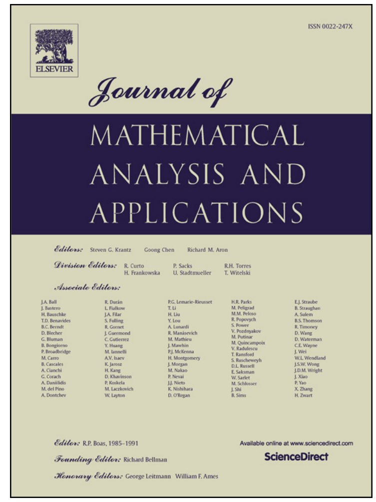

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A SUPPORT THEOREM FOR GENERALIZED CONVEXITY AND ITS APPLICATIONS.

ANDRZEJ OLBRYŚ

Abstract. In the present paper we introduce a notion of (ω, t)-convexity as a natural generalization of the notion of usual t-convexity, t-strongly convexity, approximate t-convexity, delta t-convexity and many other. The main result of this paper establishes the necessary and sufficient conditions under which an (ω, t)- convex map can be supported at a given point by an (ω, t)-affine support function. Several applications of this support theorem are presented. For instance, new characterizations of inner product spaces among normed spaces involving the notion of (ω, t)-convexity are given.

1. Introduction and terminology

Let $t \in (0, 1)$ be a fixed number and let $\mathbb{Q}(t)$ be the smallest field containing the singleton $\{t\}$. Throughout the whole paper (unless explicitly stated otherwise) X denotes a linear space over the field $\mathbb{K}$, where $\mathbb{Q}(t) \subseteq \mathbb{K} \subseteq \mathbb{R}$ and D stands for a non-empty t-convex set i.e.

$$tD + (1-t)D \subseteq D.$$

Now, for a given function $\omega : D \times D \times [0, 1] \to \mathbb{R}$ we introduce a notion of (ω, t)*-convexity.* A function $f : D \to \mathbb{R}$ is said to be:

(ω, t)-*convex*, if

$$f(tx + (1-t)z) \le tf(x) + (1-t)f(z) + \omega(x, z, t), \quad x, z \in D,$$

(ω, t)-*concave*, if

$$tf(x) + (1-t)f(z) + \omega(x, z, t) \le f(tx + (1-t)z), \quad x, z \in D.$$

If f is at the same time (ω, t)-*convex* and (ω, t)-*concave* then we say that it is an (ω, t)-*affine*. In this case f satisfies the following functional equation

$$tf(x) + (1-t)f(z) + \omega(x, z, t) = f(tx + (1-t)z), \quad x, z \in D.$$

If $t = \frac{1}{2}$ then f is said to be ω-*midpoint convex* (ω-*midpoint concave*, ω-*midpoint affine*). If the above inequalities are satisfied for all numbers $t \in [0, 1]$ (where D stands for a convex set) then we say that f is ω-*convex* (ω-*concave*, ω-*affine*, respectively).

The notion of ω-*convexity* is a common generalization of the notion of usual *convexity, strong- convexity, approximate-convexity, delta-convexity* and many other. The term on the left-hand side of the inequality is the same in all definitions while the right-hand side of all inequalities has different form.

Let $(X, \|\cdot\|)$ be a real normed space, D be a convex subset of X and let $c > 0$. A function $f : D \to \mathbb{R}$ is *called strongly* t*-convex* ($t \in (0, 1)$) with modulus $c > 0$ if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2,$$

for all $x, y \in D$. If the above inequality is satisfied with $t = \frac{1}{2}$ then f is said to be *strongly midpoint convex* function. If f is t*-strongly convex* function for all $t \in [0, 1]$ then we say that it is *strongly convex.*

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Strongly convex functions were introduced by Polyak in [20] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance [3], [5], [16], [19], [20], [27] and the references therein). It turns out, that if $f : I \to \mathbb{R}$ is *strongly convex* (where I stands for a real interval), then it is bounded from below, its level sets $\{x \in I : \ f(x) \leq \lambda\}$ are bounded for each λ and f has a unique minimum on every closed subinterval of I. The usual notions of *t-convexity, midpoint-convexity, convexity* correspond to the case $c = 0$. The *t-strongly convex* functions are (ω, t)*-convex* with

$$\omega(x, y, t) := -ct(1-t)\|x-y\|^2, \quad x, y \in D.$$

The notion of approximate convexity was introduced by D. H. Hyers and S. M. Ulam [9]. A function $f : D \to \mathbb{R}$ defined on a convex subset D of a real normed vector space is called ε*-convex* (where $\varepsilon > 0$) if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon, \quad x, y \in D, \ t \in [0, 1].$$

Another type of approximate convexity was introduced by S. Rolewicz [24]. A function $f : D \to \mathbb{R}$ is said to be γ*-paraconvex* if for certain $\varepsilon > 0$

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon\|x-y\|^\gamma, \quad x, y \in D, \ t \in [0, 1].$$

Rolewicz proved that if $\gamma > 2$ then every γ*-paraconvex* function is convex. This statement can be understood as a superstability phenomenon-a respective perturbation of convexity still guarantees convexity. The notion of approximate convexity has been recently successively generalized (see [8], [25]). For $c > 0, \ p > 0, \ t \in (0, 1)$ a real valued function f defined on convex subset D of a real normed space is called (c, t, p)*-convex* if it satisfies

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + ct(1-t)\|x-y\|^p, \quad x, y \in D.$$

The (c, t, p)*-convex* functions are (ω, t)*-convex* with

$$\omega(x, y, t) := ct(1-t)\|x-y\|^p, \quad x, y \in D.$$

Now, recall that for given two real normed spaces X, Y, a number $t \in (0, 1)$ and a nonempty open and convex subset $D \subseteq X$ a map $F : D \to Y$ is said to be *delta-t-convex* with a control function $f : D \to \mathbb{R}$ if

$$\|tF(x) + (1-t)F(y) - F(tx + (1-t)y)\| \leq tf(x) + (1-t)f(y) - f(tx + (1-t)y),$$

holds for all $x, y \in D$. The concept of *delta t-convex* maps generalized the concept of *delta-convex* maps which was introduced and intensively investigated by L. Veselý and L. Zajiček in [26]. Note that, the notion of *delta-convex* mappings has nice properties and seems to be the most natural generalization of functions which are representable as a difference of two convex functions. The *delta t-convex* functions are (ω, t)*-convex* with

$$\omega(x, y, t) := -\|tF(x) + (1-t)F(y) - F(tx + (1-t)y)\|, \quad x, y \in D.$$

The aim of the present note is to prove some version of a support theorem for above defined (ω, t)-convex as well as for ω-convex maps and to give its several applications. Support type theorems play a central role in the theory of convexity and have many applications. Several theorems of this type are known in the literature (see for instance [1], [5], [11], [12], [13], [14], [15], [17], [18], [21], [22], [23] and the references therein). The support theorem for t-convex functions was proved by Kuhn in [12] and it is a direct consequence of an abstract version of Hahn-Banach theorem due to Rodé [21].

2. Main result

We start our investigation with the following proposition.

Proposition 1. *Let $D \subseteq X$ be a t-convex set, and let $f : D \to \mathbb{R}$ be an (ω, t)-convex function where $\omega : D \times D \times [0,1] \to \mathbb{R}$. Then for an arbitrary $n \in \mathbb{N}$ the inequality*

$$f(t^n x + (1-t^n)y) \leq t^n f(x) + (1-t^n) f(y) + \sum_{j=0}^{n-1} t^j \omega(t^{n-1-j}x + (1-t^{n-1-j})y, y, t). \tag{1}$$

holds.

Proof. For $n = 1$ the inequality (1) follows from the definition of (ω, t)-convexity. Assume that (1) is valid for some n. Then using an (ω, t)-convexity of f and inductive assumption we obtain

$$\begin{aligned} f(t^{n+1}x + (1-t^{n+1})y) &= f(t[t^n x + (1-t^n)y] + (1-t)y) \\ &\leq t f(t^n x + (1-t^n)y) + (1-t)f(y) + \omega(t^n x + (1-t^n)y, y, t) \\ &\leq t\Big[t^n f(x) + (1-t^n)f(y) + \sum_{j=0}^{n-1} t^j \omega(t^{n-1-j}x + (1-t^{n-1-j})y, y, t)\Big] \\ &+ (1-t)f(y) + \omega(t^n x + (1-t^n)y, y, t) = t^{n+1}f(x) + (1-t^{n+1})f(y) \\ &+ \sum_{j=0}^{n} t^j \omega(t^{n-j}x + (1-t^{n-j})y, y, t). \end{aligned}$$

□

Recall that a point x_0 is said to be an *algebraically internal* (over the field $\mathbb{K} \subseteq \mathbb{R}$) for a set $A \subseteq X$ if for every $x \in X$ there exists an $\varepsilon_x > 0$ such that

$$x_0 + \alpha x \in A, \quad \text{for all } \alpha \in (-\varepsilon_x, \varepsilon_x).$$

The set of all algebraically internal points of A (over $\mathbb{K}$) will be denoted by $algint_{\mathbb{K}}(A)$. A set A is *algebraically open* (over $\mathbb{K}$) if $A = algint_{\mathbb{K}}(A)$. In the case when $\mathbb{K} = \mathbb{R}$ we will use the standard symbol $algint(D)$ instead of $algint_{\mathbb{R}}(D)$.

Following [1] where the definition of quadratic support function was given we introduce the following definition:

Definition 2. A function $a_y : D \to \mathbb{R}$ is said to be an (ω, t)-*affine support* for a function $f : D \to \mathbb{R}$ at a point $y \in D$, if $a_y(y) = f(y)$, $a_y(x) \leq f(x)$, $x \in D$, and

$$\omega(x, z, t) = a_y(tx + (1-t)z) - ta_y(x) - (1-t)a_y(z), \quad x, z \in D.$$

If the above conditions are satisfied for all $x, z \in D$ and all $t \in [0,1]$ (where D is a convex set) then f is said to be an ω-*affine support* for f at a point y.

Now, we are able to proof our main result. The following theorem generalized the celebrated support theorem for t-convex functions.

Theorem 3. *Let D be a t-convex set and let $y \in algint_{L(t)}(D)$. Assume that $f : D \to \mathbb{R}$ is an (ω, t)-convex function where $\omega : D \times D \times [0,1] \to \mathbb{R}$. Then there exists an (ω, t)-affine support function $a_y : D \to \mathbb{R}$ of f at y such that $f - a_y$ is t-convex if and only if for all $u, v, x, z \in D$, and $s \in \{t, 1-t\}$ the function ω satisfies the following three conditions:*

(a) $\omega(y, y, t) = 0$,
(b) $\omega(x, z, t) = \omega(z, x, 1-t)$,
(c) $s\omega(u, z, s) + (1-s)\omega(v, z, s) - \omega(su + (1-s)v, z, s) \leq s\omega(u, v, s) - \omega(su + (1-s)z, sv + (1-s)z, s)$.

Proof. Suppose that f is an (ω, t)-convex function where ω satisfies the conditions (a)-(c). Let consider the following family of functions

$$\mathcal{H} := \{h : D \to \mathbb{R} \mid h \text{ is an } (\omega, t)\text{-convex}, \ h \leq f, \ h(y) = f(y)\}.$$

Clearly, $\mathcal{H} \neq \emptyset$ because $f \in \mathcal{H}$. Observe that the family $\mathcal{H}$ can be partially ordered using the partial order by letting

$$h_1 \preceq h_2 \text{ if and only if } h_1(x) \leq h_2(x), \text{ for all } x \in D.$$

We shall show that any chain contained in $\mathcal{H}$ has a lower bound in $\mathcal{H}$. To prove it, fix an arbitrary chain $\mathcal{L} \subseteq \mathcal{H}$ and define the function $h_0 : D \to [-\infty, \infty)$ via the formula

$$h_0(x) := \inf\{h(x) \mid h \in \mathcal{L}\}.$$

First, we show that h_0 takes finite values. To see it, define the sequence of sets D_n by

$$D_n = \Big[\frac{y - (1 - t^n)D}{t^n}\Big] \cap D, \quad n \in \mathbb{N}.$$

Observe, that h_0 has a finite value at each point of D_n. Obviously, $h(y) > -\infty$, for all $h \in \mathcal{L}$. Fix an arbitrary $h \in \mathcal{L}$. For $x \in D_n$ there exists an $z \in D$ such that $t^n x + (1 - t^n)z = y$. Hence, in view of Proposition 1 we get

$$\begin{aligned} h(x) &\geq \frac{h(y) - (1 - t^n)h(z) - \sum_{j=0}^{n-1} t^j \omega(t^{n-1-j}x + (1 - t^{n-1-j})z, z, t)}{t^n} \\ &\geq \frac{f(y) - (1 - t^n)f(z) - \sum_{j=0}^{n-1} t^j \omega(t^{n-1-j}x + (1 - t^{n-1-j})z, z, t)}{t^n} > -\infty, \end{aligned}$$

and consequently $h_0(x) > -\infty, \ x \in D_n$.

On the other hand, we show that

$$D = \bigcup_{n=1}^{\infty} D_n.$$

For fixed $x \in D$, define the sequence $\{x_n\}_{n\in\mathbb{N}}$ by the formula

$$x_n := \frac{y - t^n x}{1 - t^n}.$$

Since $y \in algint_{L(t)}(D)$ then there exists an $n \in \mathbb{N}$ such that

$$x_n = y + \frac{t^n}{1 - t^n}(y - x) \in D,$$

hence

$$x = \frac{y - (1 - t^n)x_n}{t^n} \in \Big[\frac{y - (1 - t^n)D}{t^n}\Big] \cap D = D_n.$$

Obviously, $h_0(y) = f(y)$ and $h_0(x) \leq f(x), \ x \in D$. Note that h_0 is (ω, t)-convex in D. To see it take arbitrary $x, z \in D$ and arbitrary $c_1, c_2 \in \mathbb{R}$ such that

$$h_0(x) < c_1, \qquad h_0(z) < c_2.$$

There exist $h_1, h_2 \in \mathcal{L}$ such that

$$h_1(x) < c_1, \qquad h_2(z) < c_2.$$

Hence putting $h_3 = \min\{h_1, h_2\}$ we obtain

$$\begin{aligned} tc_1 + (1 - t)c_2 &> th_1(x) + (1 - t)h_2(z) \geq th_3(x) + (1 - t)h_3(z) \\ &\geq h_3(tx + (1 - t)z) - \omega(x, z, t) \\ &\geq h_0(tx + (1 - t)z) - \omega(x, z, t). \end{aligned}$$

Tending in the above inequalities with $c_1 \to h_0(x), \ c_2 \to h_0(z)$ we get the (ω, t)-convexity of h_0, therefore $h_0 \in \mathcal{H}$. We have shown that any chain in $\mathcal{H}$ has a lower bound in $\mathcal{H}$, so by the lemma of Kuratowski and Zorn, there exists a minimal element g of $\mathcal{H}$. We will show that g is an (ω, t)-affine function. For $z \in D$ we define the function $g_z : D \to \mathbb{R}$ by the formula

$$g_z(x) := \frac{1}{t}[g(tx + (1 - t)z) - (1 - t)g(z) - \omega(x, z, t)].$$

We shall show that $g_z \in \mathcal{H}$. Clearly, by the (ω,t)-convexity of g the inequality

$$g_z(x) \le g(x), \quad x \in D,$$

holds. To see that g_z is an (ω,t)-convex function fix $u,v \in D$ arbitrarily. By (ω,t)-convexity of g and the inequality (c) applying for $s=t$ we obtain

$$\begin{aligned}
g_z(tu+(1-t)v) &= \frac{1}{t}\Big[g(t[tu+(1-t)v]+(1-t)z) - (1-t)g(z) - \omega(tu+(1-t)v,z,t)\Big]\\
&= \frac{1}{t}\Big[g(t[tu+(1-t)z]+(1-t)[tv+(1-t)z]) - (1-t)g(z) - \omega(tu+(1-t)v,z,t)\Big]\\
&\le \frac{1}{t}\Big[tg(tu+(1-t)z)+(1-t)g(tv+(1-t)z)+\omega(tu+(1-t)z,tv+(1-t)z,t)\\
&- (1-t)g(z) - \omega(tu+(1-t)v,z,t)\Big]\\
&= t\Big[\frac{1}{t}\big(g(tu+(1-t)z)-(1-t)g(z)-\omega(u,z,t)\big)\Big] + \omega(u,z,t)\\
&+ (1-t)\Big[\frac{1}{t}\big(g(tv+(1-t)z)-(1-t)g(z)-\omega(v,z,t)\big)\Big] + \frac{1-t}{t}\omega(v,z,t)\\
&+ \frac{1}{t}\omega(tu+(1-t)z,tv+(1-t)z,t) - \frac{1}{t}\omega(tu+(1-t)v,z,t)\\
&\le tg_z(u)+(1-t)g_z(v)+\omega(u,v,t).
\end{aligned}$$

Since, in particular $g_y(y)=g(y)=f(y)$ then $g_y \in \mathcal{H}$ and by the minimality of g we infer that

$$g(tx+(1-t)y) = tg(x)+(1-t)g(y)+\omega(x,y,t), \qquad x \in D.$$

Analogously, let define the function $\overline{g}_y : D \to \mathbb{R}$ via the formula

$$\overline{g}_y(x) := \frac{1}{1-t}[g((1-t)x+ty) - tg(y) - \omega(x,y,1-t)].$$

Using similar argumentation as for function g_y and applying the inequality (c) for $s=1-t$ one can verify that $\overline{g}_y$ is an (ω,t)-convex function. Because $\overline{g}_y(y)=f(y)$ then $\overline{g}_y \in \mathcal{H}$ and according once more the minimality of g we get

$$\omega(x,y,1-t) = g((1-t)x+ty) - tg(y) - (1-t)g(x), \quad x \in D.$$

Using this, and condition (b) for any $z \in D$ we have

$$\begin{aligned}
g_z(y) &= \frac{1}{t}[g(ty+(1-t)z) - (1-t)g(z) - \omega(y,z,t)]\\
&= \frac{1}{t}[g((1-t)z+ty) - (1-t)g(z) - \omega(z,y,1-t)]\\
&= \frac{1}{t}[g((1-t)z+ty) - (1-t)g(z) - g((1-t)z+ty)\\
&+ tg(y)+(1-t)g(z)] = \frac{1}{t}tg(y) = g(y) = f(y),
\end{aligned}$$

therefore $g_z \in \mathcal{H}$ and using again the minimality of g we conclude that

$$\omega(x,z,t) = g(tx+(1-t)z) - tg(x) + (1-t)g(z), \qquad x,z \in D.$$

Obviously, $f-g$ is an (ω,t)-convex. To end the proof of necessity it is enough to put $a_y := g$.

For the proof of sufficiency assume that there exists a function $g : D \to \mathbb{R}$ such that

$$\omega(x,z,t) = g(tx+(1-t)z) - tg(x) - (1-t)g(z), \qquad x,z \in D.$$

Clearly, $\omega(y,y,t)=0$, moreover, for all $x,z\in D$ we have

$$\begin{aligned}\omega(x,z,t)&=g(tx+(1-t)z)-tg(x)-(1-t)g(z)\\&=g((1-t)z+tx)-(1-t)g(z)-tg(x)=\omega(z,x,1-t).\end{aligned}$$

On the other hand, for arbitrary $u,v,z\in D$ and $s\in\{t,1-t\}$ we get

$$\begin{aligned}&s\omega(u,z,s)+(1-s)\omega(v,z,s)-\omega(su+(1-s)v,z,s)\\&\quad=s\Big[g(su+(1-s)z)-sg(u)-(1-s)g(z)\Big]\\&\quad+(1-s)\Big[g(sv+(1-s)z)-sg(v)-(1-s)g(z)\Big]\\&\quad-g(s[su+(1-s)v]+(1-s)z)+sg(su+(1-s)v)+(1-s)g(z)\\&\quad=s\Big[g(su+(1-s)v)-sg(u)-(1-s)g(v)\Big]\\&\quad-\Big[g(s[su+(1-s)z]+(1-s)[sv+(1-s)z])\\&\quad-sg(su+(1-s)z)-(1-s)g(sv+(1-s)z)\Big]\\&\quad=s\omega(u,v,s)-\omega(su+(1-s)z,sv+(1-s)z,s).\end{aligned}$$

□

It is well-known that convex functions are characterized by having affine support at every point of their domains. An analogous result for t-convex functions is due to Kuhn [12]. The following theorem below states that the existence of a support mapping at an arbitrary point in fact also characterizes an (ω,t)-convexity. (A similar result for ω-convex maps also is true. The details are omitted.)

Theorem 4. *Assume that D is a t-convex set, and let $f:D\to\mathbb{R}$. If for every $y\in D$ there exists an (ω,t)-affine function $a_y:D\to\mathbb{R}$ such that $a_y(y)=f(y)$ and*

$$a_y(x)\le f(x),\quad x\in D.$$

then f is an (ω,t)-convex.

Proof. Fix $x,z\in D$ arbitrarily, and put $y:=tx+(1-t)z$. By our assumptions we get

$$\begin{aligned}f(tx+(1-t)z)=f(y)=a_y(y)&=a_y(tx+(1-t)z)\\&=\ ta_y(x)+(1-t)a_y(z)+\omega(x,z,t)\\&\le\ tf(x)+(1-t)f(z)+\omega(x,z,t).\end{aligned}$$

It completes the proof. □

It follows from the proof of Theorem 3 that for ω-convex functions the following theorem holds true

Theorem 5. *Assume that D is a convex subset of a real linear space and $y\in algint(D)$. Let $f:D\to\mathbb{R}$ be an ω-convex function where $\omega:D\times D\times[0,1]\to\mathbb{R}$ is a given map. Then there exists an ω-affine support function $a_y:D\to\mathbb{R}$ such that $f-a_y$ is convex if and only if for all $u,v,z\in D$ and all $s,t\in[0,1]$ the map ω satisfies the following conditions:*

(i) $\omega(y,y,t)=0$,
(ii) $\omega(x,z,t)=\omega(z,x,1-t)$,
(iii) $s\omega(u,z,t)+(1-s)\omega(v,z,t)-\omega(su+(1-s)v,z,t)\le t\omega(u,v,s)-\omega(tu+(1-t)z,tv+(1-t)z,s)$.

Observation 6. *Since the function $\omega\equiv 0$ satisfies the conditions (a)-(c) and (i)-(iii) then the well-known classical support theorem for t-convex functions as well as for convex functions are a consequence of our main result.*

From Theorem 3 (Theorem 5) immediately we obtain the following characterization of (ω,t)-convex (ω-convex) maps:

Theorem 7. *Let D be a t-convex (convex) set and let $algint_{\mathbb{Q}(t)}(D) \neq \emptyset$ ($algint(D) \neq \emptyset$). Assume that $f : D \to \mathbb{R}$ is an (ω, t)-convex (ω-convex) function where $\omega : D \times D \times [0,1] \to \mathbb{R}$. Then there exist a t-convex (convex) function $h : D \to \mathbb{R}$ and an (ω, t)-affine (ω-affine) function $a : D \to \mathbb{R}$ such that*

$$f(x) = a(x) + h(x) \quad x \in D,$$

if and only if for some point $y \in algint_{\mathbb{Q}(t)}(D)$ ($y \in algint(D)$) ω satisfies the conditions (a)-(c) ((i)-(iii)).

As an immediate consequence of Theorem 7 we obtain the following

Theorem 8. *Let D be a t-convex (convex) set, and let $f : D \to \mathbb{R}$ be an (ω, t)-convex (ω-convex) function, where $\omega : D \times D \times [0,1] \to \mathbb{R}$ satisfies the conditions (a)-(c) ((i)-(iii)) for some point $y \in algint_{\mathbb{Q}(t)}(D)$ ($y \in algint(D)$). If, moreover,*

$$\omega(x,z,t) \geq 0,\ x,z \in D, \qquad (\omega(x,z,t) \geq 0,\ x,z \in D,\ t \in [0,1])$$

then there exist a t-convex (convex) functions $g, h : D \to \mathbb{R}$ such that

$$f(x) = g(x) - h(x), \quad x \in D.$$

The next consequence of our main results reads as follows

Theorem 9. *Let D be a t-convex (convex) set, and let $\omega : D \times D \times [0,1] \to [0,\infty)$. If $y \in algint_{\mathbb{Q}(t)}(D)$ ($y \in algint(D)$) and ω satisfies the conditions (a)-(c) ((i)-(iii)) then for arbitrary $c \in \mathbb{R}$ there exists a t-concave (concave) function $g_y : D \to \mathbb{R}$ such that $g_y(y) = c,\ \ g_y(x) \leq c,\ x \in D$, and*

$$\omega(x,z,t) = g_y(tx + (1-t)z) - tg_y(x) - (1-t)g_y(z),\ x,z \in D \quad (x,z \in D,\ t \in [0,1]).$$

Proof. It is enough to apply the Theorem 3 (Theorem 5) for the function $f(x) := c,\ x \in D$. □

3. Applications

In this section we are going to give some applications of our main results. As we have already seen in section 1 the most frequently form of the function ω appearing in the literature is the following:

$$\omega(x,y,t) = h(t)g(x-y),$$

for some functions $h : [0,1] \to \mathbb{R}$ and $g : D^\star \to \mathbb{R}$, where $D^\star = D - D$. In this case the conditions (a)-(c) have the form:

(a') $h(t)g(0) = 0$,
(b') $h(t)g(x-y) = h(1-t)g(y-x), \quad x,y \in D$,
(c') $h(s)[sg(u-z) + (1-s)g(v-z) - g(s(u-z) + (1-s)(v-z))] \leq h(s)[sg(u-v) - g(s(u-v))]$, $u,v,z \in D,\ s \in \{t, 1-t\}$.

Now, we solve the system of the above conditions in the case when $t = \frac{1}{2}$. If ω is not identically equal to 0, then we can rewrite these conditions in the form:

(1) $g(0) = 0$,
(2) $g(-x) = g(x), \quad x \in D^\star$,
(3) $g(u-z) + g(v-z) - 2g(\frac{u+v}{2} - z) \leq g(u-v) - 2g(\frac{u-v}{2}), \quad u,v,z \in D$.

Let us recall in this place that the abelian group $(X,+)$ is uniquely 2-divisible, if the mapping $u : X \to X$, $u(x) = 2x$ is bijective. Then both u and u^{-1} are automorphisms of $(X,+)$, and we write $\frac{x}{2}$ for $u^{-1}(x)$. The following theorem gives a characterization of maps which satisfy the conditions (1)-(3).

Theorem 10. *Let $(X,+)$ be a 2-divisible abelian group. If $g : X \to \mathbb{R}$ satisfies the conditions (1)-(3) then it is a superquadratic function i.e.*

$$2g(u) + 2g(v) \leq g(u+v) + g(u-v), \quad u,v \in X.$$

If, moreover, $g(2x) \leq 4g(x),\ x \in X$, then g is a quadratic function i.e.

$$g(x) = A(x,x), \quad x \in X,$$

where $A : X \times X \to \mathbb{R}$ is a biadditive and symmetric map.

Proof. Putting $z = 0$ in (3) we obtain

$$g(u) + g(v) - 2g\Big(\frac{u+v}{2}\Big) \le g(u-v) - 2g\Big(\frac{u-v}{2}\Big), \quad u, v \in X.$$

Applying the above inequality with $u := x - y$ and $v := x + y$ we infer that

$$g(x-y) + g(x+y) - 2g(y) \le g(2x) - 2g(x), \quad x, y \in X.$$

Interchanging the roles of x and y and using the evenness of g leads to

$$g(x-y) + g(x+y) - 2g(x) \le g(2y) - 2g(y), \quad x, y \in X.$$

Now, summing up these last two inequalities we get the inequality

$$2g(x+y) + 2g(x-y) \le g(2y) + g(2x), \quad x, y \in X,$$

which is equivalent to the following one

$$2g\Big(\frac{x+y}{2}\Big) + 2g\Big(\frac{x-y}{2}\Big) \le g(x) + g(y), \quad x, y \in X.$$

Setting $x := u + v$ and $y := u - v$ we obtain that

$$2g(u) + 2g(v) \le g(u+v) + g(u-v), \quad u, v \in X.$$

In particular, $4g(x) \le g(2x),\ x \in X$. We have shown that

$$2g\Big(\frac{u-v}{2}\Big) + 2g\Big(\frac{u+v}{2}\Big) \le g(u) + g(v) \le \frac{g(u-v) + g(u+v)}{2}, \quad u, v \in X,$$

which together with condition $g(2x) \le 4g(x),\ x \in X$ implies that g is a quadratic function, i.e.

$$g(u+v) + g(u-v) = 2g(u) + 2g(v), \quad u, v \in X.$$

So, there exists a biadditive and symmetric functional $A : X \times X \to \mathbb{R}$ such that $f(x) = A(x,x),\ x \in X$. (see [2], Proposition 1, p. 166). This completes the proof. □

In the sequel we will use the following counterpart of the previous result.

Proposition 11. *Let X be a linear space over the field $\mathbb{K}$. If $g : X \to \mathbb{R}$ and $h : [0,1] \to \mathbb{R}$, satisfy the conditions (b') and (c') then*

$$[h(t) + h(1-t)]\Big[g(x) + g(y) - 2g\Big(\frac{x+y}{2}\Big)\Big] \le \frac{2[h(t) + h(1-t)]}{t(1-t)}\Big[tg\big(\frac{y-x}{2}\Big) - g\Big(t\frac{y-x}{2}\Big)\Big]$$

for all $x, y \in X$. If, moreover, $h(t) + h(1-t) > 0$, then

$$g(x) + g(y) - 2g\Big(\frac{x+y}{2}\Big) \le \frac{2}{t(1-t)}\Big[tg\Big(\frac{y-x}{2}\Big) - g\Big(t\frac{y-x}{2}\Big)\Big], \quad x, y \in X.$$

Proof. Using the well-known Daroczy and Páles identity of the mean $\frac{x+y}{2}$ (see [6])

$$\frac{x+y}{2} = t\Big[t\frac{x+y}{2} + (1-t)y\Big] + (1-t)\Big[tx + (1-t)\frac{x+y}{2}\Big],$$

we get

$$h(t)\Big[tg\Big(t\frac{x+y}{2} + (1-t)y\Big) + (1-t)g\Big(tx + (1-t)\frac{x+y}{2}\Big) - g\Big(\frac{x+y}{2}\Big)\Big]$$
$$\le h(t)\Big[tg\Big(\frac{y-x}{2}\Big) - g\Big(t\frac{y-x}{2}\Big)\Big],$$

for all $x, y \in X$. On the other hand, using the condition (b') we infer that

$$h(t)\Big[tg\Big(\frac{x+y}{2}\Big) + (1-t)g(y) - g\Big(t\frac{x+y}{2} + (1-t)y\Big)\Big] \le h(1-t)\Big[tg\Big(\frac{y-x}{2}\Big) - g\Big(t\frac{y-x}{2}\Big)\Big],$$

for all $x, y \in X$. Analogously,

$$h(t)\Big[tg(x) + (1-t)g\Big(\frac{x+y}{2}\Big) - g\Big(tx + (1-t)\frac{x+y}{2}\Big)\Big] \le h(1-t)\Big[tg\Big(\frac{y-x}{2}\Big) - g\Big(t\frac{y-x}{2}\Big)\Big].$$

Multiplying the second inequality by t and the third by $1-t$ and summing up these three inequalities, after simplification, we obtain

$$h(t)t(1-t)\Big[g(x) + g(y) - 2g\Big(\frac{x+y}{2}\Big)\Big] \le [h(t) + h(1-t)]\Big[tg\Big(\frac{y-x}{2}\Big) - g\Big(t\frac{y-x}{2}\Big)\Big].$$

Replacing t by $1-t$ in the above inequality leads to

$$[h(t) + h(1-t)]\Big[g(x) + g(y) - 2g\Big(\frac{x+y}{2}\Big)\Big] \le \frac{2[h(t) + h(1-t)]}{t(1-t)}\Big[tg\Big(\frac{y-x}{2}\Big) - g\Big(t\frac{y-x}{2}\Big)\Big],$$

for all $x, y \in X$.

If additionally, $h(t) + h(1-t) > 0$, then

$$g(x) + g(y) - 2g\Big(\frac{x+y}{2}\Big) \le \frac{2}{t(1-t)}\Big[tg\Big(\frac{y-x}{2}\Big) - g\Big(t\frac{y-x}{2}\Big)\Big], \quad x, y \in X,$$

which ends the proof. $\square$

Now, we apply our main result to the proof of a support theorem for strongly t-convex as well as for $(c, t, 2)$-approximately convex functions. The following theorem for $t = \frac{1}{2}$ and $c > 0$ has been proved in [5].

Theorem 12. *Let $(X, (\cdot|\cdot))$ be a real inner product space, let $D \subseteq X$ be a t-convex set such that $algint_{\mathbb{Q}(t)}(D) = D$ and let $c \in \mathbb{R}$. Then a function $f : D \to \mathbb{R}$ satisfies the inequality*

$$f(tx + (1-t)z) \le tf(x) + (1-t)f(z) + ct(1-t)\|x - z\|^2, \tag{2}$$

for all $x, z \in D$ if and only if at every point $y \in D$, f has a support $h_y : D \to \mathbb{R}$ of the form

$$h_y(x) = a(x - y) + f(y) - c\|x - y\|^2, \quad x \in D,$$

where $a : X \to \mathbb{R}$ is an additive function (depending on y) and t-homogeneous i.e. $a(tx) = ta(x)$, $x \in X$.

Proof. Suppose that f satisfies the inequality (2) and take an arbitrary point $y \in D$. Define the map $\omega : D \times D \times [0, 1] \to \mathbb{R}$ by formula

$$\omega(x, z, t) := -ct(1-t)\|x - z\|^2, \quad x, z \in D.$$

Clearly, ω satisfies the conditions (a) and (b) from Theorem 3. Observe that ω also satisfies (c). Indeed, for arbitrary $x, z, u, v \in D$ and $s \in \{t, 1-t\}$ we have

$$\begin{aligned} s\|u - z\|^2 &+ (1-s)\|x - z\|^2 - \|s(u - z) + (1-s)(x - z)\|^2 \\ &= s\|u - z\|^2 + (1-s)\|x - z\|^2 - s^2\|u - z\|^2 \\ &- (1-s)^2\|x - z\|^2 - 2s(1-s)(u - z|x - z) \\ &= s(1-s)\|u - z\|^2 + s(1-s)\|x - z\|^2 - 2s(1-s)(u - z|x - z) \\ &= s(1-s)\|u - x\|^2 = s\|u - x\|^2 - \|s(u - x)\|^2. \end{aligned}$$

On account of Theorem 3 there is an (ω, t)-affine support of f at y i.e. a function $g_y : D \to \mathbb{R}$ satisfying the following conditions: $g_y(y) = f(y)$, $g_y(x) \le f(x)$, $x \in D$ and, moreover,

$$-ct(1-t)\|x - z\|^2 = g_y(tx + (1-t)z) - tg_y(x) - (1-t)g_y(z), \quad x, z \in D.$$

On the other hand,

$$ct(1-t)\|x - z\|^2 = tc\|x\|^2 + (1-t)c\|z\|^2 - c\|tx + (1-t)z\|^2, \quad x, z \in D,$$

therefore the function $g_y - c\|\cdot\|^2$ is t-affine, so it has the form

$$g_y(x) - c\|x\|^2 = \overline{a}(x) + b, \quad x \in D,$$

where $\overline{a} : X \to \mathbb{R}$ is additive and t-homogeneous function.

Since $f(y) = g_y(y) = \overline{a}(y) + c\|y\|^2 + b$, then $b = f(y) - \overline{a}(y) - c\|y\|^2$, whence we get

$$\begin{aligned} g_y(x) &= \overline{a}(x) + c\|x\|^2 + f(y) - \overline{a}(y) - c\|y\|^2 \\ &= \overline{a}(x-y) + f(y) - c(\|y\|^2 - \|x\|^2) \\ &= f(y) + \overline{a}(x-y) - c\|x-y\|^2 + 2c(x-y|y) \\ &= f(y) + a(x-y) - c\|x-y\|^2, \end{aligned}$$

where $a(x) := \overline{a}(x) + 2c(x|y),\ x \in X$ is additive and t-homogeneous function.

Conversely, assume that f has at arbitrary point $y \in D$ a support function $g_y : D \to \mathbb{R}$ of the form

$$g_y(x) = a(x-y) + f(y) - c\|x-y\|^2, \quad x \in D,$$

where $a : X \to \mathbb{R}$ is an additive and t-homogeneous function. It is easy to check that g_y is an (ω, t)-affine function, where $\omega(x, z, t) = ct(1-t)\|x-z\|^2,\ x, z \in D$. Now, to the end of the proof it remains to apply the Theorem 4. □

Observe that using similar arguments and Theorem 5 instead of Theorem 3 one can prove the following theorem.

Theorem 13. *Let $(X, (\cdot|\cdot))$ be a real inner product space, let $D \subseteq X$ be a convex set such that $algint(D) = D$ and let $c \in \mathbb{R}$. Then a function $f : D \to \mathbb{R}$ satisfies the inequality*

$$f(tx + (1-t)z) \le tf(x) + (1-t)f(z) + ct(1-t)\|x-z\|^2,$$

for all $x, z \in D$ and all $t \in [0,1]$ if and only if at every point $y \in D$, f has a support $h_y : D \to \mathbb{R}$ of the form

$$h_y(x) = a(x-y) + f(y) - c\|x-y\|^2, \quad x \in D,$$

where $a : X \to \mathbb{R}$ is a linear function (depending on y) i.e. additive and homogeneous ($a(tx) = ta(x),\ x \in X,\ t \in \mathbb{R}$).

The next consequence of our main result is a theorem which gives a characterization of inner product spaces. In the literature there are a number of paper which gives conditions under which the norm in a real-linear space can be defined from an inner product. The first result of this type is due to Jordan and von Neumann [10] who showed that a linear metric space X is an inner product space if and only if it satisfies the parallelogram low i.e.

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, \qquad x, y \in X.$$

A rich collection of such characterizations is contained in the celebrated book of Amir [4].

Theorem 14. *Let $(X, \|\cdot\|)$ be a real normed space. The following conditions are equivalent to each other:*

(α) *The map $\omega : D \times D \times [0,1] \to \mathbb{R}$ of the form*

$$\omega(x, y, t) = ct(1-t)\|x-y\|^2, \quad x, y \in X$$

satisfies the inequalities (c) from Theorem 3, for some $c > 0$ and $t \in (0,1)$.

(β) *There exist a number $t \in (0,1)$ and a function $g : X \to \mathbb{R}$ such that*

$$\|x-y\|^2 = tg(x) + (1-t)g(y) - g(tx + (1-t)y), \quad x, y \in X.$$

(γ) *$(X, \|\cdot\|)$ is an inner product space.*

Proof. The implication (α) $\Rightarrow$ (β) follows from Theorem 3, because for non-negative ω there is always a function f which is an (ω, t)-convex (for example any constant function has this property).

To show (β) $\Rightarrow$ (γ), note that, on account of the proof of Theorem 3 the map

$$\omega(x, y, s) := \|x-y\|^2,\ x, y \in X,\ s \in [0,1],$$

satisfies conditions $(a)-(c)$ from Theorem 3. It follows from Proposition 11 that

$$\|x\|^2+\|y\|^2-2\Big\|\frac{x+y}{2}\Big\|^2\le\frac{2t}{t(1-t)}\Big\|\frac{y-x}{2}\Big\|^2-\Big\|t\frac{y-x}{2}\Big\|^2,\quad x,y\in X,$$

which is equivalent to

$$2\|x\|^2+2\|y\|^2\le\|x-y\|^2+\|x+y\|^2,\quad x,y\in X.$$

Now, putting $u:=x+y$ and $v:=x-y$ we get

$$\|u+v\|^2+\|u-v\|^2\le 2\|u\|^2+2\|v\|^2,\quad u,v\in X.$$

Since the norm satisfies the two above inequalities, it satisfies the parallelogram low, which implies that $(X,\|\cdot\|)$ is an inner product space.

For the proof of implication $(\gamma)\Rightarrow(\alpha)$ take arbitrary $c\in\mathbb{R}$ and $t\in(0,1)$. Put $f(x):=-c\|x\|^2,\ x\in X$. Because the norm is defined from an inner product then

$$f(tx+(1-t)y)-tf(x)-(1-t)f(y)=ct(1-t)\|x-y\|^2,\quad x,y\in X,$$

which finishes our proof. □

Now, we apply the Theorem 3 to obtain the negative result concerning some inequality which is in the spirit of the concept of t-strongly convexity. In the proof of this result we use the following statement which is a particular case of result proved by Bruce Ebanks [7, Corollary 7].

Theorem 15. *Let G be a uniquely 2-divisible abelian group, and let X be a rational vector space. Then the map $\Delta: G\times G\to X$ satisfies conditions:*

$\Delta(x,x)=0,\quad x\in G,$

$\Delta(x,y)=\Delta(y,x),\quad x,y\in G,$

$\Delta(x,y)+\Delta(z,w)+2\Delta\Big(\frac{x+y}{2},\frac{z+w}{2}\Big)=\Delta(x,z)+\Delta(y,w)+2\Delta\Big(\frac{x+z}{2},\frac{y+w}{2}\Big),\quad x,y,w,z\in G.$

if and only if there exists a function $f: G\to X$ such that

$$\Delta(x,y)=f(x)+f(y)-2f\Big(\frac{x+y}{2}\Big),\quad x,y\in G.$$

We are able to formulated our last result

Theorem 16. *Let $(X,\|\cdot\|)$ be a real normed space. Then for arbitrary $t\in(0,1)$ there is no function $f: X\to\mathbb{R}$ satisfying the inequality:*

$$f(tx+(1-t)y)\le tf(x)+(1-t)f(y)-c\|x-y\|,\quad x,y\in X, \tag{3}$$

where $c>0$.

Proof. Assume to the contrary that there exist a number $t\in(0,1)$ and a function f satisfying the inequality (3). From the identity of Daróczy and Páles [6] and the inequality (3) we obtain

$$\begin{aligned}f\Big(\frac{x+y}{2}\Big)&\le tf\Big(t\frac{x+y}{2}+(1-t)y\Big)+(1-t)f\Big(tx+(1-t)\frac{x+y}{2}\Big)-\frac{c}{2}\|x-y\|\\&\le t\Big[tf\Big(\frac{x+y}{2}\Big)+(1-t)f(y)-\frac{c}{2}\|x-y\|\Big]\\&\quad+(1-t)\Big[tf(x)+(1-t)f\Big(\frac{x+y}{2}\Big)-\frac{c}{2}\|x-y\|\Big]-\frac{c}{2}\|x-y\|,\end{aligned}$$

which means that

$$t(1-t)f\Big(\frac{x+y}{2}\Big)\le t(1-t)\frac{f(x)+f(y)}{2}-\frac{c}{2}\|x-y\|,$$

and consequently,

$$f\Big(\frac{x+y}{2}\Big)\le\frac{f(x)+f(y)}{2}-\frac{c}{2t(1-t)}\|x-y\|.$$

Now, we apply the Theorem 3 for the map $\omega : D \times D \times [0,1] \to \mathbb{R}$ given by the formula

$$\omega(x,y,s) := -\frac{c}{2t(1-t)}\|x-y\|, \quad x,y \in X,\ s \in [0,1].$$

It is easy to check that ω satisfies conditions (a)-(c) from Theorem 3. By virtue of this theorem there exists a function $g : X \to \mathbb{R}$ such that

$$\|x-y\| = \frac{g(x)+g(y)}{2} - g\Big(\frac{x+y}{2}\Big), \quad x,y \in X.$$

On account of Ebanks theorem the following identity

$$\|x-y\| + \|z-w\| + 2\Big\|\frac{x-z+y-w}{2}\Big\| = \|x-z\| + \|y-w\| + 2\Big\|\frac{x-y+z-w}{2}\Big\|,$$

holds for all $x,y,w \in X$. Substituting in the above equality $z = w$ we get

$$\|x-y\| + \|x+y-2w\| = \|x-w\| + \|y-w\| + \|x-y\|,$$

hence,

$$\|(x-w)+(y-w)\| = \|x-w\| + \|y-w\|, \quad x,y,w \in X,$$

or equivalently,

$$\|a+b\| = \|a\| + \|b\|, \quad a,b \in X.$$

This contradiction ends the proof.

□

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