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Schauder estimates for stochastic transport-diffusion equations with Lévy processes

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ABSTRACT

In this study, we consider a transport-diffusion equation with Lévy noise and Hölder continuous coefficients. By using the heat kernel estimates, we derive the Schauder estimates for the mild solutions. Moreover, when the transport term vanishes and $p = 2$, we show that the Hölder index in the space variable is optimal.

1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a filtered complete probability space with the right continuous filtration $\mathcal{F}_t$. Denote $\{W_t\}_{t\geqslant 0}$ as a scalar Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. Let E be a ball $B_c(0) - \{0\}$ of radius c without the center. Moreover, $\tilde{N}$ is a time-homogeneous compensated Poisson random measure defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ (defined in Definition 2.2), which is independent of $\{W_t\}_{t\geqslant 0}$ and with an intensity measure $\nu \times \lambda$ on $E \times \mathbb{R}_+$.

In the present study, we consider the existence, uniqueness, and regularity of the mild solution for the following stochastic transport-diffusion equation:

$$du(t,x) - b(t,x) \cdot \nabla u(t,x)dt - \frac{1}{2}\Delta u(t,x)dt$$

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$$= h(t,x)dt + f(t,x)dW_t + \int_E g(t,x,v)\tilde{N}(dt,dv), \quad t > 0,\ x \in \mathbb{R}^d. \tag{1.1}$$

When the Lévy noise part is absent ($g = 0$), this stochastic partial differential equation (SPDE) (1.1) has been studied widely. When $g = h = 0$, $b = 0$, and the initial datum vanishes, (1.1) becomes:

$$du(t,x) - \frac{1}{2}\Delta u(t,x)dt = f(t,x)dW_t, \quad t > 0,\ x \in \mathbb{R}^d, \quad u(t,x)|_{t=0} = 0. \tag{1.2}$$

For example, Krylov [26] obtained the following estimate for the solution of the Cauchy problem (1.2) for $p \geqslant 2$:

$$\mathbb{E}\|\nabla u\|^p_{L^p((0,T)\times\mathbb{R}^d)} \leqslant C(d,p)\mathbb{E}\|f\|^p_{L^p((0,T)\times\mathbb{R}^d)}, \tag{1.3}$$

using a variant of the Littlewood–Paley inequality. This result was extended by Neerven, Veraar, and Weis [36] to the case when the Laplace operator Δ is replaced by a linear operator A, which admits a bounded H^∞-calculus for an angle less than $\pi/2$. For further details of the L^p-theory for linear SPDEs, please refer to previous studies [20,25,27,28], and for the L^p-theory for nonlinear SPDEs, refer to other studies [9,8,22,39]. For $p = \infty$, an analogue and interesting estimate of (1.3) for (1.2) was also derived by Denis, Matoussi, and Stoica [10]. By using Moser's iteration scheme developed by Aronson and Serrin, they derived space–time L^∞ estimates for certain nonlinear SPDEs. Moreover, after introducing a notion of stochastic BMO spaces, Kim [21] obtained a BMO estimate for ∇u, which is controlled by $\|f\|_{L^\infty}$. For further details regarding this topic, please refer to previous studies [29,32].

Some Schauder estimates also exist for solutions of (1.1) when Lévy noise is absent ($g = 0$). The time and space $\mathcal{C}^\alpha$ estimates were discussed by Kuksin, Nadirashvili, and Piatnitski [30] when $f(t,\cdot)$ belongs to L^p with a sufficiently large p (or $p = \infty$) and $\mathbb{R}^d$ is replaced by a bounded domain Q (with a smooth boundary). This result was further supported by [19], for general Hölder estimates of the generalized solutions with $L_q(L_p)$ coefficients. Subsequently, Du and Liu [12] extended the result on bounded domains to $\mathbb{R}^d$ and constructed the $C^{2+\alpha}$-theory. The $\mathcal{C}^\alpha$ estimates were also derived by Hsu, Wang and Wang [17] when f and h are dependent on u (nonlinear SPDE case). They used a stochastic De Giorgi iteration technique and proved that the solution is almost surely $\mathcal{C}^\alpha$ in both space and time. Some regularity results have been reported for when u takes values in a Hilbert space [8,35].

When the Lévy noise part is present ($g \neq 0$), Kotelenez [24], and Albeverio, Wu, and Zhang [2] studied the L^2-theory for the SPDE (1.1). Moreover, an L^p theory was developed by Marinelli, Prévôt, and Röckner [33].

However, to the best of our knowledge, very few studies have considered the Schauder estimates for (1.1). Thus, in the present study, we address this deficiency and derive the Schauder estimates for the mild solutions.

The remainder of this paper is organized as follows. After introducing some notions and stating the main result in Section 2, we present several useful lemmas in Section 3. In Section 4, we prove the main result. Finally, we conclude with some remarks regarding the regularity of the mild solutions to problem (1.1) in Section 5.

Notations Denote $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ as the ball centered at x with radius r. $a \wedge b = \min\{a,b\}$, $a \vee b = \max\{a,b\}$. $\mathbb{R}_+ = \{r \in \mathbb{R},\ r \geqslant 0\}$. The letter C denotes a positive constant with values that may change in different places. The Lebesgue measure is denoted by λ, or by dt if there is no confusion. $\mathbb{N}$ is the set of natural numbers. Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\overline{\mathbb{N}} := \mathbb{N}_0 \cup \{\infty\}$. $\mathcal{B}(E)$ is the Borel σ-algebra on E. We denote $M_+(E)$ as the family of all σ-finite positive measures on E and $\mathcal{M}_+(E)$ as the σ-field on $M_+(E)$ generated by the functions $i_B : M_+(E) \ni \mu \to \mu(B) \in \mathbb{R}_+, B \in \mathcal{B}(E)$.

2. Main result

Let E and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be as stated in the previous section. First, we recall the notion of a Poisson random measure.

Definition 2.1. A time-homogeneous Poisson random measure N on $(E, \mathcal{B}(E))$ over the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ with an intensity measure $\nu \times \lambda$ is a measurable function $N : (\Omega, \mathcal{F}) \to (M_+(E \times \mathbb{R}_+), \mathcal{M}_+(E \times \mathbb{R}_+))$ such that:

(i) For each $B \times I \in \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}_+)$, if $\nu(B) < \infty$, then $N(B \times I)$ is a Poisson random variable with parameter $\nu(B)\lambda(I)$;

(ii) N is independently scattered, i.e., if the sets $E_j \times I_j \in \mathcal{B}(E) \times \mathcal{B}(\mathbb{R}_+)$, $j = 1, \dots, n$ are pairwise disjoint, then the random variables $N(B_j \times I_j)$, $j = 1, \dots, n$ are mutually independent; and

(iii) For each $U \in \mathcal{B}(E)$, the $\overline{\mathbb{N}}$-valued process $\{N((0,t], U)\}_{t>0}$ is $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted and its increments are independent of the past.

Remark 2.1. In this definition, ν is called a Lévy measure and it satisfies the following condition:

$$\int_E 1 \wedge |v|^2 \nu(dv) < \infty.$$

Definition 2.2. Let N be a homogeneous Poisson random measure on $(E, \mathcal{B}(E))$ over the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$. The $\mathbb{R}$-valued process $\{\tilde{N}((0,t], A)\}_{t>0}$ defined by:

$$\tilde{N}((0,t], A) = N((0,t], A) - \nu(A)t, \quad t > 0,\ A \in \mathcal{B}(E),$$

is called a compensator Poisson random measure. In addition, $\{\tilde{N}((0,t], A)\}_{t>0}$ is a martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$.

In the present study, we focus on Schauder estimates of the mild solutions for (1.1). To formulate the Cauchy problem, we assume that the initial value vanishes. The mild solution is defined as follows.

Definition 2.3. Let u be a $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{F}$ measurable function. We refer to u as a mild solution of (1.1) when the initial data vanish if the following properties hold:

(1) u is $\mathcal{F}_t$-adapted;

(2) $\{u(t,x,\cdot)\}_{t\geqslant 0}$ comprises a family of $L^2(\Omega, \mathcal{F}, \mathbb{P})$-valued random variables, which are right continuous and have left limits in the variable $t \in [0,\infty)$, i.e.:

$$u(t-, x, \cdot) = L^2(\Omega) - \lim_{s\uparrow t} u(s,x,\cdot),\ t \in [0,\infty); \tag{2.1}$$

(3) $u \in L^\infty_{loc}([0,\infty); W^{1,\infty}(\mathbb{R}^d; L^2(\Omega)))$;

(4) for every $t > 0$, the following equation holds almost surely:

$$\begin{aligned} u(t,x) = & \int_0^t P_{t-r}(b(r,\cdot)\cdot\nabla u(r,\cdot))(x)dr + \int_0^t P_{t-r}h(r,\cdot)(x)dr + \int_0^t P_{t-r}f(r,\cdot)(x)dW_r \\ & + \int_{(0,t]}\int_E P_{t-r}g(r,\cdot,v)(x)\tilde{N}(dr,dv), \end{aligned} \tag{2.2}$$

where the stochastic integral in (2.2) is interpreted in Itô's and P_t denotes the forward heat semigroup, i.e.:

$$P_t\varphi(x) = \frac{1}{(2\pi t)^{d/2}} \int\limits_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{2t}} \varphi(y)dy, \quad \varphi \in L^\infty(\mathbb{R}^d). \tag{2.3}$$

Remark 2.2. The definition is inspired by Marinelli, Prévôt, and Röckner [33, Definition 2.1], and the definition given by [2].

Before stating our main result, we recall some notations for function spaces. For $T > 0$, $\alpha > 0$ and $p \geqslant 2$, and we define $L^\infty([0,T]; \mathcal{C}_b^\alpha(\mathbb{R}^d))$ as the set of all $\mathcal{C}_b^\alpha(\mathbb{R}^d)$-valued essentially bounded functions u such that:

$$\|u\|_{T,\infty,\alpha} := \operatorname*{esssup}_{0\leqslant t\leqslant T} \max_{x\in\mathbb{R}^d} |u(t,x)| + \operatorname*{esssup}_{0\leqslant t\leqslant T} \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|u(t,x)-u(t,y)|}{|x-y|^\alpha} < \infty.$$

When $\alpha = 0$, $\|u\|_{T,\infty,0}$ is written as $\|u\|_{T,\infty}$ for short and $\|u\|_\infty := \max_{x\in\mathbb{R}^d} |u(x)|$. Correspondingly, $L^\infty([0,T]; \mathcal{C}_b^{1,\alpha}(\mathbb{R}^d))$ is the set of all functions in $L^\infty([0,T]; \mathcal{C}_b^\alpha(\mathbb{R}^d))$ such that:

$$\begin{aligned}\|u\|_{T,\infty,1+\alpha} := {} & \operatorname*{esssup}_{0\leqslant t\leqslant T} \max_{x\in\mathbb{R}^d} |u(t,x)| + \operatorname*{esssup}_{0\leqslant t\leqslant T} \max_{x\in\mathbb{R}^d} |Du(t,x)| \\ & + \operatorname*{esssup}_{0\leqslant t\leqslant T} \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|Du(t,x)-Du(t,y)|}{|x-y|^\alpha} < \infty.\end{aligned}$$

Similarly, we can define the spaces $L^\infty([0,T]; \mathcal{C}_b^\alpha(\mathbb{R}^d; L^p(\Omega)))$ and $L^\infty([0,T]; L^p(E,\nu; \mathcal{C}_b^\alpha(\mathbb{R}^d)))$. For $h \in L^\infty([0,T]; \mathcal{C}_b^\alpha(\mathbb{R}^d; L^p(\Omega)))$ and $g \in L^\infty([0,T]; L^p(E,\nu; \mathcal{C}_b^\alpha(\mathbb{R}^d)))$, the norms are given by:

$$\|h\|_{T,\infty,\alpha,p} := \operatorname*{esssup}_{0\leqslant t\leqslant T} \max_{x\in\mathbb{R}^d} \|h(t,x)\|_{L^p(\Omega)} + \operatorname*{esssup}_{0\leqslant t\leqslant T} \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{\|h(t,x)-h(t,y)\|_{L^p(\Omega)}}{|x-y|^\alpha} < \infty$$

and

$$\begin{aligned}\|g\|_{T,\infty,p,E,\alpha} := {} & \operatorname*{esssup}_{0\leqslant t\leqslant T} \| \max_{x\in\mathbb{R}^d} |g(t,x,\cdot)| \|_{L^p(E,\nu)} \\ & + \operatorname*{esssup}_{0\leqslant t\leqslant T} \Big\| \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|g(t,x,\cdot)-g(t,y,\cdot)|}{|x-y|^\alpha} \Big\|_{L^p(E,\nu)} < \infty,\end{aligned}$$

respectively.

Our main result is as follows.

Theorem 2.1. *Let b, h, f, and g be measurable functions. We consider the stochastic transport-diffusion equation (1.1) with zero initial data. For $\alpha > 0$, $p > 2$, we assume that:*

$$0 < \alpha + \frac{2}{p} - 1 =: \gamma, \tag{2.4}$$

and

$$f \in L^\infty_{loc}([0,\infty); \mathcal{C}_b^\alpha(\mathbb{R}^d)),\ \ g \in L^\infty_{loc}([0,\infty); L^{p+}(E,\nu; \mathcal{C}_b^\alpha(\mathbb{R}^d)))\ g(t,x,\cdot) \text{ vanishes near } 0. \tag{2.5}$$

In addition, we assume that a real number $0 < \beta < \gamma$ exists such that:

$$b \in L^\infty_{loc}([0,\infty); \mathcal{C}_b^\beta(\mathbb{R}^d; \mathbb{R}^d)),\ \ h \in L^\infty_{loc}([0,\infty); \mathcal{C}_b^\beta(\mathbb{R}^d; L^p(\Omega))). \tag{2.6}$$

Then, a unique mild solution u exists for the equation (1.1). Moreover, u is in the class of $L^\infty_{loc}([0,\infty); \mathcal{C}_b^{1+\gamma-}(\mathbb{R}^d; L^p(\Omega)))$ and for every $t > 0$, $C(p,t,\|b\|_{t,\infty,\beta}) > 0$ (independent of u, h, f, and g) exists such that:

$$\|u\|_{t,\infty,1+\gamma-,p} \leqslant C(p,t,\|b\|_{t,\infty,\beta})\Big[\|h\|_{t,\infty,\beta,p} + \|f\|_{t,\infty,\alpha} + \|g\|_{t,\infty,p+,E,\alpha}\Big], \tag{2.7}$$

where

$$\mathcal{C}_b^{1+\gamma-}(\mathbb{R}^d) = \lim_{\varepsilon\to 0+} \mathcal{C}_b^{1+\gamma-\epsilon}(\mathbb{R}^d) = \cap_{0<r<\gamma}\mathcal{C}_b^{1+r}(\mathbb{R}^d), \quad L^{p+}(E,\nu) = \lim_{\epsilon\to 0+} L^{p+\epsilon}(E,\nu).$$

Remark 2.3. (i) Let k be a measurable function on (E,ν) and k vanishes near 0. For any $1 \leqslant r_1 \leqslant r_2$, if $k \in L^{r_2}(E,\nu)$, then $k \in L^{r_1}(E,\nu)$ and $\|k\|_{r_1,E} \leqslant C\|k\|_{r_2,E}$. We note that $g(t,x,\cdot)$ vanishes near 0, $g(t,x,\cdot) \in L^{p+}(E,\nu)$, and we have $g(t,x,\cdot) \in \cup_{r>p} L^r(E,\nu)$, which implies that a positive real number $\epsilon > 0$ exists such that $g(t,x,\cdot) \in L^{p+\epsilon}(E,\nu)$. Therefore, (2.7) can be understood as for every sufficiently small $\epsilon_1 > 0$, a sufficiently small positive real number ϵ_2 ($\epsilon_2 \leqslant \epsilon$) exists, and for every $t > 0$, $C(p,t,\|b\|_{t,\infty,\beta}) > 0$ (independent of u, h, f, and g) exists such that:

$$\|u\|_{t,\infty,1+\gamma-\epsilon_1,p} \leqslant C(p,t,\|b\|_{t,\infty,\beta})\Big[\|h\|_{t,\infty,\beta,p} + \|f\|_{t,\infty,\alpha} + \|g\|_{t,\infty,p+\epsilon_2,E,\alpha}\Big]. \tag{2.8}$$

(ii) From the proof, when the vector field b vanishes, we can also assert that for every $p \geqslant 2$ and $g \in L^\infty_{loc}([0,\infty); L^p(E,\nu;\mathcal{C}_b^\alpha(\mathbb{R}^d)))$ with $g(t,x,\cdot) = 0$ near 0, there is a unique mild solution u to (1.1). Moreover, $u \in L^\infty_{loc}([0,\infty); \mathcal{C}_b^{1+\gamma}(\mathbb{R}^d; L^p(\Omega)))$ and for every $t > 0$, $C > 0$ exists such that:

$$\|u\|_{t,\infty,1+\gamma,p} \leqslant C(p,t)\Big[\|h\|_{t,\infty,\beta,p} + \|f\|_{t,\infty,\alpha} + \|g\|_{t,\infty,p,E,\alpha}\Big].$$

(iii) The assumption that g vanishes near 0 is not necessary if $g \in L^\infty_{loc}([0,\infty); L^2 \cap L^{p+}(E,\nu;\mathcal{C}_b^\alpha(\mathbb{R}^d)))$. There are no suitable techniques for obtaining the solution except for some unnecessary and tedious calculations, and thus for simplicity, we focus on g that vanishes near 0.

3. Useful lemmas

Next, we present several lemmas that are needed for the proof of the main theorem.

Lemma 3.1. *(Minkowski inequality [38]) Assume that $(S_1,\mathcal{F}_1,\mu_1)$ and $(S_2,\mathcal{F}_2,\mu_2)$ are two measure spaces and that $G : S_1 \times S_2 \to \mathbb{R}$ is measurable. For given real numbers $1 \leqslant p_1 \leqslant p_2$, we also assume that $G \in L^{p_1}(S_1; L^{p_2}(S_2))$. Then, $G \in L^{p_2}(S_2; L^{p_1}(S_1))$ and:*

$$\Big[\int_{S_2}\Big(\int_{S_1}|G(x,y)|^{p_1}\mu_1(dx)\Big)^{\frac{p_2}{p_1}}\mu_2(dy)\Big]^{\frac{1}{p_2}} \leqslant \Big[\int_{S_1}\Big(\int_{S_2}|G(x,y)|^{p_2}\mu_2(dy)\Big)^{\frac{p_1}{p_2}}\mu_1(dx)\Big]^{\frac{1}{p_1}}. \tag{3.1}$$

The next lemmas have important roles in the estimation of the stochastic integrals.

Lemma 3.2. *(Burkholder's inequality [3, Theorem 4.4.21]) Let F be an $\{\mathcal{F}_t\}_{t\geqslant 0}$ adapted stochastic process. Suppose that $\{M_t\}_{t\geqslant 0}$ is a Brownian type integral of the form:*

$$M_t = \int_0^t F(r)dW_r,$$

for which $F \in L^p(\Omega; L^2_{loc}([0,\infty)))$. Then, for any $p \geqslant 2$, a positive constant $C(p) > 0$ exists such that for each $t \geqslant 0$:

$$\mathbb{E}[|M_t|^p] \leqslant C(p)\mathbb{E}\Big[\int_0^t |F(r)|^2 dr\Big]^{\frac{p}{2}}.$$

Corollary 3.1. *Let F be a $\mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^d)$-measurable function. Suppose that $\{M_t(x)\}_{t\geqslant 0}$ is a Brownian type integral of the form:*

$$M_t(x) = \int_0^t F(t,r,x)dW_r,$$

for which:

$$\int_0^t |F(t,r,x)|^2 dr < \infty, \quad \text{for almost everywhere } \; x \in \mathbb{R}^d. \tag{3.2}$$

Then, for any $p \geqslant 2$, a positive constant $C(p) > 0$ that is independent of x exists such that for each $t \geqslant 0$:

$$\mathbb{E}[|M_t(x)|^p] \leqslant C(p)\Big[\int_0^t |F(t,r,x)|^2 dr\Big]^{\frac{p}{2}}. \tag{3.3}$$

Proof. First, we assume that F has the following form:

$$F(t,r,x) = \sum_{j=1}^m F_j(t,x)1_{(t_{j-1},t_j]}(r), \tag{3.4}$$

where $m \in \mathbb{N}$, F_j are $(\mathbb{R}_+ \times \mathbb{R}^d; \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^d))$-measurable, and $0 = t_0 < t_1 < t_2 <...< t_m = t$.

Using Lemma 3.2, for $p = 2$, we obtain:

$$\mathbb{E}|M_t(x)|^2 = \mathbb{E}|\sum_{j=1}^m (W_{t_j} - W_{t_{j-1}})F_j(t,x)|^2 = \sum_{j=1}^m |F_j(t,x)|^2(t_j - t_{j-1}) = \int_0^t |F(t,r,x)|^2 dr. \tag{3.5}$$

For $p = 4$, we also have:

$$\begin{aligned}
&\mathbb{E}|M_t(x)|^4 \\
&= \mathbb{E}|\sum_{j=1}^m (W_{t_j} - W_{t_{j-1}})F_j(t,x)|^4 \\
&= \sum_{j=1}^m \mathbb{E}|W_{t_j} - W_{t_{j-1}}|^4 |F_j(t,x)|^4 + 6\sum_{i\neq j} \mathbb{E}|W_{t_i} - W_{t_{i-1}}|^2 \mathbb{E}|W_{t_j} - W_{t_{j-1}}|^2 |F_i(t,x)|^2 |F_j(t,x)|^2 \\
&= 6\Big[\sum_{j=1}^m |t_j - t_{j-1}|^2 |F_j(t,x)|^4 + \sum_{i\neq j}(t_i - t_{i-1})(t_j - t_{j-1})|F_i(t,x)|^2 |F_j(t,x)|^2\Big] \\
&\leqslant 6\Big[\sum_{j=1}^m |t_j - t_{j-1}|\,|F_j(t,x)|^2\Big]^2 \\
&= 6\Big[\int_0^t |F(t,r,x)|^2 dr\Big]^2.
\end{aligned} \tag{3.6}$$

In addition, we have the L^p-interpolating formulation:

$$F \in L^{p_1} \cap L^{p_3} \implies \|F\|_{L^{p_2}} \leqslant \|F\|_{L^{p_1}}^{\frac{(p_3-p_2)p_1}{(p_3-p_1)p_2}} \|F\|_{L^{p_3}}^{\frac{(p_2-p_1)p_3}{(p_3-p_1)p_2}}, \quad \forall \ \ p_1 \leqslant p_2 \leqslant p_3. \tag{3.7}$$

By combining (3.5), (3.6), and (3.7) for $p \in (2,4)$, we conclude that a $C(p) > 0$ that is independent of m exists such that:

$$\mathbb{E}|M_t(x)|^p \leqslant C(p)\Big[\int_0^t |F(t,r,x)|^2 dr\Big]^{\frac{p}{2}}. \tag{3.8}$$

We observe that the functions that satisfy the condition (3.2) can be approximated by the step functions of the form (3.4), and for $p \in [2,4]$, (3.2) holds for step functions, so we have completed the proof for $p \in [2,4]$.

Analogously, we can prove that (3.3) holds for every even number and every step function of the form (3.4). Given (3.7), we can derive an inequality of (3.8) for every $p > 4$. Then, we complete the proof by an approximating argument.

Remark 3.1. When $F(t,r,x) = F(t-r,x) = e^{(t-r)A}f(r,\cdot)(x)$ (where A is the generator of a strongly continuous semigroup), we obtain a Burkholder type inequality for a stochastic convolution. This estimate was considered by Kotelenez [23] for square integral martingales where the stochastic convolution takes values in a Hilbert space. The Hilbert space where the Burkholder inequality holds is then generalized to 2-uniformly smooth Banach spaces, and thus in the Lebesgue spaces in particular, $W^{k,q}(\mathbb{R}^d)$ $(2 \leqslant q < \infty)$, which follows from [37, Proposition 2.4], [31, Lemma 1.1]. Other regularities and related problems for stochastic convolution taking values in 2-uniformly smooth Banach spaces were described in previous studies (see [4,6,7,15,34,36] and the references cited therein). $L^\infty(\mathbb{R}^d)$ is not a 2-uniformly smooth Banach space, so the following inequality is generally not true:

$$\mathbb{E}\Big[\Big\|\int_0^t e^{(t-r)A}f(r,\cdot)dW_r\Big\|_{L^\infty(\mathbb{R}^d)}^p\Big] \leqslant C(p)\Big[\int_0^t \Big\|\int_0^t e^{(t-r)A}f(r,\cdot)\Big\|_{L^\infty(\mathbb{R}^d)}^2 dr\Big]^{\frac{p}{2}}. \tag{3.9}$$

However, instead of (3.9), as a consequence of (3.3), we can obtain:

$$\Big\|\mathbb{E}\Big|\int_0^t e^{(t-r)A}f(r,\cdot)dW_r\Big|^p\Big\|_{L^\infty(\mathbb{R}^d)} \leqslant C(p)\Big\|\Big[\int_0^t |e^{(t-r)A}f(r,\cdot)|^2 dr\Big]^{\frac{p}{2}}\Big\|_{L^\infty(\mathbb{R}^d)}. \tag{3.10}$$

Lemma 3.3. *(Kunita's first inequality [3, Theorem 4.4.23]) Let $E = B_c(0) - \{0\}$ $(0 < c \in \mathbb{R})$. Suppose that $H \in L^p(\Omega; L^2_{loc}([0,\infty); L^2(E,\nu)) \cap L^p_{loc}([0,\infty); L^p(E,\nu)))$ is an $\{\mathcal{F}_t\}_{t\geqslant 0}$ adapted stochastic process and:*

$$I_t = \int_{(0,t]}\int_E H(r,v)\tilde{N}(dr,dv).$$

Then, for every $p \geqslant 2$ and $t \geqslant 0$, a positive constant $C(p) > 0$ exists such that:

$$\mathbb{E}[|I_t|^p] \leqslant C(p)\Big\{\mathbb{E}\Big[\int_0^t\int_E |H(r,v)|^2\nu(dv)dr\Big]^{\frac{p}{2}} + \mathbb{E}\int_0^t\int_E |H(r,v)|^p\nu(dv)dr\Big\}.$$

From the lemma given above, by combining with a similar manipulation of Corollary 3.1, we derive the following

Corollary 3.2. *Let H be a $\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(E)$-measurable function. Suppose that $\{I_t(x)\}_{t\geqslant 0}$ is a Poisson type integral of the form:*

$$I_t(x) = \int\limits_{(0,t]}\int\limits_E H(t,r,x,v)\tilde{N}(dr,dv), \tag{3.11}$$

for which $H(t,r,x,\cdot)$ vanishes near 0 and:

$$\int\limits_0^t\int\limits_E |H(t,r,x,v)|^p\nu(dv)dr < \infty, \text{ for almost everywhere } x\in\mathbb{R}^d.$$

Then, for any $p\geqslant 2$, a positive constant $C(p)>0$ that is independent of x exists such that for each $t\geqslant 0$:

$$\mathbb{E}[|I_t(x)|^p] \leqslant C(p)\int\limits_0^t\int\limits_E |H(t,r,x,v)|^p\nu(dv)dr. \tag{3.12}$$

Proof. First, suppose that H has the following form:

$$H(t,r,x,v) = \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} H_{i,j}(t,x)1_{(t_{i-1},t_i]}(r)1_{E_j}(v), \tag{3.13}$$

where $m_1, m_2\in\mathbb{N}$, $H_{i,j}$ are $(\mathbb{R}_+\times\mathbb{R}^d;\mathcal{B}(\mathbb{R}_+)\times\mathcal{B}(\mathbb{R}^d))$-measurable, $0=t_0<t_1<t_2<\ldots<t_{m_1}=t$, $E_j\in\mathcal{B}(E)$, and $E_{j_1}\cap E_{j_2}=\varnothing$ $(j_1\neq j_2)$.

Using Lemma 3.3 and the property of an independently scattered Poisson random measure (see Definition 2.1 (ii)), after combining with an argument analogous to Corollary 3.1, for $p=2$, we have:

$$\mathbb{E}[|I_t(x)|^2] = \int\limits_0^t\int\limits_E |H(t,r,x,v)|^2\nu(dv)dr. \tag{3.14}$$

For $p=4$, we also have:

$$\mathbb{E}[|I_t(x)|^4] \leqslant C\Big\{\Big[\int\limits_0^t\int\limits_E |H(t,r,x,v)|^2\nu(dv)dr\Big]^2 + \int\limits_0^t\int\limits_E |H(t,r,x,v)|^4\nu(dv)dr\Big\}. \tag{3.15}$$

$H(t,r,x,\cdot)$ vanishes near 0, so by employing Hölder's inequality, from (3.15), we have:

$$\mathbb{E}[|I_t(x)|^4] \leqslant C\int\limits_0^t\int\limits_E |H(t,r,x,v)|^4\nu(dv)dr. \tag{3.16}$$

From (3.14) and (3.16), we can conclude that for every $t>0$, the linear operator given by (3.11) is bounded from $L^2([0,t]\times E)$ to $L^2(\Omega)$ and also bounded from $L^4([0,t]\times E)$ to $L^4(\Omega)$. Then, the Marcinkiewicz interpolation theorem ([1, Theorem 2.58]) is applied and for every $p\in(2,4)$, (3.12) holds for step functions, and thus we have completed the proof for $p\in[2,4]$ based on an approximating argument.

The remaining part is similar to the proof of Corollary 3.1. Thus, the proof is complete.

Remark 3.2. (i) When $H(t,r,x,v)$ is replaced by $U(t,r)h(r-)$ (U is an evolution operator), the estimate was discussed by Kotelenez [24] initially for a square integral martingale where the stochastic convolution takes values in a Hilbert space. The estimate was then strengthened by Ichikawa [18], and Hamedani and Zangeneh [13]. Some other extensions were given by [5,11,14,16], where these results considered the stochastic evolution by taking values in Banach spaces of martingale type $1 < p < \infty$. As noted in [14, Remark 2.11], $L^\infty(\mathbb{R}^d)$ is not a Banach space of martingale type p for any $p > 1$, so the estimate of (3.9) for Poisson random measure is generally not true. However, as a consequence of (3.12), if $H(t,r,x,v) = e^{(t-r)A}h(r,\cdot,v)(x)$ (A is the generator of a strongly continuous semigroup), then we can obtain:

$$\begin{aligned}&\Big\|\mathbb{E}\Big|\int_0^t\int_E e^{(t-r)A}h(r,\cdot,v)\tilde{N}(dr,dv)\Big|^p\Big\|_{L^\infty(\mathbb{R}^d)}\\&\leqslant C\Big\|\int_0^t\int_E |e^{(t-r)A}h(r,\cdot,v)|^p\nu(dv)dr\Big\|_{L^\infty(\mathbb{R}^d)}.\end{aligned} \tag{3.17}$$

This estimate (3.17) plays an important role in proving the Schauder estimates later in this study.

(ii) The assumption that H vanishes near 0 is not necessary if for almost everywhere $x \in \mathbb{R}^d$:

$$\int_0^t\int_E |H(t,r,x,v)|^p\nu(dv)dr + \int_0^t\int_E |H(t,r,x,v)|^2\nu(dv)dr < \infty.$$

Now, from (3.15) and (3.16), we can conclude that for every $t > 0$, the linear operator given by (3.11) is bounded from $L^2([0,t]\times E)$ to $L^2(\Omega)$ and also bounded from $L^2 \cap L^4([0,t]\times E)$ to $L^4(\Omega)$. Next, if the Marcinkiewicz interpolation theorem ([1, Theorem 2.58]) is applied for every $p \in (2,4)$, then Lemma 3.3 holds for step functions, and thus we have completed the proof for $p \geqslant 2$ using an approximating argument. Based on this result and the proof of Theorem 2.1, we prove Remark 2.3 (iii) in the next section.

4. Proof of Theorem 2.1

Proof. We divide the proof into three parts: uniqueness, existence, and regularity.

(Uniqueness). The stochastic transport-diffusion equation (1.1) is linear, so in order to show the uniqueness, it is sufficient to show that a mild solution with $h = f = g = 0$ vanishes identically. When $h = f = g = 0$, the equation becomes a deterministic equation. Due to the classical Schauder estimates, we find that $u = 0$, so the mild solution is unique.

To demonstrate the existence and regularity, we first assume that $b = 0$.

(Existence). The result follows by using the explicit formula:

$$u(t,x) = \int_0^t P_{t-r}h(r,\cdot)(x)dr + \int_0^t P_{t-r}f(r,\cdot)(x)dW_r + \int_{(0,t]}\int_E P_{t-r}g(r,\cdot,v)(x)\tilde{N}(dr,dv), \tag{4.1}$$

where P_t is defined by (2.3).

According to this obvious representation, u satisfies properties (1), (2), and (4) in Definition 2.3 (for further details, refer to [2]). To prove the existence of mild solutions, we need to show that $u \in L^\infty_{loc}([0,\infty); W^{1,\infty}(\mathbb{R}^d; L^2(\Omega)))$.

For every $t > 0$, from (4.1) and by using (3.3) and (3.12), we can deduce that:

$$\|u\|^2_{t,\infty,0,2} \leqslant C(t)\Big[\|h\|^2_{t,\infty,0,2} + \|f\|^2_{t,\infty} + \int_0^t\int_E \|g(r,\cdot,z)\|^2_\infty \nu(dz)dr\Big].$$

Now, let us verify that $u \in L^\infty_{loc}([0,\infty); W^{1,\infty}(\mathbb{R}^d; L^p(\Omega)))$. Denote $K(r,x) = \frac{1}{(2\pi r)^{d/2}} e^{-\frac{|x|^2}{2r}}$, and if we use Corollary 3.1 and Corollary 3.2, then for a given p:

$$\begin{aligned}
&\mathbb{E}|u(t,x)|^p \\
&\leqslant C(p)\mathbb{E}\Big|\int_0^t\int_{\mathbb{R}^d} K(t-r,x-z)h(r,z)dzdr\Big|^p \\
&\quad + C(p)\Big[\int_0^t\Big|\int_{\mathbb{R}^d} K(t-r,x-z)f(r,z)dz\Big|^2 dr\Big]^{\frac{p}{2}} \\
&\quad + C(p)\int_0^t\int_E\Big|\int_{\mathbb{R}^d} K(t-r,x-z)g(r,z,v)dz\Big|^p \nu(dv)dr \\
&\leqslant C(p)\Big[t^p\|h\|^p_{t,\infty,0,p} + t^{\frac{p}{2}}\|f\|^p_{t,\infty} + t\|g\|^p_{t,\infty,p,E,0}\Big] \\
&\leqslant C(p,t)\Big[\|h\|^p_{t,\infty,0,p} + \|f\|^p_{t,\infty} + \|g\|^p_{t,\infty,p,E,0}\Big],
\end{aligned} \tag{4.2}$$

where we have used Lemma 3.1 in the second inequality and the Hölder inequality in the last inequality. Moreover, $C(p,t)$ is continuous and non-decreasing in t, and $C(p,t) \to 0$ as $t \to 0$.

Now, let us calculate $|Du|$. For every $1 \leqslant i \leqslant d$:

$$\begin{aligned}
&\partial_{x_i} u(t,x) \\
&= \int_0^t dr \int_{\mathbb{R}^d} \partial_{x_i} K(t-r,x-z)h(r,z)dz + \int_0^t dW_r \int_{\mathbb{R}^d} \partial_{x_i} K(t-r,x-z)f(r,z)dz \\
&\quad + \int_{(0,t]}\int_E\int_{\mathbb{R}^d} \partial_{x_i} K(t-r,x-z)g(r,z,v)dz\tilde{N}(dr,dv) \\
&= \int_0^t dr \int_{\mathbb{R}^d} \partial_{x_i} K(t-r,x-z)[h(r,z)-h(r,x)]dz \\
&\quad + \int_0^t dW_r \int_{\mathbb{R}^d} \partial_{x_i} K(t-r,x-z)[f(r,z)-f(r,x)]dz \\
&\quad + \int_{(0,t]}\int_E\int_{\mathbb{R}^d} \partial_{x_i} K(t-r,x-z)[g(r,z,v)-g(r,x,v)]dz\tilde{N}(dr,dv).
\end{aligned}$$

Thus, by Corollary 3.1 and Corollary 3.2, we conclude that:

$$\mathbb{E}|\partial_{x_i} u(t,x)|^p$$

$$\leqslant C(p)\mathbb{E}\Big|\int_0^t\int_{\mathbb{R}^d}\partial_{x_i}K(t-r,x-z)[h(r,z)-h(r,x)]dzdr\Big|^p$$
$$+C(p)\Big[\int_0^t\Big|\int_{\mathbb{R}^d}\partial_{x_i}K(t-r,x-z)[f(r,z)-f(r,x)]dz\Big|^2dr\Big]^{\frac{p}{2}}$$
$$+C(p)\int_0^t\int_E\Big|\int_{\mathbb{R}^d}\partial_{x_i}K(t-r,x-z)[g(r,z,v)-g(r,x,v)]dz\Big|^p\nu(dv)dr. \tag{4.3}$$

From assumptions (2.5) and (2.6), $h\in L^\infty_{loc}([0,\infty);\mathcal{C}_b^\beta(\mathbb{R}^d;L^p(\Omega)))$, $f\in L^\infty_{loc}([0,\infty);\mathcal{C}_b^\alpha(\mathbb{R}^d;\mathbb{R}^d))$ and $g\in L^\infty_{loc}([0,\infty);L^{p+}(E,\nu;\mathcal{C}_b^\alpha(\mathbb{R}^d)))$, and from (4.3), we obtain:

$$\mathbb{E}|\partial_{x_i}u(t,x)|^p$$
$$\leqslant C(p)[h]^p_{t,\infty,\beta,p}\Big|\int_0^t r^{\frac{\beta-1}{2}}dr\int_{\mathbb{R}^d}e^{-\frac{|z|^2}{2}}|z|^\beta dz\Big|^p+C(p)[f]^p_{t,\infty,\alpha}\Big[\int_0^t r^{\alpha-1}dr\Big]^{\frac{p}{2}}\Big[\int_{\mathbb{R}^d}e^{-\frac{|z|^2}{2}}|z|^\alpha dz\Big]^p$$
$$+C(p)[g]^p_{t,\infty,p,E,\alpha}\int_0^t r^{\frac{(\alpha-1)p}{2}}dr\Big[\int_{\mathbb{R}^d}e^{-\frac{|z|^2}{2}}|z|^\alpha dz\Big]^p. \tag{4.4}$$

Due to $\alpha,\beta,\gamma>0$, the first two terms on the right-hand side of (4.4) are finite. Moreover, by (2.4), $\alpha+2/p>1$, so the last term on the right-hand side of (4.4) is also finite. $g(t,x,\cdot)$ vanishes near 0, and thus for every $t>0$:

$$\mathbb{E}|\partial_{x_i}u(t,x)|^p\leqslant C(p)\Big[t^{\frac{(\beta+1)p}{2}}[h]^p_{t,\infty,\beta,p}+t^{\frac{p}{2}}[f]^p_{t,\infty,\alpha}+t^{\frac{(\alpha-1)p+1}{2}}[g]^p_{t,\infty,p,E,\alpha}\Big]$$
$$\leqslant C(p,t)\Big[[h]^p_{t,\infty,\beta,p}+[f]^p_{t,\infty,\alpha}+[g]^p_{t,\infty,p,E,\alpha}\Big], \tag{4.5}$$

where $C(p,t)$ is continuous and non-decreasing in t, and $C(p,t)\to 0$ as $t\to 0$.

(Regularity). We still need to show the Hölder estimate for Du. We demonstrate that $Du\in L^\infty_{loc}([0,\infty);\mathcal{C}_b^{\gamma-}(\mathbb{R}^d;L^p(\Omega)))$ and (2.7) holds. We observe that $g(t,x,\cdot)$ vanishes near 0 and $g\in L^\infty_{loc}([0,\infty);L^{p+}(E,\nu;\mathcal{C}_b^\alpha(\mathbb{R}^d)))$. According to Remark 2.3 (i), we need to prove that for every sufficiently small $\epsilon_1>0$, a sufficiently small positive real number $\epsilon_2(\epsilon_1)$ ($\epsilon_2\leqslant\epsilon$) exists, and for every $t>0$, $C(p,t,\|b\|_{t,\infty,\beta})>0$ (independent of u,h,f, and g) exists such that:

$$\|Du\|_{t,\infty,\gamma-\epsilon_1,p}\leqslant C(p,t,\|b\|_{t,\infty,\beta})\Big[\|h\|_{t,\infty,\beta,p}+\|f\|_{t,\infty,\alpha}+\|g\|_{t,\infty,p+\epsilon_2,E,\alpha}\Big]. \tag{4.6}$$

For every $x,y\in\mathbb{R}^d$ and $1\leqslant i\leqslant d$:

$$\partial_{x_i}u(t,x)-\partial_{y_i}u(t,y)$$
$$=\int_0^t dr\int_{|x-z|\leqslant 2|x-y|}\partial_{x_i}K(t-r,x-z)[h(r,z)-h(r,x)]dz$$
$$-\int_0^t dr\int_{|x-z|\leqslant 2|x-y|}\partial_{y_i}K(t-r,y-z)[h(r,z)-h(r,y)]dz$$

$$
\begin{aligned}
&+\int_0^t dr \int_{|x-z|>2|x-y|} \partial_{y_i} K(t-r, y-z)[h(r,y)-h(r,x)]dz \\
&+\int_0^t dr \int_{|x-z|>2|x-y|} [\partial_{x_i} K(t-r, x-z)-\partial_{y_i} K(t-r, y-z)][h(r,z)-h(r,x)]dz \\
&+\int_0^t dW_r \int_{|x-z|\leqslant 2|x-y|} \partial_{x_i} K(t-r, x-z)[f(r,z)-f(r,x)]dz \\
&-\int_0^t dW_r \int_{|x-z|\leqslant 2|x-y|} \partial_{y_i} K(t-r, y-z)[f(r,z)-f(r,y)]dz \\
&+\int_0^t dW_r \int_{|x-z|>2|x-y|} \partial_{y_i} K(t-r, y-z)[f(r,y)-f(r,x)]dz \\
&+\int_0^t dW_r \int_{|x-z|>2|x-y|} [\partial_{x_i} K(t-r, x-z)-\partial_{y_i} K(t-r, y-z)][f(r,z)-f(r,x)]dz \\
&+\int_{(0,t]}\int_E \int_{|x-z|\leqslant 2|x-y|} \partial_{x_i} K(t-r, x-z)[g(r,z,v)-g(r,x,v)]dz\tilde{N}(dr,dv) \\
&-\int_{(0,t]}\int_E \int_{|x-z|\leqslant 2|x-y|} \partial_{y_i} K(t-r, y-z)[g(r,z,v)-g(r,y,v)]dz\tilde{N}(dr,dv) \\
&+\int_{(0,t]}\int_E \int_{|x-z|>2|x-y|} \partial_{y_i} K(t-r, y-z)[g(r,y,v)-g(r,x,v)]dz\tilde{N}(dr,dv) \\
&+\int_{(0,t]}\int_E \int_{|x-z|>2|x-y|} [\partial_{x_i} K(t-r, x-z)-\partial_{y_i} K(t-r, y-z)][g(r,z,v)-g(r,x,v)]dz\tilde{N}(dr,dv) \\
&=: I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t)+I_6(t)+I_7(t)+I_8(t)+I_9(t)+I_{10}(t)+I_{11}(t)+I_{12}(t).
\end{aligned}
$$

Let us estimate $I_1 - I_{12}$. First, we manipulate the terms $I_1 - I_4$. To simplify the calculations, we set $p_1 = 2p/(\alpha p - \beta p - \epsilon_1 p + 2)$, and then $1 < p_1 < p$ and $\beta - 1 + 2/p_1 = \gamma - \epsilon_1$.

By using condition (2.6) and Lemma 3.1:

$$
\begin{aligned}
\mathbb{E}|I_1(t)|^p &\leqslant C(p)\mathbb{E}\Big|\int_0^t \int_{|x-z|\leqslant 2|x-y|} |\partial_{x_i} K(t-r, x-z)|\,|h(r,z)-h(r,x)|dzdr\Big|^p \\
&\leqslant C(p)[h]^p_{t,\infty,\beta,p}\Big|\int_0^t \int_{|x-z|\leqslant 2|x-y|} r^{-\frac{d+1}{2}} e^{-\frac{|x-z|^2}{2r}} |x-z|^\beta dzdr\Big|^p. \qquad (4.7)
\end{aligned}
$$

By utilizing the Hölder inequality and (3.1), from (4.7), we have:

$$
\begin{aligned}
\mathbb{E}|I_1(t)|^p &\leqslant C(p)t^{\frac{(p_1-1)p}{p_1}}[h]^p_{t,\infty,\beta,p}\Big[\int_0^t \Big|\int_{|x-z|\leqslant 2|x-y|} r^{-\frac{d+1}{2}}e^{-\frac{|x-z|^2}{2r}}|x-z|^\beta dz\Big|^{p_1}dr\Big]^{\frac{p}{p_1}} \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}\Big[\int_{|x-z|\leqslant 2|x-y|}\Big|\int_0^t r^{-\frac{(d+1)p_1}{2}}e^{-\frac{p_1|x-z|^2}{2r}}dr\Big|^{\frac{1}{p_1}}|x-z|^\beta dz\Big]^p \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}\Big[\int_{|x-z|\leqslant 2|x-y|}\Big|\int_0^\infty r^{\frac{(d+1)p_1}{2}-2}e^{-\frac{p_1 r}{2}}dr\Big|^{\frac{1}{p_1}}|x-z|^{\beta-d-1+\frac{2}{p_1}}dz\Big]^p \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^{(\beta-1+\frac{2}{p_1})p} \\
&= C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^{(\gamma-\epsilon_1)p}.
\end{aligned}
\tag{4.8}
$$

Analogously:

$$
\mathbb{E}|I_2(t)|^p \leqslant C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^{(\gamma-\epsilon_1)p}. \tag{4.9}
$$

For I_3, we employ the Gauss–Green formula mainly to obtain:

$$
I_3(t) = \int_0^t dr \int_{|y-z|=2|x-y|} K(t-r,y-z)n_i[h(r,y)-h(r,x)]dS. \tag{4.10}
$$

From (4.10), due to the Minkowski and Hölder inequalities, we obtain:

$$
\begin{aligned}
\mathbb{E}|I_3(t)|^p &\leqslant \Big[\int_0^t \int_{|y-z|=2|x-y|} K(t-r,y-z)\|h(r,y)-h(r,x)\|_{L^p(\Omega)}dSdr\Big]^p \\
&\leqslant C(p)t^{\frac{(p_1-1)p}{p_1}}[h]^p_{t,\infty,\beta,p}|x-y|^{\beta p}\Big[\int_0^t\Big|\int_{|x-z|=2|x-y|}K(t-r,y-z)dS\Big|^{p_1}dr\Big]^{\frac{p}{p_1}} \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^{\beta p}\Big[\int_0^t\Big|\int_{|x-z|=2|x-y|} r^{-\frac{d}{2}}e^{-\frac{|y-z|^2}{2r}}dS\Big|^{p_1}dr\Big]^{\frac{p}{p_1}}.
\end{aligned}
\tag{4.11}
$$

By employing Minkowski's inequality, from (4.11), we have:

$$
\begin{aligned}
&\mathbb{E}|I_3(t)|^p \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^{\beta p}\Big[\int_{|x-z|=2|x-y|}\Big(\int_0^\infty r^{-\frac{p_1 d}{2}}e^{-\frac{p_1|y-z|^2}{2r}}dr\Big)^{\frac{1}{p_1}}dS\Big]^p \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^{\beta p}\Big[\int_{|x-z|=2|x-y|}|y-z|^{-d+\frac{2}{p_1}}dz\Big]^p\Big(\int_0^\infty r^{\frac{p_1 d}{2}-2}e^{-\frac{p_1 r}{2}}dr\Big)^{\frac{p}{p_1}} \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^{(\gamma-\epsilon_1)p}.
\end{aligned}
\tag{4.12}
$$

To calculate I_4, we first use (3.1), followed by the Hölder inequality and (3.1) to obtain:

$$
\begin{aligned}
&\mathbb{E}|I_4(t)|^p \\
&\leqslant C(p)t^{\frac{(p_1-1)p}{p_1}}[h]^p_{t,\infty,\beta,p}\Big[\int_0^t\Big|\int_{|x-z|>2|x-y|}|x-z|^\beta|\partial_{x_i}K(t-r,x-z)-\partial_{y_i}K(t-r,y-z)|dz\Big|^{p_1}dr\Big]^{\frac{p}{p_1}} \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}\Big[\int_{|x-z|>2|x-y|}|x-z|^\beta\Big(\int_0^t|\partial_{x_i}K(r,x-z)-\partial_{y_i}K(r,y-z)|^{p_1}dr\Big)^{\frac{1}{p_1}}dz\Big]^p.
\end{aligned}
$$

We note that $|x-z|>2|x-y|$. Thus, for every $\xi\in[x,y]$:

$$\frac{1}{2}|x-z|\leqslant|\xi-z|\leqslant2|x-z|.$$

Due to the mean value inequality, we have:

$$
\begin{aligned}
&\mathbb{E}|I_4(t)|^p \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^p\Big[\int_{|x-z|>2|x-y|}|x-z|^\beta\Big(\int_0^t r^{-\frac{(d+2)p_1}{2}}e^{-\frac{p_1|x-z|^2}{8r}}dr\Big)^{\frac{1}{p_1}}dz\Big]^p \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^p\Big[\int_{|x-z|>2|x-y|}|x-z|^{\beta-d-2+\frac{2}{p_1}}\Big(\int_0^\infty r^{\frac{(d+2)p_1}{2}-2}e^{-\frac{p_1r}{8}}dr\Big)^{\frac{1}{p_1}}dz\Big]^p \\
&\leqslant C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^{(\gamma-\epsilon_1)p}.
\end{aligned}
\tag{4.13}
$$

Let us estimate I_5-I_8. Now, we set $p_2=2p/(2-\epsilon_1p)$, and then $2<p_2<p$ and $\alpha-1+2/p_2=\gamma-\epsilon_1$. To calculate the term I_5, we first use (3.3) to derive:

$$
\begin{aligned}
\mathbb{E}|I_5(t)|^p &\leqslant C(p)\Big[\int_0^t\Big|\int_{|x-z|\leqslant2|x-y|}\partial_{x_i}K(t-r,x-z)[f(r,z)-f(r,x)]dz\Big|^2dr\Big]^{\frac{p}{2}} \\
&\leqslant C(p)[f]^p_{t,\infty,\alpha}\Big[\int_0^t\Big|\int_{|x-z|\leqslant2|x-y|}r^{-\frac{d+1}{2}}e^{-\frac{|x-z|^2}{2r}}|x-z|^\alpha dz\Big|^2dr\Big]^{\frac{p}{2}}.
\end{aligned}
\tag{4.14}
$$

Then, we apply the Hölder inequality and for every $t>0$, from (4.14), we obtain:

$$\mathbb{E}|I_5(t)|^p\leqslant C(p)t^{\frac{(p_2-2)p}{2p_2}}[f]^p_{t,\infty,\alpha}\Big[\int_0^t\Big|\int_{|x-z|\leqslant2|x-y|}r^{-\frac{d+1}{2}}e^{-\frac{|x-z|^2}{2r}}|x-z|^\alpha dz\Big|^{p_2}dr\Big]^{\frac{p}{p_2}}.$$

$p_2>2$, so by using the Minkowski inequality, we find that:

$$
\begin{aligned}
&\mathbb{E}|I_5(t)|^p \\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}\Big[\int_{|x-z|\leqslant2|x-y|}\Big(\int_0^t r^{-\frac{(d+1)p_2}{2}}e^{-\frac{p_2|x-z|^2}{2r}}dr\Big)^{\frac{1}{p_2}}|x-z|^\alpha dz\Big]^p
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}\Big[\int\limits_{|x-z|\leqslant 2|x-y|}\Big(\int\limits_0^\infty r^{\frac{(d+1)p_2}{2}-2}e^{-\frac{p_2 r}{2}}drdr\Big)^{\frac{1}{p_2}}|x-z|^{\alpha+\frac{2}{p_2}-d-1}dz\Big]^p\\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^{(\alpha-1+\frac{2}{p_2})p}.\\
&= C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^{(\gamma-\epsilon_1)p}.
\end{aligned}
\tag{4.15}
$$

Similar calculations also imply that:

$$
\mathbb{E}|I_6(t)|^p \leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^{(\gamma-\epsilon_1)p}. \tag{4.16}
$$

For I_7, we first employ the Gauss–Green formula to obtain:

$$
I_7(t)=\int\limits_0^t dW_r\int\limits_{|y-z|=2|x-y|}K(t-r,y-z)n_i[f(r,y)-f(r,x)]dz \tag{4.17}
$$

From (4.17), after applying Corollary 3.1 and the Hölder inequality, we have:

$$
\begin{aligned}
\mathbb{E}|I_7(t)|^p &\leqslant C(p)\Big[\int\limits_0^t\Big|\int\limits_{|x-z|=2|x-y|}K(t-r,y-z)n_i[f(r,y)-f(r,x)]dS\Big|^2dr\Big]^{\frac{p}{2}}\\
&\leqslant C(p)t^{\frac{(p_2-2)p}{2p_2}}\Big[\int\limits_0^t\Big|\int\limits_{|x-z|=2|x-y|}K(t-r,y-z)n_i[f(r,y)-f(r,x)]dS\Big|^{p_2}dr\Big]^{\frac{p}{p_2}}\\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^{\alpha p}\Big[\int\limits_0^t\Big|\int\limits_{|x-z|=2|x-y|}r^{-\frac{d}{2}}e^{-\frac{|y-z|^2}{2r}}dS\Big|^{p_2}dr\Big]^{\frac{p}{p_2}},
\end{aligned}
\tag{4.18}
$$

which also suggests that:

$$
\begin{aligned}
&\mathbb{E}|I_7(t)|^p\\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^{\alpha p}\Big[\int\limits_{|x-z|=2|x-y|}\Big(\int\limits_0^\infty r^{-\frac{p_2 d}{2}}e^{-\frac{p_2|y-z|^2}{2r}}dr\Big)^{\frac{1}{p_2}}dS\Big]^p\\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^{\alpha p}\Big[\int\limits_{|x-z|=2|x-y|}|y-z|^{-d+\frac{2}{p_2}}dz\Big]^p\Big(\int\limits_0^\infty r^{\frac{p_2 d}{2}-2}e^{-\frac{p_2 r}{2}}dr\Big)^{\frac{p}{p_2}}\\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^{(\alpha+\frac{2}{p_2}-1)p}\Big(\int\limits_0^\infty r^{\frac{p_2 d}{2}-2}e^{-\frac{p_2 r}{2}}dr\Big)^{\frac{p}{p_2}}.\\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^{(\gamma-\epsilon_1)p},
\end{aligned}
\tag{4.19}
$$

by using Minkowski's inequality, where we have used $p_2>2$ in the last inequality.

First, we estimate I_8 in terms of Corollary 3.1, the Hölder inequality, and then the Minkowski inequality to obtain:

$$
\begin{aligned}
&\mathbb{E}|I_8(t)|^p\\
&\leqslant C(p)\Big[\int_0^t \Big|\int_{|x-z|>2|x-y|} [\partial_{x_i}K(t-r,x-z)-\partial_{y_i}K(t-r,y-z)][f(r,z)-f(r,x)]dz\Big|^2 dr\Big]^{\frac{p}{2}}\\
&\leqslant C(p)t^{\frac{(p_2-2)p}{2p_2}}[f]^p_{t,\infty,\alpha}\Big[\int_0^t \Big|\int_{|x-z|>2|x-y|} |x-z|^\alpha|\partial_{x_i}K(t-r,x-z)-\partial_{y_i}K(t-r,y-z)|dz\Big|^{p_2} dr\Big]^{\frac{p}{p_2}}\\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}\Big[\int_{|x-z|>2|x-y|} |x-z|^\alpha\Big(\int_0^t |\partial_{x_i}K(r,x-z)-\partial_{y_i}K(r,y-z)|^{p_2}dr\Big)^{\frac{1}{p_2}}dz\Big]^p.
\end{aligned}
$$

We note that $|x-z|>2|x-y|$, so for every $\xi\in[x,y]$:

$$
\frac{1}{2}|x-z|\leqslant|\xi-z|\leqslant 2|x-z|.
$$

Thus, due to the mean value inequality, we have:

$$
\begin{aligned}
&\mathbb{E}|I_8(t)|^p\\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^p\Big[\int_{|x-z|>2|x-y|} |x-z|^\alpha\Big(\int_0^t r^{-\frac{(d+2)p_2}{2}}e^{-\frac{p_2|x-z|^2}{8r}}dr\Big)^{\frac{1}{p_2}}dz\Big]^p\\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^p\Big[\int_{|x-z|>2|x-y|} |x-z|^{\alpha-d-2+\frac{2}{p_2}}\Big(\int_0^\infty r^{\frac{(d+2)p_2}{2}-2}e^{-\frac{p_2 r}{8}}dr\Big)^{\frac{1}{p_2}}dz\Big]^p\\
&\leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^{(\gamma-\epsilon_1)p}. \qquad (4.20)
\end{aligned}
$$

Now, let us calculate I_9-I_{12}. First, we apply analogous manipulations of I_5-I_8 and according to Corollary 3.2, we find that:

$$
\begin{aligned}
&\mathbb{E}|I_9(t)|^p\\
&\leqslant C(p)\int_0^t\int_E \Big|\int_{|x-z|\leqslant2|x-y|} \partial_{x_i}K(t-r,x-z)[g(r,z,v)-g(r,x,v)]dz\Big|^p\nu(dv)dr\\
&\leqslant C(p)t^{\frac{\epsilon_2}{p+\epsilon_2}}\Big[\int_0^t\int_E \Big|\int_{|x-z|\leqslant2|x-y|} \partial_{x_i}K(t-r,x-z)[g(r,z,v)-g(r,x,v)]dz\Big|^{p+\epsilon_2}\nu(dv)dr\Big]^{\frac{p}{p+\epsilon_2}}\\
&\leqslant C(p,t)\Big[\int_0^t\int_E \Big|\int_{|x-z|\leqslant2|x-y|} |\partial_{x_i}K(t-r,x-z)||x-z|^\alpha dz\Big|^{p+\epsilon_2}[g(r,\cdot,v)]^{p+\epsilon_2}_\alpha\nu(dv)dr\Big]^{\frac{p}{p+\epsilon_2}}\\
&\leqslant C(p,t)[g]^p_{t,\infty,p+\epsilon_2,E,\alpha}\Big[\int_0^t \Big|\int_{|x-z|\leqslant2|x-y|} |\partial_{x_i}K(t-r,x-z)||x-z|^\alpha dz\Big|^{p+\epsilon_2}dr\Big]^{\frac{p}{p+\epsilon_2}}. \qquad (4.21)
\end{aligned}
$$

After applying the calculations from (4.14) to (4.15) again, we have:

$$
\mathbb{E}|I_9(t)|^p\leqslant C(p,t)[g]^p_{t,\infty,p+\epsilon_2,E,\alpha}|x-y|^{(\alpha+\frac{2}{p+\epsilon_2}-1)p}. \qquad (4.22)
$$

By setting $\epsilon_2 = \epsilon_1 p^2/(2-\epsilon_1 p)$, then as $\epsilon_1 \to 0$, $\epsilon_2 \to 0$, and thus if ϵ_1 is sufficiently small, $\epsilon_2 < \epsilon$. Moreover, $\alpha + 2/(p+\varepsilon_2) - 1 = \gamma - \epsilon_1$, so we conclude that:

$$\mathbb{E}|I_9(t)|^p \leqslant C(p,t)[g]^p_{t,\infty,p+\epsilon_2,E,\alpha}|x-y|^{(\gamma-\epsilon_1)p}. \tag{4.23}$$

By the previous argument, we conclude that:

$$\mathbb{E}|I_{10}(t)|^p \vee \mathbb{E}|I_{11}(t)|^p \vee \mathbb{E}|I_{12}(t)|^p \leqslant C(p)t^{\frac{\epsilon_2}{p+\epsilon_2}}[g]^p_{t,\infty,p+\epsilon_2,E,\alpha}|x-y|^{(\gamma-\epsilon_1)p}. \tag{4.24}$$

By combining (4.8)–(4.9), (4.12)–(4.13), (4.15)–(4.16), (4.19)–(4.20), and (4.23)–(4.24), we obtain the estimate (4.6). According to Remarks 2.3 (i) and (2.8), and (4.2) and (4.5), this also implies that:

$$\|u\|_{t,\infty,1+\gamma-,p} \leqslant C(p,t)\Big[\|h\|_{t,\infty,\beta,p} + \|f\|_{t,\infty,\alpha} + \|g\|_{t,\infty,p+,E,\alpha}\Big], \tag{4.25}$$

where $C(p,t)$ is continuous and non-decreasing in t, and as $t \to 0$, $C(p,t) \to 0$.

Thus, we can complete the proof for $b = 0$. For general values of b, we use the continuity method. First, we consider the following family of equations:

$$\begin{aligned} & du(t,x) + \theta b(t,x)\cdot\nabla u(t,x)dt - \frac{1}{2}\Delta u(t,x)dt \\ & = h(t,x)dt + f(t,x)dW_t + \int_E g(t,x,v)\tilde{N}(dt,dv), \quad t>0,\ x\in\mathbb{R}^d, \end{aligned} \tag{4.26}$$

for $\theta \in [0,1]$. We refer to a value of $\theta \in [0,1]$ as “good” if for any:

$$\begin{cases} f \in L^\infty_{loc}([0,\infty);\mathcal{C}^\alpha_b(\mathbb{R}^d)),\ h \in L^\infty_{loc}([0,\infty);\mathcal{C}^\beta_b(\mathbb{R}^d;L^p(\Omega))), \\ g \in L^\infty_{loc}([0,\infty);L^{p+}(E,\nu;\mathcal{C}^\alpha_b(\mathbb{R}^d))),\ g(t,x,\cdot) \text{ vanishes near } 0, \end{cases} \tag{4.27}$$

a unique mild solution u to (4.26) exists such that (2.7) is satisfied. Moreover, if u belongs to $L^\infty_{loc}([0,\infty);\mathcal{C}^{1+\gamma-}_b(\mathbb{R}^d;L^p(\Omega)))$ and (4.26) holds in the sense of Definition 2.3, where b is in the class of $L^\infty_{loc}([0,\infty);\mathcal{C}^\beta_b(\mathbb{R}^d;\mathbb{R}^d))$, and f,h,g satisfy (4.27), and by using the estimate (4.25), for every given $t>0$, $C(p,t)>0$ exists that is continuous and non-decreasing in t such that:

$$\begin{aligned} & \|u\|_{t,\infty,1+\gamma-,p} \\ & \leqslant C(p,t)\Big[\|b\cdot\nabla u\|_{t,\infty,\beta,p} + \|h\|_{t,\infty,\beta,p} + \|f\|_{t,\infty,\alpha} + \|g\|_{t,\infty,p+,E,\alpha}\Big] \\ & \leqslant C(p,t)\Big[\|b\|_{t,\infty,\beta}\|Du\|_{t,\infty,\gamma-,p} + \|h\|_{t,\infty,\beta,p} + \|f\|_{t,\infty,\alpha} + \|g\|_{t,\infty,p+,E,\alpha}\Big]. \end{aligned} \tag{4.28}$$

From (4.28), given that $C(p,t) \to 0$ as $t \to 0$, for any given $T_0 > 0$, a $T > 0$ exists that is sufficiently small such that $2C(p,T) < 1/\|b\|_{T_0,\infty,\beta}$. Therefore:

$$\|u\|_{T,\infty,1+\gamma-,p} \leqslant C(p,T,\|b\|_{T_0,\infty,\beta})\Big[\|h\|_{T,\infty,\beta,p} + \|f\|_{T,\infty,\alpha} + \|g\|_{T,\infty,p+,E,\alpha}\Big], \tag{4.29}$$

where the constant C is independent of θ. Clearly, 0 is a “good” point.

Now, we claim that for T given above, on $[0,T]$, all the points from $[0,1]$ are “good.” To prove this claim, we take a “good” point θ_0 (say $\theta_0 = 0$) and rewrite (4.26) as:

$$
\begin{aligned}
&du(t,x)+\theta_0 b(t,x)\cdot\nabla u(t,x)dt-\frac{1}{2}\Delta u(t,x)dt\\
&=(\theta_0-\theta)b(t,x)\cdot\nabla u(t,x)dt+h(t,x)dt+f(t,x)dW_t+\int_E g(t,x,v)\tilde{N}(dt,dv).
\end{aligned}
\tag{4.30}
$$

For measurable functions f,g, and h that satisfy (4.27), we define a mapping S, which maps $u_1\in L^\infty([0,T];\mathcal{C}_b^{1+\gamma-}(\mathbb{R}^d;L^p(\Omega)))$ to the solution $u\in L^\infty([0,T];\mathcal{C}_b^{1+\gamma-}(\mathbb{R}^d;L^p(\Omega)))$ of the equation:

$$
\begin{aligned}
&du(t,x)+\theta_0 b(t,x)\cdot\nabla u(t,x)dt-\frac{1}{2}\Delta u(t,x)dt\\
&=(\theta_0-\theta)b(t,x)\cdot\nabla u_1(t,x)dt+h(t,x)dt+f(t,x)dW_t+\int_E g(t,x,v)\tilde{N}(dt,dv).
\end{aligned}
\tag{4.31}
$$

We observe that the mapping S is well defined due to our assumptions and the choice of θ_0. Estimate (4.29) shows that for any $u_1,\ u_2\in L^\infty([0,T];\mathcal{C}_b^{1+\gamma-}(\mathbb{R}^d;L^p(\Omega)))$:

$$
\|Su_1-Su_2\|_{T,\infty,1+\gamma-,p}\leqslant C(p,T,\|b\|_{T_0,\infty,\beta})|\theta-\theta_0|\|u_1-u_2\|_{T,\infty,1+\gamma-,p},
\tag{4.32}
$$

where C is independent of θ_0,θ,u_1, and u_2. Thus, it follows that an $\varepsilon>0$ exists such that for $|\theta-\theta_0|\leqslant\varepsilon$, the mapping S is contractive in $L^\infty([0,T];\mathcal{C}_b^{1+\gamma-}(\mathbb{R}^d;L^p(\Omega)))$ and it has a fixed point u that obviously satisfies (4.26). Therefore, these values of θ are "good," which certainly proves our claim on the time interval $[0,T]$.

u is given by (2.2), so it is right continuous in t. Thus, $u(T)\in\mathcal{C}_b^{1+\gamma-}(\mathbb{R}^d;L^p(\Omega))$. We then repeat the previous argument to extend our solution to the time interval $[T,2T]$. By continuing this procedure with finitely many steps, we construct a solution on the interval $[0,T_0]$ for every given $T_0>0$. T_0 is arbitrary, so our proof is complete. □

As shown by the preceding proof, we obtain a stronger result when the non-Gaussian Lévy noise is absent ($g=0$).

Corollary 4.1. *(Lévy noise is absent: $g=0$) Consider the stochastic transport-diffusion equation with Brownian noise:*

$$
du(t,x)+b(t,x)\cdot\nabla u(t,x)dt-\frac{1}{2}\Delta u(t,x)dt=h(t,x)dt+f(t,x)dW_t,\quad t>0,\ x\in\mathbb{R}^d,
\tag{4.33}
$$

with zero initial data. Assume that b and h satisfy the condition (2.6) with $\beta>0$ and $p>2$. Let f satisfy the condition (2.5) with $\alpha>0$. Then, a unique mild solution u exists to (4.33) in the space $L^\infty_{loc}([0,\infty);\mathcal{C}_b^{1+\alpha-}(\mathbb{R}^d;L^p(\Omega)))$. Moreover, for every given $t>0$, a $C(p,t,\|b\|_{t,\infty,\beta})>0$ (independent of u,h, and f) exists such that:

$$
\|u\|_{T,\infty,1+\alpha-,p}\leqslant C(p,t,\|b\|_{t,\infty,\beta})\Big[\|h\|_{t,\infty,\beta,p}+\|f\|_{t,\infty,\alpha}\Big].
\tag{4.34}
$$

Proof. If we can prove the case where $b=0$, then by using the continuity method, we will obtain the conclusion for $b\neq 0$ that satisfies the assumption (2.6). Hence, it is sufficient to prove this corollary in the case where $b=0$.

The existence and uniqueness of the mild solution in the space $L^\infty_{loc}([0,\infty);W^{1,\infty}(\mathbb{R}^d;L^p(\Omega)))$ can be deduced from (4.2) and (4.5). Thus, we still need to derive the $\mathcal{C}^{1+\alpha-}$ estimate.

Using the notations in the proof in Theorem 2.1, we can prove that for every sufficiently small $\epsilon_1>0$, a sufficiently small real positive number $\epsilon_2(\epsilon_1)$ ($\epsilon_2\leqslant\epsilon$) exists, and for every $t>0$, a $C(p,t)>0$ (continuous and non-decreasing in t, and independent of u,h,f) exists such that:

$$\|Du\|_{t,\infty,\alpha-\epsilon_1,p} \leqslant C(p,t)\Big[\|h\|_{t,\infty,\beta,p} + \|f\|_{t,\infty,\alpha}\Big]. \tag{4.35}$$

For $1 < q < p$, from (4.7) and (4.8), we obtain:

$$\mathbb{E}|I_1(t)|^p \leqslant C(p)t^{\frac{(q-1)p}{q}}[h]^p_{t,\infty,\beta,p}|x-y|^{(\beta-1+\frac{2}{q})p}. \tag{4.36}$$

Similarly, we have:

$$\mathbb{E}|I_2(t)|^p \vee \mathbb{E}|I_3(t)|^p \vee \mathbb{E}|I_4(t)|^p \leqslant C(p)t^{\frac{(q-1)p}{q}}[h]^p_{t,\infty,\beta,p}|x-y|^{(\beta-1+\frac{2}{q})p}. \tag{4.37}$$

We take $q = 2/(1+\alpha-\beta-\epsilon_1)$. Then, $1 < q < 2 < p$ and $\beta - 1 + 2/q = \alpha - \epsilon_1$. From (4.36) and (4.37), we obtain:

$$\mathbb{E}|I_1(t)|^p \vee \mathbb{E}|I_2(t)|^p \vee \mathbb{E}|I_3(t)|^p \vee \mathbb{E}|I_4(t)|^p \leqslant C(p,t)[h]^p_{t,\infty,\beta,p}|x-y|^{(\alpha-\epsilon_1)p}. \tag{4.38}$$

To calculate $I_5 - I_8$, we use (4.14)–(4.20). For every $2 < q_1 < p$, we have:

$$\mathbb{E}|I_5(t)|^p \vee \mathbb{E}|I_6(t)|^p \vee \mathbb{E}|I_7(t)|^p \vee \mathbb{E}|I_8(t)|^p \leqslant C(p)t^{\frac{(q_1-2)p}{2q_1}}[f]^p_{t,\infty,\alpha}|x-y|^{(\alpha-1+\frac{2}{q_1})p}. \tag{4.39}$$

If we take $q_1 = 2/(1-\epsilon_1)$, then $2 < q_1 < p$ and:

$$\mathbb{E}|I_5(t)|^p \vee \mathbb{E}|I_6(t)|^p \vee \mathbb{E}|I_7(t)|^p \vee \mathbb{E}|I_8(t)|^p \leqslant C(p,t)[f]^p_{t,\infty,\alpha}|x-y|^{(\alpha-\epsilon_1)p}. \tag{4.40}$$

From (4.38), (4.40), and Remark 2.3 (i), we have completed the proof. □

5. Concluding remarks

Given Remark 2.3 (ii) and Corollary 4.1, we have obtained the following result. For every $\alpha > 0$, we assume that $f \in L^\infty_{loc}([0,\infty); \mathcal{C}^\alpha_b(\mathbb{R}))$, and that a real number β (with $0 < \beta < \alpha$) exists such that $h \in L^\infty_{loc}([0,\infty); \mathcal{C}^\beta_b(\mathbb{R}; L^2(\Omega)))$. Then, the Cauchy problem:

$$du(t,x) - \frac{1}{2}\Delta u(t,x)dt = h(t,x)dt + f(t,x)dW_t, \quad t > 0, \ x \in \mathbb{R}, \quad u|_{t=0} = 0, \tag{5.1}$$

has a unique mild solution $u \in L^\infty_{loc}([0,\infty); \mathcal{C}^{1+\alpha}_b(\mathbb{R}; L^2(\Omega)))$, which is given by:

$$u(t,x) = \int_0^t dr \int_{\mathbb{R}} K(t-r, x-z)h(r,z)dz + \int_0^t dW_r \int_{\mathbb{R}} K(t-r, y-z)f(r,z)dz. \tag{5.2}$$

In $L^\infty_{loc}([0,\infty); \mathcal{C}^{1+\alpha}_b(\mathbb{R}))$, we select a non-negative and time-independent function f such that:

(i) f is non-decreasing on $\mathbb{R}$ and $\mathrm{supp} f \subset \mathbb{R}_+$;

(ii) For $x, y \in [0,1]$, $|f(x) - f(y)| \approx |x-y|^\alpha$.

For this function and $0 < x < 1$, we conclude that by using (3.5):

$$\begin{aligned}
&\mathbb{E}|\partial_x u(t,x) - \partial_x u(t,0)|^2 \\
&= \int_0^t \Big| \int_{\mathbb{R}} \partial_z K(r,z)[f(x-z) - f(-z)]dz \Big|^2 dr
\end{aligned}$$

$$= \int_0^t \Big| \int_{\mathbb{R}} \frac{z}{r} K(r,z)[f(x+z) - f(z)]dz \Big|^2 dr$$
$$= \int_0^t \Big| \int_0^\infty \frac{z}{r} K(r,z)[f(x+z) - f(z)]dz \Big|^2 dr. \tag{5.3}$$

We observe that f is non-negative and non-decreasing. For every δ (with $1 > \delta > \alpha$), from (5.3), we have:

$$\begin{aligned}
&\sup_{0<x<1} \frac{\|\partial_x u(t,x) - \partial_x u(t,0)\|_2}{x^\delta} \\
&= \sup_{0<x<1} \Big[\int_0^t \Big| \int_0^\infty \frac{z}{r} K(r,z) \frac{f(x+z) - f(z)}{x^\delta} dz \Big|^2 dr \Big]^{\frac{1}{2}} \\
&\geqslant \sup_{0<x<\frac{1}{2}} \Big[\int_0^t \Big| \int_0^{\frac{1}{2}} \frac{z}{r} K(r,z) \frac{f(x+z) - f(z)}{x^\delta} dz \Big|^2 dr \Big]^{\frac{1}{2}} \\
&\geqslant C \sup_{0<x<\frac{1}{2}} \Big[\int_0^t \Big| \int_0^{\frac{1}{2}} \frac{z}{r} K(r,z) \frac{x^\alpha}{x^\delta} dz \Big|^2 dr \Big]^{\frac{1}{2}} \\
&= \infty.
\end{aligned}$$

In addition, if we select $h(t,x,\omega) = h_1(t,x)h_2(\omega)$ with $h_1 \in L^\infty_{loc}([0,\infty); \mathcal{C}^\beta_b(\mathbb{R}))$ $h_2 \in L^2(\Omega)$, then the first term on the right-hand side of (5.2) belongs to $L^\infty_{loc}([0,\infty); \mathcal{C}^{1+\delta}_b(\mathbb{R}^d; L^2(\Omega)))$ for every δ (with $\alpha < \delta < 1$). Based on this result, we can see that for every $\delta > \alpha$, u is not in the class of $L^\infty_{loc}([0,\infty); \mathcal{C}^{1+\delta}_b(\mathbb{R}^d; L^2(\Omega)))$, i.e., the Hölder index α is now optimal.

Analogously, by taking $g(t,x,v) = f(t,x)g_1(v)$, where f satisfies the properties described above and $g_1 \in L^2(E,\nu)$, we claim the following. For every $\alpha > 0$, if a real number $0 < \beta < \alpha$ exists such that $h \in L^\infty_{loc}([0,\infty); \mathcal{C}^\beta_b(\mathbb{R}; L^2(\Omega)))$, then the following Cauchy problem:

$$du(t,x) - \frac{1}{2}\Delta u(t,x)dt = h(t,x)dt + \int_E g(t,x,v)\tilde{N}(dt,dv), \quad t>0,\ x \in \mathbb{R}, \quad u|_{t=0} = 0, \tag{5.4}$$

has a unique mild solution $u \in L^\infty_{loc}([0,\infty); \mathcal{C}^{1+\alpha}_b(\mathbb{R}; L^2(\Omega)))$, which is given by:

$$u(t,x) = \int_0^t dr \int_{\mathbb{R}} K(t-r, x-z)h(r,z)dz + \int_0^t \int_E \int_{\mathbb{R}} K(t-r, y-z)g(r,z,v)dz\tilde{N}(dt,dv). \tag{5.5}$$

Given (3.14) and (5.3), and by using (3.5), we conclude that for $0 < x < 1$:

$$\mathbb{E}|\partial_x u(t,x) - \partial_x u(t,0)|^2 = \int_0^t \Big| \int_0^\infty \frac{z}{r} K(r,z)[f(x+z) - f(z)]dz \Big|^2 dr \int_E |g_1(v)|^2 \nu(dv), \tag{5.6}$$

which shows that the Hölder index α is optimal for (5.4).

The stochastic processes $\{W_t\}_{t\geqslant 0}$ and $\{\tilde{N}_t\}_{t\geqslant 0}$ are independent, so the Wiener–Itô integral in (5.2) (interpreted as a stochastic process) and the Wiener–Lévy integral in (5.5) are also independent. By combining (5.3) and (5.6), we conclude that when $p = 2$, the Hölder index α for the Cauchy problem:

$$du(t,x) - \frac{1}{2}\Delta u(t,x)dt = h(t,x)dt + f(t,x)dW_t + \int_E g(t,x,v)\tilde{N}(dt,dv),\ t > 0, x \in \mathbb{R},\ u|_{t=0} = 0,$$

is also optimal.

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