



Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa



Global strong solutions for the incompressible nematic liquid crystal flows with density-dependent viscosity coefficient [☆]

Yang Liu ^{a,b,*}^a Department of Mathematics, Nanjing University, Nanjing 210093, PR China^b School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, PR China

ARTICLE INFO

Article history:

Received 5 March 2017

Available online xxxx

Submitted by W. Layton

Keywords:

Density-dependent viscosity
Incompressible nematic liquid
crystal flows
Strong solution
Vacuum

ABSTRACT

This paper is concerned with an initial–boundary value problem of the incompressible nematic liquid crystal flows with density-dependent viscosity in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. The global well-posedness of strong solutions with large oscillations is established in vacuum, provided $\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2}$ is suitably small with arbitrary large initial density, which extended the local strong solution by Gao et al. [12] to be a global one.

© 2018 Elsevier Inc. All rights reserved.

1. Introduction

We consider the following hydrodynamic system modeling the flow of nematic liquid crystal materials

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho u_t + \rho(u \cdot \nabla)u - \operatorname{div}(\mu(\rho)\nabla u) + \nabla P = -\lambda \operatorname{div}(\nabla d \odot \nabla d), \\ \operatorname{div} u = 0, \\ d_t + u \cdot \nabla d = \gamma(\Delta d + |\nabla d|^2 d), \end{cases} \quad (1.1)$$

in $\Omega \times (0, \infty)$, together with the initial and boundary conditions

$$(\rho, u, d)|_{t=0} = (\rho_0, u_0, d_0), \quad \text{with } |d_0| = 1, \quad \operatorname{div} u_0 = 0, \quad \text{in } \Omega, \quad (1.2)$$

[☆] This work was supported by excellent doctoral dissertation cultivation grant from Dalian University of Technology.

* Correspondence to: Department of Mathematics, Nanjing University, Nanjing 210093, PR China.

E-mail address: liuyang19850524@163.com.

$$u(x, t) = 0, \quad \frac{\partial d}{\partial \nu}(x, t) = 0, \quad \text{on } \partial\Omega \times (0, \infty), \quad (1.3)$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$ whose unit outward normal is ν . Here $u \in \mathbb{R}^3$ represents the velocity field of the flow, $d \in \mathcal{S}^2$, the unit sphere in \mathbb{R}^3 , represents the macroscopic molecular orientation of the liquid crystal material, $\rho \in \mathbb{R}^+$ and $P \in \mathbb{R}$ are scalar functions, respectively, denoting the density of the fluid and the pressure arising from the usual assumption of incompressibility $\operatorname{div} u = 0$. The positive constants λ and γ represent viscosity of fluid, competition between kinetic and potential energy, and microscopic elastic relaxation time respectively. The viscosity coefficient $\mu = \mu(\rho)$ is a general function of density, which is assumed to satisfy

$$\mu \in C^1[0, \infty) \quad \text{and} \quad 0 < \underline{\mu} \leq \mu \leq \bar{\mu} < \infty \quad \text{on } [0, \infty), \quad (1.4)$$

for some positive constant $\underline{\mu}, \bar{\mu}$. Without loss of generality, both λ and γ are normalized to 1. The symbol $\nabla d \odot \nabla d$, which exhibits the property of the anisotropy of the material, denotes the $n \times n$ matrix whose (i, j) -th entry is given by $\partial_i d \cdot \partial_j d$, for $i, j = 1, 2, 3$.

System (1.1)–(1.3) is a simplified version of the Ericksen–Leslie model, which reduces to the Osssen–Frank model in the static case, for the hydrodynamics of nematic liquid crystals developed by Ericksen [9] and Leslie [18] in the 1960's, but it still retains most important mathematical structures as well as most of the essential difficulties of the original Ericksen–Leslie model. Both the full Ericksen–Leslie model and the simplified version are the macroscopic continuum description of the time evolution of the materials, under the influence of both the flow velocity field u and the microscopic orientation configurations d of rod-like liquid crystals. Mathematically, system (1.1)–(1.3) is a strongly coupled system between the nonhomogeneous incompressible Navier–Stokes equations and the transported heat flows of harmonic map, and thus, its mathematical analysis is full of challenges.

When d is a constant vector and $|d| = 1$, the system (1.1)–(1.3) reduces to the well-known nonhomogeneous incompressible Navier–Stokes equations. In the case that the viscosity μ is a constant and the initial density has a uniform positive lower bound, Kazhikov [1, 17] established the global existence of weak solutions, and proved that there exists a unique local strong solution for arbitrary initial data with global existence of large strong solutions in \mathbb{R}^2 . However in \mathbb{R}^3 the global well-posedness results were obtained only for small solutions. These results require relatively high regularity of the density, though. It is worthwhile to emphasize that for smooth densities with vacuum states, with the initial compatibility conditions

$$-\mu \Delta u_0 + \nabla p_0 = \sqrt{\rho_0} g \quad \text{and} \quad \operatorname{div} u_0 = 0 \quad \text{in } \Omega \quad (1.5)$$

for some $(p_0, g) \in H^1 \times L^2$, Cho–Kim [3] proved the existence and uniqueness of local strong solutions in bounded domains or the whole space. Furthermore, global strong small solutions were obtained by Craig et al. [5]. Subsequently, without the initial compatibility conditions (1.5), Liang [23] proved the local strong solutions on the whole two-dimensional space with vacuum as far field density. Lv et al. [26] extended this result to global one and obtained some decay estimate of solutions. Recently, with the help of the Lagrangian formulation, Danchin–Mucha [6] get the local well-posedness with piecewise constant initial density ($\rho_0 \in L^\infty(\mathbb{R}^3)$). Under additional assumption that the initial velocity is small and the density is close enough to a positive constant, they get the unique global solution. Similar results with lower regularity can be found in [13, 27, 2]. In case of density-dependent viscosity, Lions [24, Chapter 2] established the global existence of weak solutions to nonhomogeneous Navier–Stokes equations in any space dimensions for the initial density allowing vacuum. Cho–Kim [4] used the condition

$$-\operatorname{div}(2\mu(\rho_0)\nabla u_0) + \nabla p_0 = \sqrt{\rho_0} g \quad \text{and} \quad \operatorname{div} u_0 = 0 \quad \text{in } \Omega \quad (1.6)$$

for some $(p_0, g) \in H^1 \times L^2$. They obtained existence of unique local strong solutions for two or three-dimensional bounded domains. Very recently, Huang–Wang [14,15], and independently by Zhang [29], showed the global existence of strong solutions on bounded domains under some smallness assumption.

Let's go back to the System (1.1)–(1.3). When viscosity μ is a constant, there is a huge literature on the studies about well-posedness of solutions to (1.1)–(1.3). The global existence of weak solutions to system (1.1)–(1.3), with $|\nabla d|^2 d$ being replaced by $\frac{1-|d|^2}{\varepsilon^2} d$, the Ginzburg–Landau approximation term, was established in [16,28] and [25], for each $\varepsilon > 0$. They cannot get the uniform estimates in ε , and therefore cannot take the limit, as $\varepsilon \rightarrow 0$. It's also a challenging problem to study the convergence, as ε tends to zero, for the non-homogeneous case. As for the case of $|\nabla d|^2 d$, Wen and Ding [8] obtained local existence and uniqueness of strong solutions to the Dirichlet problem in bounded domain with initial density being allowed to have vacuum. They also established the global existence and uniqueness of solutions for two dimensional case if the initial density was away from vacuum and the initial data is small. If the initial data is small, or satisfies some geometric condition, for constant density case, Li and Wang [21] obtained the global existence of strong solutions in dimension three. Li and Wang [22] considered the nonconstant but positive density case. Fan et al. [11] established the global existence and uniqueness of strong solutions to the 2D density dependent liquid crystal flows with vacuum in a bounded smooth domain and the initial data is suitably small. Recently, Li proved the global existence and uniqueness of strong solutions with initial data being of small norm for the dimensions two and three in bounded domain in [20] and the initial direction field satisfying some geometric structure for the two dimensional whole space in [19]. Fan et al. [7] proved global existence and uniqueness of the strong solutions with nonnegative ρ_0 and small initial data in three dimensional whole space. Recently, when the viscosity coefficient is a function of the density of fluid, Gao et al. [12] established the local unique strong solutions to the initial boundary value problem (1.1)–(1.3) in a bounded domain of \mathbb{R}^3 with smooth boundary. They also obtained the Serrin-type blow up criterion of the strong solutions. However, the global strong solution to the problem (1.1)–(1.3) is still unknown.

Motivated by [14,15,29], we establish global strong solutions for the 3D incompressible nematic liquid crystal flows with density-dependent viscosity, provided $\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2}$ is suitably small allowing large fluctuation of density.

Theorem 1.1. *Let Ω be a bounded smooth domain in \mathbb{R}^3 and $q \in (3, \infty)$ be a fixed constant. Suppose that μ satisfies (1.4), and the initial data (ρ_0, u_0, d_0) satisfies the regularity conditions*

$$\begin{aligned} 0 \leq \rho_0 \leq \bar{\rho}, \quad \rho_0 \in W^{1,q}, \quad \|\nabla \mu(\rho_0)\|_{L^q} \leq \mu^*, \\ u_0 \in H_{0,\sigma}^1 \cap H^2, \quad d_0 \in H^3, \quad \text{and} \quad |d_0| = 1 \quad \text{in } \Omega, \end{aligned} \quad (1.7)$$

and the compatibility condition

$$-\operatorname{div}(\mu(\rho_0)\nabla u_0) + \nabla P_0 + \operatorname{div}(\nabla d_0 \odot \nabla d_0) = \sqrt{\rho_0}g \quad \text{in } \Omega \quad (1.8)$$

for some $(P_0, g) \in H^1 \times L^2$. Then there exists some small positive constant ε_0 , depending on Ω , $\bar{\rho}$, $\underline{\mu}$, $\bar{\mu}$, μ^* , such that if

$$\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2} \leq \varepsilon_0, \quad (1.9)$$

then the initial boundary value problem (1.1)–(1.3) admits a unique global strong solution (ρ, u, d) satisfying

$$\begin{aligned} \rho &\in C([0, \infty); W^{1,q}), \quad \rho_t \in C([0, \infty); L^q), \\ u &\in C([0, \infty); H_0^1 \cap H^2) \cap L^2(0, \infty; W^{2,r}), \\ u_t &\in L^2(0, \infty; H_0^1), \quad \sqrt{\rho}u_t \in L^\infty(0, \infty; L^2), \end{aligned}$$

$$\begin{aligned}
P &\in L^\infty(0, \infty; H^1) \cap L^2(0, \infty; W^{1,r}), \\
d &\in C([0, \infty); H^3) \cap L^2(0, \infty; H^4), \quad |d| = 1 \text{ in } \Omega \times [0, \infty), \\
d_t &\in C([0, \infty); H^1) \cap L^2(0, \infty; H^2), \quad d_{tt} \in L^2(0, \infty; L^2)
\end{aligned} \tag{1.10}$$

for some r with $3 < r < \min\{6, q\}$.

Remark 1.1. Theorem 1.1 extends the result of Gao et al. [12] to the Global one. When d is a constant vector, it also generalizes the previous results for the 3D Navier–Stokes equations in [29,14,15].

The rest of this paper is organized as follows: in Section 2, we state some notations and auxiliary lemmas; in Section 3, we perform some time independent a priori estimates on (ρ, u, d) ; in Section 4, we prove the global existence of strong solutions.

2. Preliminaries

Throughout this paper, we assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. For simplicity, we denote

$$\int f dx = \int_{\Omega} f dx.$$

For $1 \leq r \leq \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way.

$$\begin{aligned}
L^r &= L^r(\Omega), \quad W^{k,r} = W^{k,r}(\Omega) = \{f \in L^r : \nabla^k f \in L^r\}, \\
H^k &= W^{k,2}, \quad C_{0,\sigma}^\infty = \{f \in C_0^\infty : \operatorname{div} f = 0\}, \\
H_0^1 &= \overline{C_{0,\sigma}^\infty}, \quad H_{0,\sigma}^1 = \overline{C_{0,\sigma}^\infty}, \quad \text{closure in the norm of } H^1.
\end{aligned}$$

The well-known Poincaré and Sobolev inequalities will be frequently used (cf. [10]).

Lemma 2.1. *There exists an absolutely positive constant C , depending only on Ω , such that*

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}, \quad \forall u \in H_0^1(\Omega), \tag{2.1}$$

$$\|u\|_{L^\infty} \leq C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{H^1}^{\frac{1}{2}}, \quad \forall u \in H_0^1(\Omega) \cap H^2(\Omega). \tag{2.2}$$

High-order a priori estimates rely on the following regularity results for density-dependent Stokes equations.

Lemma 2.2. [29,15] *Assume that μ satisfies (1.4), $\rho \in W^{1,q}$ and $0 \leq \rho \leq \bar{\rho}$ for some $3 < q < \infty$. Let $(u, P) \in H_{0,\sigma}^1 \times L^2$ be the unique weak solution to the boundary value problem:*

$$-\operatorname{div}(\mu(\rho)\nabla u) + \nabla P = F, \quad \operatorname{div} u = 0 \quad \text{in } \Omega, \quad \int P dx = 0. \tag{2.3}$$

Then the following regularity estimates hold for (u, P) :

(1) *If $F \in L^2$, then $(u, P) \in H^2 \times H^1$ and*

$$\|u\|_{H^2} + \|P\|_{H^1} \leq C \|F\|_{L^2} (1 + \|\nabla \mu(\rho)\|_{L^q})^{\frac{q}{q-3}}. \tag{2.4}$$

(2) If $F \in L^r$ for some $r \in (3, q)$, then $(u, P) \in W^{2,r} \times W^{1,r}$ such that

$$\|u\|_{W^{2,r}} + \|P\|_{W^{1,r}} \leq C\|F\|_{L^r} (1 + \|\nabla\mu(\rho)\|_{L^q})^{\frac{qr}{2(q-r)}}. \quad (2.5)$$

Here, the positive constant C depends only on $r, p, \underline{\mu}, \bar{\mu}$ and $\bar{\rho}$.

Theorem 1.1 will be proved by combining the global a priori estimates with the following local existence results (see, for example, [12]).

Lemma 2.3. For $q > 3$, assume that μ satisfies (1.4), and the initial data (ρ_0, u_0, d_0) satisfies

$$\begin{aligned} 0 \leq \rho_0 \leq \bar{\rho}, \quad \rho_0 \in W^{1,q}, \quad \|\nabla\mu(\rho_0)\|_{L^q} \leq \mu^*, \\ u_0 \in H_{0,\sigma}^1 \cap H^2, \quad d_0 \in H^3, \quad \text{and} \quad |d_0| = 1 \quad \text{in } \Omega, \end{aligned} \quad (2.6)$$

and that (1.8) holds for some $(P_0, g) \in H^1 \times L^2$. Then there exist positive small time T_0 and a unique strong solution (ρ, u, d, P) of (1.1)–(1.3) such that for $3 < r < \min\{q, 6\}$,

$$\begin{aligned} \rho &\in C([0, T_0]; W^{1,q}), \quad \rho_t \in C([0, T_0]; L^q), \\ u &\in C([0, T_0]; H_0^1 \cap H^2) \cap L^2(0, T_0; W^{2,r}), \\ u_t &\in L^2(0, T_0; H_0^1), \quad \sqrt{\rho}u_t \in L^\infty(0, T_0; L^2), \\ P &\in L^\infty(0, T_0; H^1) \cap L^2(0, T_0; W^{1,r}), \\ d &\in C([0, T_0]; H^3) \cap L^2(0, T_0; H^4), \quad |d| = 1 \quad \text{in } \Omega \times [0, T_0], \\ d_t &\in C([0, T_0]; H^1) \cap L^2(0, T_0; H^2), \quad d_{tt} \in L^2(0, T_0; L^2). \end{aligned} \quad (2.7)$$

3. A priori estimates

The proof of Theorem 1.1 is based on the local existence result (cf. Lemma 2.3) and the global a priori estimates. To do this, let (ρ, u, d, P) be a strong solution of (1.1)–(1.3) on $\Omega \times (0, T)$ for some $T > 0$. Moreover, we assume that the following a priori hypothesis for the viscosity function $\mu(\rho)$:

$$\sup_{0 \leq t \leq T} \|\nabla\mu(\rho)(t)\|_{L^q} \leq 2\mu^* \quad \text{for } q > 3. \quad (3.1)$$

In this section, we shall make full use of the a priori hypothesis (3.1) to derive some global a priori estimates of the solution (ρ, u, d, P) . For simplicity, throughout this section, the same letter C or C_i ($i = 1, 2, 3, \dots$) will be repeatedly used to denote the positive constant, which may depend on $\Omega, \bar{\rho}, \underline{\mu}, \bar{\mu}$ and μ^* , but is independent of the lower bound of density $\underline{\rho}$. It is also worth mentioning that all the positive constants throughout this paper are independent of the time T .

We begin with the nonnegativity and boundedness of density and the basic energy estimates.

Lemma 3.1 (Basic energy identity). Let (ρ, u, d, P) be a strong solution of (1.1)–(1.3) on $\Omega \times (0, T)$, and the conditions of Theorem 1.1 are satisfied. Then there is a positive constant ε_1 depending only on $\Omega, \underline{\mu}, \bar{\rho}, \bar{\mu}, \mu^*$, such that if

$$\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2} + \|\Delta d\|_{L^2}) \leq \varepsilon_1, \quad (3.2)$$

then we have the following estimate

$$0 \leq \rho(x, t) \leq \bar{\rho} \quad (3.3)$$

for all $(x, t) \in \Omega \times (0, T)$, and

$$\sup_{0 \leq t \leq T} (\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + \int_0^T (\|\nabla u\|_{L^2}^2 + \|d_t\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \leq CE_0, \quad (3.4)$$

$$\sup_{0 \leq t \leq T} t(\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + \int_0^T t(\|\nabla u\|_{L^2}^2 + \|d_t\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \leq CE_0, \quad (3.5)$$

where $E_0 \triangleq \|\nabla u_0\|_{L^2}^2 + \|\Delta d_0\|_{L^2}^2$.

Proof. Note that (3.3) follows from the transport equation (1.1)₁ and making use of (1.1)₃ (see Lions [24, Theorem 2.1]). To prove (3.4), multiplying (1.1)₂ by u , integrating by parts and adding them together, by (1.1)₁ yield that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx + \int \mu(\rho) |\nabla u|^2 dx \\ &= - \int (\nabla d \cdot \Delta d) \cdot u dx \\ &\leq C \|u\|_{L^6} \|\nabla d\|_{L^3} \|\Delta d\|_{L^2} \\ &\leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C \|\nabla d\|_{L^3}^2 \|\Delta d\|_{L^2}^2. \end{aligned} \quad (3.6)$$

Using the fact that $|d| = 1$ and integrating by parts, we infer from (1.1)₄ and Hölder's inequality that

$$\begin{aligned} & \frac{d}{dt} \int |\nabla d|^2 dx + \int (|d_t|^2 + |\Delta d|^2) dx \\ &= \int |d_t - \Delta d|^2 dx \\ &= \int |u \cdot \nabla d - |\nabla d|^2 d|^2 dx \\ &\leq C \|u\|_{L^6}^2 \|\nabla d\|_{L^3}^2 + \|\nabla d\|_{L^4}^4 \\ &\leq C \|\nabla d\|_{L^3}^2 (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2), \end{aligned} \quad (3.7)$$

which together with (3.6) and (3.7), and Poincaré inequality yield

$$\begin{aligned} & \frac{d}{dt} \int \left(\frac{1}{2} \rho |u|^2 + |\nabla d|^2 \right) dx + \int \left(\frac{3\mu}{4} |\nabla u|^2 + |d_t|^2 + |\Delta d|^2 \right) dx \\ &\leq C \|\nabla d\|_{L^3}^2 (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\ &\leq C \|\nabla d\|_{L^2} \|\nabla d\|_{H^1} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\ &\leq C \|\Delta d\|_{L^2}^2 (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\ &\leq \varepsilon_1^2 C (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2). \end{aligned} \quad (3.8)$$

Taking ε_1 sufficiently small, and we infer from (3.8) that

$$\frac{d}{dt}(\|\sqrt{\rho}u\|_{L^2} + \|\nabla d\|_{L^2}^2) + \|\nabla u\|_{L^2}^2 + \|d_t\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 \leq 0, \quad (3.9)$$

which derives (3.4). Multiplying (3.9) by t , we have

$$\frac{d}{dt}t(\|\sqrt{\rho}u\|_{L^2} + \|\nabla d\|_{L^2}^2) + t(\|\nabla u\|_{L^2}^2 + \|d_t\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \leq C(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2). \quad (3.10)$$

Then, (3.6) follows from the above inequality and (3.4). \square

Lemma 3.2. *Let $\varepsilon_1 > 0$ be the same one fixed in (3.2). Assume that (3.2) and (3.1) hold. Then*

$$\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \int_0^T (\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) dt \leq CE_0, \quad (3.11)$$

$$\sup_{0 \leq t \leq T} t(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \int_0^T t(\|\sqrt{\rho}u\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) dt \leq CE_0. \quad (3.12)$$

Proof. Due to (1.1)₁, it holds that

$$\mu(\rho)_t + u \cdot \nabla \mu(\rho) = 0, \quad (3.13)$$

so that, multiplying (1.1)₂ by u_t in L^2 and integrating by parts over Ω , by (1.1)₃ we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \mu(\rho) |\nabla u|^2 dx + \int \rho |u_t|^2 dx \\ &= - \int \rho u \cdot \nabla u \cdot u_t dx - \frac{1}{2} \int u \cdot \nabla \mu(\rho) |\nabla u|^2 dx + \int \nabla d \odot \nabla d : \nabla u_t dx \\ &= \frac{d}{dt} \int \nabla d \odot \nabla d : \nabla u dx - \int (\nabla d \odot \nabla d)_t : \nabla u dx - \int \rho u \cdot \nabla u \cdot u_t dx \\ &+ \frac{1}{2} \int \mu(\rho) u \cdot \nabla |\nabla u|^2 dx \triangleq \sum_{i=0}^3 I_i. \end{aligned} \quad (3.14)$$

By (3.1), Hölder's and Gagliardo–Nirenberg inequalities, we have

$$\begin{aligned} I_1 &\leq \|\nabla d_t\|_{L^2} \|\nabla d\|_{L^6} \|\nabla u\|_{L^3} \\ &\leq \|\nabla d_t\|_{L^2} \|\nabla^2 d\|_{L^2} \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{H^1}^{\frac{1}{2}} \\ &\leq \varepsilon \|\nabla d_t\|_{L^2}^2 + C(\varepsilon) \|\nabla^2 d\|_{L^2}^2 \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \\ &\leq \varepsilon \|\nabla d_t\|_{L^2}^2 + \delta \|\nabla u\|_{H^1} + C(\varepsilon, \delta) \|\Delta d\|_{L^2}^4 \|\nabla u\|_{L^2}^2, \end{aligned} \quad (3.15)$$

$$\begin{aligned} I_2 &\leq \|\rho\|_{L^\infty}^{\frac{1}{2}} \|\sqrt{\rho}u_t\|_{L^2} \|u\|_{L^6} \|\nabla u\|_{L^3} \\ &\leq C \|\sqrt{\rho}u_t\|_{L^2} \|\nabla u\|_{L^2}^{\frac{3}{2}} \|\nabla u\|_{H^1}^{\frac{1}{2}} \\ &\leq \varepsilon \|\sqrt{\rho}u_t\|_{L^2}^2 + \delta \|\nabla u\|_{H^1}^2 + C(\varepsilon, \delta) \|\nabla u\|_{L^2}^6, \end{aligned} \quad (3.16)$$

$$\begin{aligned} I_3 &\leq C \int |u| |\nabla u| |\nabla^2 u| dx \\ &\leq C \|u\|_{L^6} \|\nabla u\|_{L^3} \|\nabla^2 u\|_{L^2} \end{aligned}$$

$$\begin{aligned}
&\leq C\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{3}{2}} \\
&\leq \delta\|\nabla u\|_{H^1}^2 + C(\delta)\|\nabla u\|_{L^2}^6.
\end{aligned} \tag{3.17}$$

Next, the gradient operator to (1.1)₄, one obtains that

$$\nabla d_t - \nabla \Delta d = -\nabla(u \cdot \nabla d) + \nabla(|\nabla d|^2 d), \tag{3.18}$$

which together with integration by parts and Young's inequality, yields that

$$\begin{aligned}
&\frac{d}{dt}\|\Delta d\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2 \\
&= \int |\nabla d_t - \nabla \Delta d|^2 dx \\
&= \int |-\nabla(u \cdot \nabla d) + \nabla(|\nabla d|^2 d)|^2 dx \\
&\leq C \int (|\nabla d|^6 + |\nabla u|^2 |\nabla d|^2 + |u|^2 |\nabla^2 d|^2 + |\nabla d|^2 |\nabla^2 d|^2) dx \\
&\leq C(\|\nabla^2 d\|_{L^2}^6 + \|\nabla u\|_{L^3}^2 \|\nabla d\|_{L^6}^2 + \|u\|_{L^6}^2 \|\nabla^2 d\|_{L^3}^2 + \|\nabla d\|_{L^6}^2 \|\nabla^2 d\|_{L^3}^2) \\
&\leq C(\|\nabla^2 d\|_{L^2}^6 + \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \|\nabla^2 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2} (\|\Delta d\|_{L^2} + \|\nabla \Delta d\|_{L^2})) \\
&\quad + C\|\nabla^2 d\|_{L^2}^2 \|\nabla^2 d\|_{L^2} (\|\Delta d\|_{L^2} + \|\nabla \Delta d\|_{L^2}) \\
&\leq \varepsilon \|\nabla \Delta d\|_{L^2}^2 + \delta \|\nabla u\|_{H^1}^2 + C\|\Delta d\|_{L^2}^4 \|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|\Delta d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^6 \\
&\quad + C\|\nabla u\|_{L^2}^2 \|\Delta d\|_{L^2}^2 + C\|\Delta d\|_{L^2}^4.
\end{aligned} \tag{3.19}$$

Applying Lemma 2.2 with $F \triangleq \rho u_t + \rho u \cdot \nabla u + \Delta d \cdot \nabla d$, we deduce from (2.4) and (3.18) that

$$\begin{aligned}
\|\nabla u\|_{H^1} + \|\nabla P\|_{L^2} &\leq C\|F\|_{L^2} (1 + \|\nabla \mu(\rho)\|_{L^p})^{\frac{p}{p-3}} \\
&\leq C(\mu^*)(\|\rho u_t\|_{L^2} + \|\rho u \cdot \nabla u\|_{L^2} + \| |\nabla d| |\Delta d| \|_{L^2}) \\
&\leq C(\mu^*)(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla d\|_{L^6} \|\Delta d\|_{L^3} + \|u\|_{L^6} \|\nabla u\|_{L^3}) \\
&\leq C(\mu^*)(\|\sqrt{\rho} u_t\|_{L^2} + \|\nabla^2 d\|_{L^2}^{\frac{3}{2}} (\|\Delta d\|_{L^2} + \|\nabla \Delta d\|_{L^2})^{\frac{1}{2}} + \|\nabla u\|_{L^2}^{\frac{3}{2}} \|\nabla u\|_{H^1}^{\frac{1}{2}})
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
\|\nabla \Delta d\|_{L^2}^2 &\leq C(\|\nabla d_t\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^6 + \| |\nabla d| |\nabla^2 d| \|_{L^2}^2) \\
&\quad + C(\| |\nabla u| |\nabla d| \|_{L^2}^2 + \| |u| |\nabla^2 d| \|_{L^2}^2) \\
&\leq C(\|\nabla d_t\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^6 + \|\nabla^2 d\|_{L^2}^3 (\|\Delta d\|_{L^2} + \|\nabla \Delta d\|_{L^2})) \\
&\quad + C\|\nabla u\|_{L^2}^2 \|\nabla^2 d\|_{L^2} (\|\Delta d\|_{L^2} + \|\nabla \Delta d\|_{L^2}) \\
&\leq C\|\Delta d\|_{L^2}^2 (\|\Delta d\|_{L^2}^4 + \|\nabla u\|_{L^2}^4) + C\|\nabla d_t\|_{L^2}^2 + \varepsilon \|\nabla \Delta d\|_{L^2}^2 \\
&\quad + C\|\Delta d\|_{L^2}^4 + C\|\nabla u\|_{L^2}^2 \|\Delta d\|_{L^2}^2
\end{aligned} \tag{3.21}$$

which implies

$$\begin{aligned}
\|\nabla u\|_{H^1} + \|\nabla P\|_{L^2} + \|\nabla \Delta d\|_{L^2} &\leq C(\mu^*)(\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla d_t\|_{L^2}) \\
&\quad + C(\mu^*)(\|\Delta d\|_{L^2}\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^3 + \|\nabla u\|_{L^2}^3) \\
&\quad + C\|\Delta d\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\Delta d\|_{L^2}.
\end{aligned} \tag{3.22}$$

Define functions g and G as

$$g(t) = \int (|\nabla u|^2 + |\Delta d|^2 - \nabla d \odot \nabla d : \nabla u) dx, \tag{3.23}$$

$$G(t) = \frac{1}{A} \int (\rho|u_t|^2 + |\nabla d_t|^2 + |\nabla \Delta d|^2) dx, \tag{3.24}$$

for any $t \in (0, T)$ and $A \geq 1$ is a positive constant. Then, it follows from (3.19) and (3.14) that

$$\begin{aligned}
g'(t) + \frac{1}{2}G(t) &\leq C(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)^2(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \\
&\quad + C\varepsilon_1^2(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2),
\end{aligned} \tag{3.25}$$

for any $t \in (0, T)$.

By assumption, it follows from the Hölder, Sobolev and Young inequalities that

$$\begin{aligned}
\left| \int \nabla d \odot \nabla d : \nabla u dx \right| &\leq \|\nabla u\|_{L^2} \|\nabla d\|_{L^3} \|\nabla d\|_{L^6} \leq C\|\nabla u\|_{L^2} \|\nabla d\|_{L^2}^{\frac{1}{2}} \|\Delta d\|_{L^2}^{\frac{3}{2}} \\
&\leq C\|\nabla u\|_{L^2} \|\Delta d\|_{L^2}^2 \leq C\varepsilon_1(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2),
\end{aligned} \tag{3.26}$$

for any $t \in (0, T)$, from which, choosing ε_1 suitably small, and recalling the definition of $g(t)$, we have

$$\frac{1}{4}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \leq g(t) \leq \|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2. \tag{3.27}$$

Thanks to (3.27) and (3.25), we then obtain

$$g'(t) + \frac{1}{2}G(t) \leq C(\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2)^2 g(t) + C\varepsilon_1^2(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2), \tag{3.28}$$

from which, by the Gronwall inequality and Lemma 3.1, and noticing that $E_0 \leq Cg(0)$ and $E_0 \leq \sup_{0 \leq s \leq T} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)$, we have

$$\begin{aligned}
g(t) + \frac{1}{2} \int_0^t G(s) ds &\leq \exp \left\{ \int_0^t (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)^2 ds \right\} \left(g(0) + C \int_0^t (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \right) \\
&\leq \exp \left\{ CE_0 \sup_{0 \leq s \leq T} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \right\} (g(0) + CE_0) \\
&\leq e^{C\varepsilon_1^2} (g(0) + CE_0) \leq Cg(0),
\end{aligned} \tag{3.29}$$

for any $t \in (0, T)$. This combined with (3.27), implies

$$\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \int_0^t (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2) dt \leq CE_0. \tag{3.30}$$

Furthermore, we multiplying (3.28) by t , it follows from (3.5) and Gronwall's inequality that

$$\begin{aligned} & \sup_{0 \leq t \leq T} t(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + \int_0^T t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 + \|\nabla \Delta d\|_{L^2}^2)dt \\ & \leq \exp \left\{ \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)dt \right\} \\ & \quad \cdot \int_0^T t(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)dt \leq CE_0. \end{aligned} \quad (3.31)$$

The proof of Lemma 3.2 is complete. \square

Lemma 3.3. *Let $\varepsilon_1 > 0$ be the same one fixed in (3.2). Assume that (3.2) and (3.1) hold. Then there exists a positive constant $\varepsilon_2 > 0$, depending only on Ω , $\underline{\mu}$, $\bar{\mu}$, $\bar{\rho}$ and μ^* , such that if*

$$\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2} \leq \varepsilon_2, \quad (3.32)$$

then

$$\sup_{0 \leq t \leq T} t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + \int_0^T t(\|\nabla u_t\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2)dt \leq CE_0, \quad (3.33)$$

$$\sup_{0 \leq t \leq T} t^2(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + \int_0^T t^2(\|\nabla u_t\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2)dt \leq CE_0. \quad (3.34)$$

Proof. Differentiating equation (1.1)₂ in t , multiplying the resulting equation by u_t and integrating over Ω , then it follows from integrating by parts and Young inequality that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \int |\nabla u_t|^2 dx \\ & = \int (\nabla d \odot \nabla d)_t : \nabla u_t + \int u \cdot \nabla \mu(\rho) \nabla u \cdot \nabla u_t dx - 2 \int \rho u \cdot \nabla u_t \cdot u_t dx \\ & \quad - \int \rho u_t \cdot \nabla u \cdot u_t dx - \int \rho u \cdot \nabla (u \cdot \nabla u \cdot u_t) dx \\ & \leq C \int |\nabla d| |\nabla d_t| |\nabla u_t| dx + \int \rho |u_t|^2 |\nabla u| dx + \int \rho |u| |\nabla u_t| |u_t| dx \\ & \quad + \int \rho |u| |\nabla u|^2 |u_t| dx + \int \rho |u|^2 |\nabla u| |\nabla u_t| dx + \int \rho |u|^2 |\nabla^2 u| |u_t| dx \\ & \quad + \int |u| |\nabla \mu(\rho)| |\nabla u| |\nabla u_t| dx \triangleq \sum_{i=1}^7 J_i. \end{aligned} \quad (3.35)$$

Using (3.1), Hölder's, interpolation, Sobolev, and Young inequality repeatedly, we get

$$\begin{aligned} J_1 & \leq C \|\nabla u_t\|_{L^2} \|\nabla d\|_{L^6} \|\nabla d_t\|_{L^3} \\ & \leq C \|\nabla u_t\|_{L^2} \|\nabla d_t\|_{L^2}^{\frac{1}{2}} \|\nabla^2 d_t\|_{L^2}^{\frac{1}{2}} \|\nabla^2 d\|_{L^2} \end{aligned}$$

$$\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + \delta \|\nabla^2 d_t\|_{L^2}^2 + C(\varepsilon, \delta) \|\nabla^2 d\|_{L^2}^4 \|\nabla d_t\|_{L^2}^2, \quad (3.36)$$

$$\begin{aligned} J_2 &\leq C \|\sqrt{\rho} u_t\|_{L^4}^2 \|\nabla u\|_{L^2} \\ &\leq C \|\sqrt{\rho} u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{3}{2}} \|\nabla u\|_{L^2} \\ &\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \|\nabla u\|_{L^2}^4 \|\sqrt{\rho} u_t\|_{L^2}^2, \end{aligned} \quad (3.37)$$

$$\begin{aligned} J_3 &\leq C \|\sqrt{\rho} u_t\|_{L^3} \|\nabla u_t\|_{L^2} \|u\|_{L^6} \\ &\leq C \|\sqrt{\rho} u_t\|_{L^2}^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_{L^6}^{\frac{1}{2}} \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2} \\ &\leq C \|\sqrt{\rho} u_t\|_{L^2}^{\frac{1}{2}} \|\nabla u_t\|_{L^2}^{\frac{3}{2}} \|\nabla u\|_{L^2} \\ &\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \|\nabla u\|_{L^2}^4 \|\sqrt{\rho} u_t\|_{L^2}^2, \end{aligned} \quad (3.38)$$

$$\begin{aligned} \sum_{i=4}^6 J_i &\leq C \|u_t\|_{L^6} \|u\|_{L^6} \|\nabla u\|_{L^3}^2 + C \|u\|_{L^6}^2 \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2} \\ &\quad + C \|\nabla^2 u\|_{L^2} \|u_t\|_{L^6} \|u\|_{L^6}^2 \\ &\leq C \|\nabla u_t\|_{L^2} \|\nabla u\|_{L^2}^2 \|\nabla u\|_{H^1} \\ &\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C(\varepsilon) \|\nabla u\|_{L^2}^4 \|\nabla u\|_{H^1}^2, \end{aligned} \quad (3.39)$$

$$\begin{aligned} J_7 &\leq C \|u\|_{L^\infty} \|\nabla \mu(\rho)\|_{L^p} \|\nabla u\|_{L^{\frac{2p}{p-2}}} \|\nabla u_t\|_{L^2} \\ &\leq C \|\nabla u\|_{L^2}^{\frac{1}{2}} \|\nabla u\|_{H^1}^{\frac{3}{2}} \|\nabla u_t\|_{L^2} \\ &\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2} \|\nabla u\|_{H^1}^3 \\ &\leq \varepsilon \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{H^1}^4. \end{aligned} \quad (3.40)$$

Differentiate equation (1.1)₄ with respect to t , then it has

$$d_{tt} - \Delta d_t = |\nabla d|^2 d_t - (u \cdot \nabla) d_t + 2(\nabla d : \nabla d_t) d - (u_t \cdot \nabla) d. \quad (3.41)$$

Squaring both sides of this equation, and integrating over Ω , then it follows from integration by parts and the Cauchy inequality that

$$\begin{aligned} &\frac{d}{dt} \int |\nabla d_t|^2 dx + \int (|d_{tt}|^2 + |\Delta d_t|^2) dx \\ &\leq C \int (|\nabla d|^4 |d_t|^2 + |u|^2 |\nabla d_t|^2 + |\nabla d|^2 |\nabla d_t|^2 + |\nabla d|^2 |u_t|^2) dx \\ &\leq C \|\nabla d\|_{L^6}^4 \|d_t\|_{L^6}^2 + C \|u\|_{L^6}^2 \|\nabla d_t\|_{L^2} \|\nabla d_t\|_{L^6} \\ &\quad + C \|u_t\|_{L^6}^2 \|\nabla d\|_{L^2} \|\nabla d\|_{L^6} + C \|\nabla d\|_{L^6}^2 \|\nabla d_t\|_{L^2} \|\nabla d_t\|_{L^6} \\ &\leq C \|\Delta d\|_{L^2}^2 \|\Delta d_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla d_t\|_{L^2} \|\Delta d_t\|_{L^2} \\ &\quad + C \|\nabla u_t\|_{L^2}^2 \|\Delta d\|_{L^2}^2 + C \|\Delta d\|_{L^2}^2 \|\nabla d_t\|_{L^2} \|\Delta d_t\|_{L^2} \\ &\leq \varepsilon_1 \|\nabla u_t\|_{L^2}^2 + \varepsilon_1 \|\Delta d_t\|_{L^2}^2 + \delta \|\Delta d_t\|_{L^2}^2 \\ &\quad + C (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)^2 \|\nabla d_t\|_{L^2}^2. \end{aligned} \quad (3.42)$$

Thanks to the above estimate, it follows from (3.35) that

$$\begin{aligned}
 & \frac{d}{dt} (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + \|\nabla u_t\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2 \\
 & \leq C\|\Delta d\|_{L^2}^4 \|\nabla d_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|\nabla u\|_{H^1}^2 \\
 & \quad + C(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)^2 \|\nabla d_t\|_{L^2}^2 + C\|\nabla u\|_{H^1}^4 \\
 & \leq C\|\Delta d\|_{L^2}^4 \|\nabla d_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|\sqrt{\rho}u_t\|_{L^2}^2 + C\|\nabla u\|_{L^2}^4 \|\nabla d_t\|_{L^2}^2 \\
 & \quad + C\|\nabla u\|_{L^2}^{10} + C\|\nabla u\|_{L^2}^4 \|\Delta d\|_{L^2}^6 + C(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)^2 \|\nabla d_t\|_{L^2}^2 \\
 & \quad + C\|\sqrt{\rho}u_t\|_{L^2}^4 + C\|\nabla d_t\|_{L^2}^4 + C\|\Delta d\|_{L^2}^{12} + C\|\nabla u\|_{L^2}^{12} \\
 & \quad + C\|\Delta d\|_{L^2}^4 \|\nabla u\|_{L^2}^8 + C\|\Delta d\|_{L^2}^8 + C\|\Delta d\|_{L^2}^4 \|\nabla u\|_{L^2}^4 \\
 & \leq C(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2)^2 (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + C(\|\sqrt{\rho}u_t\|_{L^2}^4 + \|\nabla d_t\|_{L^2}^4) \\
 & \quad + C(\|\nabla u\|_{L^2}^{10} + \|\Delta d\|_{L^2}^{10}) + C(\|\nabla u\|_{L^2}^{12} + \|\Delta d\|_{L^2}^{12}) + C(\|\nabla u\|_{L^2}^8 + \|\Delta d\|_{L^2}^8). \tag{3.43}
 \end{aligned}$$

Multiplying (3.43) by t and applying the Gronwall's inequality, we obtain from (3.4), (3.11), (3.32) and (3.12) that

$$\begin{aligned}
 & \sup_{0 \leq t \leq T} t(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + \int_0^T t(\|\nabla u_t\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2) \\
 & \leq \left[\int_0^T \left(t(\|\nabla u\|_{L^2}^{10} + \|\Delta d\|_{L^2}^{10}) + t(\|\nabla u\|_{L^2}^{12} + \|\Delta d\|_{L^2}^{12}) + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) dt \right] \\
 & \quad \cdot \exp \left\{ \int_0^T \left(\|\nabla u\|_{L^2}^4 + \|\Delta d\|_{L^2}^4 + \|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) dt \right\} \\
 & \leq \left[\sup_{0 \leq t \leq T} t(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \cdot \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^6 + \|\Delta d\|_{L^2}^6) \int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \right. \\
 & \quad + \sup_{0 \leq t \leq T} t(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \cdot \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^8 + \|\Delta d\|_{L^2}^8) \int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \\
 & \quad + \sup_{0 \leq t \leq T} t(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \cdot \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^4 + \|\Delta d\|_{L^2}^4) \int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \\
 & \quad \left. + \int_0^T (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) dt \right] \cdot \exp \left\{ \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \right\} \\
 & \quad \cdot \exp \left\{ \int_0^T (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) dt \right\} \leq CE_0. \tag{3.44}
 \end{aligned}$$

On the other hand, multiplying (3.43) by t^2 , and applying the Gronwall's inequality again. Similarly, we obtain from (3.4), (3.11), (3.32) and (3.12) that

$$\sup_{0 \leq t \leq T} t^2 (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) + \int_0^T t^2 (\|\nabla u_t\|_{L^2}^2 + \|d_{tt}\|_{L^2}^2 + \|\Delta d_t\|_{L^2}^2)$$

$$\begin{aligned}
&\leq \left[\int_0^T \left(t^2 (\|\nabla u\|_{L^2}^{10} + \|\Delta d\|_{L^2}^{10}) + t^2 (\|\nabla u\|_{L^2}^{12} + \|\Delta d\|_{L^2}^{12}) + t (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) \right) dt \right] \\
&\quad \cdot \exp \left\{ \int_0^T \left(\|\nabla u\|_{L^2}^4 + \|\Delta d\|_{L^2}^4 + \|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2 \right) dt \right\} \\
&\leq \left[\sup_{0 \leq t \leq T} t^2 (\|\nabla u\|_{L^2}^4 + \|\Delta d\|_{L^2}^4) \cdot \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^4 + \|\Delta d\|_{L^2}^4) \int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \right. \\
&\quad + \sup_{0 \leq t \leq T} t^2 (\|\nabla u\|_{L^2}^4 + \|\Delta d\|_{L^2}^4) \cdot \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^6 + \|\Delta d\|_{L^2}^6) \int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \\
&\quad + \sup_{0 \leq t \leq T} t^2 (\|\nabla u\|_{L^2}^4 + \|\Delta d\|_{L^2}^4) \cdot \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \\
&\quad \left. + \int_0^T t (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) dt \right] \cdot \exp \left\{ \sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) \int_0^T (\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) dt \right\} \\
&\quad \cdot \exp \left\{ \int_0^T (\|\sqrt{\rho} u_t\|_{L^2}^2 + \|\nabla d_t\|_{L^2}^2) dt \right\} \leq CE_0. \tag{3.45}
\end{aligned}$$

This finishes the proof of Lemma 3.3. \square

With the t -weighted estimates in Lemma 3.1–3.3 at hand, we now deal with $\|\nabla u\|_{L^\infty}$.

Lemma 3.4. *Let (3.32) be in force. Assume that (ρ, u, d) is a strong solution of the problem (1.1)–(1.3) on $\Omega \times (0, T)$, satisfying (3.2) and (3.1). Then for any $r \in (3, \min\{q, 6\})$,*

$$\int_0^T \|\nabla u\|_{L^\infty} dt \leq C_1 (\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2}). \tag{3.46}$$

Proof. Let $F \triangleq \rho u_t + \rho u \cdot \nabla u + \nabla d \cdot \Delta d$. By virtue of Lemma 2.2, one has for $r \in (3, \min\{q, 6\})$

$$\begin{aligned}
\|\nabla u\|_{W^{1,r}} &\leq C \|F\|_{L^r} (1 + \|\nabla \mu(\rho)\|_{L^p})^{\frac{pr}{2(p-r)}} \\
&\leq C(\mu^*) (\|\rho u_t\|_{L^r} + \|\rho u \cdot \nabla u\|_{L^r} + \|\nabla d\|_{L^r} \|\Delta d\|_{L^r}) \\
&\leq C(\mu^*) \|\sqrt{\rho} u_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}} + C(\mu^*) \|u\|_{L^6} \|\nabla u\|_{L^{\frac{6r}{6-r}}} + C(\mu^*) \|\nabla d\|_{L^\infty} \|\Delta d\|_{L^r} \\
&\leq C(\mu^*) \|\sqrt{\rho} u_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}} + C(\mu^*) \|\nabla u\|_{L^2}^{\frac{6(r-1)}{5r-6}} \|\nabla u\|_{W^{1,r}}^{\frac{4r-6}{5r-6}} + C(\mu^*) \|\Delta d\|_{L^2}^{\frac{1}{2}} \|\Delta d\|_{H^1}^{\frac{3}{2}}, \tag{3.47}
\end{aligned}$$

which deduces

$$\|\nabla u\|_{W^{1,r}} \leq C (\|\sqrt{\rho} u_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}} + \|\nabla u\|_{L^2}^{\frac{6(r-1)}{5r-6}} + \|\Delta d\|_{L^2}^{\frac{1}{2}} \|\Delta d\|_{H^1}^{\frac{3}{2}}). \tag{3.48}$$

On the one hand, if $0 < T \leq 1$, we derive from (3.4), (3.11), (3.33), Sobolev's and Hölder's inequality that

$$\begin{aligned}
& \int_0^T \|\nabla u\|_{L^\infty} dt \leq C \int_0^T \|\nabla u\|_{W^{1,r}} dt \\
& \leq C \int_0^T \|\sqrt{\rho} u_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla u_t\|_{L^2}^{\frac{3r-6}{2r}} dt + C \int_0^T \|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}} dt + C \int_0^T \|\Delta d\|_{L^2}^{\frac{1}{2}} \|\Delta d\|_{H^1}^{\frac{3}{2}} dt \\
& \leq C \left(\sup_{0 \leq t \leq T} t^{\frac{1}{2}} \|\sqrt{\rho} u_t\|_{L^2} \right)^{\frac{6-r}{2r}} \left(\int_0^T t \|\nabla u_t\|_{L^2}^2 dt \right)^{\frac{3(r-2)}{4r}} \left(\int_0^T t^{-\frac{2r}{r+6}} dt \right)^{\frac{r+6}{4r}} \\
& \quad + C \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^{\frac{4r-6}{r}} \cdot \int_0^T \|\nabla u\|_{L^2}^2 dt + C \left(\int_0^T \|\Delta d\|_{L^2}^2 dt \right)^{\frac{1}{4}} \left(\int_0^T \|\Delta d\|_{H^1}^2 dt \right)^{\frac{3}{4}} \\
& \leq C_1 (\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2}).
\end{aligned} \tag{3.49}$$

On the other hand, for $T > 1$, (3.4), (3.11), (3.34), Sobolev's and Hölder's inequality leads to

$$\begin{aligned}
& \int_1^T \|\nabla u\|_{L^\infty} dt \leq C \int_1^T \|\nabla u\|_{W^{1,r}} dt \\
& \leq C \left(\sup_{0 \leq t \leq T} t \|\sqrt{\rho} u_t\|_{L^2} \right)^{\frac{6-r}{2r}} \left(\int_0^T t^2 \|\nabla u_t\|_{L^2}^2 dt \right)^{\frac{3(r-2)}{4r}} \left(\int_0^T t^{-\frac{4r}{r+6}} dt \right)^{\frac{r+6}{4r}} \\
& \quad + C \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^{\frac{4r-6}{r}} \cdot \int_0^T \|\nabla u\|_{L^2}^2 dt + C \left(\int_0^T \|\Delta d\|_{L^2}^2 dt \right)^{\frac{1}{4}} \left(\int_0^T \|\Delta d\|_{H^1}^2 dt \right)^{\frac{3}{4}} \\
& \leq C_1 (\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2}).
\end{aligned} \tag{3.50}$$

Combining (3.49) and (3.50) lead to (3.46). The proof of Lemma 3.4 is completed. \square

In view of (3.46), one can prove that the norm of $\|\nabla \mu(\rho)\|_{L^p}$ with $3 < p < \infty$ is strictly less than $2\mu^*$, provided $\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2}$ is suitably small in some sense. This particularly finishes the proof of the *a priori* assumption (3.1).

Lemma 3.5. Assume that (ρ, u, d) is a strong solution of the problem (1.1)–(1.3) on $\Omega \times (0, T)$, satisfying (3.1) and (3.2). Then there exists a positive constant $\varepsilon_3 > 0$, depending only on Ω , $\underline{\mu}$, $\bar{\mu}$, $\bar{\rho}$ and μ^* , such that if

$$\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2} \leq \varepsilon_3, \tag{3.51}$$

then

$$\sup_{0 \leq t \leq T} \|\nabla \mu(\rho)(t)\|_{L^p} \leq \frac{3\mu^*}{2}, \tag{3.52}$$

and

$$\sup_{0 \leq t \leq T} \|\nabla \rho(t)\|_{L^p} \leq C \|\nabla \rho_0\|_{L^p}. \tag{3.53}$$

Proof. Consider the x_i -derivative of the equation for $\mu(\rho)$,

$$(\partial_i \mu(\rho))_t + (\partial_i u \cdot \nabla) \mu(\rho) + u \cdot \nabla \partial_i \mu(\rho) = 0. \quad (3.54)$$

It implies that for every $t \in [0, T]$

$$\begin{aligned} \|\nabla \mu(\rho)(t)\|_{L^q} &\leq \|\nabla \mu(\rho_0)\|_{L^q} \cdot \exp \left\{ \int_0^t \|\nabla u\|_{L^\infty} dt \right\} \\ &\leq \|\nabla \mu(\rho_0)\|_{L^q} \cdot \exp \{C_1(\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2})\} \\ &\leq \frac{3}{2} \mu^*, \end{aligned} \quad (3.55)$$

provided $\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2} \leq \varepsilon_3 = \min\{\varepsilon_2, C_1^{-1} \ln \frac{3}{2}\}$. The proof of (3.53) is similar. Therefore, the proof of Lemma 3.5 is completed. \square

4. Proof of Theorem 1.1

Note that the derivations of Lemmas 3.1–3.5 in Section 3 were based on the a priori assumptions of (3.1) and (3.2). Since (3.1) has been ensured by Lemma 3.5, to complete the proofs, it remains to show that the a priori assumption (3.2) actually holds, provided the initial velocity u_0 and unit-vector field d_0 is suitably small in some sense. This will be done in the following lemma.

Lemma 4.1. *Assume that (ρ, u, d, P) , satisfying (3.1), is a strong solution of the problem (1.1)–(1.3) on $\Omega \times (0, T)$. Then there exists a positive constant ε_4 , depending only on Ω , $\underline{\mu}$, $\bar{\mu}$, $\bar{\rho}$ and μ^* , such that*

$$\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^2} + \|\Delta d\|_{L^2}) \leq \varepsilon_1, \quad (4.1)$$

provided

$$\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2} \leq \varepsilon_4, \quad (4.2)$$

where the positive constant $\varepsilon_1 > 0$ is the same one as in (3.2).

Proof. Define

$$T^* = \max\{t \in (0, T] \mid \sup_{0 \leq s \leq t} (\|\nabla u\|_{L^2} + \|\Delta d\|_{L^2}) \leq \varepsilon_1\}. \quad (4.3)$$

By Lemma 3.2, we then obtain

$$\sup_{0 \leq s \leq t} (\|\nabla u\|_{L^2} + \|\nabla d\|_{L^2}) \leq C_2(\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2}) \leq C_2 \varepsilon_4 \leq \frac{\varepsilon_1}{2} < \varepsilon_1, \quad (4.4)$$

with $\varepsilon_4 = \min\{\varepsilon_2, \varepsilon_3, \frac{\varepsilon_1}{2C_2}\}$ for any $t \in (0, T^*)$.

We claim that $T^* = T$, and as result the conclusion is proved. Suppose, by contradiction that $T^* \in (0, T)$, then the above inequality implies that there is another time $T^{**} \in (T^*, T]$, such that $\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2} \leq \varepsilon_1$, for any $t \in (0, T^{**}]$, contradicting to the definition of T^* . \square

Proof of Theorem 1.1. In view of the local existence result (cf. Lemma 2.2), we know that there exists a $T_0 > 0$ such that the problem (1.1)–(1.3) has a unique strong solution (ρ, u, d, P) on $\Omega \times (0, T_0)$. So, to prove Theorem 1.1, it suffices to show that the local solution can be extended to be a global one. To do this, we assume from now that if (1.9) holds, this is

$$\|\nabla u_0\|_{L^2} + \|\Delta d_0\|_{L^2} \leq \varepsilon_0 \triangleq \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}.$$

Thank to (1.7), one has

$$\|\nabla \mu(\rho_0)\|_{L^q} \leq \mu^*, \quad \text{with } q > 3, \quad (4.5)$$

it thus follows from the continuity arguments that there exists a $T_1 \in (0, T_0]$ such that (3.1) holds for $T = T_1$. Set

$$\tilde{T} = \sup\{T | (\rho, u, d, P) \text{ is a strong solution of (1.1)–(1.3) on } \Omega \times [0, T] \text{ satisfying (3.1)}\},$$

and

$$\tilde{T}^* = \sup\{T | (\rho, u, d, P) \text{ is a strong solution of (1.1)–(1.3) on } \Omega \times [0, T]\}.$$

Clearly, $\tilde{T} > T_1 > 0$. Due to (1.9), it follows from (3.1) and Lemma 4.1 that (3.2) holds on $(0, \tilde{T})$. With (3.1) and (3.2) at hand, we deduce from Lemma 3.1–3.5 that the global estimates stated in (3.3)–(3.5), (3.11), (3.12), (3.33), (3.34), (3.46) and (3.52) hold on $(0, \tilde{T})$. Since the initial data (u_0, d_0) satisfies the smallness condition (1.9). In particular, (3.52) together with continuity argument, implies that (3.1) in fact holds on $(0, \tilde{T}^*)$, and so does (3.2). Thus,

$$\tilde{T} = \tilde{T}^*. \quad (4.6)$$

Next, we claim that

$$\tilde{T}^* = \infty. \quad (4.7)$$

Otherwise, $0 < \tilde{T}^* < \infty$. Then, by (4.6) one gets that (3.1) holds for $T = \tilde{T}^*$. Thus, it follows from Lemmas 3.1–3.3 and the local existence result (cf. Lemma 2.2) that there is a $\tilde{T}^{**} > \tilde{T}^*$ such that (ρ, u, d, P) can be extended to be a strong solution of (1.1)–(1.3) on $\Omega \times (0, \tilde{T}^{**})$. This contradicts the definition of \tilde{T}^* , and hence, (4.7) holds. This together with (4.6) finishes the proof of Theorem 1.1. \square

References

- [1] S.A. Antontsev, A.V. Kazhikov, V.N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, North-Holland, Amsterdam, 1990.
- [2] D. Chen, Z. Zhang, W. Zhao, Fujita–Kato theorem for the 3D inhomogeneous Navier–Stokes equations, *J. Differential Equations* 261 (2016) 738–761.
- [3] H.J. Cho, H. Kim, Strong solutions of the Navier–Stokes equations for non-homogeneous incompressible fluids, *Comm. Partial Differential Equations* 28 (2003) 1183–1201.
- [4] Y. Cho, H. Kim, Unique solvability for the density-dependent Navier–Stokes equations, *Nonlinear Anal.* 59 (2004) 465–489.
- [5] W. Craig, X.D. Huang, Y. Wang, Global well-posedness for the 3D inhomogeneous incompressible Navier–Stokes equations, *J. Math. Fluid Mech.* 15 (2013) 747–758.
- [6] R. Danchin, P.B. Mucha, Incompressible flows with piecewise constant density, *Arch. Ration. Mech. Anal.* 207 (3) (2013) 991–1023.
- [7] S. Ding, J. Huang, F. Xia, Global existence of strong solutions for incompressible hydrodynamic flow of liquid crystals with vacuum, *Filomat* 27 (7) (2013) 1247–1257.

- [8] S. Ding, H. Wen, Solutions of incompressible hydrodynamic flow of liquid crystals, *Nonlinear Anal. Real World Appl.* 12 (2011) 1510–1531.
- [9] J. Ericksen, Hydrostatic theory of liquid crystals, *Arch. Ration. Mech. Anal.* 9 (1962) 371–378.
- [10] L.C. Evans, *Partial Differential Equations*, Grad. Stud. Math., vol. 19, AMS, Providence, RI, 1997.
- [11] J. Fan, F. Li, G. Nakamura, Global strong solution to the 2D density-dependent liquid crystal flows with vacuum, *Nonlinear Anal.* 97 (2014) 185–190.
- [12] J. Gao, Q. Tao, Z. Yao, Strong solutions to the density-dependent incompressible nematic liquid crystal flows, *J. Differential Equations* 260 (2016) 3691–3748.
- [13] J. Huang, M. Paicu, P. Zhang, Global well-posedness of incompressible inhomogeneous fluid systems with bounded density or non-Lipschitz velocity, *Arch. Ration. Mech. Anal.* 209 (2013) 631–682.
- [14] X.D. Huang, Y. Wang, Global strong solution with vacuum to the two dimensional density dependent Navier–Stokes system, *SIAM J. Math. Anal.* 46 (2014) 1771–1788.
- [15] X.D. Huang, Y. Wang, Global strong solution of 3D inhomogeneous Navier–Stokes equations with density-dependent viscosity, *J. Differential Equations* 259 (2015) 1606–1627.
- [16] F. Jiang, Z. Tan, Global weak solution to the flow of liquid crystal system, *Math. Methods Appl. Sci.* 32 (2009) 2243–2266.
- [17] A.V. Kazhikov, Resolution of boundary value problems for nonhomogeneous viscous fluids, *Dokl. Akad. Nauk* 216 (1974) 1008–1010.
- [18] F. Leslie, Some constitutive equations for liquid crystals, *Arch. Ration. Mech. Anal.* 28 (1968) 265–283.
- [19] J. Li, Global strong and weak solutions to inhomogeneous nematic liquid crystal flow in two dimensions, *Nonlinear Anal.* 99 (2014) 80–94.
- [20] J. Li, Global strong solutions to the inhomogeneous incompressible nematic liquid crystal flow, *Methods Appl. Anal.* 22 (2015) 201–220.
- [21] X. Li, D. Wang, Global solution to the incompressible flow of liquid crystals, *J. Differential Equations* 252 (2012) 745–767.
- [22] X. Li, D. Wang, Global strong solution to the density-dependent incompressible flow of liquid crystals, *Trans. Amer. Math. Soc.* 367 (2015) 2301–2338.
- [23] Z. Liang, Local strong solution and blow-up criterion for the 2D nonhomogeneous incompressible fluids, *J. Differential Equations* 258 (2015) 2633–2654.
- [24] P.L. Lions, *Mathematical Topics in Fluid Mechanics, Vol. I: Incompressible Models*, Oxford University Press, Oxford, 1996.
- [25] X. Liu, Z. Zhang, Existence of the flow of liquid crystal system, *Chin. Ann. Math.* 30A (2009) 1–20.
- [26] B. Lv, X. Shi, X. Zhong, Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent Navier–Stokes equations with vacuum, *arXiv:1506.03143*.
- [27] M. Paicu, P. Zhang, Z.F. Zhang, Global unique solvability of inhomogeneous Navier–Stokes equations with bounded density, *Comm. Partial Differential Equations* 38 (2013) 1208–1234.
- [28] J. Xu, Z. Tan, Global existence of the finite energy weak solution to a nematic liquid crystal model, *Math. Methods Appl. Sci.* 34 (2011) 929–938.
- [29] J. Zhang, Global well-posedness for the incompressible Navier–Stokes equations with density dependent viscosity coefficient, *J. Differential Equations* 259 (2015) 1722–1742.