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Convergence of iterates of nonexpansive mappings and orbits of nonexpansive semigroups

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ABSTRACT

Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and let C be a bounded closed and convex subset of X. Assume that C has nonempty interior and is locally uniformly rotund. Let T be a nonexpansive self-mapping of C. If T has no fixed point in the interior of C, then there exists a unique point $\tilde{x}$ on the boundary of C such that each sequence of iterates of T converges in norm to $\tilde{x}$. We also establish an analogous result for nonexpansive semigroups.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space, C a nonempty closed subset of X and let $T : C \to C$ be a nonexpansive mapping with a fixed point. The behavior of the sequence of Picard iterates of T and, in particular, its possible convergence in norm is one of the important problems in metric fixed point theory because this allows us to approximate a fixed point in the simplest way. It appears, for example, in the famous Banach Contraction Principle which has many applications in various branches of mathematics. In the case of nonexpansive mappings the situation is different because their iterates need not be strongly convergent. However, under additional assumptions they do converge. The first result of this type is due to J.-J. Moreau, who proved that if X is a Hilbert space, C is a closed subset of X and if the fixed point set $\mathrm{Fix}(T)$ of T has nonempty interior, then for each $x \in C$, the Picard iterates of T at x strongly converge to a point in $\mathrm{Fix}(T)$ ([29]). W. A. Kirk and B. Sims generalized this result to Banach spaces which are strictly convex

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and the nonempty closed subsets of which are densely proximal ([26]; see also [20], [23], [24] and [25]). In particular, they proved the following theorem.

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a reflexive and locally uniformly rotund Banach space, and let C be a nonempty closed subset of X. Suppose that $T : C \to C$ is a nonexpansive mapping and that the interior of its fixed point set* $\mathrm{Fix}(T)$ *is not empty. Then for each point $x \in C$, the Picard sequence $\{T^n(x)\}_{n=0}^{\infty}$ converges in norm to a point in* $\mathrm{Fix}(T)$ *as $n \to \infty$.*

An analogue of this result for orbits of nonexpansive semigroups was established in [16]. In this connection, see also [2], [30] and [31].

Theorem 1.2. *Let $(X, \|\cdot\|)$ be a reflexive and locally uniformly rotund Banach space, and let C be a closed subset of X. Suppose $\mathcal{S}$ is a nonexpansive semigroup on C and that the interior of its fixed point set* $\mathrm{Fix}(\mathcal{S})$ *is not empty. Then for each point $x \in C$, the trajectory $\{S(t) : 0 \le t < \infty\}$ converges in norm to a point of* $\mathrm{Fix}(\mathcal{S})$ *as $t \to \infty$.*

In the above theorems the assumption that the interior of the fixed point set $\mathrm{Fix}(T)$ or $\mathrm{Fix}(\mathcal{S})$ is not empty is essential. If this set is empty, then the only possible case in which we can obtain convergence of iterates (orbits) in a general setting is the case where the fixed point set is a singleton and lies on the boundary of C, and C is not only strictly convex, but locally uniformly rotund. Our paper is devoted to this situation and we study both iterates of nonexpansive mappings and orbits of nonexpansive semigroups. In Section 5 we prove our main theorems by using strictly geometric arguments connected with ergodic results. In Section 6 we study the case of the closed unit ball in a Hilbert space, where, using the asymptotic center method, we arrive at simpler proofs of our results. Finally, we also mention that in Section 4 of our paper we obtain convergence results for approximating sequences and curves.

2. Basic notions and facts

Throughout this paper all Banach spaces are real except for Remark 6.4.

We begin by recalling the standard notion of a strictly convex Banach space.

Definition 2.1. ([17]) A Banach space $(X, \|\cdot\|)$ is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in X$, $\|x\| \le 1$, $\|y\| \le 1$ and $x \ne y$.

Next we recall the definition of a locally uniformly rotund Banach space.

Definition 2.2. ([28]) Let $(X, \|\cdot\|)$ be a Banach space and let $S_X := \{x \in X : \|x\| = 1\}$ be its unit sphere. We say that $(X, \|\cdot\|)$ is *locally uniformly rotund* (LUR) if for each $x \in S_X$ and for each $\epsilon \in (0, 2]$, there exists $\delta(x, \epsilon) > 0$ such that for each $y \in S_X$ with $\|x - y\| \ge \epsilon$, we have

$$1 - \|\frac{x+y}{2}\| \ge \delta(x, \epsilon).$$

In his paper A. R. Lovaglia proved the following theorem regarding the ℓ^2-product of locally uniformly rotund Banach spaces.

Theorem 2.3. *([28]) Let $\{X_n\}$ be a sequence of LUR Banach spaces. Denote by $\|\cdot\|_n$ the norm in X_n. Let X be the space of all sequences, $x = \{x_n\}$, $x_n \in X_n$, for which $\sum_{n=1}^{\infty} \|x_n\|_n^2$ is convergent. Define a norm $\|\cdot\|$ on n X by $\|x\| := \left(\sum_{n=1}^{\infty} \|x_n\|_n^2\right)^{\frac{1}{2}}$. Then $(X, \|\cdot\|)$ is a locally uniformly rotund Banach space.*

We also mention the following equivalent definition of a locally uniformly rotund Banach space.

Definition 2.4. ([11]) We say that a Banach space $(X, \|\cdot\|)$ is *locally uniformly rotund* (LUR) if for each $x \in X$, every sequence $\{x_n\}_{n=1}^{\infty}$ with $\lim_n \|x_n\| = \|x\|$ and $\lim_n \|x + x_n\| = 2\|x\|$ tends strongly to x.

So it is natural to introduce the notion of a locally uniformly rotund set in the following way.

Definition 2.5. ([37]) Let $(X, \|\cdot\|)$ be a Banach space, C be a nonempty bounded closed and convex subset of X, and let C have nonempty interior, that is, $\mathrm{int}(C) \neq \emptyset$. We say that C is *locally uniformly rotund* (LUR) if for each $x \in \partial C$ and for each $\epsilon \in (0, d_x)$, where $d_x := \sup\{\|x - x'\| : x' \in C\}$, there exists $\delta(x, \epsilon) > 0$ such that for each $y \in C$ with $\|x - y\| \geq \epsilon$, we have

$$\mathrm{dist}(\frac{x+y}{2}, \partial C) := \inf\{\|\frac{x+y}{2} - x'\| : x' \in \partial C\} \geq \delta(x, \epsilon).$$

Now we also recall the classical notion of uniform convexity.

Definition 2.6. ([12]) Let $(X, \|\cdot\|)$ be a Banach space and let $\overline{B_X(0,1)} = \{x \in X : \|x\| \leq 1\}$ denote its closed unit ball. If for each $\epsilon \in (0, 2]$, there exists $\delta(\epsilon) > 0$ such that for each $x, x' \in \overline{B_X(0,1)}$ with $\|x - x'\| \geq \epsilon$, we have

$$\|\frac{x+x'}{2}\| \leq 1 - \delta(\epsilon),$$

then we say that the space $(X, \|\cdot\|)$ is *uniformly convex*.

Definition 2.7. ([37]) Let $(X, \|\cdot\|)$ be a Banach space, C be a nonempty bounded closed and convex subset of X, and let C have nonempty interior. We say that C is *uniformly convex* if for each $\epsilon \in (0, \mathrm{diam}(C))$, there exists $\eta_C(\epsilon) > 0$ such that for each $x, y \in C$ with $\|x - y\| \geq \epsilon$, we have

$$\mathrm{dist}(\frac{x+y}{2}, \partial C) \geq \eta_C(\epsilon).$$

Observe that if a Banach space $(X, \|\cdot\|)$ admits a nonempty bounded closed and convex subset which has nonempty interior and is uniformly convex, then $(X, \|\cdot\|)$ has to be reflexive (see [17]).

At this point we present a simple example of a bounded closed and convex subset of a Hilbert space, which is LUR but not uniformly convex.

Example 2.8. This example is a small modification of an example given in [28]. Let $H = \ell^2$ and let

$$C := \{x = \{x^i\} \in H = \ell^2 : \sum_{k=2}^{\infty} (|x^{2k-1}|^k + |x^{2k}|^k)^{\frac{2}{k}} \leq 1\}.$$

Then C is bounded, closed, convex and has nonempty interior. Moreover, by Theorem 2.3, the set C is LUR, but not uniformly convex.

Now we recall a few facts from nonexpansive mapping theory.

Definition 2.9. Let $(X, \|\cdot\|)$ be a Banach space and let C be a weakly compact and convex subset of X. We say that C has the *fixed point property* for nonexpansive mappings if each nonexpansive mapping $T : C \to C$ (that is, $\|Tx - Tx'\| \leq \|x - x'\|$ for every $x, x' \in C$) has a fixed point.

Theorem 2.10. *([1]; see also [14]) If $(X, \|\cdot\|)$ is a strictly convex Banach space, C is a nonempty closed and convex subset of X, $T : C \to C$ is nonexpansive, $x, x' \in C$, $x \neq x'$ and x, x' are fixed points of T, then the whole linear segment $[x, x']$ lies in the fixed point set* $\mathrm{Fix}(T)$ *of T.*

Theorem 2.11. *([3]) If $(X, \|\cdot\|)$ is a uniformly convex Banach space, C is a nonempty bounded, closed and convex subset of X, and $T : C \to C$ is nonexpansive, then the mapping T has a fixed point.*

Theorem 2.12. *([1]; see also [14]) If C is a closed and convex subset of a strictly convex Banach space $(X, \|\cdot\|)$, and if $T : C \to C$ is a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$, *then the set* $\mathrm{Fix}(T)$ *is closed and convex.*

Definition 2.13. ([4]) Let $(X, \|\cdot\|)$ be a Banach space, C a nonempty subset of X and $T : C \to C$. The mapping T is said to be *demiclosed* if for any sequence $\{x_n\}_{n=1}^{\infty}$ in C, the following implication holds:

$$w - \lim_n x_n = x \quad \text{and} \quad \lim_n \|Tx_n - y\| = 0$$

imply that

$$x \in C \quad \text{and} \quad Tx = y.$$

Theorem 2.14. *([4]) If $(X, \|\cdot\|)$ is a uniformly convex Banach space, C is a nonempty closed and convex subset of X, $T : C \to C$ is nonexpansive and $I : C \to C$ is the identity mapping, then the mapping $F = I - T$ is demiclosed on C.*

In other words, any uniformly convex Banach space admits a demiclosedness principle for nonexpansive mappings.

Theorem 2.15. *([5]) If $(X, \|\cdot\|)$ is a uniformly convex Banach space, C is a nonempty bounded closed and convex subset of X, and $T : C \to C$ is nonexpansive, then for each point $x \in C$, we have*

$$\lim_{n\to\infty} \|T(\frac{1}{n}\sum_{i=1}^{n} T^{i+k}x) - \frac{1}{n}\sum_{i=1}^{n} T^{i+k}x\| = 0,$$

uniformly in $k > 0$.

In the case of Hilbert spaces, we will also apply the asymptotic center technique, which is one of the basic tools in metric fixed point theory.

Definition 2.16. ([15]) Let $(H, \|\cdot\|)$ be a Hilbert space. For $x \in H$ and a bounded sequence $\{x_n\}_{n=1}^{\infty}$, the *asymptotic radius* of $\{x_n\}$ at x is the number

$$r(x, \{x_n\}) := \limsup_n \|x - x_n\|.$$

For a nonempty closed and convex subset C of H, the *asymptotic radius* of $\{x_n\}$ in C is the number

$$r(C, \{x_n\}) := \inf\{r(x, \{x_n\}) : x \in C\}.$$

The *asymptotic center* of $\{x_n\}$ in C is, by definition, the set

$$Ac(C, \{x_n\}) := \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

Theorem 2.17. *([15], [17] and [18]) Let $(H, \|\cdot\|)$ be a Hilbert space and C a nonempty bounded closed and convex subset of H. Then for each bounded sequence $\{x_n\}$ in H, we have*

$$Ac\,(C, \{x_n\}) = Ac\,(H, \{x_n\})$$

and $Ac\,(C, \{x_n\})$ is a singleton.

Theorem 2.18. *([15], [17] and [18]) Let $(H, \|\cdot\|)$ be a Hilbert space. If C is a nonempty bounded closed and convex subset of H, and $T : C \to C$ is nonexpansive, then for each $x \in C$, we have $Ty = y$, where $\{y\} = Ac\,(C, \{T^n x\})$.*

Next we recall the notions of a nonexpansive semigroup and a strongly measurable nonexpansive semigroup.

Definition 2.19. ([32]) Let $(X, \|\cdot\|)$ be a Banach space, C a nonempty subset of X and let $\mathcal{S} = \{S\,(t) : t \in [0, \infty)\}$ be a family of self-mappings of C. Then $\mathcal{S}$ is said to be a *nonexpansive semigroup* acting on C if the following four conditions are satisfied:
(i) $S\,(t) : C \to C$ for each $t \in [0, \infty)$;
(ii) $S\,(s + t)\,x = S\,(s)\,S\,(t)\,x$ for all $s, t \in [0, \infty)$ and $x \in C$;
(iii) $S\,(0) = I$;
(iv)

$$\|S\,(t)\,x - S\,(t)\,y\| \leq \|x - y\|$$

for all $x, y \in C$ and $t \in [0, \infty)$.

If, in addition,

(v) $S\,(t)\,x$ is strongly measurable in $t \in [0, \infty)$ for each $x \in C$, then the semigroup $\mathcal{S}$ is called a *strongly measurable nonexpansive semigroup.*

The set of common fixed points of $\mathcal{S}$ is denoted by $\mathrm{Fix}\,(\mathcal{S})$.

Theorem 2.20. *([3]) If $(X, \|\cdot\|)$ is a uniformly convex Banach space, C is a nonempty bounded closed and convex subset of X, and $\mathcal{S}$ is a nonexpansive semigroup acting on C, then the semigroup $\mathcal{S}$ has a common fixed point.*

Theorem 2.21. *([1]) If C is a closed and convex subset of a strictly convex Banach space $(X, \|\cdot\|)$, and if $\mathcal{S}$ is a nonexpansive semigroup acting on C with $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$, then the set $\mathrm{Fix}(\mathcal{S})$ is closed and convex.*

We also recall the following result, which is due to the fourth author (see the proof of the Theorem in [32], pages 548–549).

Theorem 2.22. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space, C a nonempty bounded closed and convex subset of X, $\mathcal{S}$ a strongly measurable nonexpansive semigroup acting on C, $x \in C$, $0 < t_n \to \infty$, and let the sequence $\{\frac{1}{t_n} \int_0^{t_n} S(s)x ds\}$ tend weakly to $\tilde{x} \in C$. Then $\tilde{x} \in \mathrm{Fix}(\mathcal{S})$.*

In Hilbert spaces we can also define the asymptotic center for bounded nets $\{x_t\}_{t \geq 0}$.

Definition 2.23. ([17] and [18]) Let $(H, \|\cdot\|)$ be a Hilbert space. For $x \in H$ and a bounded net $\{x_t\}_{t\geq 0}$, the *asymptotic radius* of $\{x_t\}_{t\geq 0}$ at x is the number

$$r\,(x, \{x_t\}_{t\geq 0}) := \limsup_{t\to\infty} \|x - x_t\|\,.$$

For a nonempty closed and convex subset C of H, the *asymptotic radius* of $\{x_t\}_{t\geq 0}$ in C is the number

$$r\left(C, \{x_t\}_{t\geq 0}\right) := \inf\left\{r\left(x, \{x_t\}_{t\geq 0}\right) : x \in C\right\}.$$

The *asymptotic center* of $\{x_t\}_{t\geq 0}$ in C is, by definition, the set

$$Ac\left(C, \{x_t\}_{t\geq 0}\right) := \left\{x \in C : r\left(x, \{x_t\}_{t\geq 0}\right) = r\left(C, \{x_t\}_{t\geq 0}\right)\right\}.$$

Theorem 2.24. *([17] and [18]) Let $(H, \|\cdot\|)$ be a Hilbert space and let C be a nonempty bounded closed and convex subset of H. Then for each bounded net $\{x_t\}_{t\geq 0}$ in H, we have*

$$Ac\left(C, \{x_t\}_{t\geq 0}\right) = Ac\left(H, \{x_t\}_{t\geq 0}\right)$$

and $Ac\left(C, \{x_t\}_{t\geq 0}\right)$ is a singleton.

Theorem 2.25. *([17] and [18]) Let $(H, \|\cdot\|)$ be a Hilbert space. If C is a nonempty bounded closed and convex subset of H, and $\mathcal{S}$ is a nonexpansive semigroup acting on C, then for each $x \in C$, we have $S(t)y = y$ for all $t \geq 0$, where $\{y\} = Ac\left(C, \{S(t)x\}_{t\geq 0}\right)$.*

3. An auxiliary lemma

In this section we prove a lemma which is a basic tool in our subsequent considerations.

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a Banach space and let C be a bounded closed and convex subset of X. Assume that $\operatorname{int}(C)$ is nonempty, $0 \in \operatorname{int}(C)$ and C is locally uniformly rotund. Let $\tilde{x} \in \partial C$, $x^* \in X^*$, $\|x^*\| = 1$, $k \in (0, +\infty)$ and let the hyperplane $V_{k,\tilde{x}} := \{x \in X : x^*(x) = k\}$ which supports C at the point $\tilde{x}$ be given. If $r \in (0, +\infty)$ and the set*

$$C_r := C \cap \{x \in X : \|x - \tilde{x}\| \geq r\}$$

is nonempty, then there exists $0 < k_1 < k$ such that

$$C_r \subset \{x \in X : x^*(x) \leq k_1\}.$$

Proof. Suppose to the contrary that there exists a sequence $\{x_n\}$ in C_r such that $\lim_n x^*(x_n) = k$. By our assumptions, $\tilde{x} \in \partial C$ and $x_n \in C_r \subset C$ for all $n = 1, 2, \ldots$, that is, $\|x_n - \tilde{x}\| \geq r$ for all $n = 1, 2, \ldots$. Since C is locally uniformly rotund, there exists $\delta > 0$ such that each closed ball $B(\frac{x_n+\tilde{x}}{2}, \delta)$ is a subset of C $(n = 1, 2, \ldots)$. Hence $\frac{x_n+\tilde{x}}{2} + \frac{\delta}{\|\tilde{x}\|}\tilde{x} \in C$. This, however, contradicts the assumption

$$x^*\left(\frac{x_n + \tilde{x}}{2} + \frac{\delta}{\|\tilde{x}\|}\tilde{x}\right) \leq k$$

because we have

$$\lim_n x^*\left(\frac{x_n + \tilde{x}}{2} + \frac{\delta}{\|\tilde{x}\|}\tilde{x}\right) = \left[1 + \frac{\delta}{\|\tilde{x}\|}\right] k > k.$$

The contradiction we have reached completes the proof of this lemma. $\square$

4. Convergence of approximating sequences

In this section we consider the behavior of approximating sequences of nonexpansive mappings with no fixed points in the interior of their domains. Recall that a sequence $\{x_n\}_{n=1}^{\infty} \subset C$ is said to be an approximating sequence of a nonexpansive mapping $T : C \to C$ if $\lim_n \|x_n - Tx_n\| = 0$. We begin with the following theorem.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a reflexive Banach space which admits a demiclosedness principle with respect to nonexpansive mappings. Assume that $C \subset X$ is bounded, closed and convex with nonempty interior. Assume further that C is locally uniformly rotund. Let T be a nonexpansive self-mapping of C. If T has a unique fixed point $\tilde{x}$ and $\tilde{x}$ lies on the boundary ∂C of C, then every approximating sequence $\{x_n\}_{n=1}^{\infty}$ of T tends strongly to $\tilde{x}$.*

Proof. We may assume without any loss of generality that $0 \in \mathrm{int}(C)$. Suppose to the contrary that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in C such that $\lim_n \|x_n - Tx_n\| = 0$, but $\{x_n\}$ does not strongly converge to $\tilde{x}$. Without any loss of generality we may assume that there exists $\lim_n \|x_n - \tilde{x}\| = r > 0$ and that the sequence $\{x_n\}$ is weakly convergent. By our assumption, $(X, \|\cdot\|)$ admits a demiclosedness principle with respect to nonexpansive mappings and therefore the weak limit of $\{x_n\}$ is a fixed point of T. Since the mapping T has a unique fixed point $\tilde{x}$, it follows that $w\text{-}\lim x_n = \tilde{x}$.

Next, note that for $0 < \eta < r$, there exists $n_0 \in \mathbb{N}$ such that

$$x_n \in C_{r-\eta} = C \cap \{x \in X : \|x - \tilde{x}\| \geq r - \eta\}$$

for each $n \geq n_0$. Let $x^* \in X^*$, $\|x^*\| = 1$ and $0 < k \in \mathbb{R}$ be such that the hyperplane $V_{k,\tilde{x}} = \{x \in X : x^*(x) = k\}$ supports C at the point $\tilde{x}$. By Lemma 3.1, there exists $0 < k_1 < k$ such that

$$C_{r-\eta} \subset \{x \in X : x^*(x) \leq k_1\}.$$

It follows that $\tilde{x} \in \{x \in X : x^*(x) \leq k_1\}$. But this contradicts the assumption that $x^*(\tilde{x}) = k$ because $k_1 < k$. The contradiction we have reached completes the proof. □

Directly from this theorem we get the following corollary.

Corollary 4.2. *Let $(X, \|\cdot\|)$ be a reflexive Banach space which admits a demiclosedness principle with respect to nonexpansive mappings. Assume that $C \subset X$ is bounded closed and convex with nonempty interior. Assume further that C is locally uniformly rotund. Let T be a nonexpansive self-mapping of C and let T have a unique fixed point $\tilde{x}$ which lies on the boundary ∂C of C. Then for each $x \in C$, the approximating curve $z_x : [0, 1) \to C$, defined implicitly by $z_x(s) = (1 - s)x + sTz_x(s)$, where $s \in [0, 1)$, tends strongly to $\tilde{x}$ as $s \to 1$. In addition, this convergence is uniform with respect to $x \in C$.*

5. Convergence of iterates of nonexpansive mappings and orbits of nonexpansive semigroups

This section is the main part of our paper. Our goal is to state and prove an analogue of Theorem 1.1.

Theorem 5.1. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and let C be a bounded closed and convex subset of X. Assume that C has nonempty interior and is locally uniformly rotund. Let T be a nonexpansive self-mapping of C. If T has no fixed point in $\mathrm{int}(C)$, then there exists a unique point $\tilde{x}$ on the boundary ∂C of C such that each sequence of iterates $\{T^n x\}_{n=1}^{\infty}$ of T converges strongly to $\tilde{x}$.*

Proof. Without any loss of generality we may assume that $0 \in \mathrm{int}(C)$. It follows from Theorems 2.11 and 2.12 that the nonexpansive mapping T has exactly one fixed point $\tilde{x}$ and that this point lies on the boundary ∂C of C. We claim that each sequence of iterates $\{T^n x\}$ of T converges to $\tilde{x}$ in norm. To show this, we first observe that for each point $x \in C$, the real sequence $\{\|T^n x - \tilde{x}\|\}$ is decreasing and therefore there exists $\lim_n \|T^n x - \tilde{x}\|$. Now suppose to the contrary that there exists a point $y \in C$ such that $\lim_n \|T^n y - \tilde{x}\| = r > 0$. Since the sequence $\{\|T^n y - \tilde{x}\|\}$ is decreasing, we have

$$T^n y \in C_r := C \cap \{x \in X : \|x - \tilde{x}\| \geq r\}$$

for all $n = 1, 2, \ldots$. Now let $x^* \in X^*$, $\|x^*\| = 1$, and $0 < k \in \mathbb{R}$ be such that the hyperplane $V_{k,\tilde{x}} = \{x \in X : x^*(x) = k\}$ supports C at the point $\tilde{x}$. By Lemma 3.1, there exists $0 < k_1 < k$ such that

$$C_r \subset \{x \in X : x^*(x) \leq k_1\}.$$

By Theorem 2.15, we have

$$\lim_{n\to\infty} \|T(\frac{1}{n}\sum_{i=1}^{n} T^i y) - \frac{1}{n}\sum_{i=1}^{n} T^i y\| = 0.$$

Taking

$$x_n := \frac{1}{n}\sum_{i=1}^{n} T^i y$$

for $n = 1, 2, \ldots$, we obtain an approximating sequence $\{x_n\}$ of T. Since $T^i(y) \in C_r$, for all i, it follows that

$$x^*(x_n) = x^*\left(\frac{1}{n}\sum_{i=1}^{n} T^i y\right) \leq k_1.$$

Using the strong convergence of $\{x_n\}$ to $\tilde{x}$ (see Theorem 4.1), we obtain

$$k_1 \geq \lim_n x^*(x_n) = x^*(\tilde{x}) = k > k_1.$$

The contradiction we have reached completes the proof of Theorem 5.1. □

The following example shows that the assumption that C is locally uniformly rotund is crucial.

Example 5.2. Let $H = \mathbb{R}^2$ be endowed with the standard Euclidean norm and let $C := \{(x, y) \in \mathbb{R}^2 : |x| \leq 1,\ |y| \leq 1\}$. If $T(x, y) := (1, -y)$ for $(x, y) \in C$, then T is nonexpansive and $(1, 0) \in \partial C$ is its unique fixed point, but the iterates $T^n(1, 1)$, $n = 1, 2, \ldots$, do not converge to $(1, 0)$.

Remark 5.3. Observe that Example 2.8 shows that there exist subsets of X which satisfy the conditions of Theorem 5.1.

In the second part of this section we consider orbits of nonexpansive semigroups and establish result which constitutes an analogue of the previous theorem. The proof of this result is similar to the proof of Theorem 5.1, but for the convenience of the reader we repeat it with the necessary changes.

Theorem 5.4. *Let $(X, \|\cdot\|)$ be a uniformly convex Banach space and let C be a bounded closed and convex subset of X. Assume that C has nonempty interior and is locally uniformly rotund. Let $\mathcal{S}$ be a strongly measurable nonexpansive semigroup acting on C. If $\mathcal{S}$ has no common fixed point in the interior of C, then there exists a unique point $\tilde{x}$ on the boundary ∂C of C such that each orbit $\{S(t)x : t \geq 0\}$ converges strongly to $\tilde{x}$.*

Proof. Without any loss of generality we may assume that $0 \in \operatorname{int}(C)$. By Theorems 2.20 and 2.21, the semigroup $\mathcal{S}$ has exactly one common fixed point $\tilde{x}$ and this point lies on the boundary ∂C of C. Suppose that for some point $y \in C$, its orbit $\{S(t)y : t \geq 0\}$ does not strongly converge to $\tilde{x}$. Since for this y the function $\|S(\cdot)y - \tilde{x}\|$, $t \geq 0$, is decreasing, there exists $r := \lim_{t\to\infty} \|S(t)y - \tilde{x}\| > 0$. Put

$$C_r := C \cap \{x \in X : \|x - \tilde{x}\| \geq r\}.$$

Then we have $S(t)y \in C_r$ for all $t \geq 0$. Now consider the hyperplane $V_{k,\tilde{x}} = \{x \in X : x^*(x) = k\}$ which supports C at the point $\tilde{x}$ (here $x^* \in X^*$, $\|x^*\| = 1$, and $k \in (0, +\infty)$). By Lemma 3.1, there exists $0 < k_1 < k$ such that

$$C_r \subset \{x \in X : x^*(x) \leq k_1\}.$$

Consider a weakly convergent sequence $\{\frac{1}{t_n}\int_0^{t_n} S(s)yds\}_{n=1}^{\infty}$ with $0 < t_n \to \infty$. Then by Theorem 2.22, we have

$$w\text{-}\lim_n \frac{1}{t_n}\int_0^{t_n} S(s)\,yds = \tilde{x}.$$

This, however, is impossible because $\frac{1}{t_n}\int_0^{t_n} S(s)yds \in C_r$ for each n and therefore

$$k = x^*(\tilde{x}) = \lim_n x^*(\frac{1}{t_n}\int_0^{t_n} S(s)\,yds) \leq k_1 < k.$$

The contradiction we have reached completes the proof. □

6. The case of the closed unit ball in a Hilbert space

In this section we present simple proof of a theorem regarding the strong convergence of iterates of nonexpansive mappings to a point on the unit sphere of a Hilbert ball.

Theorem 6.1. *Let $(H, \|\cdot\|)$ be a Hilbert space and let T be a nonexpansive self-mapping of the closure $\overline{B_H}$ of its open unit ball $B_H = \{x \in H : \|x\| < 1\}$. If T has no fixed point in B_H, then it has a unique fixed point $\tilde{x}$, which lies on the boundary ∂B_H of B_H, and for each point $x \in \overline{B_H}$, the sequence of its iterates $\{T^n x\}_{n=1}^{\infty}$ converges in norm to $\tilde{x}$.*

Proof. Without loss of generality we may assume that $\dim H \geq 2$ (see Remark 6.2 below). First observe that by Theorems 2.10 and 2.12, and by the strict convexity of the Hilbert space, the nonexpansive mapping T has a unique fixed point $\tilde{x}$ in $\overline{B_H}$ and this point lies on the boundary ∂B_H. Now take an arbitrary point $x \in \overline{B_H}$. We claim that the sequence of its iterates $\{T^n x\}$ is strongly convergent. Suppose to the contrary that this is not true. Then the asymptotic radius $r := r\left(\overline{B_H}, \{T^n x\}\right)$ of $\{T^n x\}$ is positive and $0 < r \leq 1$. It is obvious that r is, in fact, strictly less than 1 because by Theorems 2.17 and 2.18, one would otherwise obtain $\{0\} = Ac\left(\overline{B_H}, \{T^n x\}\right)$ and $T0 = 0$. Therefore, applying once more Theorems 2.17 and 2.18, we get $\{\tilde{x}\} = Ac\left(\overline{B_H}, \{T^n x\}\right)$. Take $r < r_1 < 1$ such that

$$r_1\sqrt{1 - \frac{r_1^2}{4}} < r.$$

Then there exists $n_1 \in \mathbb{N}$ such that

$$\|T^n x - \tilde{x}\| \le r_1$$

for each $n_1 \le n \in \mathbb{N}$. Now it is sufficient to observe that using the point $(1 - \frac{r_1^2}{2})\tilde{x}$ in B_H, which is equal to the orthogonal projection of each $x \in \partial\overline{B_H} \cap \partial\overline{B(\tilde{x}, r_1)}$ onto the line generated by $\tilde{x}$, we get

$$(1 - \frac{r_1^2}{2})\tilde{x} \ne \tilde{x}$$

and

$$\overline{B_H} \cap \overline{B(\tilde{x}, r_1)} \subset \overline{B((1 - \frac{r_1^2}{2})\tilde{x}, r_1\sqrt{1 - \frac{r_1^2}{4}})}.$$

In order to obtain the last inclusion we take $x = \alpha\tilde{x} + \beta e = [s + (1 - \frac{r_1^2}{2})]\tilde{x} + \beta e \in \partial(\overline{B_H} \cap \overline{B(\tilde{x}, r_1)})$, where $e \perp \tilde{x}$ and $\|e\| = 1$ and consider two cases.

Case 1: $x \in \partial\overline{B(\tilde{x}, r_1)})$. Then we have $\alpha = (1 - \frac{r_1^2}{2}) - s$, $|\beta| = \sqrt{r_1^2 - (\frac{r_1^2}{2} + s)^2}$, where $0 \le s \le r_1 - \frac{r_1^2}{2}$, and the function $f : [0, r_1 - \frac{r_1^2}{2}] \to [0, +\infty)$, defined by

$$f(s) = \|x - (1 - \frac{r_1^2}{2})\tilde{x}\|^2 = s^2 + r_1^2 - (\frac{r_1^2}{2} + s)^2 = r_1^2 - \frac{r_1^4}{4} - sr_1^2,$$

is strictly decreasing and

$$\max_{0 \le s \le r_1 - \frac{r_1^2}{2}} f(s) = r_1^2 - \frac{r_1^4}{4};$$

Case 2: $x \in \partial(\overline{B_H})$. Then we have $\alpha = (1 - \frac{r_1^2}{2}) + s$, $|\beta| = \sqrt{1 - (1 - \frac{r_1^2}{2} + s)^2}$, where $0 \le s \le \frac{r_1^2}{2}$, and the function $f_1 : [0, \frac{r_1^2}{2}] \to [0, +\infty)$, defined by

$$\begin{aligned} f_1(s) = \|x - (1 - \frac{r_1^2}{2})\tilde{x}\|^2 &= s^2 + 1 - (1 - \frac{r_1^2}{2} + s)^2 \\ &= r_1^2 - \frac{r_1^4}{4} - s(2 - r_1^2), \end{aligned}$$

is strictly decreasing and

$$\max_{0 \le s \le \frac{r_1^2}{2}} f_1(s) = r_1^2 - \frac{r_1^4}{4}.$$

The inclusion we have obtained, namely,

$$\overline{B_H} \cap \overline{B(\tilde{x}, r_1)} \subset \overline{B((1 - \frac{r_1^2}{2})\tilde{x}, r_1\sqrt{1 - \frac{r_1^2}{4}})},$$

implies the following contradiction:

$$r < \limsup_n \|T^n x - (1 - \frac{r_1^2}{2})\tilde{x}\| \leq r_1 \sqrt{1 - \frac{r_1^2}{4}} < r.$$

This means that $\lim_n T^n x = \tilde{x}$ in norm, as asserted. $\square$

Remark 6.2. It is generally known that the above result is valid for each continuous self-mappings of $[-1,1] \subset \mathbb{R}$ with a unique fixed point which, in addition, lies on the boundary $\partial[-1,1] = \{-1,1\}$. For example, if $T : [0,1] \to [0,1]$ and $\mathrm{Fix}(T) = \{1\}$, then for each $x \in [-1,1)$, we have $T(x) > x$ by the intermediate value property of continuous mappings and therefore $\lim_n T^n x = 1$. For more information regarding the behavior of the iterates of continuous decreasing functions, see [19].

Remark 6.3. When $\dim H \geq 2$ Theorem 6.1 is false, in general, for continuous self-mappings of $\overline{B_H}$. Indeed, assume that $H = H_1 + H_2$, where $e_1 \in H$ is fixed, $\|e_1\| = 1$, $H_1 = \mathrm{span}\{e_1\}$ and $e_1 \perp H_2$. If $x \in \overline{B_H}$ and $x = x_1 + x_2 \in H_1 + H_2$, then we set $Tx := \sqrt{1 - \|x_2\|^2} e_1 - x_2$. It is not difficult to observe that T maps $\overline{B_H}$ into the boundary ∂B_H of B_H, has the unique fixed point e_1 and for every $e \in \partial B_H \cap H_2$, we have $Te = -e$, $T(-e) = e$.

Remark 6.4. Observe that in the case of the complex Euclidean space $\mathbb{C}^n$ with its open unit ball B_n and a holomorphic fixed point free self-mapping T of B_n, the Wolff-Denjoy theorem guarantees the convergence of all iterates $T^n(x)$ to the unique fixed point $\xi \in \partial B_n$ of T (see, for example, [6], [9], [10], [13], [18], [35], [36] and the references therein). However, if H is an *infinite-dimensional* complex Hilbert space, then the Wolff-Denjoy theorem does not hold in the Hilbert ball B_H ([34]; see also [7], [8], [18], [21], [22], [27], [33] and the references therein).

Now we present two examples of self-mappings of the closed unit ball $\overline{B_H}$, which satisfy the assumptions of Theorem 6.1.

Example 6.5. Let $(H_1, \|\cdot\|_1)$ be a Hilbert space, $\dim(H_1) \geq 1$, $H = \mathbb{R} \times H_1$, and let the norm $\|\cdot\|$ in H be defined by

$$\|(t,x)\| := \sqrt{t^2 + \|x\|_1^2}$$

for $(t,x) \in H = \mathbb{R} \times H_1$. Let $\overline{B_H}$ be the closed unit ball in H, $R : H \to \overline{B_H}$ be the radial retraction and let $c \in (0,1)$. Define a nonexpansive mapping $T : \overline{B_H} \to \overline{B_H}$ by

$$T(t,x) := R((1-c)t + c, x),$$

where $(t,x) \in \overline{B_H}$. Then $(1,0) \in \partial\overline{B_H}$ is the unique fixed point of T.

We can, in fact, generalize this example in the following way.

Example 6.6. Let $(H_1, \|\cdot\|_1)$ be a Hilbert space, $\dim(H_1) \geq 1$, $H = \mathbb{R} \times H_1$, and let the norm $\|\cdot\|$ in H be defined by

$$\|(t,x)\| := \sqrt{t^2 + \|x\|_1^2}$$

for $(t,x) \in H = \mathbb{R} \times H_1$. Let $\overline{B_H}$ be the closed unit ball in H, $\overline{B_{H_1}}$ the closed unit ball in H_1, $T_1 : \overline{B_{H_1}} \to H_1$ a nonexpansive mapping, $R : H \to \overline{B_H}$ the radial retraction and let $c \in (0,1)$. Define a nonexpansive mapping $T : \overline{B_H} \to \overline{B_H}$ by

$$T(t,x) = R((1-c)t + c, T_1 x),$$

where $(t,x) \in \overline{B_H}$. It is not difficult to check that T has a unique fixed point and this point lies on the boundary ∂B_H of B_H.

Finally, observe that an analogue of Theorem 6.1 is valid for nonexpansive semigroups (in this connection, see also Theorems 2.24 and 2.25).

Theorem 6.7. *Let $(H, \|\cdot\|)$ be a Hilbert space and let $\mathcal{S}$ be a nonexpansive semigroup acting on the closure $\overline{B_H}$ of the open unit ball $B_H = \{x \in H : \|x\| < 1\}$. If $\mathcal{S}$ has no fixed point in B_H, then it has a unique fixed point $\tilde{x}$, which lies on the boundary ∂B_H of B_H, and for each point $x \in \overline{B_H}$, the orbit $\{S(t)x\}_{t\geq 0}$ converges in norm to $\tilde{x}$.*

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