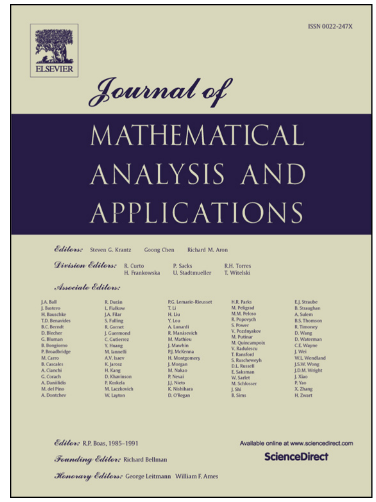
Accepted Manuscript



Nontrivial solutions of nonlocal fourth order elliptic equation of Kirchhoff type in $\mathbb{R}^3$

Anmin Mao, Wenqing Wang

PII: S0022-247X(17)30919-8
DOI: https://doi.org/10.1016/j.jmaa.2017.10.020
Reference: YJMAA 21738

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 24 February 2017

Nontrivial solutions of nonlocal fourth order elliptic equation of Kirchhoff type in $\mathbb{R}^{3*}$

Anmin Mao[†] Wenqing Wang

School of Mathematical Sciences, Qufu Normal University, Shandong 273165, P.R. China

Abstract This paper is devoted to a class of important and general nonlocal fourth order elliptic problem

$$\Delta^2 u - (1 + \lambda \int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u = f(x, u) \ \ in \ \ \mathbb{R}^3,$$

where $\Delta^2 = \Delta(\Delta)$ is the bi-harmonic operator, $\lambda \geq 0$ is a constant. We focus on the case that $f(x, u)$ involves a combination of convex and concave terms and the potential $V(x)$ is allowed to be sign-changing. By new techniques, multiplicity results of two different type of solutions are established. Our results improves and generalizes that obtained in the literature.

Key words: Bi-harmonic operator; Concave and convex terms; Sign-changing potential

2010 MSC: 35J05; 35J20; 35J60

1 Introduction and main results

In this paper, we consider a class of important fourth order elliptic equation

$$\Delta^2 u - (1 + \lambda \int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u = f(x, u) \ \ in \ \ \mathbb{R}^3. \tag{1.1}$$

Problem (1.1) is often called nonlocal because of the presence of the integral term $(\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u$, which implies that the equation (1.1) is no longer a pointwise identity. Problem (1.1) can be seen as a linear couple of elliptic equation

$$\Delta^2 u + c\Delta u + V(x)u = f_1(x, u) \ \ in \ \ \mathbb{R}^N, \tag{1.2}$$

and the Kirchhoff type equation

$$-(a + b \int_{R^N} |\nabla u|^2)\Delta u + V(x)u = f_2(x, u) \ \ in \ \ \mathbb{R}^N. \tag{1.3}$$

[*]E-mail: maoam@163.com(A.Mao), 1923226881@qq.com(W.Wang)

[†]Supported by the NSFC(11471187,11571197) and SNSFC(ZR2014AM034).

For problem (1.2), we refer the readers to [3]-[7]. When $V(x) = 0$, $\mathbb{R}^N$ is replaced by a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and set $u = \Delta u = 0$ on $\partial\Omega$, then problem (1.3) can be reduced to

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0, \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

It is well known that Problem (1.4) is related to

$$u_{tt} - (a + b\int_\Omega |\nabla u|^2 dx)\Delta u = g(x,u) \tag{1.5}$$

proposed by Kirchhoff in [8]. Some early classical studies of Kirchhoff equations were those of Bernstein [9] and Pohozaev [10]. However, (1.5) received great attention only after Lions [11] proposed an abstract framework for the problem. Some interesting results can be found in [12]-[18] and the references therein.

By fixed point theory, Ma [21] obtained positive solutions of the following nonlocal problem in one dimension

$$\begin{cases} u^4 - M\left(\int_0^1 |u'|^2 dx\right) u'' = h(x)f(x,u), \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.6}$$

Based on the work in [21], Wang et al. in [21] obtained nontrivial solutions of

$$\begin{cases} \Delta^2 u - \lambda\left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

by using the mountain pass and truncation method. Recently, Avci et al. [23] studied

$$\Delta^2 u - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + cu = f(u) \quad in \ \ \mathbb{R}^N, \tag{1.8}$$

and got at least one positive solution by using variational method and the truncation method.

Inspired by above-mentioned papers, we study

$$\Delta^2 u - (1 + \lambda\int_{\mathbb{R}^3} |\nabla u|^2 dx)\Delta u + V(x)u = f(x,u) \ \ in \ \ \mathbb{R}^3$$

and focus on the case that $f(x,u)$ involves a combination of convex and concave terms and the potential $V(x)$ is allowed to be sign-changing.

We make the following hypothesis on $V(x)$ and $f(x,u)$:

(V)*For any* $M > 0$, $meas\{x \in \mathbb{R}^3 : V(x) \le M\} < \infty$, *and there exist constants* m, a *satisfying* $0 < m < a < \frac{1}{S_2^2}$, $\inf_{x\in\mathbb{R}^3} V(x) > m - a$, *where* S_2 *is defined in (2.1).*

Setting $F(x,u) := \int_0^u f(x,s)ds$ and suppose that $F(x,u) = \bar{F}(x,u) + \alpha(x)|u|^s$, where $1 < s < 2$ and $\bar{F}, \alpha$ satisfy the following conditions:

(f_1) $\alpha(x) \in L^{\frac{2}{2-s}}(\mathbb{R}^3)$ and $\alpha(x) \geq 0$;

(f_2) $\bar{F}(x,u) \in C^1(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $\bar{F}(x,0) \equiv 0$ for all $x \in \mathbb{R}^3$ and there exist a real number $r > 4$ and two continuous bounded functions $p, q : \mathbb{R}^3 \to \mathbb{R}$ with $q > 0$ on a bounded domain Ω such that

$$p(x) \leq \frac{\bar{F}(x,u)}{|u|^r} \leq q(x) \quad for\ all\ x \in \mathbb{R}^3 \ \ and \ \ u \in \mathbb{R} \setminus \{0\},$$

and

$$\lim_{|u| \to \infty} \frac{\bar{F}(x,u)}{|u|^r} = q(x) \quad uniformly \ \ in \ \ x \in \mathbb{R}^3;$$

(f_3) there exists d_0 satisfying $0 \leq d_0 < \frac{1-aS_2^2}{4S_2^2}$ such that

$$\bar{F}(x,u) - \frac{1}{4}(\bar{f}(x,u), u) \leq d_0|u|^2 \ \ for\ all\ x \in \mathbb{R}^3\ and\ u \in \mathbb{R},$$

where $\bar{f}(x,u) = \bar{F}_u(x,u)$.

Our main results read as follows.

Theorem 1.1. *Assume (V) and $(f_1) - (f_3)$ hold. Then*

(i) problem (1.1) has at least one nontrivial mountain-pass type of solution.

(ii) problem (1.1) has at least one nontrivial local minimum type of solution.

Compared with literature, the novelty of our results lies in two aspects. One is that problem (1.1) considered here is set in whole space and the potential $V(x)$ is allowed to be sign-changing, furthermore, the nonlinearity f involves the combination of convex and concave terms which makes it very hard to check the Mountain Pass geometry for energy functional. The other is that we obtain two type of nontrivial solutions, one is obtained via the Mountain Pass lemma, the other is constructed through the local minimization. Just as stated before, our main result improves and generalizes the results obtained in [21, 22, 23].

This paper is organized as follows. In Sect.2, we state the variational framework of our problem and some preliminary setting. Sect.3 is devoted to the proof of Theorem 1.1.

2 Preliminaries and functional setting

Throughout this paper we denote by $\to$ (resp.$\rightharpoonup$) the strong (resp.weak) convergence. Let $H = H^2(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \Delta u, \nabla u \in L^2(\mathbb{R}^3)\}$ with the inner product and norm

$$(u,v)_H = \int_{\mathbb{R}^3} (\Delta u \Delta v + \nabla u \nabla v + uv), \quad \|u\|_H = (u,u)_H^{\frac{1}{2}}.$$

For $1 \leq q < +\infty$, by $|\cdot|_q$ we denote the usual L^q-norm. Define our working space

$$E = \{u \in H : \int_{\mathbb{R}^3} (|\Delta u|^2 + |\nabla u|^2 + (V(x)+a)u^2)dx < \infty\}$$

with the inner product and norm

$$(u,v) = \int_{\mathbb{R}^3} (\Delta u \Delta v + \nabla u \nabla v + (V(x)+a)uv)\, dx,\ \|u\| = (u,u)^{\frac{1}{2}}.$$

Since the embedding $E \to L^p(\mathbb{R}^3)$ is continuous for $2 \leq p < 2^*$, then there exist $S_p > 0$ such that

$$|u|_p \leq S_p \|u\|, \quad for\ all\ u \in E. \tag{2.1}$$

Lemma 2.1 ([19]) Suppose (V) holds, then embedding $E \to L^p(\mathbb{R}^3)$ is compact for $2 \leq p < 2^*$.

Define energy functional I_λ on E by

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}(\int_{\mathbb{R}^3} |\nabla u|^2 dx)^2 - \int_{\mathbb{R}^3} F(x,u)dx - \frac{a}{2}\int_{\mathbb{R}^3} u^2 dx. \tag{2.2}$$

$u \in E$ is a solution of system (1.1) if and only if $u \in E$ is a critical point of I_λ. Define $\tilde{E} = \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$ with the inner product and norm

$$(u,v)_{\tilde{E}} = \int_{\mathbb{R}^3} \nabla u \nabla v dx,\ \|u\|_{\tilde{E}} = (u,u)_{\tilde{E}}^{\frac{1}{2}},$$

then I_λ can be rewritten as

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}(\|u\|_{\tilde{E}})^4 - \int_{\mathbb{R}^3} F(x,u)dx - \frac{a}{2}\int_{\mathbb{R}^3} u^2 dx. \tag{2.3}$$

The following theorem allows us to find *Cerami* type sequence. Recall that a sequence $\{u_n\} \subset E$ is said to be a *Cerami* sequence at the level $c \in R$ ($(C)_c$-sequence for short) if $I(u_n) \to c$ and $(1 + \|u_n\|)I'(u_n) \to 0$ as $n \to \infty$. I is said to satisfy the $(C)_c$ condition if any $(C)_c$-sequence has a convergent subsequence.

Theorem 2.2.([1]) *Let E be a real Banach space with dual space E^*, and suppose that $I \in C^1(E,R)$ satisfies*

$$max\{I(0), I(e)\} \leq \mu < \eta \leq \inf_{\|u\|=\rho} I(u),$$

for some $\mu < \eta, \rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $\hat{c} \geq \eta$ be characterized by

$$\hat{c} = \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$, *then there exists a sequence* $\{u_n\} \subset E$ *such that*

$$I(u_n) \to \hat{c} \geq \eta \text{ and } (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0, \text{ as } n \to \infty.$$

Lemma 2.3.([2]) *Given a weakly lower semicontinuous functional* $I : X \to \mathbb{R}$ *on a Banach space* X *and a closed convex subset* $C \subset E$ *on which* I *is bounded from below, then we can find* $u_0 \in C$ *such that* $I(u_0) = \inf_{u \in C} I(u)$.

Lemma 2.4. *Assume* (f_2) *holds. Set*

$$\psi(u) := \int_{\mathbb{R}^3} F(x,u)dx,$$

then ψ *is weakly continuous.*

Proof. The proof is similar to lemma 2.2 in [2], we omit it.

3 Proof of Theorem 1.1

In this Section, we begin with some lemmas.

Lemma 3.1.([20]) *Let* $1 < s < 2 < r, A, B > 0$, *and consider the function*

$$\Psi_{A,B} = t^2 - At^s - Bt^r$$

for $t \geq 0$. *Then* $\max_{t\geq 0} \Psi_{A,B}(t) > 0$ *if and only if* $A^{r-2}B^{2-s} < d(r,s) := \frac{(r-2)^{r-2}(2-s)^{2-s}}{(r-s)^{r-s}}$. *Furthermore, for* $t = t_B = [\frac{2-s}{B(r-s)}]^{\frac{1}{(r-2)}}$, *one has*

$$\max_{t\geq 0} \Psi_{A,B}(t) = \Psi_{A,B}(t_B) = t_B^2[\frac{r-2}{r-s} - AB^{\frac{2-s}{r-2}} \left(\frac{r-s}{2-s}\right)^{\frac{2-s}{r-2}}] > 0.$$

Lemma 3.2. *If* (V), $(f_1) - (f_3)$ *hold, then there exists* $r > 0$ *such that* $\inf_{\|u\|=r} I_\lambda(u) > 0$.

Proof. (f_2) yields

$$\bar{F}(x,u) \leq q^+|u|^r \quad \text{for all} \quad x \in \mathbb{R}^3 \quad \text{and} \quad u \in \mathbb{R}, \tag{3.1}$$

which implies

$$
\begin{aligned}
\int_{\mathbb{R}^3} F(x,u)dx \le |q^+|_\infty \int_{\mathbb{R}^3} |u|^r dx + \int_{\mathbb{R}^3} \alpha(x)|u|^s dx \\
&\le |q^+|_\infty S_r^r \|u\|^r + (\int_{\mathbb{R}^3} |\alpha(x)|^{\frac{2}{2-s}})^{\frac{2-s}{2}} (\int_{\mathbb{R}^3} |u|^2)^{\frac{s}{2}} \\
&= C_1\|u\|^r + |\alpha|_{\frac{2}{2-s}} |u|_2^s \\
&\le C_1\|u\|^r + |\alpha|_{\frac{2}{2-s}} S_2^s \|u\|^s \\
&= C_1\|u\|^r + C_2\|u\|^s,
\end{aligned} \tag{3.2}
$$

where $C_1 = |q^+|_\infty S_r^r, C_2 = |\alpha|_{\frac{2}{2-s}} S_2^s$. Since

$$
\begin{aligned}
I_\lambda(u) =& \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}(\|u\|_{\tilde{E}})^4 - \int_{\mathbb{R}^3} F(x,u)dx - \frac{a}{2}\int_{\mathbb{R}^3} u^2 dx \\
\ge& \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}(\|u\|_{\tilde{E}})^4 - C_1\|u\|^r - C_2\|u\|^s - \frac{a}{2}S_2^2\|u\|^2 \\
=& \frac{1-aS_2^2}{2}\|u\|^2 - C_2\|u\|^s - C_1\|u\|^r + \frac{\lambda}{4}(\|u\|_{\tilde{E}})^4
\end{aligned} \tag{3.3}
$$

Lemma 3.1 together with (V) gives that for $r = t_B$ and $\|u\| = r$,

$$
I_\lambda(u) \ge \frac{1-aS_2^2}{2}\Psi_{A,B}(t_B) + \frac{\lambda}{4}\|u\|_{\tilde{E}}^4 > 0, \tag{3.4}
$$

where $A = \frac{C_2}{\frac{1-aS_2^2}{2}}$, $B = \frac{C_1}{\frac{1-aS_2^2}{2}}$, it comes to the conclusion.

Lemma 3.3. *Assume (V) and $(f_1)-(f_3)$ hold. Let $r > 0$ be as in Lemma 3.1, then there exists $e \in E$ with $\|e\| > r$ such that $I_\lambda(e) < 0$.*

Proof. Since $q > 0$ on a bounded domain Ω, we can choose a function $u \in E$ such that

$$
\int_{\mathbb{R}^3} q(x)|u|^r dx > 0.
$$

Therefore, using the condition (f_2) and Fatou's lemma, we have

$$
\begin{aligned}
\lim_{l\to+\infty} \frac{I_\lambda(lu)}{l^r} =& \limsup_{l\to+\infty}(-\int_{\mathbb{R}^3} \frac{\bar{F}(x,lu)}{l^r|u|^r}|u|^r dx) \\
\le& -\int_{\mathbb{R}^3} q(x)|u|^r dx < 0.
\end{aligned} \tag{3.5}
$$

So $I_\lambda(lu) \to -\infty$ *as* $l \to +\infty$, then there exists $e \in E$ with $\|e\| > r$ such that $I_\lambda(e) < 0$.

Thus by Theorem 2.2, we obtain that there exist a *Cerami* sequence $\{u_n\} \subset E$ such that

$$
I_\lambda(u_n) \to c > 0 \text{ and } (1 + \|u_n\|)I'_\lambda(u_n) \to 0 \text{ as } n \to \infty. \tag{3.6}
$$

Lemma 3.4. *Assume (V) and $(f_1)-(f_3)$ hold, then $\{u_n\}$ defined by (3.6) has a convergent subsequence.*

Proof. For n large enough, by $(f_1)-(f_3)$ we have

$$\begin{aligned}
c+1+\|u_n\| &\geq I_\lambda(u_n)-\frac{1}{4}(I'_\lambda(u_n),u_n)\\
&=\frac{1}{2}\|u_n\|^2+\frac{\lambda}{4}\|u_n\|^4_{\tilde{E}}-\int_{\mathbb{R}^3}F(x,u_n)dx-\frac{a}{2}\int_{\mathbb{R}^3}u_n^2dx\\
&-\frac{1}{4}\left(\|u_n\|^2+\lambda\|u_n\|^4_{\tilde{E}}-\int_{\mathbb{R}^3}(f(x,u_n),u_n)dx-a\int_{\mathbb{R}^3}u_n^2dx\right)\\
&=\frac{1}{4}\|u_n\|^2+\frac{1}{4}\int_{\mathbb{R}^3}(f(x,u_n),u_n)dx-\int_{\mathbb{R}^3}F(x,u_n)dx-\frac{a}{4}\int_{\mathbb{R}^3}u_n^2dx\\
&\geq\frac{1}{4}\|u_n\|^2+\int_{\mathbb{R}^3}\left(\frac{1}{4}(\bar{f}(x,u_n),u_n)-\bar{F}(x,u_n)\right)dx-\frac{a}{4}\int_{\mathbb{R}^3}u_n^2dx\\
&\geq\frac{1}{4}\|u_n\|^2-d_0\int_{\mathbb{R}^3}u_n^2dx-\frac{a}{4}\int_{\mathbb{R}^3}u_n^2dx\\
&\geq(\frac{1}{4}-d_0S_2^2-\frac{aS_2^2}{4})\|u_n\|^2>0,
\end{aligned}$$

which gives a boundedness for $\{u_n\}$.

Next, we prove that the sequence $\{u_n\}$ has a convergent subsequence. Going if necessary to a subsequence, we can assume that

$$\begin{aligned}
&u_n\rightharpoonup u \quad in \;\; E,\\
&u_n\to u \quad a.e. \;\; \mathbb{R}^3,\\
&u_n\to u \quad in \;\; L^s(\mathbb{R}^3), \quad 2\leq s<2^*.
\end{aligned}$$

Since $(1+\|u_n\|)I'_\lambda(u_n)\to 0$, we have

$$(I'_\lambda(u_n),u_n)=\|u_n\|^2+\lambda\|u_n\|^4_{\tilde{E}}-\int_{\mathbb{R}^3}(f(x,u_n)u_ndx-a\int_{\mathbb{R}^3}u_n^2dx=o(1), \tag{3.7}$$

$$(I'_\lambda(u_n),u)=(u_n,u)+\lambda\|u_n\|^2_{\tilde{E}}\int_{\mathbb{R}^3}\nabla u_n\nabla udx-\int_{\mathbb{R}^3}f(x,u_n)udx-a\int_{\mathbb{R}^3}u_nudx=o(1), \tag{3.8}$$

so in order to prove that $\|u_n\|\to\|u\|$, we just need to check

$$\int_{\mathbb{R}^3}f(x,u_n)u_ndx-\int_{\mathbb{R}^3}f(x,u_n)udx=o(1), \tag{3.9}$$

$$\int_{\mathbb{R}^3}\nabla u_n\nabla u_ndx-\int_{\mathbb{R}^3}\nabla u_n\nabla udx=o(1), \tag{3.10}$$

and

$$\int_{\mathbb{R}^3}u_n^2dx-\int_{\mathbb{R}^3}u_nudx=o(1). \tag{3.11}$$

In fact, by Lemma 2.1, it is easy to check (3.9) hold. (3.10) and (3.11) follow from that the embedding $E\to\tilde{E}$ is continuous.

Proof of Theorem 1.1. (i) As a consequence of Lemma 3.1-3.4, using Theorem 2.2, we get the desired result.

(ii) Since $\alpha(x) \geq 0$, it is easy to take a $\varphi \in E$ such that $\int_{\mathbb{R}^3} \alpha(x)|\varphi|^s dx > 0$, it follows from (f_2) that for $t > 0$ sufficiently small,

$$
\begin{aligned}
I_\lambda(t\varphi) = & \frac{t^2}{2}\|\varphi\|^2 + \frac{\lambda}{4}t^4\|\varphi\|_{\tilde{E}}^4 - \int_{\mathbb{R}^3} \bar{F}(x, t\varphi)dx - t^s \int_{\mathbb{R}^3} \alpha(x)|\varphi|^s dx - \frac{at^2}{2}\int_{\mathbb{R}^3} |\varphi|^2 dx \\
\leq & \frac{t^2}{2}\|\varphi\|^2 + \frac{\lambda}{4}t^4\|\varphi\|_{\tilde{E}}^4 - t^r \int_{\mathbb{R}^3} p(x)|\varphi|^r dx - t^s \int_{\mathbb{R}^3} \alpha(x)|\varphi|^s dx - \frac{at^2}{2}\int_{\mathbb{R}^3} |\varphi|^2 dx \\
< & 0.
\end{aligned}
$$

It follows from Lemma 2.3 that the minimum of the functional I_λ on any closed ball in E with center 0 and radius $\hat{r} < r$ satisfying

$$I_\lambda(u) \geq 0 \quad for\ all\ u \in E\ with\ \|u\| = \hat{r}$$

is achieved in the corresponding open ball and thus yields a nontrivial solution u_2 of (1.1) satisfying

$$I_\lambda(u_2) < 0\ and\ \|u_2\| < \hat{r} < r.$$

This completes the proof.

References

[1] I. Ekeland, Convexity Methods in Hamiltonian Mechanics, Springer, 1990.

[2] David G. Costa, An Invitation to Variational Methods in Differential Equations, Birkh*ä*user, Berlin, 2006.

[3] Claudianor O. lves, Nodal ground state sulution to a biharmonic equation via dual method, J.Differential Equations 260 (2016) 5174-5201.

[4] A. Ferrero, F. Gazzola, A partially hinged retangular plate as a model for suspension bridges, Discrete Contin. Dyn.Syst. 35 (2015) 5879-5908.

[5] J. Zhou, X. Wu, Sign-changing solutions for some fourth-order nonlinear elliptic problems, J.Math.Anal.Appl. 342 (2008) 542-558.

[6] B.T. Cheng, X.H. Tang, High energy solutions of modified quasilinear fourth-order elliptic equations with sign-changing potential, Comput. Math. Appl. 1(2017) 27-36.

[7] J. Chabrowski, J. Marcos Do O, On some fourth-order semilinear elliptic problems in R^N, Nonlinear Anal. 49 (2002) 861-884.

[8] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.

[9] S. Bernstein, Sur une classe d'equations fonctinnelles aux derivees partielles, Bull. Acad. Sci. URSS Ser.[Izv. Akad. Nauk SSSR] 4(1940) 17-26.

[10] S.I. Pohozaev, A certain class of quasilinear hyperbolic equations, Mat. Sb. (N.S.) 96 (138)(1975) 152-166, 168(in Russian).

[11] J.L. Lions, On some questions in boundary value problems of mathmatical physics, in: Contemporary Developments in Continuum Mechanics and Partial Differential Equations, Proc. Internat. Sympos. Inst. Mat. Univ. Fed. Rio de Janeiro, 1977, in: North-Holland Math. Stud. vol. 30, North-Holland, Amsterdam, New York, 1978, pp. 284-346.

[12] J.T. Sun, Tsung-fang Wu, Ground state solutions for an indefinite Kirchhoff type problem with steep potential well, J.Differential Equations, 256 (2014) 1771-1792.

[13] Z.J. Guo, Ground states for Kirchhoff equations without compact condition, J.Differentional Equations 259 (2015) 2884-2902.

[14] S. Wei, Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains, J.Differentional Equations 259 (2015) 1256-1274.

[15] Z.P. Liang, F.Y. Li, J.P. Shi, Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior, Ann.I.H.Poincar*é*-AN., 31 (2014) 155-167.

[16] X.M. He, W.M. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$, J.Differential Equations 252 (2012) 1813-1834.

[17] Y. Li, F. Li, J. Shi, Existence of a positive solution to Kirchhoff type problems without commpactness conditions, J.Differential Equations 253 (2012) 2285-2294.

[18] C. Chen, Y. Kuo, T. Wu, The Nehari manifold for a Kirchhoff type problem involving Sign-changing weight functions, J.Differentional Equations 250 (2011)1876-1908.

[19] S. Chen, Y. Liu, X. Wu, Existence and multiplicity of nontrivial solutions for a class of modified nonlinear fourth order elliptic equations on $\mathbb{R}^N$, Appl.Math.Comput. 248 (2014) 593-601.

[20] D.G.De Figueiredo, Positive solutions of semilinear elliptic problems, in: Lecture in Math, vol.957, Springer-Verlag, Berlin, 1982.

[21] T. Ma, Positive solutions for a nonlocal fourth order equation of Kirchhoff type, Discrete Contin.Dyn.Syst. (2007) 694-703.

[22] F. Wang, M. Avci, Y. An, Existence of solutions for fourth order elliptic equations of Kirchhoff type, J.Math.Anal.Appl. 409 (2014) 140-146.

[23] F. Wang, T.Q. An, Y.K. An, Existence of solutions for fourth order elliptic equations of Kirchhoff type on $\mathbb{R}^N$, Electronic J.Qual.Theory Differ.Equ. 39 (2014) 1-11.