



Effect of nonlinear diffusion on a lower bound for the blow-up time in a fully parabolic chemotaxis system



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ABSTRACT

This paper deals with a lower bound for the blow-up time for solutions of the fully parabolic chemotaxis system

$$\begin{cases} u_t = \nabla \cdot [(u + \alpha)^{m_1-1} \nabla u - \chi u (u + \alpha)^{m_2-2} \nabla v] & \text{in } \Omega \times (0, T), \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0, T) \end{cases}$$

under Neumann boundary conditions and initial conditions, where Ω is a general bounded domain in \mathbb{R}^n with smooth boundary, $\alpha > 0$, $\chi > 0$, $m_1, m_2 \in \mathbb{R}$ and $T > 0$. Recently, Anderson–Deng [1] gave a lower bound for the blow-up time in the case that $m_1 = 1$ and Ω is a convex bounded domain. The purpose of this paper is to generalize the result in [1] to the case that $m_1 \neq 1$ and Ω is a non-convex bounded domain. The key to the proof is to make a sharp estimate by using the Gagliardo–Nirenberg inequality and an inequality for boundary integrals. As a consequence, the main result of this paper reflects the effect of nonlinear diffusion and need not assume the convexity of Ω .

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1. Introduction

In this paper we consider a lower bound for the blow-up time in the following fully parabolic chemotaxis system with nonlinear diffusion:

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$$\begin{cases} u_t = \nabla \cdot [(u + \alpha)^{m_1-1} \nabla u - \chi u(u + \alpha)^{m_2-2} \nabla v] & \text{in } \Omega \times (0, T), \\ v_t = \Delta v - v + u & \text{in } \Omega \times (0, T), \\ \nabla u \cdot \nu = \nabla v \cdot \nu = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0, \quad v(\cdot, 0) = v_0 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where Ω is a general bounded domain in \mathbb{R}^n ($n \in \mathbb{N}$) with smooth boundary $\partial\Omega$ and ν is the outward normal vector to $\partial\Omega$ and $T > 0$. The initial data u_0 and v_0 are supposed to be nonnegative functions such that $u_0 \in C(\overline{\Omega})$ and $v_0 \in C^1(\overline{\Omega})$. Also we assume that

$$\alpha > 0, \quad \chi > 0, \quad m_1, m_2 \in \mathbb{R}.$$

In the system (1.1), the unknown function $u(x, t)$ represents the density of the cell population and the unknown function $v(x, t)$ shows the concentration of the signal substance at place x and time t . The system (1.1) with the simplest choices $m_1 = 1$ and $m_2 = 2$ describes a part of life cycle of cellular slime molds with chemotaxis and it was proposed by Keller–Segel [18] in 1970. After that, a quasilinear system such as (1.1) was proposed by Painter–Hillen [25]. A number of variations of the original Keller–Segel system are proposed and investigated (see e.g., Bellomo–Bellouquid–Tao–Winkler [2], Hillen–Painter [9] and Horstmann [10,11]).

According to a continuity model, the first equation in (1.1) has the flux vector $F = -[(u + \alpha)^{m_1-1} \nabla u - \chi u(u + \alpha)^{m_2-2} \nabla v]$. We can recognize that $(u + \alpha)^{m_1-1} \nabla u$ represents the diffusive flux and $-\chi u(u + \alpha)^{m_2-2} \nabla v$ represents the chemotactic flux modeling undirected cell migration and the advective flux with velocity dependent on the gradient of the signal. More precisely, when cellular slime molds plunge into hunger, they move towards higher concentrations of the chemical substance secreted by cells.

From a mathematical point of view, u in (1.1) enjoys the mass conservation property:

$$\int_{\Omega} u(\cdot, t) = \int_{\Omega} u_0 \quad (1.2)$$

for all $t \in (0, T)$. It is a meaningful question whether solutions of (1.1) remain bounded or blow up. As to this question, it is known that the borderline between boundedness and blow-up is the case that $m_2 = m_1 + \frac{2}{n}$, $m_1 \geq 1$, $m_2 \geq 2$. According to the result established by Horstmann–Winkler [12, Theorems 4.1 and 6.1] in the case $m_1 = 1$, it can be expected that (1.1) has a global bounded solution in the case that $m_2 < m_1 + \frac{2}{n}$ and a blow-up solution in the case that $m_2 > m_1 + \frac{2}{n}$. Indeed, in the case that Ω is a bounded domain and $m_2 < m_1 + \frac{2}{n}$, there exists a global bounded solution of (1.1) (see Tao–Winkler [33], Ishida–Seki–Yokota [13] and Senba–Suzuki [31]). In addition, this result was shown also for the degenerate chemotaxis system ((1.1) with $\alpha = 0$) (see Ishida–Yokota [14,16] when $\Omega = \mathbb{R}^n$ and $m_2 < m_1 + \frac{2}{n}$; [13] when Ω is a bounded domain and $m_2 < m_1 + \frac{2}{n}$; Mimura [23] when Ω is a bounded domain with Dirichlet–Neumann boundary condition, $m_2 = 2$ and $m_1 > 2 - \frac{2}{n}$). If $m_2 \geq m_1 + \frac{2}{n}$, then the results are divided by the size of initial data. For example, the system (1.1) has a global solution with small initial data when $\Omega = \mathbb{R}^n$ and $\alpha = 0$ even if $m_2 \geq m_1 + \frac{2}{n}$ (see Ishida–Yokota [15]). On the other hand, in the case that $\Omega = B_R := \{x \in \mathbb{R}^n \mid |x| < R\}$ ($R > 0$), $m_1 = 1$, $m_2 = 2$, $n \geq 3$, which implies $m_2 > m_1 + \frac{2}{n}$, there exist initial data such that the radially symmetric solution of (1.1) blows up in finite time (see Winkler [34]). The result was extended to the case that $\Omega = B_R$, $m_2 > m_1 + \frac{2}{n}$, $n \geq 2$ (see Cieřlak–Stinner [3,4] when $\alpha > 0$, Hashira–Ishida–Yokota [8] when $\alpha = 0$). In the most important case that $\Omega = B_R$, $m_1 = 1$, $m_2 = 2$, $n = 2$, which implies $m_2 = m_1 + \frac{2}{n}$, there exist initial data such that the corresponding solutions of (1.1) blow up in finite time (see Mizoguchi–Winkler [24]).

We are especially interested in a *lower bound* for the blow-up time for solutions of (1.1), because it seems to be important to know how m_1 affects the blow-up time for solutions of (1.1). The study of a lower

bound for the blow-up time seems to be interesting widely for general parabolic systems (see Payne–Schaefer [27] and Enache [5]), wave equations (see Philippin [30]) and heat equations (see Payne–Philippin–Vernier Piro [26]). Moreover, explicit lower bounds for the blow-up time for solutions of various semilinear parabolic equations were obtained by [27]. As to chemotaxis systems, Payne–Song [28,29] established a lower bound of the blow-up time for solutions of (1.1) with $m_1 = 1$ and $m_2 = 2$ in the form

$$\tilde{t}^* \geq \int_{\Phi_1(0)}^{\infty} \frac{d\xi}{V\xi^{\frac{3}{2}} + W\xi^2} \quad (n = 2)$$

and

$$\tilde{t}^* \geq \int_{\Phi_1(0)}^{\infty} \frac{d\xi}{X\xi^{\frac{3}{2}} + Y\xi^3} \quad (n = 3);$$

note that \tilde{t}^* means the blow-up time in Φ_1 -measure, i.e., $\lim_{t \nearrow \tilde{t}^*} \Phi_1(t) = \infty$, where $\Phi_1(t)$ is defined as

$$\Phi_1(t) := \kappa \int_{\Omega} u(\cdot, t)^2 + \int_{\Omega} |\Delta v(\cdot, t)|^2 \quad (t > 0) \quad (1.3)$$

with some $\kappa > 0$. When Ω is a convex bounded domain and $m_1 = 1$, Li–Zheng [19] gave a lower bound for the blow-up time for solutions of (1.1) by using (1.3) in the case that $m_2 \in (\frac{5}{3}, 2]$, $n = 3$ and in the case that $m_2 \in [2, 3)$, $n = 2$. After that, when $\Omega = B_1$, $\alpha = 1$ and $m_1 = 1$, in the case that $m_2 \in [\frac{5}{3}, 3]$ and $n = 3$, Tao–Vernier Piro [32] introduced the measure $\Phi_2(t)$ in the form

$$\Phi_2(t) := \int_{\Omega} (u(\cdot, t) + 1)^p + \int_{\Omega} |\nabla v(\cdot, t)|^{2q} \quad (t > 0) \quad (1.4)$$

for suitable $p, q > 1$ ($p = 2$ and $q = 2$ when $m_2 \in [\frac{5}{3}, 2]$; $p = 5$ and $q = 11$ when $m_2 \in (2, 3]$) from the view point of local existence of classical solutions to (1.1) and an initial datum $v_0 \in W^{1,q}(\Omega)$ (see [2, Lemma 3.1]). This restriction on m_2 and n was removed by Anderson–Deng [1] when Ω is a convex bounded domain and $m_1 = 1$. Furthermore, as a new attempt to estimating a lower bound for the blow-up time in the above sense, Marras–Vernier Piro–Vigialoro [21,22] obtained a lower bound for the blow-up time of the more generalized equation with a source term:

$$\begin{cases} u_t = \nabla \cdot [\nabla u - k_1(t)u^{m_2-1}\nabla v] + f(u) & \text{in } \Omega \times (0, T), \\ v_t = k_2(t)\Delta v - k_3(t)v + k_4(t)u & \text{in } \Omega \times (0, T) \end{cases} \quad (1.5)$$

under Neumann boundary conditions and initial conditions, where $k_i(t)$ ($i = 1, 2, 3, 4$) are nonnegative smooth functions of t , $m_2 \in [2, 3)$ when $n = 2$, $m_2 \in (\frac{5}{3}, 2)$ when $n = 3$, f satisfies $f(u) \leq cu^2$ with $c > 0$. A similar result for the parabolic–elliptic version of (1.5) was deduced by Jiao–Zeng [17].

Now we focus on the studies obtained by [32] and [1] which gave a lower bound for the blow-up time for solutions of (1.1) under the following conditions:

- “ $m_1 = 1$ ”, $m_2 \in [\frac{5}{3}, 3]$, $n = 3$, Ω is a *unit ball* $B_1 \subset \mathbb{R}^3$ ([32]);
- “ $m_1 = 1$ ”, $m_2 \in \mathbb{R}$, $n \in \mathbb{N}$, Ω is a *convex* bounded domain in \mathbb{R}^n ([1]).

Table 1

The known results on lower bounds for the blow-up time in (1.1).

	Ω : ball	Ω : convex	Ω : non-convex
Linear case ($m_1 = 1$)	Tao–Vernier Piro [32]	Anderson–Deng [1]	No work
Nonlinear case ($m_1 \neq 1$)	No work	No work	No work

However, there is still room for improvements in these results. More precisely, we cannot find any results in the nonlinear case that $m_1 \neq 1$ and Ω is a non-convex bounded domain. Hence the current situation is summarized in Table 1.

Here, if some results can be given in the nonlinear case that $m_1 \neq 1$, then the following natural question arises:

Question. How does m_1 affect the blow-up time for solutions of (1.1)?

Since the blow-up for solutions of the system (1.1) describes the aggregation of cells and strong diffusion seems to prevent the aggregation and to cause delay in the blow-up, we can intuitively conjecture the answer to this question as follows:

Conjecture. The larger m_1 is, the larger the blow-up time t^* for solutions of (1.1) is.

The first purpose of this paper is completely to fill in “**No work**” in Table 1. The second purpose of this paper is to present an answer to the above question and justify the above conjecture, that is, to give an explanation for effect of nonlinear diffusion and the chemotaxis term for the blow-up time in a parabolic–parabolic chemotaxis system.

Furthermore, we should mention how we can derive an explicit lower bound for the blow-up time for solutions of (1.1). In the previous works, the blow-up time for classical solutions of (1.1) can be obtained by using the energy function $\Phi_2(t)$ defined as (1.4). However, there is a gap between the blow-up time for $\Phi_2(t)$ with $L^p(\Omega) \times W^{1,2q}(\Omega)$ -norm of (u, v) and that for solutions in the classical sense, i.e., in the sense of $L^\infty(\Omega)$ -norm of u (for details see Definition 1.2). Indeed, assume that Ω is a bounded domain. Then we know that

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p}} \|u(\cdot, t)\|_{L^\infty(\Omega)}. \quad (1.6)$$

In view of (1.6) we see that if a solution of (1.1) blows up in $L^p(\Omega)$ -norm, then it blows up in $L^\infty(\Omega)$ -norm at the same time; however, even if a solution of (1.1) blows up in $L^\infty(\Omega)$ -norm, we cannot predicate whether the solution blows up or not in $L^p(\Omega)$ -norm. From the numerical resolution method, it seems that the blow-up time for solutions of (1.1) in Φ_2 -measure has a long delay (see Farina–Marras–Viglialoro [6, FIGURE 1]).

Another purpose of this paper is to bridge a gap between the blow-up time for solutions of (1.1) in Φ_2 -measure and that in the classical sense. The key to accomplishing this purpose is a refined extensibility criterion established by Freitag [7, Theorem 2.2].

Before stating the main result, we define classical solutions of (1.1) and the blow-up time as follows:

Definition 1.1. A pair (u, v) is called a classical solution of (1.1) if

$$\begin{aligned} u &\in C(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T)), \\ v &\in C(\overline{\Omega} \times [0, T)) \cap C^{2,1}(\overline{\Omega} \times (0, T)) \cap L_{\text{loc}}^\infty([0, T]; W^{1,\infty}(\Omega)) \end{aligned}$$

and u, v satisfy (1.1) in the classical sense.

Remark 1.1. Local existence and uniqueness of classical solutions to (1.1) are known (see Lemma 2.4 and Remark 2.2 (i) below).

Definition 1.2. Let t^* be a maximal time for which a solution of (1.1) exists for $0 \leq t < t^*$. Then t^* is called a *blow-up time in the classical sense* if $t^* < \infty$ and

$$\lim_{t \nearrow t^*} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)}) = \infty. \quad (1.7)$$

In order to state the main theorem we shall give the following conditions for the parameters $p > 1$ and $q > \frac{1}{\eta-1}$:

$$p > \max \left\{ \frac{n}{2}(m_2 - m_1), \ n(m_2 - m_1 - 1), \ n \right\}, \quad (C1)$$

$$p > \max \left\{ \frac{q(2m_2 - m_1 - 3)}{q\eta - q - 1}, \ -2m_2 + m_1 + 3, \ \frac{2q}{q\eta - q + 1}, \ \frac{\eta(m_1 - 1)}{(\eta - 1)(\eta - 2)} \right\}, \quad (C2)$$

where η is defined as

$$\begin{cases} \eta \in (1, 2) \text{ arbitrary} & (n = 1, 2), \\ \eta := \frac{n}{n-1} & (n \geq 3). \end{cases} \quad (1.8)$$

We now state the main result of this paper. The main result gives a lower bound for the blow-up time for solutions of (1.1) with nonlinear diffusion.

Theorem 1.1. Let $t^* < \infty$ be the blow-up time in the classical sense for a classical solution (u, v) of (1.1). Then there exist constants $A = A(m_1) > 0$, $B > 0$, $C = C(m_1) > 0$, $D \geq 0$ and $p > 1$, $q > \frac{1}{\eta-1}$ fulfilling (C1), (C2) such that

$$t^* \geq \int_{\Phi(0)}^{\infty} \frac{d\tau}{A\tau^{f(\eta,r)} + B\tau^{f(\eta,1)} + C\tau^\eta + D}, \quad (1.9)$$

where $\Phi > 0$, $f > 1$ and $r > 0$ are defined as

$$\Phi(t) := \frac{1}{p} \int_{\Omega} (u(\cdot, t) + \alpha)^p + \frac{1}{q} \int_{\Omega} |\nabla v(\cdot, t)|^{2q} \quad (t > 0), \quad (1.10)$$

$$f(\eta, s) := 1 + \frac{\eta - 1}{n \left(\frac{1}{n} - \frac{\eta}{2} + \frac{1}{2s} \right)} \quad (s > 0), \quad (1.11)$$

$$r = r(m_1) := \frac{p}{p + m_1 - 1}, \quad (1.12)$$

and $\eta \in (1, 2)$ is defined as (1.8).

Remark 1.2. Theorem 1.1 covers the case that Ω is a general non-convex bounded domain and $m_1 = 1$ (see [1, Theorem 1.1]). Moreover, the constants A, B, C, D in (1.9) are better than the previous. If Ω is a convex bounded domain, then we can take $D = 0$ (see Corollary 4.2 below).

Remark 1.3. The estimate (1.9) implies that the larger m_1 is, the larger the blow-up time t^* for solutions (u, v) of (1.1) is. Indeed, we shall consider the effect of m_1 . When we fix $p, q > 1$, $f(\eta, r(m_1))$ is decreasing in m_1 . This entails that if m_1 is sufficiently large, then t^* will be large. In other words, if the power of diffusion is strong, then a solution of (1.1) blows up with a delay. On the other hand, we automatically see that if the power of diffusion is weak, then a solution of (1.1) blows up early.

Remark 1.4. As to the assumption of Theorem 1.1, we need not suppose that a solution of (1.1) blows up in finite time in the proof of Theorem 1.1. Namely, this means that we can essentially estimate the “life span” for solutions of (1.1). Since we are interested in the blow-up time for solutions of (1.1), the assumption concerning the blow-up is added in Theorem 1.1.

By computing the integral appearing in (1.9), we can establish a lower bound for t^* in the simple form as follows:

Corollary 1.2. *Under the assumption of Theorem 1.1, if $\Phi(0) < 1$, then (1.9) is rewritten as follows:*

$$\begin{aligned} s \geq 1 &\implies t^* \geq \frac{1}{f(\eta, r) - 1} \cdot \frac{\Phi(0)}{A\Phi(0)^{f(\eta, r)-1} + B\Phi(0)^{f(\eta, 1)-1} + C\Phi(0)^{\eta-1} + D}, \\ s < 1 &\implies t^* \geq \frac{1}{f(\eta, 1) - 1} \cdot \frac{\Phi(0)}{B\Phi(0)^{f(\eta, 1)-1} + A\Phi(0)^{f(\eta, r)-1} + C\Phi(0)^{\eta-1} + D}. \end{aligned}$$

The strategy for the proof of Theorem 1.1 is to derive an ordinary differential inequality for $\Phi(t)$ defined as (1.10). We first construct the inequality

$$\begin{aligned} \frac{d\Phi}{dt} + \frac{p-1}{2} \left(\frac{2}{p+m_1-1} \right)^2 \int_{\Omega} \left| \nabla(u+\alpha)^{\frac{p+m_1-1}{2}} \right|^2 + \left(\frac{2(q-1)}{q^2} - \delta \right) \int_{\Omega} |\nabla|\nabla v|^q|^2 \\ \leq \frac{\chi^2(p-1)}{2} \int_{\Omega} (u+\alpha)^{p+2m_2-m_1-3} |\nabla v|^2 + \frac{4(q-1)+n}{2} \int_{\Omega} (u+\alpha)^2 |\nabla v|^{2q-2} + D_{\delta} \end{aligned}$$

for some $\delta \in (0, \frac{2(q-1)}{q^2})$ and $D_{\delta} > 0$. We next estimate the first and second terms on the right-hand side by using Young’s inequality and Hölder’s inequality to make $\int_{\Omega} (u+\alpha)^{p\eta}$. In [1] dealing with the case that $m_1 = 1$, by applying the Sobolev embedding $W^{1,1}(\Omega) \hookrightarrow L^{\eta}(\Omega)$, the quantity $\int_{\Omega} (u+\alpha)^{p\eta}$ is estimated as

$$\int_{\Omega} (u+\alpha)^{p\eta} \leq C \left(\int_{\Omega} (u+\alpha)^p + \int_{\Omega} |\nabla(u+\alpha)^p| \right)^{\eta}$$

with some $C > 0$, and hence we need an additional deformation to obtain $\int_{\Omega} |\nabla(u+\alpha)^{\frac{p}{2}}|^2$, because of the difference from $\int_{\Omega} |\nabla(u+\alpha)^p|^2$. Our technical innovation in this paper is to apply the Gagliardo–Nirenberg inequality instead of using the Sobolev embedding as

$$\int_{\Omega} (u+\alpha)^{p\eta} \leq c \left\| \nabla(u+\alpha)^{\frac{p}{2}} \right\|_{L^2(\Omega)}^{\eta} \left\| (u+\alpha)^{\frac{p}{2}} \right\|_{L^2(\Omega)}^{\eta} + \tilde{c} \left\| (u+\alpha)^{\frac{p}{2}} \right\|_{L^2(\Omega)}^{2\eta}$$

for some $c, \tilde{c} > 0$ in the case $m_1 = 1$ (we extend this inequality to the case $m_1 \neq 1$). We thus obtain the factor $\int_{\Omega} |\nabla(u+\alpha)^{\frac{p}{2}}|^2$ directly and a sharp lower bound for the blow-up time can be established. In addition, the key to removing the convexity of Ω is the estimate for $\int_{\partial\Omega} |\nabla v|^{2q-2} \nabla(|\nabla v|^2) \cdot \nu$ which is

estimated by 0 in the previous works [1] and [32]. In this paper it is estimated by a combination of the embedding $W^{\beta+\frac{1}{2}}(\Omega) \hookrightarrow L^2(\partial\Omega)$ ($\beta \in (0, \frac{1}{2})$) with the fractional Gagliardo–Nirenberg inequality.

This paper is organized as follows. In Section 2 we will collect lemmas which will be used in this paper. In Section 3 we will present an estimate for the first term of $\Phi(t)$ defined in Theorem 1.1. In Section 4 we will give an estimate for the second term of $\Phi(t)$. We will complete the proof of Theorem 1.1 in Section 5 through a series of four steps. An important thing is to obtain an ordinary differential inequality of $\Phi(t)$ without wasting effect of m_1 .

2. Preliminaries

In this section we recall some known basic results. Let us begin with the well-known Gagliardo–Nirenberg inequality (for details, see e.g., Li–Lankeit [20, Lemma 2.3]):

Lemma 2.1. *Suppose that Ω is a bounded domain in \mathbb{R}^n with smooth boundary. Let $r \geq 1$, $0 < q \leq p \leq \infty$, $s > 0$ be such that $\frac{1}{r} \leq \frac{1}{n} + \frac{1}{p}$. Then there exists $c > 0$ such that*

$$\|w\|_{L^p(\Omega)} \leq c \left(\|\nabla w\|_{L^r(\Omega)}^a \|w\|_{L^q(\Omega)}^{1-a} + \|w\|_{L^s(\Omega)} \right)$$

for all $w \in W^{1,p}(\Omega) \cap L^q(\Omega)$, where $a := \frac{\frac{1}{q} - \frac{1}{p}}{\frac{1}{q} + \frac{1}{n} - \frac{1}{r}}$.

Next we give an estimate for a particular boundary integral which enables us to cover the case of non-convex bounded domains (see [20, Lemma 2.1]).

Lemma 2.2. *Let Ω be a bounded domain in \mathbb{R}^n with smooth boundary. Suppose that $q \in [1, \infty)$ and $M > 0$. Then for all $\delta > 0$ there exists $C_\delta > 0$ independent of q such that for all $w \in C^2(\overline{\Omega})$ satisfying $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$ and $\int_\Omega |\nabla w| \leq M$ the following inequality holds:*

$$\int_{\partial\Omega} |\nabla w|^{2q-2} \frac{\partial |\nabla w|^2}{\partial \nu} \leq \delta \int_\Omega |\nabla |\nabla w|^q|^2 + C_\delta.$$

If Ω is a convex bounded domain, then the following holds (see [33, Lemma 3.2]):

Lemma 2.3. *Assume that Ω is a convex bounded domain, and that $w \in C^2(\overline{\Omega})$ satisfies $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$. Then*

$$\frac{\partial |\nabla w|^2}{\partial \nu} \leq 0 \quad \text{on } \partial\Omega.$$

We finally introduce the fundamental fact for classical solutions of (1.1) and results for the blow-up time. We recall the result for local existence of classical solutions (see [33, Lemma 1.1]).

Lemma 2.4. *Let $u_0 \in C(\overline{\Omega})$ and $v_0 \in C^1(\overline{\Omega})$. Then there exist $T_{\max} \in (0, \infty]$ and a uniquely determined pair (u, v) of nonnegative functions in $C(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ solving (1.1) classically in $\Omega \times (0, T_{\max})$. Additionally we either have*

$$T_{\max} = \infty \quad \text{or} \quad \limsup_{t \nearrow T_{\max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty.$$

The following lemma, which was proved in [12, Lemma 4.1], plays an important role in considering the blow-up time for solutions of (1.1) defined in Definition 1.2.

Lemma 2.5. Let (u, v) be a classical solution of (1.1). Suppose that there exist $p \geq 1$ and $C > 0$ such that

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T).$$

Then

$$\|v\|_{L^\infty(0, T; W^{1, q}(\Omega))} < \infty$$

for any $q \in [1, \frac{np}{(n-p)_+})$ and even $q = \infty$ if $p > n$.

Remark 2.1. Let a pair of (u, v) solve (1.1) classically. As to $\Phi(t)$ defined as (1.10), we note that it is sufficient only to deal with the blow-up time for u in $L^p(\Omega)$ -norm under the condition $p > n$ guaranteed by (C1). In other words, the blow-up time for v does not affect that for $\Phi(t)$. We should explain that the blow-up time for u in $L^p(\Omega)$ -norm is larger than or equal to that for v in $W^{1, \infty}(\Omega)$ -norm (see Definition 1.2). Indeed, by the contraposition of Lemma 2.5, we can find that if v blows up in $W^{1, \infty}(\Omega)$ -norm, then u blows up in $L^p(\Omega)$ -norm for all $p > n$, and hence the blow-up time for u in $L^p(\Omega)$ -norm is larger than or equal to that for v in $W^{1, \infty}(\Omega)$ -norm under the condition $p > n$.

Remark 2.2. The condition (1.7) in the definition of the blow-up time in the classical sense can be replaced with

$$\lim_{t \nearrow t^*} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1, k}(\Omega)}) = \infty, \quad (2.1)$$

where $k > n$ because of the condition $q > \frac{1}{\eta-1} = n-1$ in Theorem 1.1. Indeed, we see from Remark 2.1 and the continuous embedding $W^{1, \infty}(\Omega) \hookrightarrow W^{1, 2q}(\Omega)$ for all $q \geq 1$ and $W^{1, 2q}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $q > \frac{n}{2}$ that if v in $W^{1, 2q}(\Omega)$ -norm or in $L^\infty(\Omega)$ -norm blows up, then that in $W^{1, \infty}(\Omega)$ -norm also blows up for $n \geq 2$. An argument similar to that in Remark 2.1 implies that (1.7) can be replaced with (2.1).

The following lemma enables us to show that a maximal existence time results in unboundedness in L^p -spaces for smaller $p \in [1, \infty)$ (see [7, Theorem 2.2]).

Lemma 2.6. Let $u_0 \in C(\overline{\Omega})$ and $v_0 \in C^1(\overline{\Omega})$. If a solution (u, v) of (1.1) in $\Omega \times (0, T_{\max})$ has the blow-up time $T_{\max} < \infty$ in the classical sense, then there exists $p \geq 1$ fulfilling

$$p \geq \max \left\{ \frac{n}{2} (m_2 - m_1), n(m_2 - m_1 - 1) \right\}$$

such that

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^{p_0}(\Omega)} = \infty$$

for all $p_0 > p$.

In the proof of Theorem 1.1 we will use the following corollary in order to remove a gap between the blow-up time for solutions (u, v) of (1.1) in the classical sense and that in Φ -measure.

Corollary 2.7. Let $1 \leq p, q \leq \infty$. Let t^* be the blow-up time in the classical sense and $t_{p, q}^*$ the blow-up time in the measure $\Phi(t)$ defined as (1.10):

$$\lim_{t \nearrow t_{p, q}^*} \Phi(t) = \infty.$$

Then under the condition (C1), we have

$$t^* = t_{p,q}^*. \quad (2.2)$$

Proof. We obtain from the continuous embedding $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$ such as (1.6) that if $\Phi(t)$ blows up at $t = t_{p,q}^*$, then u in $L^\infty(\Omega)$ -norm also blows up, and hence it is clear that

$$t^* \leq t_{p,q}^*.$$

Here we note from Remark 2.1 that the blow-up for $\Phi(t)$ implies that for u in $L^p(\Omega)$ -norm. On the other hand, we can find from Lemma 2.6 that the blow-up in the classical sense implies that in Φ -measure for p satisfying (C1). Therefore, under the condition (C1), we can attain that

$$t^* \geq t_{p,q}^*.$$

Thus we obtain (2.2). \square

Hereafter, we assume that a pair (u, v) is a classical solution of (1.1).

3. An estimate for $\frac{1}{p} \int_{\Omega} (u(\cdot, t) + \alpha)^p$

In this section we estimate the first term of $\Phi(t)$:

$$\frac{1}{p} \int_{\Omega} (u(\cdot, t) + \alpha)^p.$$

The following lemma gives an estimate for the derivative of the first term in $\Phi(t)$.

Lemma 3.1. For $p \geq 1$, we have

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} (u + \alpha)^p + \frac{p-1}{2} \int_{\Omega} (u + \alpha)^{p+m_1-3} |\nabla u|^2 \\ & \leq \frac{\chi^2(p-1)}{2} \int_{\Omega} (u + \alpha)^{p+2m_2-m_1-3} |\nabla v|^2. \end{aligned} \quad (3.1)$$

Proof. The first equation of (1.1) and integration by parts enable us to see

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} (u + \alpha)^p &= - \int_{\Omega} \nabla (u + \alpha)^{p-1} \cdot [(u + \alpha)^{m_1-1} \nabla u - \chi u (u + \alpha)^{m_2-2} \nabla v] \\ &= -(p-1) \int_{\Omega} (u + \alpha)^{p+m_1-3} |\nabla u|^2 + \chi(p-1) \int_{\Omega} (u + \alpha)^{p+m_2-4} u \nabla u \cdot \nabla v \\ &\leq -(p-1) \int_{\Omega} (u + \alpha)^{p+m_1-3} |\nabla u|^2 + \chi(p-1) \int_{\Omega} (u + \alpha)^{p+m_2-3} |\nabla u \cdot \nabla v|. \end{aligned}$$

By using Young's inequality, we obtain

$$\begin{aligned}
& \chi(p-1)(u+\alpha)^{p+m_2-3}|\nabla u \cdot \nabla v| \\
&= \sqrt{p-1}(u+\alpha)^{\frac{p+m_1-3}{2}}|\nabla u| \cdot \chi\sqrt{p-1}(u+\alpha)^{\frac{p+2m_2-m_1-3}{2}}|\nabla v| \\
&\leq \frac{p-1}{2}(u+\alpha)^{p+m_1-3}|\nabla u|^2 + \frac{\chi^2(p-1)}{2}(u+\alpha)^{p+2m_2-m_1-3}|\nabla v|^2,
\end{aligned}$$

and hence we have

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} (u+\alpha)^p \leq -\frac{p-1}{2} \int_{\Omega} (u+\alpha)^{p+m_1-3} |\nabla u|^2 + \frac{\chi^2(p-1)}{2} \int_{\Omega} (u+\alpha)^{p+2m_2-m_1-3} |\nabla v|^2.$$

Therefore we can attain the conclusion (3.1). \square

4. An estimate for $\frac{1}{q} \int_{\Omega} |\nabla v(\cdot, t)|^{2q}$

In this section we estimate the second term of $\Phi(t)$:

$$\frac{1}{q} \int_{\Omega} |\nabla v(\cdot, t)|^{2q}.$$

Although the following lemma is proved in a similar way as in the proof of the previous work (see [1, Lemma 2.1]), we shall reconstruct the method in [1] and remove the convexity assumption. The following lemma presents an estimate for the derivative of the second term of $\Phi(t)$.

Lemma 4.1. *If $\delta \in (0, \frac{2(q-1)}{q^2})$, then there exists $D_{\delta} > 0$ such that*

$$\begin{aligned}
& \frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + \left(\frac{2(q-1)}{q^2} - \delta \right) \int_{\Omega} |\nabla |\nabla v|^q|^2 + 2 \int_{\Omega} |\nabla v|^{2q} \\
& \leq \frac{4(q-1)+n}{2} \int_{\Omega} (u+\alpha)^2 |\nabla v|^{2q-2} + D_{\delta}
\end{aligned} \tag{4.1}$$

for all $q \geq 1$.

Proof. We fix $\delta \in (0, \frac{2(q-1)}{q^2})$. Then we infer that

$$\frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} = \int_{\Omega} |\nabla v|^{2(q-1)} \frac{\partial}{\partial t} |\nabla v|^2. \tag{4.2}$$

The second equation in (1.1) entails

$$\begin{aligned}
\frac{\partial}{\partial t} |\nabla v|^2 &= 2 \nabla v \cdot \nabla v_t \\
&= 2 \nabla v \cdot \nabla [\Delta v - v + u] \\
&= 2 \nabla v \cdot \nabla \Delta v - 2 |\nabla v|^2 + 2 \nabla u \cdot \nabla v.
\end{aligned} \tag{4.3}$$

Noticing from the chain rule that

$$\begin{aligned}
\Delta|\nabla v|^2 &= \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} \left(\sum_{j=1}^n \left(\frac{\partial v}{\partial x_j} \right)^2 \right) \\
&= \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(2 \frac{\partial v}{\partial x_j} \cdot \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \\
&= 2 |D^2 v|^2 + 2 \nabla v \cdot \nabla \Delta v,
\end{aligned}$$

where $D^2 v$ denotes the Hessian matrix, we obtain

$$2 \nabla v \cdot \nabla \Delta v = \Delta |\nabla v|^2 - 2 |D^2 v|^2.$$

This together with (4.3) yields

$$\frac{\partial}{\partial t} |\nabla v|^2 = \Delta |\nabla v|^2 - 2 |D^2 v|^2 - 2 |\nabla v|^2 + 2 \nabla u \cdot \nabla v.$$

Applying this identity to (4.2), we have

$$\begin{aligned}
&\frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + 2 \int_{\Omega} |\nabla v|^{2(q-1)} |D^2 v|^2 + 2 \int_{\Omega} |\nabla v|^{2q} \\
&= \int_{\Omega} |\nabla v|^{2(q-1)} \Delta |\nabla v|^2 + 2 \int_{\Omega} |\nabla v|^{2(q-1)} \nabla u \cdot \nabla v.
\end{aligned} \tag{4.4}$$

Here we see from integration by parts that

$$\int_{\Omega} |\nabla v|^{2(q-1)} \Delta |\nabla v|^2 = \int_{\partial\Omega} |\nabla v|^{2q-2} \nabla (|\nabla v|^2) \cdot \nu - \int_{\Omega} \nabla (|\nabla v|^{2(q-1)}) \cdot \nabla (|\nabla v|^2).$$

Moreover, a combination of (1.2) and Lemma 2.5 with $p = q = 1$ yields that there exists $M > 0$ such that $\int_{\Omega} |\nabla v| \leq M$, and so if $\delta \in (0, \frac{2(q-1)}{q^2})$, then there exists $D_{\delta} > 0$ such that

$$\int_{\partial\Omega} |\nabla v|^{2q-2} \nabla (|\nabla v|^2) \cdot \nu \leq \delta \int_{\Omega} |\nabla |\nabla v|^q|^2 + D_{\delta} \tag{4.5}$$

(see Lemma 2.2), and hence we rewrite (4.4) as

$$\begin{aligned}
&\frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + \int_{\Omega} \nabla (|\nabla v|^{2(q-1)}) \cdot \nabla (|\nabla v|^2) + 2 \int_{\Omega} |\nabla v|^{2(q-1)} |D^2 v|^2 + 2 \int_{\Omega} |\nabla v|^{2q} \\
&\leq 2 \int_{\Omega} |\nabla v|^{2(q-1)} \nabla u \cdot \nabla v + \delta \int_{\Omega} |\nabla |\nabla v|^q|^2 + D_{\delta}.
\end{aligned} \tag{4.6}$$

Applying integration by parts to the first term on the right-hand side of (4.6) gives

$$\begin{aligned}
&2 \int_{\Omega} |\nabla v|^{2(q-1)} \nabla u \cdot \nabla v \\
&= -2(q-1) \int_{\Omega} u |\nabla v|^{2(q-2)} \nabla (|\nabla v|^2) \cdot \nabla v - 2 \int_{\Omega} u |\nabla v|^{2(q-1)} \Delta v.
\end{aligned} \tag{4.7}$$

Now we estimate the following quantities:

$$u|\nabla v|^{2(q-2)}\nabla(|\nabla v|^2)\cdot\nabla v, \quad u|\nabla v|^{2(q-1)}\Delta v.$$

Using the inequality $u \leq u + \alpha$, the pointwise inequality $|\Delta v|^2 \leq n|D^2v|^2$ and the Young inequality, we can notice that

$$\begin{aligned} -u|\nabla v|^{2(q-2)}\nabla(|\nabla v|^2)\cdot\nabla v &\leq 2\cdot\frac{1}{2}|\nabla v|^{q-2}|\nabla|\nabla v|^2|\cdot(u+\alpha)|\nabla v|^{q-1} \\ &\leq \frac{1}{4}(|\nabla v|^{q-2}|\nabla|\nabla v|^2|)^2 + ((u+\alpha)|\nabla v|^{q-1})^2 \end{aligned} \quad (4.8)$$

and

$$\begin{aligned} -u|\nabla v|^{2(q-1)}|\Delta v| &\leq 2\cdot\sqrt{\frac{1}{n}}|\nabla v|^{q-1}\Delta v\cdot\sqrt{\frac{n}{4}}(u+\alpha)|\nabla v|^{q-1} \\ &\leq \frac{1}{n}|\nabla v|^{2(q-1)}|\Delta v|^2 + \frac{n}{4}((u+\alpha)|\nabla v|^{q-1})^2 \\ &\leq \int_{\Omega}|\nabla v|^{2(q-1)}|D^2v|^2 + \frac{n}{4}((u+\alpha)|\nabla v|^{q-1})^2. \end{aligned} \quad (4.9)$$

Applying (4.8) and (4.9) to (4.7), we obtain that

$$\begin{aligned} 2\int_{\Omega}|\nabla v|^{2(q-1)}\nabla u\cdot\nabla v &\leq \frac{q-1}{2}\int_{\Omega}|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 + \frac{4(q-1)+n}{2}\int_{\Omega}(u+\alpha)^2|\nabla v|^{2q-2} \\ &\quad + 2\int_{\Omega}|\nabla v|^{2(q-1)}|D^2v|^2. \end{aligned} \quad (4.10)$$

From (4.6) and (4.10), by using that

$$|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2 = \frac{4}{q^2}|\nabla|\nabla v|^q|^2$$

as well as

$$\nabla(|\nabla v|^{2(q-1)})\cdot\nabla(|\nabla v|^2) = (q-1)|\nabla v|^{2(q-2)}|\nabla|\nabla v|^2|^2$$

for $q > 1$, we can confirm that

$$\begin{aligned} &\frac{1}{q}\frac{d}{dt}\int_{\Omega}|\nabla v|^{2q} + \frac{2(q-1)}{q^2}\int_{\Omega}|\nabla|\nabla v|^q|^2 + 2\int_{\Omega}|\nabla v|^{2q} \\ &\leq \frac{4(q-1)+n}{2}\int_{\Omega}(u+\alpha)^2|\nabla v|^{2q-2} + \delta\int_{\Omega}|\nabla|\nabla v|^q|^2 + D_{\delta}. \end{aligned}$$

Thus we arrive at (4.1). \square

Corollary 4.2. *If Ω is a convex bounded domain, then (4.1) is rewritten as*

$$\begin{aligned} & \frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + \frac{2(q-1)}{q^2} \int_{\Omega} |\nabla |\nabla v|^q|^2 + 2 \int_{\Omega} |\nabla v|^{2q} \\ & \leq \frac{4(q-1)+n}{2} \int_{\Omega} (u+\alpha)^2 |\nabla v|^{2q-2} \end{aligned}$$

for all $q \geq 1$. In other words, δ and D_{δ} are taken as 0 in Lemma 4.1.

Proof. Suppose that Ω is a convex bounded domain. Then we see from Lemma 2.3 that $\nabla (|\nabla v|^2) \cdot \nu \leq 0$, and so we can rewrite (4.5) as

$$\int_{\partial\Omega} |\nabla v|^{2q-2} \nabla (|\nabla v|^2) \cdot \nu \leq 0.$$

By an argument similar to the proof of Lemma 4.1, we can attain the conclusion. \square

5. Proof of the main theorem

In this section we prove Theorem 1.1. The following lemma plays an important role in the proof of Theorem 1.1.

Lemma 5.1. *Let $p, q \geq 1$, $p \neq -m_1 + 1$, $\delta \in (0, \frac{2(q-1)}{q^2})$, and let $\Phi(t)$ be defined as (1.10). Then there exists $D_{\delta} > 0$ such that*

$$\begin{aligned} & \frac{d\Phi}{dt} + \frac{p-1}{2} \left(\frac{2}{p+m_1-1} \right)^2 \int_{\Omega} \left| \nabla (u+\alpha)^{\frac{p+m_1-1}{2}} \right|^2 + \left(\frac{2(q-1)}{q^2} - \delta \right) \int_{\Omega} |\nabla |\nabla v|^q|^2 \\ & \leq \frac{\chi^2(p-1)}{2} \int_{\Omega} (u+\alpha)^{p+2m_2-m_1-3} |\nabla v|^2 + \frac{4(q-1)+n}{2} \int_{\Omega} (u+\alpha)^2 |\nabla v|^{2q-2} + D_{\delta}. \end{aligned} \quad (5.1)$$

Proof. A combination of (3.1) and (4.1) yields (5.1). In fact, due to (3.1) and (4.1), we can find that for $p, q \geq 1$, there exists $\delta \in (0, \frac{2(q-1)}{q^2})$ such that

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} (u+\alpha)^p + \frac{1}{q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} \\ & + \frac{p-1}{2} \int_{\Omega} (u+\alpha)^{p+m_1-3} |\nabla u|^2 + \left(\frac{2(q-1)}{q^2} - \delta \right) \int_{\Omega} |\nabla |\nabla v|^q|^2 \\ & \leq \frac{\chi^2(p-1)}{2} \int_{\Omega} (u+\alpha)^{p+2m_2-m_1-3} |\nabla v|^2 + \frac{4(q-1)+n}{2} \int_{\Omega} (u+\alpha)^2 |\nabla v|^{2q-2} + D_{\delta}. \end{aligned}$$

On the other hand, we notice that if $p \neq -m_1 + 1$, then

$$\int_{\Omega} (u+\alpha)^{p+m_1-3} |\nabla u|^2 = \left(\frac{2}{p+m_1-1} \right)^2 \int_{\Omega} \left| \nabla (u+\alpha)^{\frac{p+m_1-1}{2}} \right|^2.$$

This together with (1.10) clearly proves (5.1). \square

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We divide the proof into four steps:

- (Step 1): Estimates for $\int_{\Omega} (u + \alpha)^{p+2m_2-m_1-3} |\nabla v|^2$ and $\int_{\Omega} (u + \alpha)^2 |\nabla v|^{2q-2}$.
 (Step 2): Estimates for $\int_{\Omega} (u + \alpha)^{p\eta}$ and $\int_{\Omega} |\nabla v|^{2q\eta}$.
 (Step 3): Deriving an ordinary differential inequality for $\Phi(t)$.
 (Step 4): Establishing a lower bound for the blow-up time t^* in the classical sense.

By means of these processes, we can argue a method to obtain a lower bound for the blow-up time for solutions of (1.1).

(Step 1) We shall show that

$$\int_{\Omega} (u + \alpha)^{p+2m_2-m_1-3} |\nabla v|^2 \leq \frac{1}{(q\eta)'} \left(\int_{\Omega} (u + \alpha)^{p\eta} \right)^{\frac{1}{\beta_1}} |\Omega|^{\frac{1}{\beta_1'}} + \frac{1}{q\eta} \int_{\Omega} |\nabla v|^{2q\eta} \quad (5.2)$$

and

$$\int_{\Omega} (u + \alpha)^2 |\nabla v|^{2q-2} \leq \frac{1}{(q'\eta)'} \left(\int_{\Omega} (u + \alpha)^{p\eta} \right)^{\frac{1}{\beta_2}} |\Omega|^{\frac{1}{\beta_2'}} + \frac{1}{q'\eta} \int_{\Omega} |\nabla v|^{2q\eta}, \quad (5.3)$$

where η is the constant defined as (1.8) and $'$ denotes the Hölder conjugate exponent, i.e., $q' := \frac{q}{q-1}$ and

$$\begin{aligned} \beta_1 &:= \frac{p}{p+2m_2-m_1-3} \cdot \frac{q\eta-1}{q} > 1, \\ \beta_2 &:= \frac{p}{2} \cdot \frac{q\eta-q+1}{q} > 1. \end{aligned}$$

Applying Young's inequality to the first term on the left-hand side of (5.2) gives

$$\int_{\Omega} (u + \alpha)^{p+2m_2-m_1-3} |\nabla v|^2 \leq \frac{1}{(q\eta)'} \int_{\Omega} (u + \alpha)^{(p+2m_2-m_1-3)(q\eta)'} + \frac{1}{q\eta} \int_{\Omega} |\nabla v|^{2q\eta}. \quad (5.4)$$

Thanks to boundedness of Ω , using Hölder's inequality, we have

$$\int_{\Omega} (u + \alpha)^{(p+2m_2-m_1-3)(q\eta)'} \leq \left(\int_{\Omega} (u + \alpha)^{p\eta} \right)^{\frac{1}{\beta_1}} |\Omega|^{\frac{1}{\beta_1'}}, \quad (5.5)$$

where the condition (C2) enables us to take $\beta_1 > 1$ as

$$\beta_1 = \frac{p\eta}{(p+2m_2-m_1-3)(q\eta)'} = \frac{p}{p+2m_2-m_1-3} \cdot \frac{q\eta-1}{q}.$$

Plugging (5.5) into (5.4), we obtain (5.2). Similarly, combining Young's inequality with Hölder's inequality yields

$$\begin{aligned} \int_{\Omega} (u + \alpha)^2 |\nabla v|^{2q-2} &\leq \frac{1}{(q'\eta)'} \int_{\Omega} (u + \alpha)^{2(q'\eta)'} + \frac{1}{q'\eta} \int_{\Omega} |\nabla v|^{2q\eta} \\ &\leq \frac{1}{(q'\eta)'} \left(\int_{\Omega} (u + \alpha)^{p\eta} \right)^{\frac{1}{\beta_2}} |\Omega|^{\frac{1}{\beta_2'}} + \frac{1}{q'\eta} \int_{\Omega} |\nabla v|^{2q\eta}, \end{aligned}$$

where the condition (C2) enables us to take $\beta_2 > 1$ as

$$\beta_2 = \frac{p\eta}{2(q'\eta)'} = \frac{p}{2} \cdot \frac{q\eta - q + 1}{q}.$$

Therefore we arrive at (5.3).

(Step 2) The purpose of this step is to obtain the following inequalities:

$$\begin{aligned} \int_{\Omega} (u + \alpha)^{p\eta} &\leq C_1(m_1)\varepsilon \int_{\Omega} \left| \nabla(u + \alpha)^{\frac{p+m_1-1}{2}} \right|^2 \\ &\quad + \frac{C_2(m_1)}{\varepsilon^{\frac{1-ar\eta}{ar\eta}}} \left(\int_{\Omega} (u + \alpha)^p \right)^{f(\eta,r)} + C_3(m_1) \left(\int_{\Omega} (u + \alpha)^p \right)^{\eta}, \end{aligned} \quad (5.6)$$

$$\int_{\Omega} |\nabla v|^{2q\eta} \leq C_4\varepsilon \int_{\Omega} |\nabla |\nabla v|^q|^2 + \frac{C_5}{\varepsilon^{\frac{\eta}{2-\eta}}} \left(\int_{\Omega} |\nabla v|^{2q} \right)^{f(\eta,1)} + C_6 \left(\int_{\Omega} |\nabla v|^{2q} \right)^{\eta} \quad (5.7)$$

for all $\varepsilon > 0$, where η , f and r are defined as (1.8), (1.11) and (1.12), respectively, and

$$\begin{aligned} C_1(m_1) &:= 2^{2r\eta-1} ar\eta c_1^{2r\eta}, \quad C_2(m_1) := 2^{2r\eta} (1 - ar\eta) c_1^{2r\eta}, \quad C_3(m_1) := 2^{2r\eta-1} c_1^{2r\eta}, \\ C_4 &:= 2^{2\eta-1} \cdot \frac{\eta}{2} \cdot c_2^{2\eta}, \quad C_5 := 2^{2\eta-1} \cdot \frac{2-\eta}{2} \cdot c_2^{2\eta}, \quad C_6 := 2^{2\eta-1} c_2^{2\eta}. \end{aligned}$$

In order to estimate $\int_{\Omega} (u + \alpha)^{p\eta}$ and $\int_{\Omega} |\nabla v|^{2q\eta}$ without wasting the power of diffusion, we apply Lemma 2.1. Under the conditions (C1) and (C2), we can show existence of $c_1 > 0$ such that

$$\begin{aligned} &\int_{\Omega} (u + \alpha)^{p\eta} \\ &= \left\| (u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^{2r\eta}(\Omega)}^{2r\eta} \\ &\leq c_1^{2r\eta} \left(\left\| \nabla(u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^2(\Omega)}^a \left\| (u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^{2r}(\Omega)}^{1-a} + \left\| (u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^{2r}(\Omega)} \right)^{2r\eta} \\ &\leq 2^{2r\eta-1} c_1^{2r\eta} \left(\left\| \nabla(u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^2(\Omega)}^{2ar\eta} \left\| (u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^{2r}(\Omega)}^{2(1-a)r\eta} + \left\| (u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^{2r}(\Omega)}^{2r\eta} \right), \end{aligned}$$

where

$$a := \frac{\frac{1}{2r} - \frac{1}{2r\eta}}{\frac{1}{2r} + \frac{1}{n} - \frac{1}{2}} \in (0, 1).$$

Noting that $2ar\eta < 2$ which implies the condition $p > \frac{\eta(m_1-1)}{(\eta-1)(\eta-2)}$ guaranteed by (C2), thanks to the Young inequality, we can estimate the first term on the right-hand side as

$$\begin{aligned} & \left\| \nabla(u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^2(\Omega)}^{2ar\eta} \left\| (u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^{2r}(\Omega)}^{2(1-a)r\eta} \\ & \leq ar\eta \varepsilon \left\| \nabla(u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^2(\Omega)}^2 + \frac{1-ar\eta}{\varepsilon^{\frac{1-ar\eta}{ar\eta}}} \left\| (u + \alpha)^{\frac{p+m_1-1}{2}} \right\|_{L^{2r}(\Omega)}^{2r \frac{(1-a)\eta}{1-ar\eta}} \end{aligned}$$

for all $\varepsilon > 0$, and we see that $\frac{(1-a)\eta}{1-ar\eta}$ is rewritten as

$$\begin{aligned} \frac{(1-a)\eta}{1-ar\eta} &= \frac{\left(1 - \frac{\frac{1}{2r} - \frac{1}{2r\eta}}{\frac{1}{2r} + \frac{1}{n} - \frac{1}{2}}\right) \eta}{1 - \frac{\frac{1}{2r} - \frac{1}{2r\eta}}{\frac{1}{2r} + \frac{1}{n} - \frac{1}{2}} \cdot r\eta} \\ &= \frac{\left(\left(\frac{1}{2r} + \frac{1}{n} - \frac{1}{2}\right) - \left(\frac{1}{2r} - \frac{1}{2r\eta}\right)\right) \eta}{\frac{1}{2r} + \frac{1}{n} - \frac{1}{2} - \left(\frac{1}{2r} - \frac{1}{2r\eta}\right) r\eta} \\ &= 1 + \frac{\eta - 1}{n \left(\frac{1}{n} - \frac{\eta}{2} + \frac{1}{2r}\right)} = f(\eta, r). \end{aligned}$$

Thus we obtain

$$\int_{\Omega} (u + \alpha)^{p\eta} \leq C_1(m_1) \varepsilon \int_{\Omega} \left| \nabla(u + \alpha)^{\frac{p+m_1-1}{2}} \right|^2 + \frac{C_2(m_1)}{\varepsilon^{\frac{1-ar\eta}{ar\eta}}} \left(\int_{\Omega} (u + \alpha)^p \right)^{f(\eta, r)} + C_3(m_1) \left(\int_{\Omega} (u + \alpha)^p \right)^{\eta},$$

with $C_1(m_1) = 2^{2r\eta-1} ar\eta c_1^{2r\eta}$, $C_2(m_1) = 2^{2r\eta-1} (1-ar\eta) c_1^{2r\eta}$ and $C_3(m_1) = 2^{2r\eta-1} c_1^{2r\eta}$ for all $\varepsilon > 0$. In a similar way, there exists $c_2 > 0$ such that

$$\begin{aligned} \int_{\Omega} |\nabla v|^{2q\eta} &= \|\nabla v\|^q_{L^{2\eta}(\Omega)}^{2\eta} \\ &\leq c_2^{2\eta} \left(\|\nabla |\nabla v|^q\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla v\|^q_{L^2(\Omega)}^{\frac{1}{2}} + \|\nabla v\|^q_{L^2(\Omega)} \right)^{2\eta} \\ &\leq 2^{2\eta-1} c_2^{2\eta} \left(\|\nabla |\nabla v|^q\|_{L^2(\Omega)}^{\eta} \|\nabla v\|^q_{L^2(\Omega)}^{\eta} + \|\nabla v\|^{2\eta}_{L^2(\Omega)} \right) \\ &\leq 2^{2\eta-1} c_2^{2\eta} \left(\frac{\eta}{2} \varepsilon \|\nabla |\nabla v|^q\|_{L^2(\Omega)}^2 + \frac{2-\eta}{2\varepsilon^{\frac{\eta}{2-\eta}}} \|\nabla v\|^q_{L^2(\Omega)}^{\frac{2\eta}{2-\eta}} + \|\nabla v\|^{2\eta}_{L^2(\Omega)} \right) \\ &=: C_4 \varepsilon \int_{\Omega} |\nabla |\nabla v|^q|^2 + \frac{C_5}{\varepsilon^{\frac{\eta}{2-\eta}}} \left(\int_{\Omega} |\nabla v|^{2q} \right)^{f(\eta, 1)} + C_6 \left(\int_{\Omega} |\nabla v|^{2q} \right)^{\eta} \end{aligned}$$

for all $\varepsilon > 0$, where $C_4 = 2^{2\eta-1} \cdot \frac{\eta}{2} \cdot c_2^{2\eta}$, $C_5 = 2^{2\eta-1} \cdot \frac{2-\eta}{2} \cdot c_2^{2\eta}$ and $C_6 = 2^{2\eta-1} c_2^{2\eta}$. Hence we can obtain (5.6) and (5.7).

(Step 3) Plugging the results of Step 1 and Step 2 into (5.1), we shall show an ordinary differential inequality for $\Phi(t)$:

$$\frac{d\Phi}{dt} \leq A\Phi^{f(\eta, r)} + B\Phi^{f(\eta, 1)} + C\Phi^{\eta} + D. \quad (5.8)$$

To this end we first deal with the first and second terms on the right-hand side of (5.1). Applying (5.6) to $(\int_{\Omega}(u+\alpha)^{p\eta})^{\frac{1}{\beta_1}}$ and $(\int_{\Omega}(u+\alpha)^{p\eta})^{\frac{1}{\beta_2}}$ appearing in (5.2) and (5.3) yields

$$\int_{\Omega}(u+\alpha)^{p+2m_2-m_1-3}|\nabla v|^2 \leq \frac{1}{(q\eta)'} \cdot R^{\frac{1}{\beta_1}} |\Omega|^{\frac{1}{\beta_1'}} + \frac{1}{q\eta} \int_{\Omega} |\nabla v|^{2q\eta}, \quad (5.9)$$

$$\int_{\Omega}(u+\alpha)^2|\nabla v|^{2q-2} \leq \frac{1}{(q'\eta)'} \cdot R^{\frac{1}{\beta_2}} |\Omega|^{\frac{1}{\beta_2'}} + \frac{1}{q'\eta} \int_{\Omega} |\nabla v|^{2q\eta}, \quad (5.10)$$

where R is given by

$$R := C_1(m_1)\varepsilon \int_{\Omega} \left| \nabla(u+\alpha)^{\frac{p+m_1-1}{2}} \right|^2 + \frac{C_2(m_1)}{\varepsilon^{\frac{1-ar\eta}{ar\eta}}} \left(\int_{\Omega} (u+\alpha)^p \right)^{f(\eta,r)} + C_3(m_1) \left(\int_{\Omega} (u+\alpha)^p \right)^{\eta} \quad (5.11)$$

with $\varepsilon > 0$ and $C_1(m_1), C_2(m_1)$ and $C_3(m_1)$ defined in Step 2. In order to show that $R^{\frac{1}{\beta_i}} \leq R$ ($i = 1, 2$), we shall show that $R \geq 1$. Indeed, focusing on the second term on the right-hand side of (5.11), by using the inequality $u + \alpha \geq \alpha > 0$ and choosing ε small enough, we obtain

$$\frac{C_2(m_1)}{\varepsilon^{\frac{1-ar\eta}{ar\eta}}} \left(\int_{\Omega} (u+\alpha)^p \right)^{f(\eta,r)} \geq \frac{C_2(m_1)}{\varepsilon^{\frac{1-ar\eta}{ar\eta}}} (\alpha^p |\Omega|)^{f(\eta,r)} \geq 1.$$

Combining this inequality with (5.11) entails that $R \geq 1$. Therefore we arrive at

$$R^{\frac{1}{\beta_i}} \leq R \quad (i = 1, 2).$$

Plugging this inequality into (5.9) and (5.10), we obtain

$$\begin{aligned} \int_{\Omega}(u+\alpha)^{p+2m_2-m_1-3}|\nabla v|^2 &\leq \frac{1}{(q\eta)'} \cdot R |\Omega|^{\frac{1}{\beta_1'}} + \frac{1}{q\eta} \int_{\Omega} |\nabla v|^{2q\eta}, \\ \int_{\Omega}(u+\alpha)^2|\nabla v|^{2q-2} &\leq \frac{1}{(q'\eta)'} \cdot R |\Omega|^{\frac{1}{\beta_2'}} + \frac{1}{q'\eta} \int_{\Omega} |\nabla v|^{2q\eta}. \end{aligned}$$

By applying these two inequalities to (5.1), we see

$$\begin{aligned} &\frac{d\Phi}{dt} + \frac{p-1}{2} \left(\frac{2}{p+m_1-1} \right)^2 \int_{\Omega} \left| \nabla(u+\alpha)^{\frac{p+m_1-1}{2}} \right|^2 + \left(\frac{2(q-1)}{q^2} - \delta \right) \int_{\Omega} |\nabla |\nabla v|^q|^2 \\ &\leq \frac{\chi^2(p-1)}{2} \left(\frac{1}{(q\eta)'} \cdot R |\Omega|^{\frac{1}{\beta_1'}} + \frac{1}{q\eta} \int_{\Omega} |\nabla v|^{2q\eta} \right) \\ &\quad + \frac{4(q-1)+n}{2} \left(\frac{1}{(q'\eta)'} \cdot R |\Omega|^{\frac{1}{\beta_2'}} + \frac{1}{q'\eta} \int_{\Omega} |\nabla v|^{2q\eta} \right) + D_{\delta} \\ &= E_1 R + E_2 \int_{\Omega} |\nabla v|^{2q\eta} + D_{\delta}, \end{aligned} \quad (5.12)$$

where

$$E_1 := \frac{\chi^2(p-1)}{2} \cdot \frac{1}{(q\eta)'} \cdot |\Omega|^{\frac{1}{\beta_1'}} + \frac{4(q-1)+n}{2} \cdot \frac{1}{(q'\eta)'} \cdot |\Omega|^{\frac{1}{\beta_2'}},$$

$$E_2 := \frac{\chi^2(p-1)}{2} \cdot \frac{1}{q\eta} + \frac{4(q-1)+n}{2} \cdot \frac{1}{q'\eta}.$$

Plugging (5.7) into (5.12), we can rearrange (5.1) as follows:

$$\begin{aligned} & \frac{d\Phi}{dt} + \frac{p-1}{2} \left(\frac{2}{p+m_1-1} \right)^2 \int_{\Omega} \left| \nabla(u+\alpha)^{\frac{p+m_1-1}{2}} \right|^2 + \left(\frac{2(q-1)}{q^2} - \delta \right) \int_{\Omega} |\nabla|\nabla v|^q|^2 \\ & \leq E_1 R + E_2 \left(C_4 \varepsilon \int_{\Omega} |\nabla|\nabla v|^q|^2 + \frac{C_5}{\varepsilon^{\frac{\eta}{2-\eta}}} \left(\int_{\Omega} |\nabla v|^{2q} \right)^{f(\eta,1)} + C_6 \left(\int_{\Omega} |\nabla v|^{2q} \right)^{\eta} \right) + D_{\delta}. \end{aligned}$$

Recalling the definition of R (see (5.11)), we infer

$$\begin{aligned} & \frac{d\Phi}{dt} + \left(\frac{p-1}{2} \left(\frac{2}{p+m_1-1} \right)^2 - E_1 C_1(m_1) \varepsilon \right) \int_{\Omega} \left| \nabla(u+\alpha)^{\frac{p+m_1-1}{2}} \right|^2 \\ & + \left(\left(\frac{2(q-1)}{q^2} - \delta \right) - E_2 C_4 \varepsilon \right) \int_{\Omega} |\nabla|\nabla v|^q|^2 \\ & \leq E_1 C_2(m_1) \varepsilon^{-\frac{1-ar\eta}{ar\eta}} \left(\int_{\Omega} (u+\alpha)^p \right)^{f(\eta,r)} + E_2 C_5 \varepsilon^{-\frac{\eta}{2-\eta}} \left(\int_{\Omega} |\nabla v|^{2q} \right)^{f(\eta,1)} \\ & + E_1 C_3(m_1) \left(\int_{\Omega} (u+\alpha)^p \right)^{\eta} + E_2 C_6 \left(\int_{\Omega} |\nabla v|^{2q} \right)^{\eta} + D_{\delta}. \end{aligned} \quad (5.13)$$

Then it follows that the second and third terms on the left-hand side of (5.13) are nonnegative. We now fix $\delta \in (0, \frac{2(q-1)}{q^2})$ and choose ε small enough to satisfy not only (5.12) but also

$$E_1 C_1(m_1) \varepsilon \leq \frac{p-1}{2} \left(\frac{2}{p+m_1-1} \right)^2, \quad E_2 C_2(m_1) \varepsilon \leq \frac{2(p-1)}{q^2} - \delta.$$

Therefore we can rewrite (5.13) as

$$\begin{aligned} \frac{d\Phi}{dt} & \leq E_1 C_2(m_1) \varepsilon^{-\frac{1-ar\eta}{ar\eta}} \left(\int_{\Omega} (u+\alpha)^p \right)^{f(\eta,r)} + E_2 C_5 \varepsilon^{-\frac{\eta}{2-\eta}} \left(\int_{\Omega} |\nabla v|^{2q} \right)^{f(\eta,1)} \\ & + E_1 C_3(m_1) \left(\int_{\Omega} (u+\alpha)^p \right)^{\eta} + E_2 C_6 \left(\int_{\Omega} |\nabla v|^{2q} \right)^{\eta} + D_{\delta}. \end{aligned}$$

Noting that

$$\int_{\Omega} (u+\alpha)^p \leq p\Phi(t), \quad \int_{\Omega} |\nabla v|^{2q} \leq q\Phi(t),$$

we can establish (5.8) with

$$\begin{aligned} A &= A(m_1) := p^{f(\eta,r)} E_1 C_2(m_1) \varepsilon^{-\frac{1-ar\eta}{ar\eta}}, \\ B &:= q^{f(\eta,1)} E_2 C_5 \varepsilon^{-\frac{\eta}{2-\eta}}, \\ C &= C(m_1) := p^\eta E_1 C_3(m_1) + q^\eta E_2 C_6, \\ D &:= D_\delta. \end{aligned}$$

Thus (5.8) holds.

(Step 4) In this step we establish the following lower bound for the blow-up time t^* for solutions of (1.1) in the classical sense:

$$t^* \geq \int_{\Phi(0)}^{\infty} \frac{d\tau}{A\tau^{f(\eta,r)} + B\tau^{f(\eta,1)} + C\tau^\eta + D}. \quad (5.14)$$

We first show that we can estimate a lower bound for the blow-up time in Φ -measure. Indeed, we put

$$G(\Phi(t)) := A\Phi(t)^{f(\eta,r)} + B\Phi(t)^{f(\eta,1)} + C\Phi(t)^\eta + D$$

and

$$H(x) := \int_{\Phi(0)}^x \frac{d\tau}{G(\tau)} \quad (x \geq 0).$$

Since $f(\eta, s) > 1$ ($s > 0$), we notice that $\lim_{x \nearrow \infty} H(x)$ exists, and hence we obtain from the chain rule and the inequality $\frac{d\Phi(t)}{dt} \leq G(\Phi(t))$ (see (5.8)) that

$$\frac{d}{dt} [H(\Phi(t))] = \frac{1}{G(\Phi(t))} \cdot \frac{d\Phi(t)}{dt} \leq 1.$$

By integrating from 0 to $t_{p,q}^*$, we have

$$H(\Phi(t_{p,q}^*)) - H(\Phi(0)) \leq t_{p,q}^*.$$

Noting that $\lim_{t \nearrow t_{p,q}^*} \Phi(t) = \infty$ and $H(\Phi(0)) = 0$, we can attain that

$$t_{p,q}^* \geq \int_{\Phi(0)}^{\infty} \frac{d\tau}{A\tau^{f(\eta,r)} + B\tau^{f(\eta,1)} + C\tau^\eta + D}. \quad (5.15)$$

Furthermore, we can regard the blow-up time for solutions of (1.1) in Φ -measure as that in the classical sense under the condition (C1), i.e.,

$$t_{p,q}^* = t^* \quad (5.16)$$

(see Corollary 2.7). A combination of (5.15) with (5.16) yields

$$t^* = t_{p,q}^* \geq \int_{\Phi(0)}^{\infty} \frac{d\tau}{A\tau^{f(\eta,r)} + B\tau^{f(\eta,1)} + C\tau^\eta + D}.$$

Thus we arrive at (5.14). In conclusion, the proof of Theorem 1.1 is completed. \square

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