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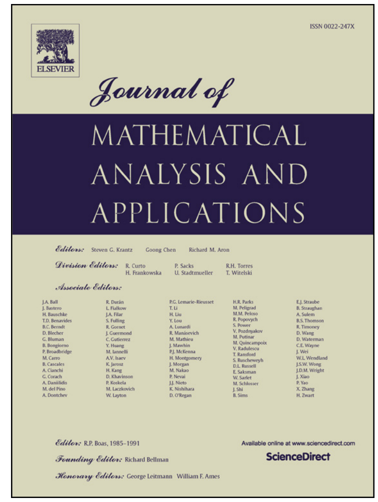

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Existence of positive solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^n$

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Abstract

In this paper, we consider the following nonlinear problem of Kirchhoff type:

$$\begin{cases} -\left(a+\lambda\int_{\mathbb{R}^n}|\nabla u|^2dx\right)\Delta u+V(x)u=|u|^{p-1}u, \quad x\in\mathbb{R}^n, \\ u\in H^1(\mathbb{R}^n), \quad u>0, \end{cases}$$

where $n\geq 3$, a, λ are positive constants, $1<p<\max(3,\frac{n+2}{n-2})$ and $V(x)$ is a positive continuous potential satisfying $V_0\leq V(x)\leq\liminf_{|x|\to\infty}V(x)=V_\infty$. Using variational methods and a cutoff technique, we prove the existence of positive solution to the above equation for all $\lambda>0$ small.

Keywords: Kirchhoff type equation, subcritical exponent, positive solution, variational method
2000 MSC: 35J50, 35B33, 35R11

1. Introduction

In this paper, we consider the following nonlinear problem of Kirchhoff type:

$$\begin{cases} -\left(a+\lambda\int_{\mathbb{R}^n}|\nabla u|^2dx\right)\Delta u+V(x)u=|u|^{p-1}u, \quad x\in\mathbb{R}^n, \\ u\in H^1(\mathbb{R}^n), \quad u>0, \end{cases} \tag{1.1}$$

where $n\geq 3$, a, λ are positive constants, $1<p<p^*:=\max(3,\frac{n+2}{n-2})$. Such problems are often referred to as being nonlocal since (1.1) is no longer a pointwise identity due to the term $(\int_{\mathbb{R}^n}|\nabla u|^2dx)\Delta u$. Certainly, it has a variational structure, and we can define a functional $I_\lambda(u)$ on $H^1(\mathbb{R}^n)$ by

$$I_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^n}(a|\nabla u|^2+V(x)u^2)dx+\frac{\lambda}{4}\left(\int_{\mathbb{R}^n}|\nabla u|^2dx\right)^2-\frac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx.$$

Problem (1.1) is a variant of the following Dirichlet problem of Kirchhoff type on bounded domains

$$\begin{cases} -\left(a+\lambda\int_{\Omega}|\nabla u|^2dx\right)\Delta u+V(x)u=f(x,u), \quad x\in\Omega, \\ u=0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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which is related to the stationary analogue of the equation

$$\begin{cases} u_{tt} - \left(a + \lambda \int_\Omega |\nabla u|^2 dx\right) \Delta u + V(x)u = f(x,u), \quad x \in \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

proposed by Kirchhoff in [8] as an existence of the classical D'Alembert's wave equations for free vibration of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. After the pioneer work of Lions [14], problem (1.1) gets much attention to mathematicians. In [1], Arosio and Panizzi studied the Cauchy-Dirichlet type problem related to (1.3) in the Hadamard sense as a special case of an abstract second-order Cauchy problem in a Hilbert space. In [3, 4], He and Zou obtained infinitely many solutions of (1.2) by Fountain Theorem. Multiple solutions and concentration phenomena were also observed in [6] for Kirchhoff type equation with critical growth by He and Zou. For more results , we refer to [1, 5, 13, 17, 18, 22] and the references therein.

When $a = 1$ and $\lambda = 0$, problem (1.2) reduces to the well known Schrödinger equation

$$-\Delta u + V(x) = f(u) \quad \text{in} \quad \mathbb{R}^n,$$

for which, the existence of positive solutions have been extensively studied in recent years via variational methods. When $\lambda > 0$, the competing effect of the nonlocal term $(\int_{\mathbb{R}^n} |\nabla u|^2 dx)\Delta u$ with the nonlinearity $f(u)$ makes the problem more complicated. Generally, either 4-superlinear condition at infinity or an Ambrosetti-Rabinowitz type condition is needed to ensure boundedness of Palais-Smale sequences. This usually requires that $f(u)$ has a growth bigger than u^p for some $p > 3$ at infinity. See for example [5, 7]. When $1 < p < 3$, it becomes much more difficult to get a bounded Palais-Smale sequence. In [15], Li et al. studied the following Kirchhoff type problem

$$\left(a + \epsilon \int_{\mathbb{R}^n} (|\nabla u|^2 + b|u|^2) dx\right) [-\Delta u + bu] = f(u), \quad \text{in} \quad \mathbb{R}^n,$$

where $n \geq 3$, a, b are positive constants and $f(u)$ is subcritical and only has superlinear growth condition as origin and infinity. They showed that when $\epsilon \geq 0$ small, the above problem has at least one positive solution by using a cut-off technique together with a monotonicity method introduced by Jeanjean [10]. In the case $n = 3$ and $2 < p < 5$, Li showed in [12] that problem (1.1) has a ground state solution for all $\lambda > 0$ under some mild conditions on potential $V(x)$ and $\nabla V(x)$. Their proof depends on a global compactness lemma and on comparison among the energy of functionals I_λ, J_λ and J_λ^∞ (see the section 3 below for the definitions of those functionals). Generally, non-constant potential function $V(x)$ makes the problem more difficult to deal with.

In this paper, we assume that $V(x)$ satisfies the following condition:

(V) $V(x)$ is a continuous non-constant function in $\mathbb{R}^n$, such that

$$0 < V_0 := \inf_{x \in \mathbb{R}^n} V(x) \leq V(x) \leq \liminf_{|x| \to \infty} V(x) = V_\infty < \infty.$$

Our main theorem is as follows:

Theorem 1.1. *Assume that $n \geq 3$, a is a positive constant, $1 < p < p*$ and $\lambda \geq 0$ is a parameter. If the function $V(x)$ satisfies the condition (V), then there exists $\lambda_0 > 0$ small such that for any $\lambda \in [0, \lambda_0)$, equation (1.1) has at least one positive solution.*

To prove Theorem 1.1, there are two difficulties that we need to overcome. First, we need to show the existence of bounded Palais-Smale(PS for short) sequence. We use the cut-off technique in [15] to get a uniform boundedness of PS sequence for I_λ independent of λ for all small λ.

Second, we need to show the sequential convergence of the bounded PS sequence. Since $V(x)$ is not a constant, we can not work in the space of radial functions. To show the convergence of PS sequence, we compare the mountain pass critical level c_λ of I_λ with the mountain pass critical level c_λ^∞ of the energy of the limit equation at infinity I_λ^∞ as in [12]. In order to do this, it is important to show that $c_\lambda < c_\lambda^\infty$. We can show this fact only when λ is small. The smallness of λ would be understood since it can be viewed as the perturbation away from $\lambda = 0$. In [12], $c_\lambda < c_\lambda^\infty$ can be proved easily when $n = 3$ and $2 < p < 5$. But the method doesn't work for all of the cases here.

Remark 1.1. *The solution of the limit equation* (2.1) *is related to the solution of* $-\Delta u + u = u^p$*. In Lemma 2.3, we give the solution of* (2.1) *and the exact value of* c_λ^∞ *explicitly. This fact will be used to show the sequential convergence of* PS *sequence for* (1.1)*. Since* $V(x)$ *is not a constant, our method only works for* $f(u) = u^p$*. However, when* $V(x)$ *is a positive constant,* $f(u)$ *allows to be more general, such as* $f(u)$ *is superlinear at* 0 *and subcritical at* ∞ *due to the fact that* $H_r^1(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *for* $2 < p < \frac{2n}{n-2}$ *is compact.*

Remark 1.2. *When* $n > 4$*, for* λ *small,* (2.1) *has two radial solutions. One corresponds to mountain pass critical value and the other corresponds to the global minimum (which tends to* $-\infty$ *as* $\lambda \to 0$*) of* I_λ^∞*. The solution we get in Theorem 1.1 for small* λ *actually is the mountain pass critical point of* I_λ*. For the following nonlinear Schrödinger-Poisson system*

$$\begin{cases} -\Delta u + u + \lambda \phi u = |u|^{p-1}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

Ruiz proved in [19] that when $1 < p \le 2$*, the above system has at least two nontrivial solutions when* $\lambda > 0$ *small. Similar multiple solutions for Kirchhoff type equation can be found in [11, 16, 20]. It should be interesting to see whether* (1.1) *has two nontrivial solutions when* $n > 4$ *and* $\lambda > 0$ *is sufficiently small.*

Remark 1.3. *In [12], some additional conditions are assumed for* $\nabla V(x)$ *to prove the existence of nontrivial solution of* (1.1) *for* $2 < p < 5$ *and* $\lambda > 0$ *in dimension three. Here, we don't need any condition for* $\nabla V(x)$ *due to* λ *small.*

Remark 1.4. *When* $n = 3$ *and* $3 \le p < 5$*, it is easy to show that any* $(PS)_c$ *sequence is bounded, so we only consider the case* $1 < p < p^*$*.*

Next, we give a nonexistence result for λ large in the case $n > 3$:

Theorem 1.2. *Under the same conditions as in Theorem 1.1 and* $n > 3$*. There exists* $\lambda_1 > 0$*, such that equation* (1.1) *has no nontrivial solution when* $\lambda \ge \lambda_1$*.*

The paper is organized as follows. In Section 2, we present some preliminary results. In Section 3, we prove Theorem 1.1. In Section 4, we give the proof of Theorem 1.2.

2. Preliminary results

Let $H^1(\mathbb{R}^n)$ be the usual Sobolev space equipped with the following inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^n} (a\nabla u \cdot \nabla v + V(x)uv)dx, \quad \|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

By the condition (V), $\|\cdot\|$ is equivalent to the standard norm on $H^1(\mathbb{R}^n)$. We know that the functional $I_\lambda(u)$ given in section 1 is well defined on $H^1(\mathbb{R}^n)$ and is of C^1 for all $\lambda \geq 0$, and

$$\langle I'_\lambda(u), v\rangle = (a + \lambda \int_{\mathbb{R}^n} |\nabla u|^2 dx) \int_{\mathbb{R}^n} \nabla u \cdot \nabla v + \int_{\mathbb{R}^n} V(x)uv - \int_{\mathbb{R}^n} |u|^{p-1}uv,$$

for all u ,$v \in H^1(\mathbb{R}^n)$. It is standard to verify that the critical points of the functional I_λ are the weak solutions of (1.1).

We first show that I_λ satisfies the mountain path geometry when λ is small.

Lemma 2.1. *For $\lambda \geq 0$ small, the functional I_λ satisfies the following conditions.*

(i) There exists $\alpha > 0$, $\rho > 0$ such that $I_\lambda(u) \geq \alpha$ for $\|u\| = \rho$.

(ii) There exists an $e \in B^c_\rho(0)$ such that $I_\lambda(e) < 0$.

Proof. (i) By the Sobolev embedding $H^1(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for $2 \leq q \leq \frac{2n}{n-2}$, we have

$$I_\lambda(u) \geq I_0(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|^{p+1}.$$

Therefore, we can take some $\alpha > 0$, $\rho > 0$ such that $I_\lambda(u) \geq \alpha$ for $\|u\| = \rho$.

(ii) Take $e \in H^1(\mathbb{R}^n)$ such that $I_0(e) < 0$, clearly $e \in B^c_\rho(0)$. Then for $0 \leq \lambda \leq \frac{-2I_0(e)}{(\int_{\mathbb{R}^n} |\nabla e|^2 dx)^2}$,

$$I_\lambda(e) = I_0(e) + \frac{\lambda}{4}(\int_{\mathbb{R}^n} |\nabla e|^2 dx)^2 \leq \frac{1}{2} I_0(e) < 0.$$

□

Remark 2.1. *In [12], when $V(x)$ is a constant, $\lambda \geq 0$ and $2 < p < 5$, the authors show that for any $u \neq 0$ in $H^1(\mathbb{R}^3)$, $I_\lambda(tu(t^{-1}x))$ first increases then decreases for $t > 0$ and tends to $-\infty$ as $t \to \infty$. Thus I_λ processes a mountain pass geometry for all $\lambda \geq 0$. The method fails for $n \geq 4$. Actually, when $n = 4$ with λ large or when $n \geq 5$, I_λ has a lower bound.*

Lemma 2.2. *When $n = 4$ with λ large or when $n \geq 5$ with $\lambda > 0$, I_λ is bounded from below in $H^1(\mathbb{R}^n)$.*

Proof. By Hölder's inequality, we have

$$\|u\|_{L^{p+1}} \leq \|u\|^r_{L^2}\|u\|^{1-r}_{L^{\frac{2n}{n-2}}}, \quad \text{for} \quad u \in L^2(\mathbb{R}^n) \cap L^{\frac{2n}{n-2}}(\mathbb{R}^n),$$

where $\frac{1}{p+1} = \frac{r}{2} + \frac{(n-2)(1-r)}{2n}$. By Sobolev imbedding,

$$\|u\|_{L^{\frac{2n}{n-2}}} \leq C\|\nabla u\|_{L^2}, \quad \text{for} \quad u \in H^1(\mathbb{R}^n).$$

From above two inequalities, we can get

$$\int_{\mathbb{R}^n} |u|^{p+1}dx \leq C\|u\|_{L^2}^{\frac{n+2-(n-2)p}{2}} \|\nabla u\|_{L^2}^{\frac{n(p-1)}{2}}, \quad \text{for} \quad u \in H^1(\mathbb{R}^n).$$

Since $\frac{n+2-(n-2)p}{2} < 2$ when $p > 1$, by Young's Inequality, for any $\epsilon > 0$, there exists a constant $C(\epsilon) > 0$, such that

$$\int_{\mathbb{R}^n} |u|^{p+1}dx \leq \epsilon \int_{\mathbb{R}^n} u^2 dx + C(\epsilon) \left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{\frac{n}{n-2}}.$$

If we choose $\epsilon = \frac{V_0}{4}$, then

$$\begin{aligned} I_\lambda(u) &\geq \tfrac{a}{2}\|\nabla u\|_{L^2}^2 + \tfrac{1}{2}\int_{\mathbb{R}^n} V(x)u^2dx + \tfrac{\lambda}{4}\|\nabla u\|_{L^2}^4 - \tfrac{V_0}{4}\|u\|_{L^2}^2 - C\|\nabla u\|_{L^2}^{\frac{2n}{n-2}} \\ &\geq \tfrac{a}{2}\|\nabla u\|_{L^2}^2 + \tfrac{V_0}{4}\|u\|_{L^2}^2 + \tfrac{\lambda}{4}\|\nabla u\|_{L^2}^4 - C\|\nabla u\|_{L^2}^{\frac{2n}{n-2}}. \end{aligned}$$

When $n = 4$, if $\frac{\lambda}{4} \geq C$, then $I_\lambda(u) \geq 0$ for all $u \in H^1(\mathbb{R}^n)$. When $n \geq 5$, since $\frac{2n}{n-2} < 4$, $I_\lambda(u) \geq 0$ when $\|\nabla u\|_{L^2} \geq \left(\frac{4C}{\lambda}\right)^{\frac{n-2}{2(n-4)}}$. Therefore, it is easy to get that $I_\lambda(u)$ is bounded from below when $n \geq 5$ for all $\lambda > 0$. □

For $\lambda > 0$ small, define the mountain path critical value c_λ of I_λ by

$$c_\lambda = \inf_{g\in\Gamma} \sup_{t\in[0,1]} I_\lambda(g(t)) > 0,$$

where

$$\Gamma := \{g \in C([0,1], H^1(\mathbb{R}^n)) | g(0) = 0, I_\lambda(g(1)) < 0\}.$$

Clearly, c_λ is non-decreasing in λ.

As we will see, it is important to compare c_λ with the mountain path critical level of the autonomous problem

$$\begin{cases} -\left(a + \lambda \int_{\mathbb{R}^n} |\nabla u|^2 dx\right) \Delta u + V_\infty u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n, \\ u \in H^1(\mathbb{R}^n), \quad u(x) > 0, \quad \forall x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

The solutions to problem (2.1) are precisely the critical points of the functional defined by

$$I_\lambda^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^n} (a|\nabla u|^2 + V_\infty u^2)dx + \frac{1}{4}\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx$$

The solution of (2.1) is related to the following well known Schrödinger equation

$$-\Delta u + u = u^p, \quad u > 0 \quad \text{in} \quad \mathbb{R}^n. \tag{2.2}$$

Let w be the unique radial solution of equation (2.2)(see [9]). By simple dilation and scaling argument, we can get the radial solutions to (2.1).

Lemma 2.3. *Let w_λ be the nontrivial radial solution of (2.1). Let $A^2(\lambda) = \int_{\mathbb{R}^n} |\nabla w_\lambda|^2 dx$, $S = V_\infty^{\frac{2}{p-1}-\frac{n-2}{2}} \int_{\mathbb{R}^n} |\nabla w|^2 dx$. Then we have*

$$w_\lambda(x) = V_\infty^{\frac{1}{p-1}} w\left(\sqrt{\frac{V_\infty}{a+\lambda A^2}}x\right), \tag{2.3}$$

with A^2 determined by the following nonlinear equality:

$$A^2 = (a + \lambda A^2)^{\frac{n-2}{2}} S, \tag{2.4}$$

which can be divided to the following three cases according to n:

(1) When $n = 3$, there is a unique nontrivial radial solution of (2.1) for all $\lambda \geq 0$ with

$$A^2(\lambda) = \frac{\lambda S^2 + \sqrt{\lambda^2 S^4 + 4aS^2}}{2}.$$

(2) When $n = 4$, if $\lambda \geq \frac{1}{S}$, (2.1) has no nontrivial solution, if $\lambda < \frac{1}{S}$, there is a unique nontrivial radial solution, in this case

$$A^2(\lambda) = \frac{aS}{1 - \lambda S}.$$

(3) When $n \geq 5$, if $\lambda > \bar{\lambda} = \frac{2}{(n-2)S}\left(\frac{n-4}{(n-2)a}\right)^{\frac{n-4}{2}}$, (2.1) has no nontrivial solution; if $\lambda = \bar{\lambda}$, (2.1) has a unique nontrivial radial solution and when $0 < \lambda < \bar{\lambda}$, it has two nontrivial radial solutions. Moreover, if we denote the two A values by $A_1(\lambda) < A_2(\lambda)$, then we have $0 < A_1(\lambda) < A(\bar{\lambda}) < A_2(\lambda)$ and

$$\lim_{\lambda \to 0^+} A_1^2(\lambda) = a^{\frac{n-2}{2}} S \quad \text{and} \quad \lim_{\lambda \to 0^+} \lambda A_2^2(\lambda) = +\infty. \tag{2.5}$$

Proof. The solutions of (2.1) have to be the solutions of the following equation

$$-(a + \lambda A^2)\Delta u + V_\infty u = u^p, \quad u > 0.$$

Therefore (2.3) and (2.4) are from simple scaling argument and calculations from the unique radial solution w of (2.2). The results can be derived easily from (2.4) for $n = 3,\ 4$. When $n \geq 5$, if we let $t = a + \lambda A^2$, then (2.4) can be written as

$$t - a = \lambda S t^{\frac{n-2}{2}}.$$

The solution to the above equation is just the intersection points of the line $y = t - a$ and the curve $y = \lambda S t^{\frac{n-2}{2}}$. When λ is large, the two curves don't intersect; If the curves intersect only at one point, they are tangent to each other at the intersection point, from which we get

$$\begin{cases} t - a = \lambda S t^{\frac{n-2}{2}}, \\ \frac{(n-2)\lambda S}{2} t^{\frac{n-4}{2}} = 1. \end{cases}$$

We get $t = \frac{(n-2)a}{n-4}$ and $\lambda = \bar{\lambda} = \frac{2}{(n-2)S}\left(\frac{n-4}{(n-2)a}\right)^{\frac{n-4}{2}}$. In this case, (2.1) only has one unique nontrivial radial solution with $A^2(\bar{\lambda}) = \frac{t-a}{\bar{\lambda}} = S\left(\frac{(n-2)a}{n-4}\right)^{\frac{n-2}{2}}$. If $\lambda > \bar{\lambda}$, the two curves don't intersect, so (2.1) has no nontrivial solution.

Now when $0 < \lambda < \bar{\lambda}$, the two curves intersect at two points. If we denote the two A values by $A_1(\lambda)$ and $A_2(\lambda)$ with $0 < A_1 < A_2$, then from the graph of the two curves, it is easy to see that for all $0 < \lambda < \bar{\lambda}$,

$$\lambda A_1^2(\lambda) < \bar{\lambda} A^2(\bar{\lambda}) = \frac{2a}{n-4} < \lambda A_2^2(\lambda),$$

together with (2.4),

$$\lim_{\lambda \to 0^+} \lambda A_1^2(\lambda) = \lim_{\lambda \to 0^+} \lambda (a + \lambda A_1^2)^{\frac{n-2}{2}} S = 0,$$

therefore

$$\lim_{\lambda \to 0^+} A_1^2(\lambda) = \lim_{\lambda \to 0^+} (a + \lambda A_1^2)^{\frac{n-2}{2}} S = a^{\frac{n-2}{2}} S.$$

If $\lambda A_2^2(\lambda) < \infty$ as $\lambda \to 0^+$, then by (2.4),

$$\lim_{\lambda\to 0^+} \lambda A_2^2(\lambda) = \lim_{\lambda\to 0^+} \lambda(a+\lambda A_2^2)^{\frac{n-2}{2}} S = 0,$$

which contradicts the fact $\lambda A_2^2(\lambda) > \frac{2a}{n-4}$. Therefore

$$\lim_{\lambda\to 0^+} \lambda A_2^2(\lambda) \to +\infty.$$

□

Remark 2.2. *When $n \geq 5$ and $0 < \lambda \leq \bar{\lambda}$, by the above argument, it is easy to see that $\lambda A_1^2(\lambda)$ increases in λ and $\lambda A_2^2(\lambda)$ decreases in λ. Then (2.4) shows that $A_1(\lambda)$ increases in λ and $A_2(\lambda)$ decreases in λ for $0 < \lambda \leq \bar{\lambda}$.*

For $\lambda \geq 0$ small, let c_λ^∞ be the mountain path critical value of I_λ^∞. Since the solution of (2.1) satisfies the Pohožaev identity

$$\frac{(n-2)}{2}\left(a+\lambda\int_{\mathbb{R}^n}|\nabla u|^2dx\right)\int_{\mathbb{R}^n}|\nabla u|^2dx + \frac{n}{2}\int_{\mathbb{R}^n}V_\infty u^2dx - \frac{n}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx = 0.$$

We can get that, for λ small,

$$c_\lambda^\infty = \frac{a}{n}A_0^2(\lambda) + \frac{(4-n)\lambda}{4n}A_0^4(\lambda), \tag{2.6}$$

where

$$A_0^2(\lambda) = \begin{cases} \frac{\lambda S^2+\sqrt{\lambda^2S^4+4aS^2}}{2} & \text{for} \quad n=3, \\ \frac{aS}{1-\lambda S} & \text{for} \quad n=4, \\ A_1^2(\lambda) & \text{for} \quad n\geq 5. \end{cases}$$

Remark 2.3. *According to [12], for $n=3$ and $2<p<5$, the solution w_λ is the solution corresponding to the mountain pass critical value of I_λ^∞ for all $\lambda > 0$. When $n=3$, $1<p\leq 2$ or $n=4$, if λ is small, the solution w_λ is the solution corresponding to c_λ^∞. When $n\geq 5$ with λ small, (2.1) has two radial solutions: one corresponds to the mountain path critical value c_λ^∞ and the other corresponds to the minimum of I_λ given by $\frac{a}{n}A_2^2(\lambda)+\frac{(4-n)\lambda}{4n}A_2^4(\lambda)$ (this value is negative when λ is small and approaches $-\infty$ when $\lambda\to 0$ due to the estimates on $\lambda A_2^2(\lambda)$ in (2.5)).*

When $\lambda = 0$, by condition (V), it is clearly that

$$0 < c_0 < c_0^\infty.$$

Due to competing effect of the term $(\int_{\mathbb{R}^n}|\nabla u|^2dx)^2$ with $\int_{\mathbb{R}^n}|u|^{p+1}$ when $1<p<3$, it is not clear whether $c_\lambda < c_\lambda^\infty$ for all possible λ where $I_\lambda(u)$ and $I_\lambda^\infty(u)$ both have a mountain pass geometry. However for λ small enough, we can show that $c_\lambda < c_\lambda^\infty$.

Lemma 2.4. *Suppose $a(\lambda)$, $b(\lambda)$ and $c(\lambda)$ are positive continuous increase functions defined in $\lambda \in [0,\epsilon]$. For $p \in (1,3)$, let*

$$f(t) = \frac{a(\lambda)t^2}{2} + \frac{\lambda b(\lambda)t^4}{4} - \frac{c(\lambda)t^{p+1}}{p+1}, \quad t \geq 0.$$

Assume for some $M > 0$ such that $f(M) < 0$ for all $\lambda \in [0,\epsilon]$ and $f'(1) = a(\lambda) + \lambda b(\lambda) - c(\lambda) = 0$. Then there exists a $0 < \lambda_1 \leq \epsilon$, such that $f(t)$ gets its maximum value in $[0,M]$ at $t = 1$.

Proof. This is a simple calculus lemma. By direct calculation,

$$\begin{aligned}
f'(t) &= a(\lambda)t + \lambda b(\lambda)t^3 - c(\lambda)t^p \\
f''(t) &= a(\lambda) + 3\lambda b(\lambda)t^2 - pc(\lambda)t^{p-1} \\
f'''(t) &= t^{p-2}(6\lambda b(\lambda)t^{3-p} - c(\lambda)p(p-1)).
\end{aligned}$$

Therefore for all λ sufficiently small, we get $f'''(t) < 0$ for all $t \in (0,M)$. Since $f'(t)$ changes sign in $[0,M]$, then we can see that $f''(t)$ changes sign exactly once in $[0,M]$ and we can see that $f'(t) \geq 0$ in $[0,1]$ and $f'(t) < 0$ in $(1,M]$. So $f(t)$ gets its maximum in $[0,M]$ at $t = 1$. □

Now for $\lambda > 0$ small, we let

$$w_\lambda(x) = V_\infty^{\frac{1}{p-1}} w\left(\sqrt{\frac{V_\infty}{a + \lambda A_0(\lambda)^2}}\, x\right).$$

Take $\epsilon_0 > 0$ sufficiently small, we can choose a large $M > 1$, such that $I_\lambda^\infty(Mw_\lambda) < 0$ for all $\lambda \in [0,\epsilon_0]$. Let

$$f(t) = I_\lambda^\infty(tw_\lambda), \quad t \in [0,M] \quad \text{and} \quad \lambda \in [0,\epsilon_0].$$

Then all the conditions of Lemma 2.4 is satisfied. We conclude that there exists a $\lambda_1 \in (0,\epsilon_0)$, such that for all $\lambda \in [0,\lambda_1)$, we get that

$$c_\lambda^\infty = I_\lambda^\infty(w_\lambda) = \max_{t\in[0,M]} I_\lambda^\infty(tw_\lambda).$$

By condition (V), $I_\lambda(Mw_\lambda) < I_\lambda^\infty(Mw_\lambda) < 0$, therefore for all $\lambda \in [0,\lambda_1)$,

$$\begin{aligned}
c_\lambda \leq &\max_{t\in[0,M]} I_\lambda(tw_\lambda) = I_\lambda(t_0 w_\lambda) \quad \text{for some} \quad t_0 \in (0,M) \\
<&I_\lambda^\infty(t_0 w_\lambda) \leq I_\lambda^\infty(w_\lambda) = c_\lambda^\infty.
\end{aligned}$$

Therefore, we get

$$c_\lambda < c_\lambda^\infty, \quad \text{for all} \quad \lambda \in [0,\lambda_1). \tag{2.7}$$

To overcome the difficulty of finding the bounded (PS) sequences for the functional I_λ, we follow the idea of [10, 15] and use a cut-off function $\Phi \in C^\infty(\mathbb{R}_+,\mathbb{R})$ satisfying

$$\begin{cases}
\Phi(t) = 1, & t \in [0,1], \\
0 \leq \Phi(t) \leq 1, & t \in (1,2), \\
\Phi(t) = 0, & t \in [2,\infty), \\
0 \leq \Phi'(t) \leq 2.
\end{cases}$$

For any $T > 0$, we modify the original functional and define a new functional I_λ^T by:

$$I_\lambda^T(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\lambda\Phi(\frac{\int_{\mathbb{R}^n}|\nabla u|^2dx}{T^2})\left(\int_{\mathbb{R}^n}|\nabla u|^2dx\right)^2 - \frac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx.$$

Since the term $\Phi(\frac{\int_{\mathbb{R}^n}|\nabla u|^2dx}{T^2})\left(\int_{\mathbb{R}^n}|\nabla u|^2dx\right)^2 \le 4T^4$, it is clear that for any fixed $T > 0$, the functional I_λ^T has a mountain path critical level. For this penalization, for $T > 0$ large and for λ small, we are able to find a bounded (PS) sequence $\{u_i\}$ of I_λ^T, which is also a (PS) sequence for I_λ, i. e., $\int_{\mathbb{R}^n}|\nabla u_i|^2 \le T^2$ for all i large.

For any $T > 0$, denote c_λ^T be the mountain path critical level for I_λ^T, recall the notation c_λ, c_λ^∞ defined earlier, clearly we have

$$c_0 \le c_\lambda^T \le c_0 + \lambda T^4 \le c_0^\infty + \lambda T^4.$$

Lemma 2.5. *Let $\{u_i\}$ be a $(PS)_{c_\lambda^T}$ sequence of I_λ^T. Then for $T > 0$ sufficiently large, there exists a $\lambda_2 = \lambda_2(T) = \frac{(p-1)a}{16T^4}$, such that for any $\lambda \in [0, \lambda_2)$, $\int_{\mathbb{R}^n}|\nabla u_i|^2dx \le T^2$ for all i large. Consequently, subject a subsequence, $\{u_i\}$ is a bounded $(PS)_{c_\lambda^T}$ sequence for both I_λ and I_λ^T.*

Proof. Since $\{u_i\}$ is a $(PS)_{c_\lambda^T}$ sequence of I_λ^T. For i large,

$$\begin{aligned} & c_\lambda^T + 1 + \|u_i\| \ge I_\lambda^T(u_i) - \tfrac{1}{p+1}\langle (I_\lambda^T)'(u_i), u_i\rangle \\ & = \tfrac{p-1}{2(p+1)}\|u_i\|^2 + \tfrac{\lambda(p-3)}{4(p+1)}\Phi(\tfrac{\int_{\mathbb{R}^n}|\nabla u_i|^2dx}{T^2})\left(\int_{\mathbb{R}^n}|\nabla u_i|^2dx\right)^2 \\ & -\tfrac{\lambda}{2(p+1)T^2}\Phi'(\tfrac{\int_{\mathbb{R}^n}|\nabla u_i|^2dx}{T^2})(\int_{\mathbb{R}^n}|\nabla u_i|^2dx)^3 \\ & \ge \tfrac{p-1}{2(p+1)}\|u_i\|^2 - \tfrac{T^4\lambda(3-p)}{p+1}. \end{aligned} \tag{2.8}$$

Therefore, we get

$$\begin{aligned} \tfrac{p-1}{2(p+1)}\|u_i\|^2 & \le c_\lambda^T + 1 + \|u_i\| + \tfrac{T^4\lambda(3-p)}{p+1} \\ & \le c_0^\infty + 1 + \|u_i\| + \tfrac{4\lambda T^4}{p+1}. \end{aligned}$$

If $\int_{\mathbb{R}^n}|\nabla u_i|^2dx > T^2$, since $a\int_{\mathbb{R}^n}|\nabla u_i|^2dx \le \|u_i\|^2$, we get $aT^2 \le \|u_i\|^2$. If we choose $T \ge \frac{4(p+1)}{\sqrt{a}(p-1)}$, then

$$\frac{p-1}{2(p+1)}\|u_i\|^2 - \|u_i\| \ge \frac{p-1}{4(p+1)}\|u_i\|^2.$$

Thus we have

$$\frac{a(p-1)T^2}{4(p+1)} \le \frac{p-1}{4(p+1)}\|u_i\|^2 \le c_0^\infty + 1 + \frac{4\lambda T^4}{p+1},$$

we get

$$T^2 \le \left(\frac{4(p+1)(c_0^\infty + 1)}{(p-1)a} + \frac{16\lambda T^4}{(p-1)a}\right). \tag{2.9}$$

Now if we choose $T^2 \ge \max\left(\frac{4(p+1)(c_0^\infty+1)}{(p-1)a} + 1, \frac{16(p+1)^2}{a(p-1)^2}\right)$ and then for any $0 < \lambda < \lambda_2(T) = \frac{(p-1)a}{16T^4}$, we get a contradiction with (2.9). Therefore $\int_{\mathbb{R}^n}|\nabla u|^2dx \le T^2$ for all i large. From (2.8), we can see that $\{u_i\}$ is bounded in $H^1(\mathbb{R}^n)$. Therefore subject to a subsequence if necessary, $\{u_i\}$ is a bounded $(PS)_{c_\lambda^T}$ sequence for both I_λ and I_λ^T. $\square$

3. Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1. We first give the following version of a global compactness lemma given in [12].

Lemma 3.1 (Lemma 3.4 of [12])**.** *Assume that (V) holds and $1 < p < p^*$. For $c > 0$, let $\{u_i\} \subset H^1(\mathbb{R}^n)$ be a bounded $(PS)_c$ sequence for I_λ, then there exists a $u \in H^1(\mathbb{R}^n)$ and $A \in \mathbb{R}$ such that $J'_\lambda(u) = 0$, where*

$$J_\lambda(u) = \frac{a+\lambda A^2}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} V(x)u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx, \tag{3.1}$$

and either

(i) $u_i \to u$ in $H^1(\mathbb{R}^n)$, or

(ii) there exists an positive integer $l \in \mathbb{N}$ and $\{y_i^k\} \in \mathbb{R}^n$ with $|y_i^k| \to \infty$ for each $1 \le k \le l$, nontrivial solutions w^1, ..., w^l of the following problem

$$-(a+\lambda A^2)\Delta u + V_\infty u = |u|^{p-1}u, \tag{3.2}$$

such that

$$c + \frac{\lambda A^4}{4} = J_\lambda(u) + \sum_{k=1}^{l} J_\lambda^\infty(w^k),$$

where

$$J_\lambda^\infty(u) = \frac{a+\lambda A^2}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{V_\infty}{2}\int_{\mathbb{R}^n} u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx, \tag{3.3}$$

moreover

$$\|u_i - u - \sum_{k=1}^{l} w^k(\cdot - y_i^k)\| \to 0,$$

$$A^2 = \|\nabla u\|_{L^2}^2 + \sum_{k=1}^{l} \|\nabla w^k\|_{L^2}^2.$$

Proof. The proof is standard, we omit it (see [2, 12] for detail). □

Proof of Theorem 1.1. From Lemma 2.5, if we choose

$$T_0^2 = \max\Big(\frac{4(p+1)(c_0^\infty+1)}{(p-1)a} + 1, \frac{16(p+1)^2}{a(p-1)^2}\Big),$$

then for all $\lambda < \frac{(p-1)a}{16T_0^4}$, I_λ has bounded $(PS)_{c_\lambda^{T_0}}$ sequence $\{u_i\}$ with $\|\nabla u_i\|_{L^2} \le T_0^2$. We may assume that $\lim\limits_{i\to\infty} \|\nabla u_i\|_{L^2} = A^2 \le T_0^2$. Applying Lemma 3.1, there exists a $u_\lambda \in H^1(\mathbb{R}^n)$, such that

$$u_i \rightharpoonup u_\lambda \quad \text{in} \quad H^1(\mathbb{R}^n), \qquad J'_\lambda(u_\lambda) = 0,$$

and either (i) or (ii) occurs.

If (ii) occurs, i. e., there exists an positive integer $l \in \mathbb{N}$ and $\{y_i^k\} \in \mathbb{R}^n$ with $|y_i^k| \to \infty$ for each $1 \le k \le l$, nontrivial solutions w^1, ..., w^l of (3.2) such that

$$\|u_i - u_\lambda - \sum_{k=1}^{l} w^k(\cdot - y_i^k)\| \to 0,$$

$$A^2 = \|\nabla u_\lambda\|_{L^2}^2 + \sum_{k=1}^{l} \|\nabla w^k\|_{L^2}^2.$$

and

$$\begin{aligned} c_\lambda^{T_0} &= J_\lambda(u_\lambda) + \textstyle\sum_{k=1}^{l} J_\lambda^\infty(w^k) - \frac{\lambda A^4}{4}, \\ &= \left(J_\lambda(u_\lambda) - \frac{\lambda A^2}{4}\|\nabla u_\lambda\|_{L^2}^2\right) + \textstyle\sum_{k=1}^{l}\left(J_\lambda^\infty(w^k) - \frac{\lambda A^2}{4}\|\nabla w^k\|_{L^2}^2\right). \end{aligned}$$

Next we will show that for λ small,

$$J_\lambda(u_\lambda) - \frac{\lambda A^2}{4}\|\nabla u_\lambda\|_{L^2}^2 \ge 0,$$

and

$$J_\lambda^\infty(w^k) - \frac{\lambda A^2}{4}\|\nabla w^k\|_{L^2}^2 \ge c_\lambda^\infty \quad \text{for all} \quad 1 \le k \le l.$$

This will lead to a contradiction when $l \ge 1$.

Since w^k satisfies (3.2), then similar to the proof of Lemma 2.3,

$$w^k(x) = V_\infty^{\frac{1}{p-1}} w\left(\sqrt{\frac{V_\infty}{a+\lambda A^2}}x\right).$$

Since $A^2 \ge \int_{\mathbb{R}^n} |\nabla w^k|^2 dx$ and $\int_{\mathbb{R}^n} |\nabla w^k|^2 dx = (a+\lambda A^2)^{\frac{n-2}{2}} S$ (S is given in Lemma 2.3), we get

$$A^2 \ge (a+\lambda A^2)^{\frac{n-2}{2}} S.$$

When $n = 3, 4$, it is easy to see that $A^2 \ge A_0^2$ given in (2.6). In the case $n \ge 5$, by the argument of Lemma 2.3, we can see that $A_0^2 = A_1^2(\lambda) \le A^2 \le A_2^2(\lambda)$.

Using the Pohoẑaev identity for the equation (3.2):

$$\frac{n-2}{2}(a+\lambda A^2)\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{nV_\infty}{2}\int_{\mathbb{R}^n} u^2 dx - \frac{n}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx = 0,$$

we derive that

$$\begin{aligned} J_\lambda^\infty(w^k) - \frac{\lambda A^2}{4}\|\nabla w^k\|_{L^2}^2 &= \frac{a+\lambda A^2}{n}\|\nabla w^k\|_{L^2}^2 - \frac{\lambda A^2}{4}\|\nabla w^k\|_{L^2}^2 \\ &= \frac{4a+(4-n)\lambda A^2}{4n}(a+\lambda A^2)^{\frac{n-2}{2}} S. \end{aligned}$$

Consider the function $f(t) = \frac{4a+(4-n)t}{4n}(a+t)^{\frac{n-2}{2}}S$. When $n = 3, 4$, $f(t)$ is an increase function for $t \geq 0$. As $A^2 \geq A_0^2(\lambda)$, we get

$$\begin{aligned} J_\lambda^\infty(w^k) - \tfrac{\lambda A^2}{4}\|\nabla w^k\|_{L^2}^2 &= f(\lambda A^2) \geq f(\lambda A_0^2) \\ &= \tfrac{4a+(4-n)\lambda A_0^2}{4n}(a+\lambda A_0^2)^{\frac{n-2}{2}}S \\ &= \tfrac{4a+(4-n)\lambda A_0^2}{4n}A_0^2 \quad \text{by} \quad (2.4), \\ &= c_\lambda^\infty. \end{aligned}$$

When $n \geq 5$, since

$$f'(t) = \left(\frac{a}{4} + \frac{(4-n)t}{8}\right)(a+t)^{\frac{n-4}{2}}S.$$

$f(t)$ increases in the interval $[0, \frac{2a}{n-4}]$. Therefore, if $\lambda T_0^2 \leq \frac{2a}{n-4}$, $\lambda A^2 \leq \frac{2a}{n-4}$. Similarly, we get

$$J_\lambda^\infty(w^k) - \frac{\lambda A^2}{4}\|\nabla w^k\|_{L^2}^2 = f(\lambda A^2) \geq f(\lambda A_1^2(\lambda)) = c_\lambda^\infty.$$

Next, since u_λ satisfies the equation (3.1), we get

$$J_\lambda(u_\lambda) = \frac{p-1}{2(p+1)}\left((a+\lambda A^2)\int_{\mathbb{R}^n}|\nabla u_\lambda|^2dx + \int_{\mathbb{R}^n}V(x)u_\lambda^2dx\right).$$

So

$$\begin{aligned} J_\lambda(u_\lambda) - \tfrac{\lambda A^2}{4}\|\nabla u_\lambda\|_{L^2}^2 &\geq \tfrac{p-1}{2(p+1)}(a+\lambda A^2)\|\nabla u_\lambda\|_{L^2}^2 - \tfrac{\lambda A^2}{4}\|\nabla u_\lambda\|_{L^2}^2 \\ &= \tfrac{2a(p-1)+(p-3)\lambda A^2}{4(p+1)}\|\nabla u_\lambda\|_{L^2}^2 \geq 0, \end{aligned}$$

if $\lambda T_0^2 \leq \frac{2a(p-1)}{3-p}$.

If we take

$$\lambda_0 = \begin{cases} \min\left\{\frac{(p-1)a}{16T_0^4}, \frac{2a(p-1)}{(3-p)T_0^2}, \lambda_1\right\}, & n = 3, 4 \\ \min\left\{\frac{(p-1)a}{16T_0^4}, \frac{2a(p-1)}{(3-p)T_0^2}, \frac{2a}{(n-4)T_0^2}, \lambda_1\right\}, & n \geq 5. \end{cases} \tag{3.4}$$

Then for all $\lambda \in [0, \lambda_0)$,

$$\begin{aligned} c_\lambda \geq c_\lambda^{T_0} &= J_\lambda(u_\lambda) + \textstyle\sum_{k=1}^l J_\lambda^\infty(w^k) - \tfrac{\lambda A^4}{4}, \\ &\geq lc_\lambda^\infty \geq c_\lambda^\infty. \end{aligned}$$

This contradicts with (2.7) when $l \geq 1$. Therefore, (ii) doesn't occur and $u_i \to u_\lambda$ in $H^1(\mathbb{R}^n)$. So u_λ is a nontrivial critical point of I_λ with $I_\lambda(u_\lambda) = c_\lambda^{T_0} \geq c_0$ for $\lambda \geq 0$ small enough. By standard regularity argument, we see that u_λ is a positive solution to (1.1). □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Assume that $n \geq 4$ and $u \in H^1(\mathbb{R}^n)$ is a nontrivial solution to (1.1), then multiply (1.1) by u and integrate by parts, we get

$$a\|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^n} V(x)u^2 dx + \lambda\|\nabla u\|_{L^2}^4 - \|u\|_{L^{p+1}}^{p+1} = 0.$$

As in the proof of Lemma 2.2, if we choose $\epsilon = \frac{V_0}{2}$, there exists a $C > 0$, such that

$$\int_{\mathbb{R}^n} |u|^{p+1} dx \leq \frac{V_0}{2}\int_{\mathbb{R}^n} u^2 dx + C\left(\int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{\frac{n}{n-2}}.$$

We get that

$$a\|\nabla u\|_{L^2}^2 + \int_{\mathbb{R}^n} V(x)u^2 dx + \lambda\|\nabla u\|_{L^2}^4 \leq \frac{V_0}{2}\int_{\mathbb{R}^n} u^2 dx + C\|\nabla u\|_{L^2}^{\frac{2n}{n-2}}. \tag{4.1}$$

Since $2 < \frac{2n}{n-2} \leq 4$ for $n \geq 4$, by Young's inequality, it is easy to see that

$$C\|\nabla u\|_{L^2}^{\frac{2n}{n-2}} \leq a\|\nabla u\|_{L^2}^2 + C_1\|\nabla u\|_{L^2}^4.$$

Therefore by (4.1), we have

$$\int_{\mathbb{R}^n} V(x)u^2 dx + \lambda\|\nabla u\|_{L^2}^4 \leq \frac{V_0}{2}\int_{\mathbb{R}^n} u^2 dx + C_1\|\nabla u\|_{L^2}^4.$$

If we take $\lambda \geq C_1$, we get

$$\|\nabla u\|_{L^2} = \|u\|_{L^2} = 0.$$

So (1.1) has no nontrivial solution when $n \geq 4$ and λ chosen large. □

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