l can be written as $l = \gamma\mathbf{v} + \delta A\mathbf{v}$ where $\gamma = \sum_{i=1}^{\infty} b_i\alpha_i$ and $\delta = \sum_{i=1}^{\infty} b_i\beta_i$. Thus

$$|\gamma| \leqslant \max_i |b_i|\alpha \quad \text{and} \quad |\delta| \leqslant \max_i |b_i|\beta. \tag{2.3}$$

Lemma 2.4. *Suppose* $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$ *for some integer* k. *Then* $kA\mathbf{v} - b_1\mathbf{v} \in T - T$; $(-kp - b_1)A\mathbf{v} + (-kq - b_2)\mathbf{v} \in T - T$; *and* $(kp^2 - kq + pb_1 - b_2)A\mathbf{v} + (kpq + b_1q - b_3)\mathbf{v} \in T - T$.

Proof. Multiplying $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v}$ by A and then moving the first term on the left to the right, we get $kA\mathbf{v} - b_1\mathbf{v} = \sum_{i=1}^{\infty} b_{i+1}A^{-i}\mathbf{v} \in T - T$. Applying the same procedure and replacing A^2 with $-pA - qI$ (using $f(A) = 0$), we can get the other two relations. $\square$

Our crucial part of the proof in Section 3 is to find contradictions with (2.3) by making use of (2.2) and Lemmas 2.3, 2.4.

Lemma 2.5. *Let* $T_1 = T(A, \mathcal{D})$ *and* $T_2 = T(-A, \mathcal{D})$. *Then* $T_1 + l$ *is a neighbor of* T_1 *if and only if* $T_2 + l$ *is a neighbor of* T_2. *Hence* T_1 *is connected if and only if* T_2 *is connected.*

Proof. If $l \in T_1 - T_1$, then

$$l = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} = \sum_{i=1}^{\infty} b_{2i}(-A)^{-2i}\mathbf{v} + \sum_{i=1}^{\infty}(-b_{2i-1})(-A)^{-2i+1}\mathbf{v}.$$

Hence $l \in T_2 - T_2$ and vice versa. The second part follows from the first one and Theorem 4.3 of [12]. □

The last lemma is a special case of Theorem 1.3 in [19].

Lemma 2.6. *Under the same assumption of Theorem 1.2. If $p = 0$, then T is connected if and only if $\mathcal{D} = \{0, 1, \ldots, |q| - 1\}\mathbf{v}$.*

3. Proof of the main theorem

The proof of Theorem 1.2, which consists of five parts, is given in this section. Parts I and II deal with the case when $m \geqslant 1$ while parts III to V the case $m = 0$. In view of Lemmas 2.5, 2.6 and Theorem 1.1, it suffices to show the theorem under the assumption that $|q| \geqslant 4$ and $p \geqslant 1$.

• Part I

In this part we assume that $f(x) = x^2 + px + q$ where $q \geqslant 4$ and $p \geqslant 1$. Let $D = \{0, 1, 2, \ldots, q-2, q+m\}$ where $m \geqslant 1$. The proof here is then divided into two cases: $\Delta = p^2 - 4q \geqslant 0$ and $\Delta < 0$. Moreover, in Part I, we will explain carefully how to use Lemma 2.4 to get contradictions, then we shall omit the details in the remaining parts for simplification.

Suppose

$$(m+c)\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T, \quad \text{where } \ c \geqslant 2,\ b_i \in \triangle D. \tag{3.1}$$

By Lemma 2.4, it follows from (3.1) that $(m+c)A\mathbf{v} - b_1\mathbf{v} \in T - T$ and

$$-[p(m+c) + b_1]A\mathbf{v} - [q(m+c) + b_2]\mathbf{v} \in T - T. \tag{3.2}$$

<u>**Case A**: $\Delta \geqslant 0$</u>

Since A is expanding, $q \geqslant p$ by (1.2). Lemma 2.3 and (2.3) imply that if $l := \gamma\mathbf{v} + \delta A\mathbf{v} \in T - T$, then

$$|\delta| \leqslant \frac{q+m}{q-p+1}. \tag{3.3}$$

If $(m+c)p - c < (m+c-1)q$, then $m + c > \dfrac{q+m}{q-p+1}$ contradicting (3.3). Hence $(m+c)\mathbf{v} \notin T - T$. Now assume

$$(m+c-1)q \leqslant (m+c)p - c. \tag{3.4}$$

By (3.3),

$$|p(m+c) + b_1| \leqslant \frac{q+m}{q-p+1}. \tag{3.5}$$

When $q = p$, (3.5) becomes $|q(m+c) + b_1| \leqslant q + m$. So

$$b_1 \leqslant q + m - q(m+c) = -q - m(q-1) - q(c-2) < -q - m,$$

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which implies $b_1 \notin \triangle D$ and thus $(m+c)\mathbf{v} \notin T-T$. When $q \geqslant p+1$, we can deduce from (3.5) that $|p(m+c)+b_1| \leqslant \dfrac{q+m}{2}$. So

$$\begin{aligned} b_1 &\leqslant \frac{q+m}{2} - p(m+c) \\ &\leqslant \frac{q+m}{2} - q(m+c-1) - c \quad \text{(by (3.4))} \\ &= -q - q\left(m+c-\frac{5}{2}\right) + \frac{m}{2} - c \\ &\leqslant -q - 4\left(m+c-\frac{5}{2}\right) + \frac{m}{2} - c \\ &= -q - \frac{7m}{2} - 5(c-2) \\ &< -q-m, \end{aligned}$$

implying $b_1 \notin \triangle D$. Thus $(m+c)\mathbf{v} \notin T-T$ and T is disconnected by Lemma 2.1.

Case B: $\Delta < 0$

Since the conditions $\Delta < 0$ and $q = p$ imply $q < 4$, we only need to consider the case that $q \geqslant p+1$. When $q = p+1$, there are two possibilities: $(p,q) = (3,4)$ or $(4,5)$.

In the case that $(p,q) = (3,4)$, we have $\beta < 0.56$ (see Appendix A). Then $|\delta| < 0.56(4+m)$. From (3.1), we get $(m+c)A\mathbf{v} - b_1\mathbf{v} \in T-T$. But the assumption of (3.1) is invalid because for any $m \geqslant 1$ and any $c \geqslant 2$ we have $0.56(4+m) < m+c$, contradicting (2.3). Hence $(m+c)\mathbf{v} \notin T-T$.

In the case that $(p,q) = (4,5)$, we find $\beta < 0.6$ (see Appendix A). We can deduce from (3.2) that

$$-(4(m+c)+b_1)A\mathbf{v} - (5(m+c)+b_2)\mathbf{v} \in T-T.$$

Then $|4(m+c)+b_1| < 0.6(5+m)$ by (2.3). So $b_1 < -5-m$ as $c \geqslant 2$, implying $b_1 \notin \triangle D$. Hence $(m+c)\mathbf{v} \notin T-T$.

When $q = p+2$, the possible (p,q)'s are: $(2,4), (3,5), (4,6)$ and $(5,7)$. In the case that $(p,q) = (2,4)$, then $\beta < 0.5$ (see Appendix A) and so $|\delta| < 0.5(4+m)$. But $0.5(4+m) < m+c$ holds for any $m \geqslant 1$ and $c \geqslant 2$, contradicting (2.3). Hence $(m+c)\mathbf{v} \notin T-T$. Similarly we do the other three cases by choosing the bounds of β to be $0.4, 0.4, 0.34$, respectively (see Appendix A). We also get $(m+c)\mathbf{v} \notin T-T$.

When $q \geqslant p+3$, if $\gamma\mathbf{v} + \delta A\mathbf{v} \in T-T$ we claim that

$$|\delta| \leqslant \frac{q+m}{q-p-1} \tag{3.6}$$

and

$$|\gamma| \leqslant \begin{cases} \dfrac{(q+m)(p-1)}{q-p-1} & \text{if } p^2 \geqslant q \\ \dfrac{(q+m)(qp-2p^2+q)}{q(q-p-1)} & \text{if } p^2 < q. \end{cases} \tag{3.7}$$

In order to derive (3.6) and (3.7), it suffices to show $\beta \leqslant \dfrac{1}{q-p-1}$ and

$$\alpha \leqslant \frac{p-1}{q-p-1} \quad \text{if } p^2 \geqslant q; \quad \alpha \leqslant \frac{qp-2p^2+q}{q(q-p-1)} \quad \text{if } p^2 < q.$$

To see it, by Lemma 2.2, we have $q\beta_{i+2} + p\beta_{i+1} + \beta_i = 0$, and hence $q|\beta_{i+2}| \le p|\beta_{i+1}| + |\beta_i|$. Summing the inequality for $i = 1, 2, \ldots$, we get $q(\beta - |\beta_1| - |\beta_2|) \le p(\beta - |\beta_1|) + \beta$. Note that $|\beta_1| = 1/q, |\beta_2| = p/q^2$. It follows that

$$(q - p - 1)\beta \le (q - p)|\beta_1| + q|\beta_2| = 1.$$

Using a similar method, we obtain $(q - p - 1)\alpha \leqslant (q - p)|\alpha_1| + q|\alpha_2|$. By substituting $\alpha_1 = -p/q$ and $\alpha_2 = (p^2 - q)/q^2$ into the inequality, we will get the desired upper bound for α.

(a) $p^2 \geqslant q$.

If $2p < q$, then $\dfrac{p-1}{q-p-1} < 1$. It follows from (3.2) and (3.7) that $q(m + c) + b_2 < q + m$. So $b_2 < -m(q - 1) - q(c - 1) < -m - q$, implying $b_2 \notin \Delta D$. Hence $(m + c)\mathbf{v} \notin T - T$.

Assume $q \leqslant 2p$. Notice first that $0 < \dfrac{p^2}{4} < q \leqslant 2p$ and $q \geqslant p + 3$, hence the possible (p, q)'s are

$$(3, 6), (4, 7), (4, 8), (5, 8), (5, 9), (5, 10), (6, 10), (6, 11), (6, 12), (7, 13), (7, 14).$$

By examining the bounds of β's carefully in Appendix A, we have $\beta < 0.3$ for the first five (p, q)'s; $\beta < 0.21$ for $(6, 10)$; and $\beta < 0.2$ for the remaining ones. Thus for each case $|\delta| < m + c$ always holds. Hence $(m + c)\mathbf{v} \notin T - T$.

(b) $p^2 < q$.

(i) $p = 1$. From (3.6), we have

$$|\delta| \leqslant \frac{q + m}{q - 2} \leqslant 1 + \frac{m + 2}{2} < m + c.$$

So $(m + c)\mathbf{v} \notin T - T$.

(ii) $p = 2$. From (3.7), we have

$$|\gamma| \leqslant \frac{(q + m)(3q - 8)}{q(q - 3)} = 3 + \frac{3m + 1}{q - 3} - \frac{8m}{q(q - 3)} < 3 + 3m.$$

Hence $|q(m + c) + b_2| < 3 + 3m$ and thereby $b_2 < 3 - qc - m(q - 3) < -q - m$, implying $b_2 \notin \Delta D$. So $(m + c)\mathbf{v} \notin T - T$.

(iii) $p \geqslant 3$. From (3.7), we have

$$\begin{aligned}
|\gamma| \leqslant \frac{(q + m)\left[q(p + 1) - 2p^2\right]}{q(q - p - 1)} &= \frac{(p + 1)q^2 + \left[(p + 1)m - 2p^2\right]q - 2p^2 m}{q(q - p - 1)} \\
&= \frac{(p + 1)q^2 - (p + 1)^2 q + \left[(p + 1)m - 2p^2 + (p + 1)^2\right]q - 2p^2 m}{q(q - p - 1)} \\
&= p + 1 + \frac{(p + 1)m - p^2 + 2p + 1}{q - p - 1} - \frac{2p^2 m}{q(q - p - 1)} \\
&< p + 1 + \frac{(p + 1)m + 2(p + 1)}{q - p - 1} \\
&< p + 1 + m + 2 \quad (\text{as } q > p^2 > 2p + 2 \text{ for } p \geqslant 3, \text{ i.e. } p + 1 < q - (p + 1)) \\
&\leqslant q + m.
\end{aligned}$$

Hence $|q(m + c) + b_2| \leqslant q + m$ and thereby $b_2 \leqslant -m(q - 1) - q(c - 1) < -m - q$, implying $b_2 \notin \Delta D$. So $(m + c)\mathbf{v} \notin T - T$. We proved that T is disconnected by Lemma 2.1.

• **Part II**

In this part we assume that $f(x) = x^2 + px - q$ where $q \geqslant 4$ and $p \geqslant 1$. Since A is expanding, $p \leqslant q-2$ by (1.2). We also see that $\Delta = p^2 + 4q \geqslant 0$ always holds. Let $D = \{0, 1, \dots, q-2, q+m\}$ where $m \geqslant 1$. Note that

$$\alpha = \frac{p+1}{q-p-1}, \quad \beta = \frac{1}{q-p-1}.$$

If $\gamma\mathbf{v} + \delta A\mathbf{v} \in T - T$, then by (2.3), we have

$$|\gamma| \leqslant \frac{(q+m)(p+1)}{q-p-1}, \quad |\delta| \leqslant \frac{q+m}{q-p-1}.$$

Assume $(m+c)\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T-T$ where $c \geqslant 2$. From Lemma 2.4, it follows that $(m+c)A\mathbf{v} - b_1\mathbf{v} \in T-T$. If $p(m+c)+(2m+c) < q(m+c-1)$ then $|\delta| \leq \dfrac{q+m}{q-p-1} < m+c$, impossible. Thus $(m+c)\mathbf{v} \notin T-T$. Now we consider the case

$$q(m+c-1) \leqslant p(m+2) + (2m+c). \tag{3.8}$$

Analogously to Part I, by Lemma 2.4, we can deduce from $(m+c)\mathbf{v} \in T-T$ that

$$-[(m+c)p + b_1]A\mathbf{v} + [(m+c)q - b_2]\mathbf{v} \in T-T.$$

Thus

$$|(m+c)p + b_1| \leqslant \frac{q+m}{q-p-1} \quad \text{and} \quad |(m+c)q - b_2| \leqslant \frac{(q+m)(p+1)}{q-p-1}.$$

There are two subcases to be considered here: (a) $q = p+2$ and (b) $q \geqslant p+3$.

(a) In this case $|\gamma| \leqslant (q+m)(p+1) = (q+m)(q-1)$ and $|\delta| \leqslant q+m$. By Lemma 2.4,

$$[(m+c)(p^2+q) + pb_1 - b_2]A\mathbf{v} - [qp(m+c) + qb_1 + b_3]\mathbf{v} \in T-T. \tag{3.9}$$

As $|(m+c)(p^2+q) + pb_1 - b_2| \leqslant q+m$, so

$$\begin{aligned} pb_1 &\leqslant q+m-(m+c)(p^2+q)+b_2 \\ &\leqslant q+m-(m+c)p^2-(m+c)q+q+m \\ &\leqslant -mq-(m+2)p^2+2m. \end{aligned}$$

Since $q \geqslant 4$, we have

$$b_1 \leqslant \frac{-mq}{p} - (m+2)p + \frac{2m}{p} = \frac{-mq}{q-2} - (m+2)(q-2) + \frac{2m}{q-2} < -q-m,$$

implying $b_1 \notin \Delta D$. Hence $(m+c)\mathbf{v} \notin T-T$ for all integers $c \geqslant 2$.

(b) In this case

$$|\gamma| \leqslant \frac{(q+m)(p+1)}{q-p-1} \leqslant \frac{(q+m)(p+1)}{2} \quad \text{and} \quad |\delta| \leqslant \frac{q+m}{q-p-1} \leqslant \frac{q+m}{2}.$$

From (3.9), we have $|(m+c)(p^2+q) + pb_1 - b_2| \leqslant \dfrac{q+m}{2}$. So

$$\begin{aligned}
pb_1 &\leqslant \frac{q+m}{2} - (m+c)(p^2+q) + b_2 \\
&\leqslant \frac{q+m}{2} - (m+2)(p^2+q) + b_2 \\
&\leqslant -\left(m+\frac{1}{2}\right)q - (m+2)p^2 + \frac{3m}{2} \quad (\text{as } b_2 \leqslant q+m).
\end{aligned}$$

Since $m \geqslant 1$ and $q \geqslant 4$, we have

$$\begin{aligned}
b_1 &\leqslant \frac{-\left(m+\frac{1}{2}\right)(p+3)}{p} - (m+2)p + \frac{3m}{2p} \\
&\leqslant -\left(m+\frac{1}{2}\right) - \frac{3\left(m+\frac{1}{2}\right)}{p} - (m+1)q + 2(m+1) + \frac{3m}{2p} \qquad \text{by (3.8)} \\
&< m + \frac{3}{2} - \frac{3m+1}{2p} - (m+1)q \\
&< m + \frac{3}{2} - (m+1)q \\
&< -q - m,
\end{aligned}$$

implying $b_1 \notin \Delta D$. Therefore, $(m+c)\mathbf{v} \notin T - T$ for all integers $c \geqslant 2$.

• **Part III**

In this part we assume that $m = 0$ and $f(x) = x^2 + px + q$ with $\Delta = p^2 - 4q \geqslant 0$, where $1 \leqslant p \leqslant q$ and $q \geqslant 4$. Let $D = \{0, 1, \ldots, q-2, q\}$, then $\triangle D = \{0, \pm 1, \ldots, \pm(q-1), \pm q\}$. Here we consider the following cases one by one:

(a) $q = 2p - 2$ and $p \geqslant 3$
(b) $q > 2p - 2$
(c) $q = p \geqslant 4$
(d) $2p - 2 > q > p \geqslant 3$

(a) Notice first that $0 = f(A) = A^2 + pA + qI = (A+I)[A+(p-1)I] + (p-1)I$. So $A + (p-1)I = -(p-1)(A+I)^{-1}$ and then

$$A\mathbf{v} + (p-1)\mathbf{v} = -(p-1)(A^{-1} - A^{-2} + A^{-3} - \cdots)\mathbf{v}.$$

Hence

$$\mathbf{v} = -(p-1)A^{-1}\mathbf{v} + \sum_{i=2}^{\infty}(-1)^{i-1}(p-1)A^{-i}\mathbf{v}, \tag{3.10}$$

which implies that $\mathbf{v} \in T - T$. Moreover, by $q = 2(p-1)$, we get

$$2\mathbf{v} = -qA^{-1}\mathbf{v} + \sum_{i=2}^{\infty}(-1)^{i-1}qA^{-i}\mathbf{v} \in T - T.$$

Hence T is connected by Lemma 2.1. Since the assumption $\Delta \geqslant 0$ is not used here, the proof also applies to those $f(x)$ with $\Delta < 0$. This implies that T is also connected if $(p,q) = (5,8)$ or $(6,10)$.

(b) By Lemma 2.3, we have that $\alpha = \dfrac{p-1}{q-p+1}$ and $\beta = \dfrac{1}{q-p+1}$. If $\gamma\mathbf{v} + \delta A\mathbf{v} \in T - T$, then by (2.3), we have $|\gamma| \leqslant \dfrac{q(p-1)}{q-p+1}$ and $|\delta| \leqslant \dfrac{q}{q-p+1} < 2$.

Suppose $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$ for $k \geqslant 2$. Then by Lemma 2.4, $kA\mathbf{v} - b_1\mathbf{v} = \sum_{i=2}^{\infty} b_i A^{1-i}\mathbf{v} \in T - T$. This is impossible as $|\delta| < 2$. Hence $k\mathbf{v} \notin T - T$, and T is disconnected by Lemma 2.1.

(c) In this case, if $\gamma\mathbf{v} + \delta A\mathbf{v} \in T - T$, then $|\gamma| \leqslant q(q-1)$ and $|\delta| \leqslant q$. Suppose $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$ where $k \geqslant 2$. By Lemma 2.4,

$$-(kq + b_1)A\mathbf{v} - (kq + b_2)\mathbf{v} \in T - T.$$

$|kq + b_1| \leqslant q$ is only possible when $k = 2$. In this case $b_1 = -q$, so $-qA\mathbf{v} - (2q + b_2)\mathbf{v} \in T - T$. By using Lemma 2.4 again, we have

$$(q^2 - 2q - b_2)A\mathbf{v} + (q^2 - b_3)\mathbf{v} \in T - T.$$

Since $|q^2 - 2q - b_2| \leqslant q$ and $|q^2 - b_3| \leqslant q(q-1)$, we have $q^2 - q \geqslant b_2 \geqslant q^2 - 3q = q(q-3) \geqslant q$, where the last equality holds when $q = 4$. Thus if $q = p \geqslant 5$, then $b_2 > q$, impossible. So $2\mathbf{v} \notin T - T$.

If $q = p = 4$, then by $0 = f(A)(A - I) = A^3 + 3A^2 - 4I$ we have $A^2 + 2A - 2I = 2(A + I)^{-1}$. It follows that

$$\mathbf{v} = -2A^{-1}\mathbf{v} + 2A^{-2}\mathbf{v} + 2A^{-3}\mathbf{v} - 2A^{-4}\mathbf{v} + 2A^{-5}\mathbf{v} - 2A^{-6}\mathbf{v} + \cdots \in T - T$$

and

$$2\mathbf{v} = -4A^{-1}\mathbf{v} + 4A^{-2}\mathbf{v} + 4A^{-3}\mathbf{v} - 4A^{-4}\mathbf{v} + 4A^{-5}\mathbf{v} - 4A^{-6}\mathbf{v} + \cdots \in T - T.$$

Hence T is connected by Lemma 2.1.

(d) Set $r := q - p \geqslant 1$. In this case, if $\gamma\mathbf{v} + \delta A\mathbf{v} \in T - T$, then $|\gamma| \leqslant \dfrac{q(q-r-1)}{r+1}$ and $|\delta| \leqslant \dfrac{q}{r+1}$. Suppose $2\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$. By Lemma 2.4,

$$-(2p + b_1)A\mathbf{v} - (2q + b_2)\mathbf{v} \in T - T.$$

Since $|2p + b_1| \leqslant \dfrac{q}{r+1}$, we have $-\dfrac{q}{r+1} \leqslant 2(q-r) + b_1 \leqslant \dfrac{q}{r+1}$. Thus

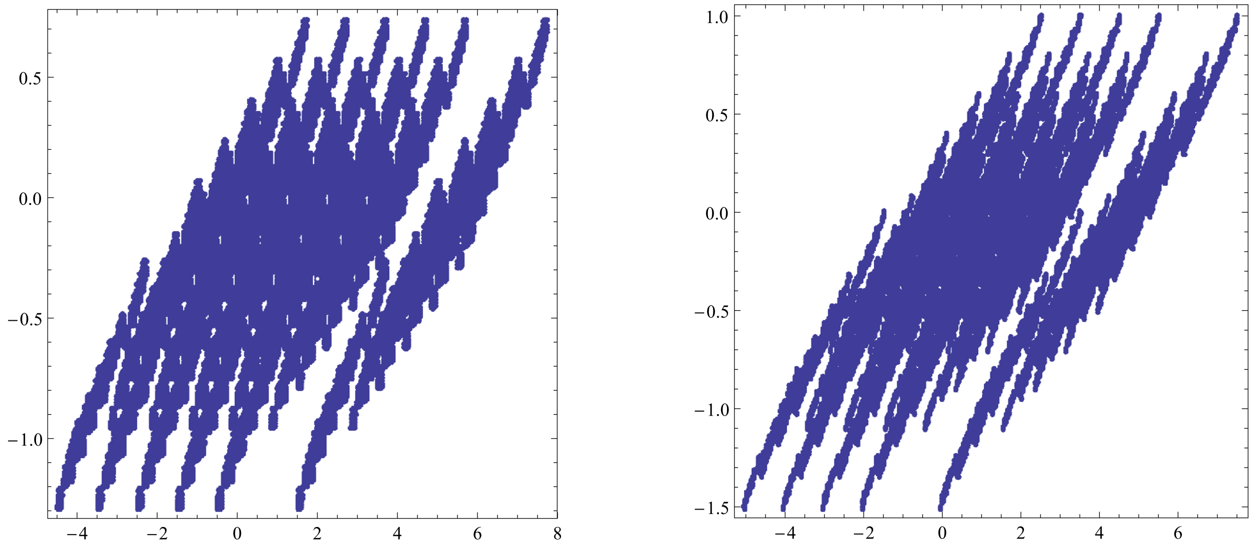
$$b_1 \leq \frac{q}{r+1} - 2(q-r) = \frac{-(2r+1)q}{r+1} + 2r < -q,$$

as $2(r+1) < q$ (i.e., $q < 2p - 2$). That is impossible, hence $2\mathbf{v} \notin T - T$.

Now suppose $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$ for $k \geqslant 3$. As $|kp + b_1| \leqslant \dfrac{q}{r+1}$, we have

$$\begin{aligned} b_1 &\leqslant \frac{q}{r+1} - kp = \frac{q}{r+1} - k(q-r) \\ &< \frac{q}{2} - kq + k\left(\frac{q}{2} - 1\right) \quad (\text{as } q > 2(r+1)) \\ &\leqslant -q - k < -q, \end{aligned}$$

which is impossible. So $k\mathbf{v} \notin T - T$. Hence T is disconnected by Lemma 2.1.



(a) $(p, q) = (4, 6)$ (b) $(p, q) = (4, 5)$

Fig. 2. (a) is connected while (b) is not connected.

• Part IV

In this part we assume that $m = 0$ and $f(x) = x^2 + px + q$ with $\Delta = p^2 - 4q < 0$, where $q \geqslant p \geqslant 1$ and $q \geqslant 4$. We need not consider the case $p = q$ as it implies $q = 2, 3$. Let $D = \{0, 1, \dots, q - 2, q\}$, then $\triangle D = \{0, \pm 1, \dots, \pm(q - 1), \pm q\}$. The proof of this part is divided into the following three cases:

$$(a)\ q = p + 1; \quad (b)\ q = p + 2; \quad \text{and} \quad (c)\ q > p + 2.$$

(a) Notice that $0 > p^2 - 4(p + 1) = (p - 2)^2 - 8$. Now $p = 1, 2, 3, 4$. The corresponding q's are $2, 3, 4, 5$. As the cases $(p, q) = (1, 2)$ and $(2, 3)$ have been solved, we need only study $(p, q) = (3, 4)$ and $(4, 5)$.

When $(p, q) = (3, 4)$, we can deduce from $0 = f(A) = A^2 + 3A + 4I$ that $A + 2I = -2(A + I)^{-1}$. It in turn implies that

$$\mathbf{v} = -2A^{-1}\mathbf{v} - 2A^{-2}\mathbf{v} + 2A^{-3}\mathbf{v} - 2A^{-4}\mathbf{v} + 2A^{-5}\mathbf{v} - 2A^{-6}\mathbf{v} + \cdots \in T - T$$

and

$$2\mathbf{v} = -4A^{-1}\mathbf{v} - 4A^{-2}\mathbf{v} + 4A^{-3}\mathbf{v} - 4A^{-4}\mathbf{v} + 4A^{-5}\mathbf{v} - 4A^{-6}\mathbf{v} + \cdots \in T - T.$$

Hence T is connected.

When $(p, q) = (4, 5)$, we have $\beta < 0.6$ (see Appendix A) and $|\delta| < 3$. Now suppose $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$ where $k \geqslant 2$. By Lemma 2.4, $-(4k + b_1)A\mathbf{v} - (5k + b_2)\mathbf{v} \in T - T$, in which $|\delta| = |4k + b_1| < 3$. But it is impossible for $k \geqslant 2$ and $|b_1| \leqslant 5$. Hence $k\mathbf{v} \notin T - T$, and T is not connected. (See Fig. 2(b).)

(b) $\Delta = p^2 - 4q < 0$ gives $0 > p^2 - 4(p + 2) = (p - 2)^2 - 12$. Thus $p = 2, 3, 4, 5$ and so $q = 4, 5, 6, 7$, respectively. We now discuss them in detail.

In the case that $(p, q) = (2, 4)$, we get $\beta < 0.43$ (see Appendix A) and $|\delta| < 1.72$. Now suppose $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$. Then $kA\mathbf{v} - b_1\mathbf{v} \in T - T$, while $|\delta| < 2$, a contradiction. Thus $k\mathbf{v} \notin T - T$. Similarly for the case $(p, q) = (3, 5)$, we get $\beta < 0.38$ (see Appendix A) and $|\delta| < 1.9 < 2$. Hence $k\mathbf{v} \notin T - T$.

For the case $(p, q) = (4, 6)$, it follows from $0 = f(A) = A^2 + 4A + 6I$ that $A + 3I = -3(A + I)^{-1}$. Then we have

$$\mathbf{v} = -3A^{-1}\mathbf{v} - 3A^{-2}\mathbf{v} + 3A^{-3}\mathbf{v} - 3A^{-4}\mathbf{v} + 3A^{-5}\mathbf{v} - \cdots \in T - T$$

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and

$$2\mathbf{v} = -6A^{-1}\mathbf{v} - 6A^{-2}\mathbf{v} + 6A^{-3}\mathbf{v} - 6A^{-4}\mathbf{v} + 6A^{-5}\mathbf{v} - \cdots \in T - T.$$

So T is connected by Lemma 2.1. (See Fig. 2(a).)

For the case $(p,q) = (5,7)$, we have $\beta < 0.4$ (see Appendix A) and $|\delta| < 2.8$. Now suppose $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$. By Lemma 2.4, $-(5k + b_1)A\mathbf{v} - (7k + b_2)\mathbf{v} \in T - T$. Since $|5k + b_1| < 2.8$, we have $b_1 < -7.2$, so $b_1 \notin \triangle D$. Thus $k\mathbf{v} \notin T - T$.

(c) Notice that if $\gamma\mathbf{v} + \delta A\mathbf{v} \in T - T$, then by (2.3), we have

$$|\delta| \leqslant \frac{q}{q-p-1} \quad \text{and} \quad |\gamma| \leqslant \begin{cases} \dfrac{q(p-1)}{q-p-1} & \text{if } p^2 - q \geqslant 0 \\ \dfrac{qp - 2p^2 + q}{q-p-1} & \text{if } p^2 - q < 0. \end{cases} \tag{3.11}$$

Now suppose $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$. As before, by Lemma 2.4,

$$-(kp + b_1)A\mathbf{v} - (kq + b_2)\mathbf{v} \in T - T. \tag{3.12}$$

Since $q > p + 2$, from (3.11), we obtain $|kp + b_1| \leqslant \frac{q}{q-p-1} \leqslant \frac{q}{2}$. Hence $k\mathbf{v} \notin T - T$ provided that $q < \dfrac{4p}{3}$. For the case $q \geqslant \dfrac{4p}{3}$. Here we consider two subcases: (i) $p^2 \geqslant q$ and (ii) $p^2 < q$.

(i) From (3.11), it follows that $|kq + b_2| \leqslant \dfrac{q(p-1)}{q-p-1}$. If $q > 2p$ then $b_2 \leqslant \dfrac{q(p-1)}{q-p-1} - 2q < -q$. Thus $k\mathbf{v} \notin T - T$ and T is disconnected. If $2p \geqslant q \geqslant \dfrac{4p}{3}$, together with $\Delta < 0$, then $p \leqslant 7$. We can find all possible (p,q)'s as follows:

$$(2,4), (3,4), (3,6), (4,6), (4,7), (4,8), (5,7), (5,8),$$
$$(5,9), (5,10), (6,10), (6,11), (6,12), (7,13), (7,14).$$

Except the solved cases, we only need to study the following ones:

$$(3,6), (4,7), (4,8), (5,9), (5,10), (6,11), (6,12), (7,13), (7,14).$$

By choosing the bounds of β's carefully in Appendix A, we take $\beta < 0.3$ for $(3,6)$; $\beta < 0.26$ for $(4,7)$; $\beta < 0.21$ for $(4,8), (5,9)$; $\beta < 0.17$ for $(5,10), (6,11)$; $\beta < 0.15$ for $(6,12), (7,13)$; $\beta < 0.13$ for $(7,14)$. While in each case $|\delta| < 2$ always holds. Thus $k\mathbf{v} \notin T - T$.

(ii) From (3.11), we have $|kq + b_2| \leqslant \dfrac{qp - 2p^2 + q}{q-p-1}$. So $b_2 \leqslant \dfrac{qp - 2p^2 + q}{q-p-1} - kq \leqslant \dfrac{qp - 2p^2 + q}{q-p-1} - 2q$. It can be shown that

$$\frac{qp - 2p^2 + q}{q-p-1} - 2q < -q \quad \Leftrightarrow \quad 2 + 2p\left(1 - \frac{p}{q}\right) < q.$$

Now for any $p \geqslant 3$ we have $p^2 - \left[2 + 2p\left(1 - \dfrac{p}{q}\right)\right] = (p-1)^2 - 3 + \dfrac{2p^2}{q} > 0$. So $2 + 2p\left(1 - \dfrac{p}{q}\right) < p^2 < q$. Thus $k\mathbf{v} \notin T - T$ and T is disconnected. We then study what will happen when $p = 1$ or 2.

Consider the case $p = 1$. Then $p^2 - q = 1 - q < 0$ and $q \geqslant \dfrac{4p}{3} = \dfrac{4}{3}$. Then

$$|\gamma| \leqslant \frac{2q-2}{q-2} = 2 + \frac{2}{q-2} \leqslant 3$$

for any $q \geqslant 4$ whereas the equality holds when $q = 4$. Suppose $k\mathbf{v} \in T - T$. By (3.12), it follows from $|kq + b_2| \leqslant 3$ that $b_2 \leqslant 3 - kq \leqslant 3 - 2q < -q$, then $b_2 \notin \triangle D$. Hence $k\mathbf{v} \notin T - T$ and T is disconnected.

Consider the case $p = 2$. Then $p^2 - q = 4 - q < 0$. So $q \geqslant 5$. Then $|\gamma| \leqslant \dfrac{3q-8}{q-3} < 4$. Suppose $k\mathbf{v} \in T - T$. By (3.12),

$$-(2k + b_1)A\mathbf{v} - (kq + b_2)\mathbf{v} \in T - T.$$

From $|kq + b_2| < 4$, it follows that $b_2 < 4 - kq \leqslant 4 - 2q < -q$. Then $b_2 \notin \triangle D$. Hence $k\mathbf{v} \notin T - T$ and T is disconnected.

• Part V

In this part we assume that $m = 0$ and $f(x) = x^2 + px - q$ where $p \geqslant 1$ and $q \geqslant 4$. By (1.2), we only consider the case $q \geqslant p + 2$. Let $D = \{0, 1, \ldots, q - 2, q\}$. By Lemma 2.3, we have $\alpha = \dfrac{p+1}{q-p-1}$ and $\beta = \dfrac{1}{q-p-1}$. If $\gamma\mathbf{v} + \delta A\mathbf{v} \in T - T$, then by (2.3), we have

$$|\gamma| \leqslant \frac{q(p+1)}{q-p-1} \quad \text{and} \quad |\delta| \leqslant \frac{q}{q-p-1}.$$

We divide the proof into the following cases:

$$(a)\ q = 2p + 2; \quad (b)\ q > 2p + 2; \quad \text{and} \quad (c)\ 2p + 2 > q \geqslant p + 2.$$

(a) Notice that $0 = f(A) = A^2 + pA - (2p + 2)I$. So

$$\begin{aligned} &(p+1)I = [A + (p+1)I](A - I) \\ \Rightarrow\quad & A + (p+1)I = (p+1)(A-I)^{-1} = (p+1)\sum_{i=1}^{\infty} A^{-i} \\ \Rightarrow\quad & \mathbf{v} = -(p+1)A^{-1}\mathbf{v} + \sum_{i=1}^{\infty}(p+1)A^{-i-1}\mathbf{v} \in T - T. \end{aligned}$$

Moreover, $2\mathbf{v} = -qA^{-1}\mathbf{v} + \sum_{i=2}^{\infty} qA^{-i}\mathbf{v} \in T - T$. Thus T is connected.

(b) Notice that $q - p - 1 > p + 1$. We get

$$|\gamma| < q. \tag{3.13}$$

Now suppose $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$ where $k \geqslant 2$. By Lemma 2.4,

$$-(kp + b_1)A\mathbf{v} + (kq - b_2)\mathbf{v} \in T - T.$$

By (3.13), we know $|kq - b_2| < q$. Then $|b_2| > (k-1)q \geqslant q$, contradicting that $b_2 \in \triangle D$. Thus $k\mathbf{v} \notin T - T$.

(c) In this case $|\gamma| \leqslant q(p+1)$ and $|\delta| \leqslant q$. Suppose $k\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T - T$ for any $k \geqslant 2$. Lemma 2.4 implies that

$$[p(kp + b_1) + (kq - b_2)]A\mathbf{v} - [q(kp + b_1) + b_3]\mathbf{v} \in T - T.$$

Since $|p(kp+b_1)+(kq-b_2)| \leqslant q$, we have $p(2p+b_1)+q \leqslant p(kp+b_1)+q \leqslant b_2$. When $q < 2p$, as $-2p < -q \leqslant b_1$, then $q < b_2$, contradicting $b_2 \in \triangle D$. Thus $k\mathbf{v} \notin T - T$. Assume $2p \leqslant q < 2p + 2$, which means (i) $q = 2p$ or (ii) $q = 2p + 1$.

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(i) As $|p(kp+b_1)+(kq-b_2)| \leqslant q$, by $q=2p$, we have $\left|\frac{q}{2}(\frac{kq}{2}+b_1)+(kq-b_2)\right| \leqslant q$. So for $k \geqslant 3$, it follows that

$$\frac{q}{2}(\frac{3q}{2}+b_1)+q < \frac{q}{2}(\frac{kq}{2}+b_1)+(k-1)q \leqslant b_2.$$

Since $b_1 \geqslant -q$, we have $q < b_2$, and $b_2 \notin \triangle D$. Hence $k\mathbf{v} \notin T-T$ for any $k \geqslant 3$.

For $k=2$, it holds that $b_2 \geqslant \frac{q}{2}(q+b_1)+q$, implying $b_1=-q, b_2=q$. Using Lemma 2.4 for $2\mathbf{v} = \sum_{i=1}^{\infty} b_i A^{-i}\mathbf{v} \in T-T$, we have $-(2p+b_1)A\mathbf{v}+(2q-b_2)\mathbf{v} \in T-T$, that means, $q\mathbf{v} \in T-T$ where $q \geqslant 4$. That contradicts the previous argument. Hence $2\mathbf{v} \notin T-T$.

(ii) Notice that $|p(kp+b_1)+(kq-b_2)| \leqslant \dfrac{q}{q-p-1} = 2+\dfrac{2}{q-1}$. Seeing that $\left|\dfrac{2}{q-1}\right| < 1$, we have $|p(kp+b_1)+(kq-b_2)| \leqslant 2$. Now

$$\begin{aligned}
& p(2p+b_1)+2q-b_2 \leqslant p(kp+b_1)+kq-b_2 \leqslant 2 \\
\Rightarrow\ & \frac{q-1}{2}(q-1+b_1)+2q-b_2 \leqslant 2 \\
\Rightarrow\ & \frac{(q-1)^2}{2}+\frac{q-1}{2}b_1+2q-2 \leqslant b_2 \\
\Rightarrow\ & \frac{(q-1)^2}{2}-\frac{q(q-1)}{2}+2q-2 \leqslant b_2 \quad (as\ -q \leq b_1) \\
\Rightarrow\ & \frac{3q}{2}-\frac{3}{2} \leqslant b_2 \\
\Rightarrow\ & q < b_2 \quad (\text{as } q > 3),
\end{aligned}$$

and $b_2 \notin \triangle D$. Therefore, $k\mathbf{v} \notin T-T$ and T is not connected.

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Appendix A. Bounds of α and β

MATLAB Program:

function [alpha, beta] = bound(p,q,n)
The polynomial is $f(x) = x^2+px+q$.
n: the iterations.
if $n > 100$ or $p^2 > 4q$
alpha= 0; beta= 0;
return;
end
$x_1 = -p/q; \quad y_1 = -1/q;$
$x_2 = (p^2-q)/q^2; \quad y_2 = p/q^2;$
$x = |x_2|+|x_1|; \quad y = |y_2|+|y_1|;$
for $i = 1 : n-3$
$x_{i+2} = (-p/q)x_{i+1}+(-1/q)x_i; \quad x = x+|x_{i+2}|;$
$y_{i+2} = (-p/q)y_{i+1}+(-1/q)y_i; \quad y = y+|y_{i+2}|;$
end

alpha $= x + \frac{2q^{-(n-1)/2}}{(1-q^{-1/2})(4q-p^2)^{1/2}}$; beta $= y + \frac{2q^{-n/2}}{(1-q^{-1/2})(4q-p^2)^{1/2}}$;
end

MATLAB Output: bound(p,q,n)=[alpha, beta]

bound(3,4,99) = [1.2301, 0.5575]; bound(4,5,99) = [1.5235, 0.5047];
bound(2,4,99) = [0.7143, 0.4286]; bound(3,5,99) = [0.8631, 0.3726];
bound(4,6,99) = [1.0437, 0.3406]; bound(5,7,99) = [1.3339, 0.3334];
bound(3,6,99) = [0.6740, 0.2790]; bound(4,7,99) = [0.8076, 0.2582];
bound(4,8,99) = [0.6508, 0.2063]; bound(5,8,99) = [1.0057, 0.2507];
bound(5,9,99) = [0.8117, 0.2013]; bound(5,10,99) = [0.6834, 0.1683];
bound(6,10,99) = [1.0001, 0.2000]; bound(6,11,99) = [0.8341, 0.1667];
bound(6,12,99) = [0.7163, 0.1430]; bound(7,13,99) = [0.8571, 0.1429];
bound(7,14,99) = [0.7500, 0.1250]; bound(4,5,99) = [1.5235, 0.5047];
bound(2,4,99) = [0.7143, 0.4286]; bound(3,5,99) = [0.8631, 0.3726];
bound(5,7,99) = [1.3339, 0.3334]; bound(3,6,99) = [0.6740, 0.2790];
bound(4,7,99) = [0.8076, 0.2582]; bound(4,8,99) = [0.6508, 0.2063];
bound(5,9,99) = [0.8117, 0.2013]; bound(5,10,99) = [0.6834, 0.1683];
bound(6,11,99) = [0.8341, 0.1667]; bound(6,12,99) = [0.7163, 0.1430];
bound(7,13,99) = [0.8571, 0.1429]; bound(7,14,99) = [0.7500, 0.1250].

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