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# A $q$ -analogue of a Ramanujan-type supercongruence involving central binomial coefficients

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## ABSTRACT

Motivated by Zudilin's work, we give a  $q$ -analogue of a Ramanujan-type supercongruence of van Hamme and Mortenson via the  $q$ -WZ method. Meanwhile, we give a  $q$ -analogue of a related congruence of Sun in the same way. We also propose several related conjectures on congruences involving central  $q$ -binomial coefficients.

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## 1. Introduction

In 1914, Ramanujan [25] obtained several fast approximations of  $1/\pi$ . One such example, though not in the list of [25], is the identity

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 = \frac{2}{\pi}, \quad (1.1)$$

which was originally proved by Bauer [3] in 1859. A computer proof of (1.1) applying the Wilf–Zeilberger method was given by Ekhad and Zeilberger [6]. It was conjectured by van Hamme [32] that several Ramanujan-like formulas for  $1/\pi$ , including (1.1), have nice  $p$ -adic analogues, such as

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 \equiv p(-1)^{\frac{p-1}{2}} \pmod{p^3}. \quad (1.2)$$

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The congruence (1.2) was first proved by Mortenson [23] using a  ${}_6F_5$  transformation, and later reproved by Zudilin [34] via the Wilf–Zeilberger method.

On the other hand,  $q$ -analogues of some arithmetic congruences have caught interest of different authors (see [2,5,7,11–15,17,19,20,24,26,27,30,31]).

Motivated by Zudilin's work, we shall give a  $q$ -analogue of (1.2). We follow the standard notation from [8]. The  $q$ -shifted factorial is defined by  $(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$  for  $n \geq 1$  and  $(a; q)_0 = 1$ , while the  $q$ -integer is defined as  $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$ . Moreover, the  $q$ -binomial coefficients  $\begin{bmatrix} x \\ k \end{bmatrix}$  are defined by

$$\begin{bmatrix} x \\ k \end{bmatrix} = \begin{bmatrix} x \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^{x-k+1}; q)_k}{(q; q)_k} & \text{if } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$(-1)^k q^{k^2} \begin{bmatrix} -\frac{1}{2} \\ k \end{bmatrix}_{q^2} = \frac{(q; q^2)_k}{(q^2; q^2)_k} = \frac{1}{(-q; q)_k^2} \begin{bmatrix} 2k \\ k \end{bmatrix}.$$

Our  $q$ -version of (1.2) can be stated as follows.

**Theorem 1.1.** *Let  $p$  be an odd prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k q^{k^2} [4k+1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \equiv [p] q^{\frac{(p-1)^2}{4}} (-1)^{\frac{p-1}{2}} \pmod{[p]^3}. \quad (1.3)$$

Let  $\Phi_n(q)$  be the  $n$ -th cyclotomic polynomial in  $q$ , which may be defined as

$$\Phi_n(q) := \prod_{\substack{1 \leq k \leq n \\ \gcd(n, k) = 1}} (q - \zeta^k),$$

where  $\zeta$  is an  $n$ -th primitive root of unity. It is well known that  $\Phi_n(q)$  is an irreducible polynomial with integer coefficients. We shall further give a generalization of (1.3).

**Theorem 1.2.** *Let  $n > 1$  be an odd integer. Then*

$$\sum_{k=0}^{\frac{n-1}{2}} (-1)^k q^{k^2} [4k+1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \equiv [n] q^{\frac{(n-1)^2}{4}} (-1)^{\frac{n-1}{2}} \pmod{[n] \Phi_n(q)^2}. \quad (1.4)$$

Letting  $n = p^r$  be an odd prime power in Theorem 1.2, and noticing that  $\Phi_{p^r}(q) = [p]_{q^{p^{r-1}}}$ , we obtain the following conclusion.

**Corollary 1.3.** *Let  $p$  be an odd prime and  $r$  a positive integer. Then*

$$\sum_{k=0}^{\frac{p^r-1}{2}} (-1)^k q^{k^2} [4k+1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \equiv [p^r] q^{\frac{(p^r-1)^2}{4}} (-1)^{\frac{(p-1)r}{2}} \pmod{[p^r] [p]_{q^{p^{r-1}}}^2}.$$

In particular, the  $q = 1$  case gives

$$\sum_{k=0}^{\frac{p^r-1}{2}} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 \equiv p^r (-1)^{\frac{(p-1)r}{2}} \pmod{p^{r+2}}. \quad (1.5)$$

Note that the  $r = 2$  case of (1.5) confirms the  $r = 2$  case of the third congruence in [29, (B.3)]. It is worth mentioning that He [16, (1.1)] also “gives” this congruence, as a corollary of [16, Theorem 1.1]. Here we must point out that the proof of [16, Theorem 1.1] is not completed, because the reason why the congruence (3.2) in [16] holds is not clear. We think that the proof of [16, (3.2)] might be very difficult.

By establishing some congruences for binomial coefficients and then using the same WZ-pair as Zudilin [34], Z.-W. Sun [28] proves the following interesting generalization of (1.2):

$$\sum_{k=0}^n (4k+1) \binom{2k}{k}^3 (-64)^{n-k} \equiv 0 \pmod{4(2n+1) \binom{2n}{n}}. \quad (1.6)$$

In this paper, we shall give a  $q$ -analogue of (1.6) as follows.

**Theorem 1.4.** *Let  $n$  be a positive integer. Then*

$$\sum_{k=0}^n (-1)^k q^{k^2} [4k+1] \left[ \begin{matrix} 2k \\ k \end{matrix} \right]^3 (-q^{k+1}; q)_{n-k}^6 \equiv 0 \pmod{(1+q^n)^2 [2n+1] \left[ \begin{matrix} 2n \\ n \end{matrix} \right]}. \quad (1.7)$$

The paper is organized as follows. In Section 2, we give a proof of Theorem 1.1 by the  $q$ -WZ method. In Section 3, we prove Theorem 1.2 similarly. The proof of Theorem 1.4 will be given in Section 4, along with two divisibility results on  $q$ -binomial coefficients. We put forward several open problems in Section 5.

## 2. Proof of Theorem 1.1

Introduce the rational functions in  $q^n$  and  $q^k$ :

$$F(n, k) = (-1)^{n+k} q^{(n-k)^2} [4n+1] \frac{(q; q^2)_n^2 (q; q^2)_{n+k}}{(q^2; q^2)_n^2 (q^2; q^2)_{n-k} (q; q^2)_k^2},$$

$$G(n, k) = (-1)^{n+k} q^{(n-k)^2} \frac{(q; q^2)_n^2 (q; q^2)_{n+k-1}}{(1-q)(q^2; q^2)_{n-1}^2 (q^2; q^2)_{n-k} (q; q^2)_k^2},$$

where we use the convention that  $1/(q^2; q^2)_m = 0$  for any negative integer  $m$ . The functions  $F(n, k)$  and  $G(n, k)$  form a  $q$ -WZ pair. Namely, they satisfy the relation

$$F(n, k-1) - F(n, k) = G(n+1, k) - G(n, k). \quad (2.1)$$

Indeed, we have the following fractions:

$$\frac{F(n, k-1)}{G(n, k)} = -\frac{q^{2n-2k+1}(1-q^{4n+1})(1-q^{2k-1})^2}{(1-q^{2n-2k+2})(1-q^{2n})^2},$$

$$\frac{F(n, k)}{G(n, k)} = \frac{(1-q^{4n+1})(1-q^{2n+2k-1})}{(1-q^{2n})^2},$$

$$\frac{G(n+1, k)}{G(n, k)} = -\frac{q^{2n-2k+1}(1-q^{2n+1})^2(1-q^{2n+2k-1})}{(1-q^{2n})^2(1-q^{2n-2k+2})}.$$

Then it is straightforward to check the identity

$$\begin{aligned}
& - \frac{q^{2n-2k+1}(1-q^{4n+1})(1-q^{2k-1})^2}{(1-q^{2n-2k+2})(1-q^{2n})^2} - \frac{(1-q^{4n+1})(1-q^{2n+2k-1})}{(1-q^{2n})^2} \\
& = - \frac{q^{2n-2k+1}(1-q^{2n+1})^2(1-q^{2n+2k-1})}{(1-q^{2n})^2(1-q^{2n-2k+2})} - 1,
\end{aligned}$$

which is the resulting form of (2.1) by dividing both sides by  $G(n, k)$ .

Summing (2.1) over  $n$  from 0 to  $\frac{p-1}{2}$ , we obtain

$$\sum_{n=0}^{\frac{p-1}{2}} F(n, k-1) - \sum_{n=0}^{\frac{p-1}{2}} F(n, k) = G\left(\frac{p+1}{2}, k\right) - G(0, k) = G\left(\frac{p+1}{2}, k\right). \quad (2.2)$$

Note that, for  $k = 1, 2, \dots, \frac{p-1}{2}$ , we have

$$\begin{aligned}
G\left(\frac{p+1}{2}, k\right) &= (-1)^{(p+1)/2+k} q^{((p+1)/2-k)^2} \frac{(q; q^2)_{(p+1)/2}^2 (q; q^2)_{(p+1)/2+k-1}}{(1-q)(q^2; q^2)_{(p-1)/2}^2 (q^2; q^2)_{(p+1)/2-k} (q; q^2)_k^2} \\
&= (-1)^{(p+1)/2+k} q^{((p+1)/2-k)^2} \frac{(1-q)[p]^2 (q; q^2)_{(p-1)/2}^2 (q; q^2)_{(p+1)/2+k-1}}{(q^2; q^2)_{(p-1)/2}^2 (q^2; q^2)_{(p+1)/2-k} (q; q^2)_k^2} \\
&\equiv 0 \pmod{[p]^3},
\end{aligned} \quad (2.3)$$

since  $(q; q^2)_{(p+1)/2+k-1}$  is divisible by  $[p]$ , while the denominator is relatively prime to  $[p]$ . Combining (2.2) and (2.3), we see that

$$\sum_{n=0}^{\frac{p-1}{2}} F(n, 0) \equiv \sum_{n=0}^{\frac{p-1}{2}} F(n, 1) \equiv \sum_{n=0}^{\frac{p-1}{2}} F(n, 2) \equiv \dots \equiv \sum_{n=0}^{\frac{p-1}{2}} F\left(n, \frac{p-1}{2}\right) \pmod{[p]^3}. \quad (2.4)$$

Furthermore, we have

$$\begin{aligned}
\sum_{n=0}^{\frac{p-1}{2}} F\left(n, \frac{p-1}{2}\right) &= F\left(\frac{p-1}{2}, \frac{p-1}{2}\right) = [2p-1] \frac{(q; q^2)_{(p-1)/2}^2 (q; q^2)_{p-1}}{(q^2; q^2)_{(p-1)/2}^2 (q; q^2)_{(p-1)/2}^2} \\
&= \frac{[p]}{(-q; q)_{p-1}^2} \left[ \frac{p-1}{2} \right]_{q^2} \left[ \frac{p-1}{2} \right]_{q^2}.
\end{aligned} \quad (2.5)$$

By a  $q$ -analogue of Wolstenholme's binomial congruence due to Straub [27, Lemma 5], it is easy to see that

$$\left[ \frac{2p-1}{p-1} \right] \equiv \frac{1+q^{p^2}}{1+q^p} \equiv q^{\frac{p(p-1)}{2}} \pmod{[p]^2}, \quad (2.6)$$

and by a  $q$ -analogue of Morley's congruence due to Pan [24, Theorem 1.2], we have

$$\left[ \frac{p-1}{2} \right]_{q^2} \equiv (-1)^{\frac{p-1}{2}} q^{\frac{1-p^2}{4}} (-q; q)_{p-1}^2 \pmod{[p]^2}. \quad (2.7)$$

Substituting (2.6) and (2.7) into (2.5) and noticing (2.4), we complete the proof of (1.3).

*Remark.* The  $q$ -WZ method has also been used by Tauraso [31] to prove some other  $q$ -analogues of congruences involving central binomial coefficients. The functions  $F(n, k)$  and  $G(n, k)$  are not easy to find,

but once they are guessed correctly they can hopefully be proved via the  $q$ -WZ method (see, for example, [22,33]).

### 3. Proof of Theorem 1.2

The congruence (2.6) can be generalized as follows:

$$\begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix} \equiv (-1)^{n-1} q^{\binom{n}{2}} \pmod{\Phi_n(q)^2}. \quad (3.1)$$

Indeed, by the  $q$ -Chu–Vandermonde summation formula (see, for example, [1, (3.3.10)]), we have

$$\begin{bmatrix} 2n \\ n \end{bmatrix} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}^2 q^{k^2} \equiv (1 + q^{n^2}) \pmod{\Phi_n(q)^2},$$

since  $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{1-q^n}{1-q^k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \equiv 0 \pmod{\Phi_n(q)}$  for  $1 \leq k \leq n-1$ . Noticing that  $\begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix} = \frac{1}{1+q^n} \begin{bmatrix} 2n \\ n \end{bmatrix}$  and

$$\left(1 + (-1)^n q^{\binom{n}{2}}\right) \left(1 + (-1)^n q^{\binom{n+1}{2}}\right) \equiv 0 \pmod{\Phi_n(q)^2},$$

we complete the proof of (3.1).

For any odd prime  $p$  and positive integer  $r$ , a special case of [20, (1.5)], due to Pan, gives

$$\begin{bmatrix} n-1 \\ \frac{n-1}{2} \end{bmatrix}_{q^2} \equiv (-1)^{\frac{n-1}{2}} q^{\frac{1-n^2}{4}} (-q; q)_{n-1}^2 \pmod{\Phi_n(q)^2}. \quad (3.2)$$

Let  $m > 1$  be an odd integer. Summing (2.1) over  $n$  from 0 to  $\frac{m-1}{2}$ , noticing that

$$\frac{(1-q)[m]^2 (q; q^2)_{(m-1)/2}^2 (q; q^2)_{(m+1)/2+k-1}}{(q^2; q^2)_{(m-1)/2}^2 (q^2; q^2)_{(m+1)/2-k} (q; q^2)_k^2} \equiv 0 \pmod{[m]\Phi_m(q)^2}$$

for  $k = 1, 2, \dots, \frac{m-1}{2}$ , and then using (3.1) and (3.2), we immediately obtain (1.4) for  $n = m$ .

### 4. Proof of Theorem 1.4

We first give two divisibility results on  $q$ -binomial coefficients.

**Lemma 4.1.** *Let  $n$  be a positive integer. Then*

$$(-q; q)_n^3 \begin{bmatrix} 4n+1 \\ 2n \end{bmatrix} \equiv 0 \pmod{(1+q^n)^2 (-q; q)_{2n}}. \quad (4.1)$$

**Proof.** Let  $[x]$  denote the greatest integer less than or equal to  $x$ . We have the following factorization of  $q$ -binomial coefficients (see, for example, [18, (10)] or [4,12]):

$$\begin{bmatrix} 4n+1 \\ 2n \end{bmatrix} = \prod_{d=1}^{4n+1} \Phi_d(q)^{\lfloor \frac{4n+1}{d} \rfloor - \lfloor \frac{2n+1}{d} \rfloor - \lfloor \frac{2n}{d} \rfloor}.$$

Moreover, it is well known that

$$\begin{aligned}
q^n - 1 &= \prod_{d|n} \Phi_d(q), \\
(q; q)_n &= (-1)^n \prod_{d=1}^n \Phi_d(q)^{\lfloor \frac{n}{d} \rfloor}, \\
(q^2; q^2)_n &= (-1)^n \left( \prod_{d=1}^n \Phi_{2d}(q)^{\lfloor \frac{n}{d} \rfloor} \right) \left( \prod_{d=1}^n \Phi_{2d-1}(q)^{\lfloor \frac{n}{2d-1} \rfloor} \right), \tag{4.2}
\end{aligned}$$

and so

$$\begin{aligned}
1 + q^n &= \frac{q^{2n} - 1}{q^n - 1} = \prod_{d|2n, d \nmid n} \Phi_d(q) = \prod_{d|n, 2d \nmid n} \Phi_{2d}(q), \\
\frac{(-q; q)_n^3}{(-q; q)_{2n}} &= \frac{(q^2; q^2)_n^3 (q; q)_{2n}}{(q^2; q^2)_{2n} (q; q)_n^3} = \prod_{d=1}^{2n} \Phi_{2d}(q)^{4 \lfloor \frac{n}{d} \rfloor - \lfloor \frac{2n}{d} \rfloor - 3 \lfloor \frac{n}{2d} \rfloor}.
\end{aligned}$$

To prove (4.1), it suffices to show that

$$\left\lfloor \frac{4n+1}{2d} \right\rfloor - \left\lfloor \frac{2n+1}{2d} \right\rfloor - \left\lfloor \frac{2n}{2d} \right\rfloor + 4 \left\lfloor \frac{n}{d} \right\rfloor - \left\lfloor \frac{2n}{d} \right\rfloor - 3 \left\lfloor \frac{n}{2d} \right\rfloor - 2\chi(d|n, 2d \nmid n) \geq 0,$$

where  $\chi(P) = 1$  if  $P$  is true and  $\chi(P) = 0$  otherwise. But this inequality is clearly equivalent to

$$2 \left\lfloor \frac{n}{d} \right\rfloor - 3 \left\lfloor \frac{n}{2d} \right\rfloor - 2\chi(d|n, 2d \nmid n) \geq 0,$$

which can be verified easily.  $\square$

**Lemma 4.2.** *Let  $n$  and  $k$  be positive integers with  $k \leq n+1$ . Then*

$$\frac{(q; q^2)_{n+1}^2 (q; q^2)_{n+k} (-q; q)_n^6}{(1-q)(q^2; q^2)_n^2 (q^2; q^2)_{n-k+1} (q; q^2)_k^2} \equiv 0 \pmod{(1+q^n)^2 [2n+1] \begin{bmatrix} 2n \\ n \end{bmatrix}}. \tag{4.3}$$

**Proof.** It is easy to see that

$$\frac{(q; q^2)_{n+1}^2}{(1-q)(q^2; q^2)_n^2} = (1-q) \frac{[2n+1]^2}{(-q; q)_n^4} \begin{bmatrix} 2n \\ n \end{bmatrix}^2,$$

and so (4.3) is equivalent to say that

$$(1-q)[2n+1] \begin{bmatrix} 2n \\ n \end{bmatrix} \frac{(q; q^2)_{n+k} (-q; q)_{n-1}^2}{(q^2; q^2)_{n-k+1} (q; q^2)_k^2} \tag{4.4}$$

is a polynomial in  $q$  with integer coefficients. Noticing (4.2) and

$$\begin{aligned}
(q; q^2)_n &= \frac{(q; q)_{2n}}{(q^2; q^2)_n} = (-1)^n \left( \prod_{d=1}^n \Phi_{2d-1}(q)^{\lfloor \frac{2n}{2d-1} \rfloor - \lfloor \frac{n}{2d-1} \rfloor} \right), \\
(-q; q)_{n-1} &= \frac{(q; q)_{n-1}}{(q^2; q^2)_{n-1}} = \prod_{d=1}^n \Phi_{2d}(q)^{\lfloor \frac{n-1}{d} \rfloor - \lfloor \frac{n-1}{2d} \rfloor},
\end{aligned}$$

we see that the expression (4.4) can be factorized into

$$\left( \prod_{d=1}^n \Phi_{2d}(q)^{\lfloor \frac{n}{d} \rfloor + 2 \lfloor \frac{n-1}{2d} \rfloor - 2 \lfloor \frac{n}{2d} \rfloor - 2 \lfloor \frac{n-1}{2d} \rfloor - \lfloor \frac{n-k+1}{d} \rfloor} \right) \\ \times \left( \prod_{d=2}^{n+k} \Phi_{2d-1}(q)^{\chi(2d-1|2n+1) + \lfloor \frac{2n}{2d-1} \rfloor + \lfloor \frac{2n+2k}{2d-1} \rfloor + 2 \lfloor \frac{k}{2d-1} \rfloor - 2 \lfloor \frac{n}{2d-1} \rfloor - \lfloor \frac{n+k}{2d-1} \rfloor - \lfloor \frac{n-k+1}{2d-1} \rfloor - 2 \lfloor \frac{2k}{2d-1} \rfloor} \right).$$

It is clear that  $\lfloor \frac{n}{d} \rfloor - \lfloor \frac{n-k+1}{d} \rfloor \geq 0$  since  $k \geq 1$ , and

$$\left\lfloor \frac{n-1}{d} \right\rfloor - \left\lfloor \frac{n}{2d} \right\rfloor - \left\lfloor \frac{n-1}{2d} \right\rfloor \geq 0.$$

So, the exponent of  $\Phi_{2d}(q)$  is non-negative. Moreover, by Sun [28, Theorem 12], we know that

$$\left\lfloor \frac{2n+1}{2d-1} \right\rfloor + \left\lfloor \frac{2n+2k}{2d-1} \right\rfloor + 2 \left\lfloor \frac{k}{2d-1} \right\rfloor \\ \geq 2 \left\lfloor \frac{n}{2d-1} \right\rfloor + \left\lfloor \frac{n+k}{2d-1} \right\rfloor + \left\lfloor \frac{n-k+1}{2d-1} \right\rfloor + 2 \left\lfloor \frac{2k}{2d-1} \right\rfloor.$$

But it is obvious that  $\chi(2d-1|2n+1) + \lfloor \frac{2n}{2d-1} \rfloor = \lfloor \frac{2n+1}{2d-1} \rfloor$ . This proves that the exponent of  $\Phi_{2d-1}(q)$  is also non-negative and therefore (4.2) is a product of cyclotomic polynomials.  $\square$

**Proof of Theorem 1.4.** Similarly as before, summing (2.1) over  $n$  from 0 to  $N$ , we obtain

$$\sum_{n=0}^N F(n, k-1) - \sum_{n=0}^N F(n, k) = G(N+1, k). \quad (4.5)$$

Furthermore, summing (4.5) over  $k$  from 1 to  $N$ , we get

$$\sum_{n=0}^N F(n, 0) - \sum_{n=0}^N F(n, N) = \sum_{k=1}^N G(N+1, k). \quad (4.6)$$

Since

$$\sum_{n=0}^N F(n, N) = F(N, N) = [4N+1] \frac{(q; q^2)_{2N}}{(q^2; q^2)_N^2} = \frac{[4N+1]}{(-q; q)_{2N}^2} \begin{bmatrix} 2N \\ N \end{bmatrix}_{q^2} \begin{bmatrix} 4N \\ 2N \end{bmatrix},$$

by Lemma 4.1, we see that

$$(-q; q)_N^6 \sum_{n=0}^N F(n, N) = (-q; q)_N^6 \frac{[2N+1]}{(-q; q)_{2N}^2} \begin{bmatrix} 2N \\ N \end{bmatrix}_{q^2} \begin{bmatrix} 4N+1 \\ 2N \end{bmatrix} \\ \equiv 0 \pmod{(1+q^N)^2 [2N+1] \begin{bmatrix} 2N \\ N \end{bmatrix}}.$$

Also, by Lemma 4.2, for  $1 \leq k \leq N$  we have

$$(-q; q)_N^6 G(N+1, k) \equiv 0 \pmod{(1+q^N)^2 [2N+1] \begin{bmatrix} 2N \\ N \end{bmatrix}}.$$

Therefore, by (4.6), we obtain

$$(-q; q)_N^6 \sum_{n=0}^N F(n, 0) \equiv 0 \pmod{(1+q^N)^2 [2N+1] \begin{bmatrix} 2N \\ N \end{bmatrix}}.$$

Namely, the congruence (1.7) holds for  $n = N$ .  $\square$

## 5. Concluding remarks and open problems

We point out that the following special case of (1.3):

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k q^{k^2} [4k+1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \equiv 0 \pmod{[p]} \quad (5.1)$$

can also be deduced from Jackson's  ${}_6\phi_5$  summation (see [8, Appendix (II.21)]):

$${}_6\phi_5 \left[ \begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1} \end{matrix}; q, \frac{aq^{n+1}}{bc} \right] = \frac{(aq; q)_n (aq/bc; q)_n}{(aq/b; q)_n (aq/c; q)_n}, \quad (5.2)$$

where the *basic hypergeometric series*  ${}_{r+1}\phi_r$  is defined as

$${}_{r+1}\phi_r \left[ \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1; q)_k (a_2; q)_k \cdots (a_{r+1}; q)_k z^k}{(q; q)_k (b_1; q)_k (b_2; q)_k \cdots (b_r; q)_k}.$$

To see this, we first let  $c \rightarrow 0$  in (5.2) to produce

$$\sum_{n=0}^N \frac{(1-aq^{2k})(a; q)_k (b; q)_k (q^{-n}; q)_k q^{nk-\binom{k}{2}}}{(1-a)(q; q)_k (aq/b; q)_k (aq^{n+1}; q)_k b^k} (-1)^k = \frac{(aq; q)_n}{(aq/b; q)_n b^n}. \quad (5.3)$$

Then putting  $n = \frac{p-1}{2}$ ,  $q \rightarrow q^2$  and  $a = b = q$  in (5.3), we are led to

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k q^{-k^2-k} [4k+1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \equiv 0 \pmod{[p]}, \quad (5.4)$$

which is in fact the dual form of (5.1) ( $q \rightarrow q^{-1}$ ). It is natural to ask whether Theorem 1.1 can be reproved from Jackson's  ${}_8\phi_7$  summation (see [8, Appendix (II.22)]) or other more complicated summation and transformation formulas.

Zudilin [34] has proved more Ramanujan-like supercongruences by using the WZ-pairs borrowed from Guillera [9,10]. Another question is whether those Ramanujan-like supercongruences have similar  $q$ -analogues.

It seems that the range of summation in (1.4) can be adjusted and the result remains unchanged. Specifically, we have

**Conjecture 5.1.** *Let  $n > 1$  be an odd integer. Then*

$$\sum_{k=0}^{n-1} (-1)^k q^{k^2} [4k+1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \equiv [n] q^{\frac{(n-1)^2}{4}} (-1)^{\frac{n-1}{2}} \pmod{[n] \Phi_n(q)^2}.$$



In particular, if  $p$  is an odd prime and  $r \geq 1$ , then

$$\sum_{k=0}^{p^r-1} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 \equiv p^r (-1)^{\frac{(p-1)r}{2}} \pmod{p^{r+2}}.$$

It is easy to see that [Conjecture 5.1](#) is true for  $n = p$ , since  $\frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \equiv 0 \pmod{[p]^3}$  for  $\frac{p+1}{2} \leq k \leq p-1$ . Long [\[21, Theorem 1.1\]](#) has proved that

$$\sum_{k=0}^{\frac{p-1}{2}} \frac{4k+1}{256^k} \binom{2k}{k}^4 \equiv p \pmod{p^4} \quad (5.5)$$

for any prime  $p > 3$ . We have the following conjectural  $q$ -congruence related to [\(5.5\)](#).

**Conjecture 5.2.** Let  $n > 1$  be an odd integer. Then

$$\begin{aligned} \sum_{k=0}^{\frac{n-1}{2}} [4k+1] \frac{(q; q^2)_k^4}{(q^2; q^2)_k^4} &\equiv [n] q^{\frac{1-n}{2}} \pmod{[n] \Phi_n(q)^2}, \\ \sum_{k=0}^{n-1} [4k+1] \frac{(q; q^2)_k^4}{(q^2; q^2)_k^4} &\equiv [n] q^{\frac{1-n}{2}} \pmod{[n] \Phi_n(q)^2}. \end{aligned}$$

In particular, if  $p$  is an odd prime, then

$$\sum_{k=0}^{\frac{p-1}{2}} [4k+1] \frac{(q; q^2)_k^4}{(q^2; q^2)_k^4} \equiv [p] q^{\frac{1-p}{2}} \pmod{[p]^3}. \quad (5.6)$$

Letting  $n = \frac{p-1}{2}$ ,  $q \rightarrow q^2$  and  $a = b = c = q$  in [\(5.2\)](#), we see that the congruence [\(5.6\)](#) holds modulo  $[p]$ .

Finally, we would like to propose two different conjectures on generalizations of [Theorem 1.4](#). The first one is

**Conjecture 5.3.** Let  $n$  and  $r$  be positive integers. Then

$$\sum_{k=0}^n (-1)^{rk} q^{-r(k+1)^2} [(4k+1)r] \begin{bmatrix} 2k \\ k \end{bmatrix}^r (-q^{k+1}; q)_{n-k}^{2r} \equiv 0 \pmod{(1+q^n)^{r-1} [2n+1] \begin{bmatrix} 2n \\ n \end{bmatrix}}.$$

Note that [Conjecture 5.3](#) is true for  $r = 1$ . In fact, we have the following identity:

$$\sum_{k=0}^n (-1)^{n-k} q^{-(k+1)^2} [4k+1] \begin{bmatrix} 2k \\ k \end{bmatrix} (-q^{k+1}; q)_{n-k}^2 = q^{-(n+1)^2} [2n+1] \begin{bmatrix} 2n \\ n \end{bmatrix}, \quad (5.7)$$

which can be easily proved by induction on  $n$ .

The second one is

**Conjecture 5.4.** Let  $n$  and  $r$  be positive integers. Then

$$\begin{aligned} \sum_{k=0}^n (-1)^k q^{k^2+(r-2)k} [4k+1] \begin{bmatrix} 2k \\ k \end{bmatrix}^{2r-1} (-q^{k+1}; q)_{n-k}^{4r-2} &\equiv 0 \pmod{(1+q^n)^{2r-2} [2n+1] \begin{bmatrix} 2n \\ n \end{bmatrix}}, \\ \sum_{k=0}^n q^{(r-2)k} [4k+1] \begin{bmatrix} 2k \\ k \end{bmatrix}^{2r} (-q^{k+1}; q)_{n-k}^{4r} &\equiv 0 \pmod{(1+q^n)^{2r-1} [2n+1] \begin{bmatrix} 2n \\ n \end{bmatrix}}. \end{aligned}$$

It is easy to see that [Theorem 1.4](#) is just the  $r = 2$  case of the first congruence in [Conjecture 5.4](#). It should also be mentioned that [Conjecture 5.4](#) is true for  $r = 1$  and there hold the following identities:

$$\sum_{k=0}^n (-1)^{n-k} q^{k^2-k} [4k+1] \begin{bmatrix} 2k \\ k \end{bmatrix} (-q^{k+1}; q)_{n-k}^2 = q^{n(n+1)} [2n+1] \begin{bmatrix} 2n \\ n \end{bmatrix},$$

$$\sum_{k=0}^n q^{-k} [4k+1] \begin{bmatrix} 2k \\ k \end{bmatrix}^2 (-q^{k+1}; q)_{n-k}^4 = q^{-n} [2n+1]^2 \begin{bmatrix} 2n \\ n \end{bmatrix}^2,$$

which can be easily proved by induction on  $n$ . The first identity is the dual form of [\(5.7\)](#), and the second one may be deemed a  $q$ -analogue of the following identity observed by Tauraso:

$$\sum_{k=0}^n \frac{4k+1}{16^k} \binom{2k}{k}^2 = \frac{(2n+1)^2}{16^n} \binom{2n}{n}^2.$$

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