J. Math. Anal. Appl. 477 (2019) 153–162

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the study of the economic equilibrium problem through preference relations

Monica Milasi [a], Alfio Puglisi [a], Carmela Vitanza [b,*]

[a] *Department of Economics, University of Messina, Italy*
[b] *Department MIFT, University of Messina, Italy*

ARTICLE INFO

Article history:
Received 8 September 2017
Available online 11 April 2019
Submitted by M. Quincampoix

Keywords:
Competitive equilibrium problems
Preference relations
Convex preference
Set-valued maps
Variational inequality problems

ABSTRACT

In this paper we consider a competitive economic equilibrium problem where preferences of consumers are expressed by means of a binary relation. The aim is to find a suitable quasi-variational inequality which characterizes the equilibria and, by using tools of variational theory, to study such equilibria. The novelty of this paper consists in the study of an economic equilibrium problem by a variational approach without the need of representing the consumer's preferences by a utility function.

1. Introduction and motivation

The variational inequality theory was introduced by Fichera and Stampacchia, in the early 1960s, in connection with several equilibrium problems originating from mathematical physics. Thanks to this theory, a large class of equilibrium problems have been analyzed such as the traffic equilibrium, the financial equilibrium, the pollution control, the vaccination problem, the oligopolistic market, see, for instance, [2,5,9–11,16,17]. In particular, with this tool the authors (see e.g. [4,14,15], and the book [8] with its references) provide a qualitative analysis of the problems in terms of existence and uniqueness of solutions, stability and sensitivity analysis.

It is well-known that a minimization problem of a continuous and convex function can be reformulated as a variational inequality, with the operator the subdifferential of the object function. However, a central assumption in economic theory is the convexity of preferences, which corresponds to the quasiconcavity of utility functions. In the setting of quasiconvex functions, in [1] and [3], the authors introduced a Stampacchia variational inequality which involves the concept of normal operator. This variational problem represents a necessary and sufficient condition to a minimization problem when the object function is continuous and

* Corresponding author.
E-mail addresses: mmilasi@unime.it (M. Milasi), puglisia@unime.it (A. Puglisi), vitanzac@gmail.com (C. Vitanza).

https://doi.org/10.1016/j.jmaa.2019.04.022
0022-247X/

quasiconvex. Such approach has been successfully applied to the study of the generalized Nash equilibrium (see e.g. [2]), the competitive equilibrium for exchange economy (see e.g. [9]), and the allocation-price equilibrium for production economy (see e.g. [10]).

These topics are well suited to solve economic equilibrium problems by means of a variational inequality without using representation of the consumer's preferences by a utility function. A key role is represented by an adapted operator to the normal cone. The aim is to apply the variational inequality theory to study an economic equilibrium problem with uncertainty, where consumers' preferences are expressed by means of a binary relation.

The problem considered here is a competitive economic equilibrium problem, which incorporates uncertainty, so that several states of the world are possible. Uncertainty, introduced into the equilibrium theory by Arrow and Debreu, is represented by assuming that technologies, endowments, and preferences depend on the *state of the world*.

The paper is organized as follows. Section 2 is devoted to the preference relation: we introduce definitions and properties which will be useful to the variational approach. In Section 3 a competitive equilibrium model with uncertainty is then introduced. We characterize such a problem in terms of a suitable quasi-variational inequality and, by using arguments of variational theory and set-valued analysis, the existence of equilibrium is finally obtained.

2. Preference relation

This section is devoted to definitions and basic properties in order to introduce the preference relations of a single agent. An individual is characterized by a nonempty subset X of $\mathbb{R}^n$ and a binary relation $\succeq$ on X. The set X represents the alternatives from which she can choose; the preference relation $\succeq$ describes the taste of individual, so that she can compare any pair of elements available on X.
By $x \succeq y$ we denote that *the bundle y is at least as desired by the consumer as x*; by the strict inequality $x \succ y$ we denote that *x is strictly preferred to y*, i.e., $x \succeq y$ *but not* $y \succeq x$. Finally, by $x \sim y$ we denote the case that *x is indifferent to y*, that is, $x \succeq y$ and $y \succeq x$.

The subsequent definitions and results are mostly taken from [13]. To keep the paper self-contained, we restate them adapted to the context of this research.

Definition 2.1. The preference relation $\succeq$ is said to be:

- *complete* iff for all x, $y \in X$, one has that $x \succeq y$ or $y \succeq x$ (or both);
- *transitive* iff for all x, y, $z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$;
- *reflexive* iff for all $x \in X$, one has $x \succeq x$.

Moreover, $\succeq$ is *rational* iff it is complete and transitive.

The assumption that $\succeq$ is complete says that the individual has well-defined preference between any two possible alternatives. Note that transitivity implies that preferences can not cycle.
From now on, let us consider (X, d) a metric space and $\succeq$ a rational and reflexive preference relation on X.

Definition 2.2. Given a preference relation $\succeq$ on a set X, for all $x \in X$, the *strict upper contour set*, $U^s(x)$, is the set of bundles strictly preferred to x, and the *upper contour set*, $U(x)$, is the set of all bundles that are at least as good as x:

$$U^s(x) := \{y \in X : \ y \succ x\}, \quad U(x) := \{y \in X : \ y \succeq x\}.$$

Clearly, for all $x \in X$ one has $U^s(x) \subset U(x)$.

Definition 2.3. The preference relation $\succeq$ is said to be:

- *upper semicontinuous* iff for each x the set $U(x)$ is closed,
- *lower semicontinuous* iff for each x the set $U^s(x)$ is open.

A preference relation $\succeq$ is continuous iff it is upper and lower semicontinuous.

As observed in [13] a preference relation $\succeq$ is continuous if and only if: if the sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ converges to (x, y) and $y_n \succeq x_n$ for each n, then $y \succeq x$.

A central assumption in the economic theory is the convexity of preference. This hypothesis ensures that, despite the individual is indifferent between the two bundles x and y, the mixes are at least desired as much as the extremes.

Definition 2.4. The preference $\succeq$ is said to be:

- *convex* iff for any x, $y \in X$ s.t. $x \succeq y$ one has

$$\lambda x + (1-\lambda)y \succeq y \qquad \forall \lambda \in [0,1]\,;$$

- *semistrictly convex* iff for any x, $y \in X$ s.t. $x \succ y$ one has

$$\lambda x + (1-\lambda)y \succ y \qquad \forall \lambda \in (0,1]\,;$$

- *strictly convex* iff for any x, $y \in X$ s.t. $x \succeq y$ one has

$$\lambda x + (1-\lambda)y \succ y \qquad \forall \lambda \in (0,1)\,.$$

If $\succeq$ is strictly convex, then it is semistrictly convex and convex. In general, the semistrict convexity does not imply the convexity; but if $\succeq$ is semistrictly convex and upper semicontinuous, then it is convex (see [13]). Other important aspect of the preference concerns the desirability.

Definition 2.5. The preference relation $\succeq$ is said to be:

- *locally non-satiated* iff for every $x \in X$ and every $\varepsilon > 0$ there is $y \in X$ such that $\|y - x\| \leq \varepsilon$ and $y \succ x$;
- *non-satiated* iff for every $x \in X$ there is $y \in X$ such that $y \succ x$.

If the preference $\succeq$ is locally non-satiated, then it is non-satiated. The Definition 2.5 can be reformulated by means the strict upper contour sets.

Proposition 2.1. *The preference relation $\succeq$ is*

- locally non-satiated *iff $\forall x \in X$, $\forall \varepsilon > 0$ one has $U^s(x) \cap B(x, \varepsilon) \neq \emptyset$;*
- non-satiated *iff for every $x \in X$ one has $U^s(x) \neq \emptyset$.*

Proposition 2.2. *Following properties are satisfied:*

(i) *$\succeq$ is convex if and only if $U(x)$ is convex for all $x \in X$ if and only if $U^s(x)$ is convex for all $x \in X$.*
(ii) *If $\succeq$ is locally non-satiated and upper semi-continuous, then, for all $x \in X$, $U(x) = \overline{U^s(x)}$.*

Proof. (i) It follows from definition of convexity of $\succeq$.

(ii) $U(x) \subseteq \overline{U^s(x)}$: fixed $y \in U(x)$. For all $n \in \mathbb{N}$, let $\varepsilon = \frac{1}{n} > 0$, $\succeq$ being locally non-satiated, there exists $y_n \in B(y, \frac{1}{n}) \cap X$ such that $y_n \succ y$. Hence, being $y_n \succ y$ and $y \succeq x$, from transitivity of $\succeq$ one has $y_n \succ x$. Then $\{y_n\}_{n\in\mathbb{N}} \subseteq U^s(x)$ and $y_n \to y$, namely $y \in \overline{U^s(x)}$.

$\overline{U^s(x)} \subseteq U(x)$: obvious. $\square$

Theorem 2.1. *Let $\succeq$ be a continuous relation preference. Then, $\succeq$ is locally non-satiated and convex if and only if it is non-satiated and semistrictly convex.*

Proof. $\Rightarrow$) It is sufficient to show that $\succeq$ is convex. Let $x, y \in X$ be such that $y \succ x$. Since $\succeq$ lower semicontinuous, $U^s(x)$ is an open and convex set then, from item (ii) of Proposition 2.2, $y \in U^s(x) = \operatorname{int} U(x)$. Then, for all $\lambda \in [0, 1)$ one has $\lambda x + (1-\lambda)y \in \operatorname{int} U(x)$, then $\lambda x + (1-\lambda)y \succ x$. Hence, $\succeq$ is semistrictly convex.

$\Leftarrow$) From upper semicontinuity and semistrict convexity it follows that $\succeq$ is convex. Let $x \in X$ and $\varepsilon > 0$. Since $\succeq$ is non-satiated, there exists $x' \in X$ such that $x' \succ x$. Let $z := \lambda x' + (1-\lambda)x$, with $\lambda \in \left(0, \min\left\{\frac{\varepsilon}{\|x'-x\|}, 1\right\}\right)$. One has $z \in B(x, \varepsilon)$ and from semistrictly convexity of $\succeq$, $z \succ x$. Then, $\succeq$ is locally non-satiated. $\square$

Now, a suitable operator is introduced. This map will have a central role in characterizing the maximum for the preference by means of a variational inequality.

Let $\succeq$ be a convex and non-satiated preference relation; let $N : X \rightrightarrows \mathbb{R}^n$ such that, for all $x \in X$

$$N(x) := \{h \in \mathbb{R}^n : \langle h, z - x\rangle \leq 0 \quad \forall z \in U^s(x)\}.$$

Since $\succeq$ is convex and non-satiated, for all $x \in X$ the set $U^s(x)$ is convex and nonempty, then $N(x)$ coincides with the normal cone of the strict upper contour set $U^s(x)$. The following properties on the set-valued map N will be useful in the sequel.

Proposition 2.3. *Let $\succeq$ be a continuous preference relation. Then, the set-valued map $N(\cdot)$ is closed and with convex and closed values.*

Proof. From definition of N and from continuity of the inner product, one has that for all $x \in X$, $N(x)$ is a convex and closed set. Let $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ and $\{h_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^n$ be two sequences such that $h_n \in N(x_n)$, $x_n \to x$ and $h_n \to h$. It is necessary to prove that $h \in N(x)$. To this aim let us prove that for all $z \in U^s(x)$, $z \succ x_n$. Indeed, if there exist $z \in U^s(x)$ and $\nu \in \mathbb{N}$ such that $x_n \succeq z$ for all $n > \nu$; one has $\{x_n\} \subseteq U(z)$. From continuity of $\succeq$, one has that $U(z)$ is a closed set, then, since $x_n \to x$, it follows $x \succeq z$, which contradicts the assumption $z \in U^s(x)$. Hence $z \in U^s(x_n)$. Since $h_n \in N(x_n)$ and $z \in U^s(x_n)$, one has $\langle h_n, z - x_n\rangle \leq 0$; passing to the limit $\langle h, z - x\rangle \leq 0$. $\square$

Proposition 2.4. *Let $\succeq$ be a lower semicontinuous preference relation and $x \in X$. For all $h \in N(x)$ with $h \neq 0$, one has*

$$\langle h, z - x\rangle < 0 \qquad \forall z \in U^s(x).$$

Proof. From definition of N one has $\langle h, z - x\rangle \leq 0$ for all $z \in U^s(x)$. We suppose there exists $\widetilde{z} \in U^s(x)$ such that $\langle h, \widetilde{z} - x\rangle = 0$. Since $\succeq$ is lower semicontinuous $U^s(x)$ is an open set, then $\widetilde{z} \in U^s(x) = \operatorname{int} U^s(x)$: there exists $\varepsilon > 0$ such that $B(\widetilde{z}, \varepsilon) \subseteq U^s(x)$. Set $z := \widetilde{z} + \alpha h$, with $\alpha \in \left(0, \frac{\varepsilon}{\|h\|}\right)$. One has $z \in U^s(x)$, and $\langle h, z - x\rangle = \alpha\|h\|^2 \leq 0$, that is $\|h\| = 0$. A contradiction with $h \neq 0$. $\square$

3. General equilibrium under uncertainty

In this Section a market economy under uncertainty is introduced. Agents of economy produce, trade, and consume L commodities; let $\mathcal{L}$ be the set of L physical commodities $\mathcal{L} := \{1, \dots, l, \dots, L\}$. In an economy with uncertainty, commodities are to be distinguished, not only by their physical characteristics and the location and dates of their availability and/or use, but also by the environmental event in which they are made available and/or used. $\mathcal{S} := \{1, \dots, s, \dots, S\}$ denotes the finite set of S states of the world, is given. The dependence of the characteristics of economic agents on the state of the world is expressed by introducing the concept of contingent commodities vectors.

Definition 3.1. For every physical commodity $l \in \mathcal{L}$ and state $s \in \mathcal{S}$ a unit of state-contingent commodity ls is a title to receive a unit of the physical good l if and only if s occurs. Accordingly, a state-contingent commodity vector is specified by

$$x := (x_{11}, \dots, x_{L1}, \dots, x_{1S}, \dots, x_{LS}) \in \mathbb{R}^{LS},$$

and is understood as an entitlement to receive the commodity vector $(x_{1s}, \dots, x_{Ls})$ if state s occurs.

In the market, there are two kinds of agents, I consumers and J producers; $\mathcal{I} := \{1, \dots, i, \dots, I\}$ and $\mathcal{J} := \{1, \dots, j, \dots, J\}$ are, respectively, the sets of consumers and the producers.

The system price To each state-contingent commodity is assigned a nonnegative price, denoted by $p \in \mathbb{R}^{LS}_+$. Given a state-contingent commodity vector x, the inner product $\langle p, x \rangle$ denotes its value at the current price p.

The producers Each producer j is characterized by a set $Y_j \subset \mathbb{R}^{LS}$ of possible production plans. y_j denotes a state-contingent commodity vector which lies in Y_j: the positive components of y_j denote the output of commodity, while the negative components represent the inputs. The product $Y := \prod_{j \in \mathcal{J}} Y_j$ represents all possible output-input schedules for the production sector. Given the production plan $y \in Y_j$, the inner product $\langle p, y \rangle$ is the profit of the agent at the system price p. Each producer acts in the market to maximize his profit. For all $j \in \mathcal{J}$ the following assumptions hold:

Assumption 3.1. Y_j are closed, convex, bounded, and $0 \in Y_j$.

The consumer Each consumer i is characterized by a consumption set X_i, a preference relation $\succeq_i$ on X_i, and an endowment ω_i. The set X_i is a non-empty subset of $\mathbb{R}^{LS}_+$ and represents the set of all possible state-contingent commodities which consumer i can choose to consume. $x_i \in X_i$ denotes the consumption plan of agent i. The preference relation $\succeq_i$ is assumed to be complete, reflexive, and transitive and describes the tastes of the consumer i on the consumption set X_i. Finally, the contingent commodity vector $\omega_i \in \mathbb{R}^{LS}$ specifies the quantity of initial endowment for each one of the commodities and of the state. For all consumer $i \in \mathcal{I}$ the following assumptions hold:

Assumption 3.2. The consumption set X_i is closed and convex.

Assumption 3.3. The preference relation $\succeq_i$ is continuous, non-satiated, and semistrictly convex.

Assumption 3.4. There exists $\widehat{x}_i \in X_i$ such that $\omega_i >> \widehat{x}_i$.[1]

[1] It means $\omega_i^l > \widehat{x}_i^l$ for all $l = 1, \dots, LS$.

Given a price system p and his wealth w_i the i-th consumer tries to satisfy her preferences $\succeq_i$ in the subset of X_i defined by the wealth constraint $\langle p, x_i\rangle \leq w_i$. Each consumer i receives a share of total production $\sum_{j\in\mathcal{J}} \theta_{ij} y_j$, determined by a system of fixed weights θ_{ij}, where $\theta_{ij} \geq 0$ and $\sum_{i\in\mathcal{I}} \theta_{ij} = 1$. Given a price system p and production y_j the wealth of i-th consumer is $w_i := \langle p, \omega_i\rangle + \sum_{j\in\mathcal{J}} \theta_{ij}\langle p, y_j\rangle$.

In summary, an economy Σ is described by the I-list $(X_i, \succeq_i, \omega_i)$, by the IJ-shares (θ_{ij}), and the J-list (Y_j), defining thus the economy $\Sigma := ((X_i, \succeq_i, \omega_i), (\theta_{ij}), (Y_j))$.

In the market there is an equilibrium if, simultaneously, each producer maximizes her profit, each consumer maximize her preferences on the budget set, and for each commodity the excess of demand over supply is less or equal to zero. Mathematically, one has the following definition.

Definition 3.2. An allocation $(\widetilde{x}_1, \ldots, \widetilde{x}_l, \widetilde{y}_1, \ldots, \widetilde{y}_J) \in X_1 \times \ldots X_I \times Y_1 \times \ldots Y_J \subset \mathbb{R}^{LS(I+J)}$ and a system of prices for the contingent commodities $\widetilde{p} = (\widetilde{p}_{11}, \ldots, \widetilde{p}_{LS}) \in \mathbb{R}^{LS}$ constitute an Arrow Debreu equilibrium for the economy Σ if

(i) for every $j \in \mathcal{J}$, $\widetilde{y}_j$ satisfies $\langle \widetilde{p}, \widetilde{y}_j\rangle \geq \langle \widetilde{p}, y_j\rangle$ for all $y_j \in Y_j$;

(ii) for every $i \in \mathcal{I}$, $\widetilde{x}_i$ is maximal for $\succeq_i$ in the budget set

$$K_i(\widetilde{p}, \widetilde{y}) := \{x_i \in X_i : \langle \widetilde{p}, x_i\rangle \leq \langle \widetilde{p}_i, \omega_i\rangle + \sum_{j\in\mathcal{J}} \theta_{ij}\langle \widetilde{p}, \widetilde{y}_j\rangle\}.$$

(iii) for every $l \in \mathcal{L}$ and $s \in \mathcal{S}$:

$$\sum_{i\in\mathcal{I}} \widetilde{x}_i{}^{ls} \leq \sum_{j\in\mathcal{J}} \widetilde{y}_j{}^{ls} + \sum_{i\in\mathcal{I}} \omega_i^{ls}$$
$$\langle \widetilde{p}, \sum_{i\in\mathcal{I}} \widetilde{x}_i - \sum_{j\in\mathcal{J}} \widetilde{y}_j - \sum_{i\in\mathcal{I}} \omega_i\rangle = 0$$

Without loss of generality, we can consider prices that lie in the simplex set of $\mathbb{R}^{LS}_+$, denoted by $P := \{p \in \mathbb{R}^{LS}_+ : \sum_{l\in\mathcal{L},\, s\in\mathcal{S}} p^{ls} = 1\}$.

The first aim is to rewrite the Arrow Debreu equilibrium in terms of a suitable variational inequality. For all $i \in \mathcal{I}$, let $U_i^s(x_i)$ and $U_i(x_i)$ be, respectively, the strict upper contour set and the upper contour set at the consumption plan x_i; let $N_i : X_i \rightrightarrows \mathbb{R}^n$ be the set-valued map defined in the Section 2 and let $K(\widetilde{p}, \widetilde{y}) := \prod_{i\in\mathcal{I}} K_i(\widetilde{p}, \widetilde{y}) \times Y \times P$. Let us introduce the following quasi-variational inequality $QVI(N, K)$:

Find $(\widetilde{x}, \widetilde{y}, \widetilde{p}) \in K(\widetilde{p}, \widetilde{y})$ and $h := (h_i)_{i\in\mathcal{I}} \in N(\widetilde{x}) := \prod_{i\in\mathcal{I}} N_i(\widetilde{x}_i) \setminus \{0\}$ s.t.

$$\sum_{i\in\mathcal{I}}\langle h_i, x_i - \widetilde{x}_i\rangle - \sum_{j\in\mathcal{J}}\langle \widetilde{p}, y_j - \widetilde{y}_j\rangle - \langle \sum_{i\in\mathcal{I}} \widetilde{x}_i - \sum_{j\in\mathcal{J}} \widetilde{y}_j - \sum_{i\in\mathcal{I}} \omega_i, p - \widetilde{p}\rangle \geq 0$$
$$\forall (x, y, p) \in K(\widetilde{p}, \widetilde{y}). \tag{1}$$

Lemma 3.1. *Let $\succeq_i$ be continuous, semistrictly convex and non-satiated. Let $(p, y) \in P \times Y$ and $\widetilde{x} := (\widetilde{x}_i)_{i\in\mathcal{I}}$ be such that for all $i \in \mathcal{I}$, $\widetilde{x}_i$ is maximal for $\succeq_i$ in $K_i(p, y)$. Then:*

$$\langle p, \sum_{i\in\mathcal{I}} \widetilde{x}_i\rangle = \langle p, \sum_{i\in\mathcal{I}} \omega_i\rangle + \langle p, \sum_{j\in\mathcal{J}} y_j\rangle. \tag{2}$$

Proof. Since for any j, $\sum_{i\in\mathcal{I}}\theta_{ij}=1$, it is sufficient to prove that, for all $i\in\mathcal{I}$:

$$\langle p,\widetilde{x}_i\rangle=\langle p,\omega_i\rangle+\sum_{j\in\mathcal{J}}\theta_{ij}\langle p,y_j\rangle\,. \tag{3}$$

Let us assume that for some $i\in\mathcal{I}$ one has $\langle p,\widetilde{x}_i\rangle<\langle p,\omega_i\rangle+\sum_{j\in\mathcal{J}}\theta_{ij}\langle p,y_j\rangle$. Then, there exists $\overline{\varepsilon}>0$ such that for all $z\in B(\widetilde{x}_i,\overline{\varepsilon})\cap X_i$ one has $\langle p,z\rangle<\langle p,\omega_i\rangle+\sum_{j\in\mathcal{J}}\theta_{ij}\langle p,y_j\rangle$, that is $B(\widetilde{x}_i,\overline{\varepsilon})\cap X_i\subseteq K_i(p,y)$. Since the preference relation $\succeq_i$ is continuous, semistrictly convex and non-satiated, from Theorem 2.1 it follows that $\succeq_i$ is locally non-satiated: there exists $\widetilde{z}\in B(\widetilde{x}_i,\overline{\varepsilon})\cap X_i$ such that $\widetilde{z}\succ\widetilde{x}_i$. Since $B(\widetilde{x}_i,\overline{\varepsilon})\cap X_i\subseteq K_i(p,y)$, this contradicts the fact that $\widetilde{x}_i$ is maximal in $K_i(p,y)$. Hence, equation (3) holds for all $i\in\mathcal{I}$. □

Theorem 3.1. *Let Σ be an economy which satisfies Assumptions 3.1, 3.2, 3.3, and 3.4. Then, $(\widetilde{x},\widetilde{y},\widetilde{p})$ is a solution to $QVI(N,K)$ (1) if and only if it is an Arrow Debreu equilibrium for the economy Σ.*

Proof. First, let us observe that:
$i)$ the vector $(\widetilde{x},\widetilde{y},\widetilde{p})$ is a solution to $QVI(N,K)$ (1) if and only if, simultaneously there exists $h=(h_i)_{i\in\mathcal{I}}\in N(\widetilde{x})$ such that:

$$\text{for every}\quad i\in\mathcal{I}\qquad\langle h_i,x_i-\widetilde{x}_i\rangle\geq 0\qquad\forall x_i\in K_i(\widetilde{p},\widetilde{y})\,; \tag{4}$$

$$\text{for every}\quad j\in\mathcal{J}\qquad\langle -\widetilde{p},y_j-\widetilde{y}_j\rangle\geq 0\qquad\forall y_j\in Y_j\,; \tag{5}$$

$$\langle\sum_{j\in\mathcal{J}}\widetilde{y}_j+\sum_{i\in\mathcal{I}}\omega_i-\sum_{i\in\mathcal{I}}\widetilde{x}_i,p-\widetilde{p}\rangle\geq 0\qquad\forall p\in P\,. \tag{6}$$

$ii)$ VI (5) is equivalent to condition (i) of Definition 3.2.

Claim A). *For any $i\in\mathcal{I}$, $\widetilde{x}_i$ is a solution to VI (4) if and only if $\widetilde{x}_i$ is maximal for $\succeq_i$ in $K_i(\widetilde{p},\widetilde{y})$.*

Let $\widetilde{x}_i$ be a solution to VI (4). We suppose that $\widetilde{x}_i$ is not maximal for $\succeq_i$ in $K_i(\widetilde{p},\widetilde{y})$: there exists $\widetilde{z}_i\in K_i(\widetilde{p},\widetilde{y})$ such that $\widetilde{z}_i\succ\widetilde{x}_i$. Since $\widetilde{z}_i\in U_i^s(\widetilde{x}_i)$, $\succeq_i$ is continuous and $h_i\in N_i(\widetilde{x}_i)\backslash\{0\}$, from Proposition 2.4 one has $\langle h_i,\widetilde{z}_i-\widetilde{x}_i\rangle<0$ which contradicts the fact that $\widetilde{x}_i$ is a solution to VI (4). Hence, condition (ii) of Definition 3.2 holds.

Let $\widetilde{x}_i$ be maximal for $\succeq_i$ in $K_i(\widetilde{p},\widetilde{y})$. From item $i)$ of Proposition 2.2 $U_i^s(\widetilde{x}_i)\neq\emptyset$, and $\operatorname{int}U_i^s(\widetilde{x}_i)\cap K_i(\widetilde{p},\widetilde{y})=\emptyset$; then from separation theorem, there exists $h_i\in\mathbb{R}^n\setminus\{0\}$ such that

$$\langle h_i,s\rangle\leq\langle h_i,t\rangle\qquad\forall s\in U_i^s(\widetilde{x}_i),\quad\forall t\in K_i(\widetilde{p},\widetilde{y})\,. \tag{7}$$

Observing that relation $\succeq$ is semistrictly convex, non-satiated and continuous, it is also, according to Theorem 2.1, locally non-satiated and convex and thus, as a consequence of Proposition 2.2 $iii)$, $\widetilde{x}_i\in\overline{U_i^s(\widetilde{x}_i)}$. Combining with (7) it gives

$$\langle h_i,t-\widetilde{x}_i\rangle\geq 0\qquad\forall t\in K_i(\widetilde{p},\widetilde{y})\,.$$

Moreover, replacing in (7), $t=\widetilde{x}_i$ one has

$$\langle h_i,s-\widetilde{x}_i\rangle\leq 0\qquad\forall s\in U_i^s(\widetilde{x}_i)\,.$$

Then, $h_i \in N_i(\widetilde{x}_i) \setminus \{0\}$ and it is such that $\widetilde{x}_i$ is a solution to (4).

Claim B). *If $(\widetilde{x}, \widetilde{y}, \widetilde{p})$ is a solution of $QVI(N, K)$ (1), it is an Arrow Debreu equilibrium.*

Thanks to step ii) and Claim A), it remains to prove condition (iii) of Definition 3.2. Since $\widetilde{x}_i$ is maximal for $\succeq_i$ in $K_i(\widetilde{p}, \widetilde{y})$, from Lemma 3.1, it follows the second condition of (iii), and moreover, from VI (6) and equality (2), one has:

$$\langle \sum_{j \in \mathcal{J}} \widetilde{y}_j + \sum_{i \in \mathcal{I}} \omega_i - \sum_{i \in \mathcal{I}} \widetilde{x}_i, p \rangle \geq 0 \qquad \forall p \in P\,. \tag{8}$$

Fixed $\widetilde{l} \in \mathcal{L}$ and $\widetilde{s} \in \mathcal{S}$, let $p \in P$ be such that

$$p^{\,ls} := \begin{cases} 0 & \text{for} \quad ls \neq \widetilde{l}\widetilde{s} \\ 1 & \text{for} \quad ls = \widetilde{l}\widetilde{s} \end{cases}$$

Replacing p in (8), one has:

$$\sum_{i \in \mathcal{I}} \widetilde{x}_i^{\,\widetilde{l}\widetilde{s}} \leq \sum_{j \in \mathcal{J}} \widetilde{y}_j^{\,\widetilde{l}\widetilde{s}} + \sum_{i \in \mathcal{I}} \omega_i^{\,\widetilde{l}\widetilde{s}}\,.$$

It follows that conditions (iii) of Definition 3.2 hold.

Claim C). *If $(\widetilde{x}, \widetilde{y}, \widetilde{p})$ is an Arrow Debreu equilibrium, it is a solution to $QVI(N, K)$ (1).*

Thesis follows from steps ii), Claims A) and B) and from the fact that condition (iii) of Definition 3.2 implies (6). □

In view of the characterization of the equilibrium as a solution to Problem (1), it is possible to study the equilibrium problem by means of the variational theory. In particular, we show the existence of the equilibrium. Let us observe that in (1) the map $K(\cdot)$ has not bounded values and, it is difficult to obtain the upper semicontinuity of the operator N (since it is unbounded). The latter facts represent the difficulty to achieve the existence of the solution. We overcame such difficulties by introducing a compact set and the normalized normal operator.

Theorem 3.2. *Let Assumptions 3.1, 3.2, 3.3, and 3.4 be satisfied for the economy Σ. Then, there exists an Arrow Debreu equilibrium for the economy Σ.*

Proof. Thanks to Theorem 3.1, it is sufficient to prove that there exists a solution to $QVI(N, K)$ (1). To this aim we consider the variational problem (1), where we replace to N and K by suitable set-valued maps. Since Y is a compact set, there exists $M_1 > 0$ such that $Y \subset B(0, M_1)$; set $M := M_1 + \sum_{i \in \mathcal{I}} \omega_i$ and $\widetilde{K}(y, p) := \prod_{i \in \mathcal{I}} \Big(K_i(y, p) \cap B(0, M) \Big) \times Y \times P$. We define the set-valued map $\widetilde{N}_i : (x) = conv\,(N_i(x) \cap S)$, where S is the boundary of the unit ball of $\mathbb{R}_+^{LS}$. Now, we consider the variational problem $QVI(\widetilde{N}, \widetilde{K})$ (1).

◀ *There exists the solution of the $QVI(\widetilde{N}, \widetilde{K})$.*

With similar techniques used in Theorem 9 of [6] and Theorem 4.2 of [12], also in the contingent commodities case one has that the set-valued map $\widetilde{K}(\cdot)$ is closed, lower semicontinuous and with nonempty, convex, compact values. Clearly, $\widetilde{N}$ is nonempty, convex and compact values and, compact graph since $\widetilde{N}_i(X_i)$ is compact. Moreover, $N_i(\cdot)$ being a closed map (from Proposition 2.3), it follows that $\widetilde{N}_i$ is closed.

Hence, since $\widetilde{N}_i$ is closed and compact graph, it is upper semicontinuous (see [7]). From existence Theorem in [18], we can conclude that $QVI(\widetilde{N}, \widetilde{K})$ admits at least a solution.

◀ *Any solution of the* $QVI(\widetilde{N}, \widetilde{K})$ *is a solution of* $QVI(N, K)$.

First, let us prove that $h_i \neq 0$. Indeed, from $h_i \in \widetilde{N}_i(\widetilde{x}_i) = conv\,(N_i(\widetilde{x}_i) \cap S)$ there exist $v_i^k \in (N_i(\widetilde{x}_i) \cap S)$ and $\lambda^k \in [0, 1]$, with $k = 1, \dots, LS + 1$, such that $\sum_{i=1}^{LS+1} \lambda^k = 1$ and $h_i = \sum_{k=1}^{LS+1} \lambda^k v_i^k$. If $h_i = 0$, since $v_i^k \neq 0$, for all k, and there exists $\lambda^{\overline{k}} \neq 0$, one has:

$$-v_i^{\overline{k}} = \sum_{k=1, k\neq\overline{k}}^{LS+1} \frac{\lambda^k}{\lambda^{\overline{k}}} v_i^k\,.$$

Since $N_i(\widetilde{x})$ a convex cone, it follows $-v_i^{\overline{k}} \in N_i(\widetilde{x})$. Hence, from Proposition 2.3, from $v_i^{\overline{k}},\ -v_i^{\overline{k}} \in N_i(\widetilde{x}) \setminus \{0\}$, it follows $\langle v_i^{\overline{k}}, z - x\rangle < 0$ and $\langle -v_i^{\overline{k}}, z - x\rangle < 0$ for all $z \in U^s(x)$, which is a contradiction.

Hence $h_i \in N_i(\widetilde{x}_i) \setminus \{0\}$. It remains to prove that $(\widetilde{x}, \widetilde{y}, \widetilde{p})$ is a solution in K. We suppose that there exists $(x, y, p) \in K(\widetilde{p}, \widetilde{y}) \times Y \times P$ such that

$$\sum_{i\in\mathcal{I}} \langle h_i, x_i - \widetilde{x}_i\rangle - \sum_{j\in\mathcal{J}} \langle \widetilde{p}, y_j - \widetilde{y}_j\rangle - \langle \sum_{i\in\mathcal{I}} \widetilde{x}_i - \sum_{j\in\mathcal{J}} \widetilde{y}_j - \sum_{i\in\mathcal{I}} \omega_i, p - \widetilde{p}\rangle < 0\,.$$

Let $(\overline{x}, \overline{y}, \overline{p}) := \lambda(x, y, p) + (1 - \lambda)(\widetilde{x}, \widetilde{y}, \widetilde{p})$ with $0 < \lambda < \dfrac{M - \|\widetilde{x}\|}{\|x - \widetilde{x}\|}$. Since $K(\widetilde{p}, \widetilde{y})$ is a convex set and from $\|\widetilde{x}\| < M$, one has $(\overline{x}, \overline{y}, \overline{p}) \in \widetilde{K}(\widetilde{x}, \widetilde{y}, \widetilde{p})$ and

$$\sum_{i\in\mathcal{I}} \langle h_i, \overline{x}_i - \widetilde{x}_i\rangle - \sum_{j\in\mathcal{J}} \langle \widetilde{p}, \overline{y}_j - \widetilde{y}_j\rangle - \langle \sum_{i\in\mathcal{I}} \widetilde{x}_i - \sum_{j\in\mathcal{J}} \widetilde{y}_j - \sum_{i\in\mathcal{I}} \omega_i, \overline{p} - \widetilde{p}\rangle < 0\,.$$

This contradicts the fact that $(\widetilde{x}, \widetilde{y}, \widetilde{p})$ is a solution to $QVI(\widetilde{N}, \widetilde{K})$ (1). □

Acknowledgment

The authors are grateful to the referee for number of comments and useful suggestions which led to an improved version of this manuscript.

References

[1] D. Aussel, New developments in quasiconvex optimization, in: S.A.R. Al-Mezel, F.R.M. Al-Solamy, Q.H.Ansari (Eds.), Fixed Point Theory, Variational Analysis, and Optimization, Chapman and Hall, CRC Press, 2014.

[2] D. Aussel, J. Dutta, Generalized Nash equilibrium problem, variational inequality and quasiconvexity, Oper. Res. Lett. 36 (2008) 461–464.

[3] D. Aussel, N. Hadjisavvas, Adjusted sublevel sets, normal operator and quasiconvex programming, SIAM J. Optim. 16 (2005) 358–367.

[4] A. Barbagallo, Regularity results for time-dependent variational and quasi-variational inequalities and application to the calculation of dynamic traffic network, Math. Models Methods Appl. Sci. 17 (2) (2007) 277–304.

[5] A. Barbagallo, M. Cojocaru, Dynamic vaccination games and variational inequalities on time-dependent sets, J. Biol. Dyn. 4 (6) (2010) 539–558.

[6] I. Benedetti, M.B. Donato, M. Milasi, Existence for competitive equilibrium by means of generalized quasivariational inequalities, Abstr. Appl. Anal. 2013 (2013) 648986.

[7] C. Berge, Topological Spaces, The Macmillan Company, New York, 1963.

[8] P. Daniele, Dynamic Networks and Evolutionary Variational Inequalities, Edward Elgar Publishing, 2006.

[9] M.B. Donato, M. Milasi, C. Vitanza, Variational problem, generalized convexity, and application to a competitive equilibrium problem, Numer. Funct. Anal. Optim. 35 (2014) 962–983.

[10] M.B. Donato, M. Milasi, C. Vitanza, Generalized variational inequality and general equilibrium problem, J. Convex Anal. 25 (2) (2018) 515–527.

[11] F. Facchinei, J.S. Pang, Nash equilibria: the variational approach, in: Convex Optimization in Signal Processing and Communications, 2009, pp. 443–493.
[12] R. Lucchetti, M. Milasi, Semistrictly quasiconcave approximation and an application to general equilibrium theory, J. Math. Anal. Appl. 428 (2015) 445–456.
[13] A. Mas-Colell, M.D. Whinston, J.R. Green, Microeconomic Theory, Oxford University Press, 1995.
[14] A. Maugeri, L. Scrimali, Global lipschitz continuity of solutions to parameterized variational inequalities, Boll. Unione Mat. Ital. 2 (1) (2009) 45–69.
[15] M. Milasi, Existence theorem for a class of generalized quasi-variational inequalities, J. Global Optim. 60 (4) (2014) 679–688.
[16] M. Milasi, C. Vitanza, Variational inequality and evolutionary market disequilibria: the case of quantity formulation, in: F. Giannessi, A. Maugeri (Eds.), Variational Analysis and Applications, in: Nonconvex Optimization and Its Applications, vol. 79, 2005, pp. 681–696.
[17] L. Scrimali, Pollution control quasi-equilibrium problems with joint implementation of environmental projects, Appl. Math. Lett. 25 (3) (2012) 385–392.
[18] N.X. Tan, Quasi-variational inequality in topological linear locally convex Hausdorff spaces, Math. Nachr. 122 (1985) 231–245.