J. Math. Anal. Appl. 485 (2020) 123790

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the orthogonal exponential functions of a class of planar self-affine measures ☆

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ARTICLE INFO

Article history:
Received 17 June 2019
Available online 17 December 2019
Submitted by P.G. Lemarie-Rieusset

Keywords:
Iterated function system
Self-affine measure
Orthogonality
Cardinality

ABSTRACT

For an expanding real matrix $M = \begin{bmatrix} \rho^{-1} & \mathcal{C} \\ 0 & \rho^{-1} \end{bmatrix} \in M_2(\mathbb{R})$ and a digit set $D = \{(0,0)^t, (1,0)^t, (0,1)^t\}$, let $\mu_{M,D}$ be the self-affine measure generated by M and D. In this paper, we show that $L^2(\mu_{M,D})$ admits an infinite orthogonal set of exponential functions if and only if $|\rho| = (q/p)^{\frac{1}{r}}$ and $\mathcal{C} = \kappa\rho^{-1}$ for some positive integers p, q, r with $p \in 3\mathbb{Z}$, $\gcd(p, q) = 1$ and $\kappa \in \mathbb{Q}$. Moreover, if $L^2(\mu_{M,D})$ does not admit any infinite orthogonal set of exponential functions, we estimate the number of orthogonal exponential functions in $L^2(\mu_{M,D})$ and give the exact maximal cardinality.

1. Introduction

A Borel probability measure μ on $\mathbb{R}^n$ is called a *spectral measure* if we can find a countable set $\Lambda \subset \mathbb{R}^n$ such that $E_\Lambda = \{e^{2\pi i\langle \boldsymbol{\lambda}, \boldsymbol{x}\rangle} : \boldsymbol{\lambda} \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. If such Λ exists, then Λ is called a *spectrum* of μ, we also say that (μ, Λ) is a *spectral pair*. For the special case that the spectral measure is the restriction of the Lebesgue measure on a bounded Borel subset Ω, we call Ω a *spectral set*. The research of spectral measures was originated from Fuglede [17], whose famous conjecture asserted that Ω is a spectral set if and only if Ω is a translational tile, that is, there exists a discrete set $\Gamma \subset \mathbb{R}^n$ such that $\cup_{\boldsymbol{\gamma}\in\Gamma}(\Omega + \boldsymbol{\gamma})$ covers $\mathbb{R}^n$ without overlaps and up to a zero set of Lebesgue measure. The conjecture was disproved eventually in both directions on $\mathbb{R}^n$ for $n \geq 3$ [33,21,22], but it is still open in dimensions $n = 1$ and $n = 2$.

☆ The research is supported in part by the NNSF of China (Nos. 11771457 and 11971500), the Hunan Provincial NSF (No. 2019JJ20012), the SRF of Hunan Provincial Education Department (No. 17B158) and the Hunan Provincial Innovation Foundation for Postgraduate (No. CX20190385).

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https://doi.org/10.1016/j.jmaa.2019.123790
0022-247X/

Since then, the study of the spectrality of measures becomes an active research topic, especially for fractal measures, such as self-similar/self-affine measures. Let $\{\phi_{\boldsymbol{d}}(\boldsymbol{x})\}_{\boldsymbol{d}\in D}$ be an *iterated function system* (IFS) defined by

$$\phi_{\boldsymbol{d}}(\boldsymbol{x}) = M^{-1}(\boldsymbol{x}+\boldsymbol{d}), \quad \boldsymbol{x}\in\mathbb{R}^n, \quad \boldsymbol{d}\in D,$$

where $M \in M_n(\mathbb{R})$ is an $n\times n$ expanding real matrix (that is, all the eigenvalues of M are greater than 1 in module), and $D\subset\mathbb{R}^n$ is a finite digit set. It is well known that there exists a unique nonempty compact set $T := T(M,D)$ such that $T=\bigcup_{\boldsymbol{d}\in D}\phi_{\boldsymbol{d}}(T)$ [18]. Also, there exists a unique probability measure $\mu := \mu_{M,D}$ supported on T satisfying

$$\mu = \frac{1}{\#D}\sum_{\boldsymbol{d}\in D}\mu\circ\phi_{\boldsymbol{d}}^{-1}, \tag{1.1}$$

where $\#D$ is the cardinality of D. The set T and the measure $\mu_{M,D}$ are called *self-affine set* (or *attractor*) and *self-affine measure*, respectively. In particular, if M is a multiple of an orthonormal matrix, then T and $\mu_{M,D}$ are called *self-similar set* and *self-similar measure*, respectively.

In 1998, Jorgensen and Pedersen [20] discovered that the standard middle-fourth Cantor measure is a spectral measure. It is the first spectral measure that is non-atomic and singular to the Lebesgue measure ever discovered. In the same paper, they also showed that the middle-third Cantor measure is not a spectral measure. Following this discovery, there has been more research on new spectral measures (see [1–4,6–8,11,9,10,12,14,15,13,16,19,23,26,28,30] and the references therein), as well as the convergence properties of the associated Fourier series [31,32]. Among these results, Hu and Lau [19] first studied the spectrality of Bernoulli convolutions μ_ρ, and Dai [6] completely settled the problem that $\mu_{1/(2k)}$ is the only spectral measure. The more general N-Bernoulli convolution was completely characterized in [7,8]. For higher dimensional cases, Dutkay, Haussermann and Lai [13], and Liu and Luo [30] studied the spectral problem for self-affine measures, which generated by an expanding integer matrix and an integer digit set. In particular, Deng and Lau [11] first considered the planar self-affine measure generated by an expanding real matrix, they proved the following theorem.

Theorem A. *[11, Theorems 1.1 and 1.2] Let* $M=\begin{bmatrix}\rho^{-1} & 0\\ 0 & \rho^{-1}\end{bmatrix}\in M_2(\mathbb{R})$ *be an expanding real matrix with* $0<|\rho|<1$, *the digit set* $D=\{(0,0)^t,(1,0)^t,(0,1)^t\}$, *and let* $\mu_{M,D}$ *be defined by* (1.1). *Then*

(i) $L^2(\mu_{M,D})$ *admits an infinite orthogonal set of exponential functions if and only if* $|\rho|=(q/p)^{\frac{1}{r}}$ *for some positive integers* p,q,r *with* $p\in 3\mathbb{Z}$ *and* $\gcd(p,q)=1$.
(ii) $L^2(\mu_{M,D})$ *is a spectral measure if and only if* $|\rho|=1/p$ *for some* $p\in 3\mathbb{Z}$.

In the opposite direction, the study of the non-spectrality of self-affine measures $\mu_{M,D}$ has drawn considerable attentions, and some interesting results have been obtained, see [5,14,24,25,27,29]) and the references therein. The non-spectral problem consists of the following two types of questions: (I) There are at most a finite number of orthogonal exponential functions in $L^2(\mu_{M,D})$. The main questions here are to estimate the number of orthogonal exponential functions in $L^2(\mu_{M,D})$ and to find them; (II) There are infinite families of orthogonal exponential functions, but none of them forms an orthogonal basis in $L^2(\mu_{M,D})$. Relating to question (I), there are many results about the self-affine measure $\mu_{M,D}$, which generated by an expanding integer matrix $M\in M_2(\mathbb{Z})$ and the digit set $D=\{(0,0)^t,(1,0)^t,(0,1)^t\}$. For examples, Dutkay and Jorgensen [14] showed that if $M=\begin{bmatrix}p & 0\\ 0 & p\end{bmatrix}$ with $p\in\mathbb{Z}\setminus 3\mathbb{Z}$ and $p\geq 2$, then there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. Later on, Li [24] proved that if $M=\begin{bmatrix}a & b\\ 0 & c\end{bmatrix}$ with $ac\notin 3\mathbb{Z}$, then there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 3 is the best. Recently, Liu, Dong and Li [29] considered the matrix $M = \begin{bmatrix} a & b \\ d & c \end{bmatrix}$ with $ac - bd \notin 3\mathbb{Z}$, they showed that there exist at most 9 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 9 is the best. For the case that M is an expanding integer matrix with $\det(M) \notin 3\mathbb{Z}$ and an integer digit set $D = \{(0,0)^t, (\alpha_1, \alpha_2)^t, (\beta_1, \beta_2)^t\}$ with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$, the first author of this paper and Liu [5] also studied the non-spectral problem for the self-affine measure $\mu_{M,D}$.

Motivated by the above results, we will study the spectrality and the non-spectrality of the planar self-affine measure $\mu_{M,D}$, which generated by an expanding real matrix $M \in M_2(\mathbb{R})$ and the digit set $D \subset \mathbb{R}^2$ with

$$M = \begin{bmatrix} \rho^{-1} & \mathcal{C} \\ 0 & \rho^{-1} \end{bmatrix} \quad (0 < |\rho| < 1) \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \tag{1.2}$$

Throughout the paper, we will use the following notations: $\beta := \rho^{-1}$, $\Theta := \Theta_1 \cup \Theta_2$ with $\Theta_1 = 3\mathbb{Z}^2 + (1,2)^t$ and $\Theta_2 = 3\mathbb{Z}^2 + (2,1)^t$, and $\mathbb{N}$ denotes the set of positive integers.

Our first main result is the following theorem.

Theorem 1.1. *Let $\mu_{M,D}$, (M,D) be defined by (1.1) and (1.2), respectively. Then $L^2(\mu_{M,D})$ admits an infinite orthogonal set of exponential functions if and only if $|\beta| = (p/q)^{\frac{1}{r}}$ and $\mathcal{C} = \kappa\beta$ for some $p, q, r \in \mathbb{N}$ with $p \in 3\mathbb{Z}$, $\gcd(p,q) = 1$ and $\kappa \in \mathbb{Q}$.*

Obviously, Theorem 1.1 is an extension of Theorem A(i). The proof depends mainly on the characterization of the zero set of the Fourier transform of $\mu_{M,D}$.

Furthermore, if $L^2(\mu_{M,D})$ does not admit infinite orthogonal set of exponential functions, we will estimate the number of orthogonal exponential functions in $L^2(\mu_{M,D})$ and give the exact maximal cardinality. In this paper, we mainly consider the case where β is the r-th root of a rational, that is, $|\beta| = (p/q)^{\frac{1}{r}}$ for some $p, q, r \in \mathbb{N}$ with $\gcd(p,q) = 1$. By Theorem 1.1, there exist two cases: (i) $\mathcal{C} \neq \kappa\beta$ for any $\kappa \in \mathbb{Q}$; (ii) $\mathcal{C} = \kappa\beta$ for some $\kappa \in \mathbb{Q}$ and $p \notin 3\mathbb{Z}$. Now we state our second main theorem.

Theorem 1.2. *Let $\mu_{M,D}$, (M,D) be defined by (1.1) and (1.2), respectively. Suppose that $|\beta| = (p/q)^{\frac{1}{r}}$ for some $p, q, r \in \mathbb{N}$ with $\gcd(p,q) = 1$. Then the following statements hold.*

- (i) *If $\mathcal{C} \neq \kappa\beta$ for any $\kappa \in \mathbb{Q}$, then there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 3 is the best.*
- (ii) *If $\mathcal{C} = \kappa\beta$ for some $\kappa \in \mathbb{Q}$ and $p \notin 3\mathbb{Z}$, then*
  - (a) *If $q \notin 3\mathbb{Z}$, then there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 3 is the best.*
  - (b) *If $q \in 3\mathbb{Z}$, then there are any number of orthogonal exponential functions in $L^2(\mu_{M,D})$.*

We remark that if β is the r-th root of a rational, Theorem 1.2 completely answered the non-spectral problem of question (I) for the self-affine measure $\mu_{M,D}$ in (1.2). To some extent, Theorem 1.2 is an extension of the results in [14,24,29] for non-spectral self-affine measures. The proof of Theorem 1.2 is through some algebraic identities induced by the orthogonality, and the technique to prove the best number is to construct a suitable orthogonal set for each case. It should be noted that there does not exist any infinite orthogonal set of exponential functions in the case (b) of Theorem 1.2(ii), although it can admit any number of orthogonal exponential functions.

Remark 1.3. For the case that β is not the r-th root of a rational, which is either transcendental or algebraic. In this case, Theorem 1.1 shows that $L^2(\mu_{M,D})$ cannot have any infinite orthogonal set of exponential functions.

(i) If β is a transcendental number, by using the similar method used in the proof of Theorem 1.2, we can show that there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 3 is the best. We omit the proof here.

(ii) If β is an algebraic number, then the number 3 is not necessarily the best. For example, let $\beta = (\sqrt{5}+1)/2$ and $\mathcal{C} = 0$, then for any $\boldsymbol{\alpha} \in \Theta/3$,

$$E_\Lambda = \left\{ e^{2\pi i \langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle} : \boldsymbol{\lambda} \in \Lambda = \{\mathbf{0}, M^{*2}\boldsymbol{\alpha}, M^{*3}\boldsymbol{\alpha}, M^{*4}\boldsymbol{\alpha}, M^{*5}\boldsymbol{\alpha}\} \right\}$$

is an orthogonal set in $L^2(\mu_{M,D})$. In fact, by observing that $M^{*\ell}\boldsymbol{\alpha} = \beta^\ell \boldsymbol{\alpha} \in \mathcal{Z}(\hat{\mu}_{M,D})$ (see (2.4)), we have

$$\begin{aligned} M^{*(\ell+1)}\boldsymbol{\alpha} - M^{*\ell}\boldsymbol{\alpha} &= \beta^{\ell-1}(\beta^2 - \beta)\boldsymbol{\alpha} = \beta^{\ell-1}\boldsymbol{\alpha} = M^{*(\ell-1)}\boldsymbol{\alpha} \in \mathcal{Z}(\hat{\mu}_{M,D}), \\ M^{*4}\boldsymbol{\alpha} - M^{*2}\boldsymbol{\alpha} &= \beta^2(\beta^2 - 1)\boldsymbol{\alpha} = \beta^3\boldsymbol{\alpha} = M^{*3}\boldsymbol{\alpha} \in \mathcal{Z}(\hat{\mu}_{M,D}), \\ M^{*5}\boldsymbol{\alpha} - M^{*2}\boldsymbol{\alpha} &= \beta^3(\beta^2 - \beta^{-1})\boldsymbol{\alpha} = \beta^3(2\boldsymbol{\alpha}) = M^{*3}(2\boldsymbol{\alpha}) \in \mathcal{Z}(\hat{\mu}_{M,D}). \end{aligned}$$

By (2.2), the assertion follows. Unfortunately, it is difficult to give the exact maximal cardinality.

The paper is organized as follows. In Section 2, we introduce some basic definitions and results that will be needed in the proof of our main results. In Section 3, we prove Theorem 1.1 and Theorem 1.2. Finally, we give some remarks and open questions in Section 4.

2. Preliminaries

Let μ be a probability measure with compact support on $\mathbb{R}^n$, its Fourier transform is defined as usual,

$$\hat{\mu}(\boldsymbol{\xi}) = \int e^{2\pi i \langle \boldsymbol{x}, \boldsymbol{\xi} \rangle} d\mu(\boldsymbol{x}), \qquad \boldsymbol{\xi} \in \mathbb{R}^n.$$

Let $\mu_{M,D}$, (M, D) be defined by (1.1) and (1.2), respectively. It follows from [14] that

$$\hat{\mu}_{M,D}(\boldsymbol{\xi}) = \prod_{j=1}^{\infty} m_D(M^{*-j}\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in \mathbb{R}^2, \tag{2.1}$$

where M^* denotes the transpose of M, and

$$m_D(\boldsymbol{x}) = \frac{1}{\#D} \sum_{\boldsymbol{d} \in D} e^{2\pi i \langle \boldsymbol{d}, \boldsymbol{x} \rangle} = \frac{1}{3}(1 + e^{2\pi i x_1} + e^{2\pi i x_2}), \qquad \boldsymbol{x} = (x_1, x_2)^t \in \mathbb{R}^2.$$

Let $\mathcal{Z}(f)$ denote the zeros of a function f. It is known that

$$\mathcal{Z}(m_D) = \frac{1}{3}(\Theta_1 \cup \Theta_2).$$

For any $\boldsymbol{\lambda_1} \neq \boldsymbol{\lambda_2} \in \mathbb{R}^2$, the orthogonality condition

$$0 = \langle e^{2\pi i \langle \boldsymbol{\lambda_1}, \boldsymbol{x} \rangle}, e^{2\pi i \langle \boldsymbol{\lambda_2}, \boldsymbol{x} \rangle} \rangle_{L^2(\mu_{M,D})} = \int e^{2\pi i \langle \boldsymbol{\lambda_1} - \boldsymbol{\lambda_2}, \boldsymbol{x} \rangle} d\mu_{M,D}(\boldsymbol{x}) = \hat{\mu}_{M,D}(\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2})$$

relates to the zero set $\mathcal{Z}(\hat{\mu}_{M,D})$ directly. It is easy to show that for a countable set $\Lambda \subset \mathbb{R}^2$, $E_\Lambda = \{e^{2\pi i \langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle} : \boldsymbol{\lambda} \in \Lambda\}$ is an orthogonal set of $L^2(\mu_{M,D})$ if and only if

$$(\Lambda - \Lambda) \setminus \{\mathbf{0}\} \subset \mathcal{Z}(\hat{\mu}_{M,D}). \tag{2.2}$$

In this case, Λ is called a *bi-zero set* of $\mu_{M,D}$.

For the matrix M given by (1.2), it is easy to check that

$$M^{*j} = \begin{bmatrix} \beta^j & 0 \\ \mathcal{C} j \beta^{j-1} & \beta^j \end{bmatrix}. \tag{2.3}$$

It follows from (2.1) that

$$\mathcal{Z}(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} M^{*j}(\mathcal{Z}(m_D)) = \bigcup_{j=1}^{\infty} \frac{1}{3} M^{*j}(\Theta_1 \cup \Theta_2) := \mathcal{Z}_{\Theta_1} \cup \mathcal{Z}_{\Theta_2}, \tag{2.4}$$

where

$$\begin{aligned} \mathcal{Z}_{\Theta_1} &= \bigcup_{j=1}^{\infty} M^{*j} \frac{\Theta_1}{3} = \left\{ \bigcup_{j=1}^{\infty} \beta^j \begin{pmatrix} \frac{\alpha_1}{3} \\ \frac{\mathcal{C} j \beta^{-1} \alpha_1 + \alpha_2}{3} \end{pmatrix} : (\alpha_1, \alpha_2)^t \in \Theta_1 \right\}, \\ \mathcal{Z}_{\Theta_2} &= \bigcup_{j=1}^{\infty} M^{*j} \frac{\Theta_2}{3} = \left\{ \bigcup_{j=1}^{\infty} \beta^j \begin{pmatrix} \frac{\alpha_3}{3} \\ \frac{\mathcal{C} j \beta^{-1} \alpha_3 + \alpha_4}{3} \end{pmatrix} : (\alpha_3, \alpha_4)^t \in \Theta_2 \right\}. \end{aligned} \tag{2.5}$$

Throughout the paper, we shall denote the number $(p/q)^{\frac{1}{r}}$ by the reduced representation. For example, if $(p/q)^{\frac{1}{2}} = (18/8)^{\frac{1}{2}} = (9/4)^{\frac{1}{2}} = 3/2$, we take $(p/q)^{\frac{1}{2}} = 3/2$. It is well known that $\beta \in \{\pm(p/q)^{\frac{1}{r}} : p, q, r \in \mathbb{N}\}$ if and only if $|\beta|$ is an algebraic rational with a minimal polynomial $qx^r - p$ for some $p, q, r \in \mathbb{N}$. The following lemma is useful for proving our main results.

Lemma 2.1. *Suppose that $\beta \in \{\pm(p/q)^{\frac{1}{r}} : p, q, r \in \mathbb{N}\}$ satisfies $a_1\beta^i + a_2\beta^j = a_3\beta^k$, where i, j, k are nonnegative integers. Then the following statements hold.*

(i) *If $a_1, a_2, a_3 \in \mathbb{Z} \setminus \{0\}$, then $i \equiv j \equiv k \pmod r$.*
(ii) *If $a_1, a_2, a_3 \in \mathbb{Z} \setminus 3\mathbb{Z}$, and $p \in 3\mathbb{Z}$ or $q \in 3\mathbb{Z}$, then at least two of the i, j, k are equal.*

Proof. We only prove the case $\beta = (p/q)^{\frac{1}{r}}$, because the other case $\beta = -(p/q)^{\frac{1}{r}}$ can be proved similarly.

The proof of (i) is essentially identical to that of [9, Lemma 2.5], we write it down for completeness. Let $i = i_1 r + s_1, j = j_1 r + s_2, k = k_1 r + s_3$ with $0 \le s_1, s_2, s_3 \le r - 1$. Note that $q\beta^r - p = 0$, so there exist some integers $b_1, b_2, b_3 \neq 0$ such that $b_1\beta^{s_1} + b_2\beta^{s_2} = b_3\beta^{s_3}$. As $qx^r - p$ is the minimal integer polynomial of β, it follows from $0 \le s_1, s_2, s_3 \le r - 1$ that $s_1 = s_2 = s_3$. Therefore, we have $i \equiv j \equiv k \pmod r$.

For (ii), suppose on the contrary that i, j, k are different nonnegative integers. Without loss of generality, we assume that $i > j > k$. According to (i), we can let $i = i_1 r + s, j = j_1 r + s$ and $k = k_1 r + s$, where $i_1 > j_1 > k_1$ and $0 \le s \le r - 1$. Multiplying $a_1\beta^i + a_2\beta^j = a_3\beta^k$ by $q^{i_1 - k_1}\beta^{-k}$, we get that

$$\begin{aligned} p^{j_1 - k_1}[a_1 p^{i_1 - j_1} + a_2 q^{i_1 - j_1}] &= a_3 q^{i_1 - k_1}, \\ q^{i_1 - j_1}[a_3 q^{j_1 - k_1} - a_2 p^{j_1 - k_1}] &= a_1 p^{i_1 - k_1}. \end{aligned}$$

Note that $\gcd(p,q)=1$, if $p\in 3\mathbb{Z}$, the first equation implies that $a_3\in 3\mathbb{Z}$; if $q\in 3\mathbb{Z}$, the second equation implies that $a_1\in 3\mathbb{Z}$. They contradict the conditions $a_1,a_3\notin 3\mathbb{Z}$, and hence the assertion follows. □

For the one dimensional self-similar measure

$$\mu_{\rho,m}(\cdot)=\frac{1}{m}\sum_{j=0}^{m-1}\mu_{\rho,m}(\rho^{-1}(\cdot)-j), \tag{2.6}$$

where $0<|\rho|<1$ and $m\in\mathbb{N}\setminus\{1\}$. Deng [9] studied the existence of infinite orthogonal set of exponential functions in $L^2(\mu_{\rho,m})$ and obtained the following result, which will be used to prove Theorem 1.1.

Theorem 2.2. *[9, Theorem 1.2] Let $\mu_{\rho,m}$ be defined by (2.6). If m is a prime, then $L^2(\mu_{\rho,m})$ admits an infinite orthogonal set of exponential functions if and only if $|\rho|=(q/p)^{\frac{1}{r}}$ for some $p,q,r\in\mathbb{N}$ with $p\in m\mathbb{Z}$ and* $\gcd(p,q)=1$.

3. Proof of the main results

This section is devoted to proving our main theorems. We first prove Theorem 1.1 by using Lemma 2.1 and Theorem 2.2, and then complete the proof of Theorem 1.2. Let Λ be a bi-zero set of $\mu_{M,D}$, and let $E_\Lambda=\{e^{2\pi i\langle\boldsymbol{\lambda},\boldsymbol{x}\rangle}:\boldsymbol{\lambda}\in\Lambda\}$. Without loss of generality, we may assume that $\mathbf{0}\in\Lambda$.

Proof of Theorem 1.1. First, we prove the necessity. Assume that $L^2(\mu_{M,D})$ admits an infinite orthogonal set E_Λ. By (2.2) and (2.4), we have $\Lambda\setminus\{\mathbf{0}\},(\Lambda-\Lambda)\setminus\{\mathbf{0}\}\subset\mathcal{Z}_{\Theta_1}\cup\mathcal{Z}_{\Theta_2}$, where $\mathcal{Z}_{\Theta_1},\mathcal{Z}_{\Theta_2}$ are defined in (2.5). For $s=1$ or 2, define $\Lambda_s=\{\lambda_{is}:\boldsymbol{\lambda_i}=(\lambda_{i1}.\lambda_{i2})^t\in\Lambda\}$. We first claim that Λ_1 is infinite. Suppose otherwise, Λ_1 is finite, then Λ_2 must be infinite. By the pigeonhole principle, there exist $\boldsymbol{\lambda_1}\neq\boldsymbol{\lambda_2}\in\Lambda\setminus\{\mathbf{0}\}$ such that $\lambda_{11}=\lambda_{21}$, where $\boldsymbol{\lambda_i}=(\lambda_{i1},\lambda_{i2})^t$ for $i=1,2$, and hence

$$\boldsymbol{\lambda_1}-\boldsymbol{\lambda_2}=\begin{pmatrix}0\\ \lambda_{12}-\lambda_{22}\end{pmatrix}\notin\mathcal{Z}_{\Theta_1}\cup\mathcal{Z}_{\Theta_2}=\mathcal{Z}(\hat{\mu}_{M,D}),$$

which is a contradiction. Thus Λ_1 is an infinite set satisfying

$$\Lambda_1\setminus\{0\},(\Lambda_1-\Lambda_1)\setminus\{0\}\subset\bigcup_{j=1}^{\infty}\beta^j(\mathbb{Z}\pm 1/3)=\mathcal{Z}(\hat{\mu}_{\rho,3}),$$

where $\mu_{\rho,3}$ is defined by (2.6). It follows from (2.2) that $E_{\Lambda_1}=\{e^{2\pi i\lambda x}:\lambda\in\Lambda_1\}$ is an infinite orthogonal set of $L^2(\mu_{\rho,3})$. According to Theorem 2.2, we have $|\beta|=(p/q)^{\frac{1}{r}}$ for some $p,q,r\in\mathbb{N}$ with $p\in 3\mathbb{Z}$ and $\gcd(p,q)=1$.

We now prove that $\mathcal{C}=\kappa\beta$ with $\kappa\in\mathbb{Q}$. Since $\#\Lambda=\infty$, without loss of generality, there must exist $\boldsymbol{\lambda_1}\neq\boldsymbol{\lambda_2}\in\Lambda\setminus\{\mathbf{0}\}$ such that $\boldsymbol{\lambda_1},\boldsymbol{\lambda_2}\in\mathcal{Z}_{\Theta_1}$. From the definition of $\mathcal{Z}_{\Theta_1}$ and the orthogonality of Λ, we can write

$$\boldsymbol{\lambda_1}=\beta^n\begin{pmatrix}\frac{\alpha_{11}}{3}\\ \frac{\mathcal{C}n\beta^{-1}\alpha_{11}+\alpha_{12}}{3}\end{pmatrix},\quad \boldsymbol{\lambda_2}=\beta^m\begin{pmatrix}\frac{\alpha_{21}}{3}\\ \frac{\mathcal{C}m\beta^{-1}\alpha_{21}+\alpha_{22}}{3}\end{pmatrix}\quad\text{and}\quad \boldsymbol{\lambda_1}-\boldsymbol{\lambda_2}=\beta^w\begin{pmatrix}\frac{\alpha_{31}}{3}\\ \frac{\mathcal{C}w\beta^{-1}\alpha_{31}+\alpha_{32}}{3}\end{pmatrix}, \tag{3.1}$$

where $n,m,w\in\mathbb{N}$, $(\alpha_{11},\alpha_{12})^t,(\alpha_{21},\alpha_{22})^t\in\Theta_1$ and $(\alpha_{31},\alpha_{32})^t\in\Theta$. Consequently,

$$\alpha_{11}\beta^n-\alpha_{21}\beta^m=\alpha_{31}\beta^w \tag{3.2}$$

and

$$\mathcal{C}\beta^{-1}(n\alpha_{11}\beta^n - m\alpha_{21}\beta^m - w\alpha_{31}\beta^w) = -\alpha_{12}\beta^n + \alpha_{22}\beta^m + \alpha_{32}\beta^w. \tag{3.3}$$

Note that $|\beta| = (p/q)^{\frac{1}{r}}$ and $p \in 3\mathbb{Z}$, by Lemma 2.1, we get that $n \equiv m \equiv w \pmod r$ and at least two of the n, m, w are equal. As $\alpha_{11} - \alpha_{21} \in 3\mathbb{Z}$ and $\alpha_{31} \notin 3\mathbb{Z}$, it follows from (3.2) that n, m, w are not all equal. Without loss of generality, we assume that $n \neq m = w$. By (3.2), we have $n\alpha_{11}\beta^n - m\alpha_{21}\beta^m - w\alpha_{31}\beta^w = (n-m)\alpha_{11}\beta^n$. Therefore, (3.3) implies that

$$\mathcal{C} = \frac{\alpha_{22} + \alpha_{32} - \alpha_{12}\beta^{n-m}}{(n-m)\alpha_{11}\beta^{n-m}}\beta := \kappa\beta.$$

It follows from $n \equiv m \pmod r$, $n \neq m$ and $|\beta| = (p/q)^{\frac{1}{r}}$ that $\kappa \in \mathbb{Q}$. The necessity follows.

Second, we prove the sufficiency. Suppose that $|\beta| = (p/q)^{\frac{1}{r}}$ and $\mathcal{C} = \kappa\beta$ for some $p, q, r \in \mathbb{N}$ with $p \in 3\mathbb{Z}$, $\gcd(p, q) = 1$ and $\kappa \in \mathbb{Q}$, it is clear that $\gcd(q, 3) = 1$. Write $\kappa = v/u$ for some $u \in \mathbb{N}$, $v \in \mathbb{Z}$ with $\gcd(u, v) = 1$, and let

$$\Lambda = \left\{ \begin{pmatrix} \frac{p^{3u\ell}}{3} \\ \frac{2p^{3u\ell}}{3} \end{pmatrix} : \ell \in \mathbb{N} \right\}.$$

We will show that $\Lambda \setminus \{\mathbf{0}\}, (\Lambda - \Lambda) \setminus \{\mathbf{0}\} \subset \mathcal{Z}(\hat{\mu}_{M,D})$. Note that $p = q|\beta|^r$, we have

$$\begin{pmatrix} \frac{p^{3u\ell}}{3} \\ \frac{2p^{3u\ell}}{3} \end{pmatrix} = |\beta|^{3ur\ell} \begin{pmatrix} \frac{q^{3u\ell}}{3} \\ \frac{2q^{3u\ell}}{3} \end{pmatrix} = |\beta|^{3ur\ell} \begin{pmatrix} \frac{q^{3u\ell}}{3} \\ \frac{3\kappa ur\ell q^{3u\ell} + (2-3\kappa ur\ell)q^{3u\ell}}{3} \end{pmatrix} = M^{*3ur\ell} \begin{pmatrix} \frac{\alpha_1}{3} \\ \frac{\alpha_2}{3} \end{pmatrix}, \tag{3.4}$$

where $\alpha_1 = q^{3u\ell}$ and $\alpha_2 = (2 - 3\kappa ur\ell)q^{3u\ell}$ if $|\beta|^{3ur\ell} = \beta^{3ur\ell}$, or $\alpha_1 = -q^{3u\ell}$ and $\alpha_2 = (3\kappa ur\ell - 2)q^{3u\ell}$ if $|\beta|^{3ur\ell} = -\beta^{3ur\ell}$. It follows from $\kappa = v/u$ and $\gcd(q, 3) = 1$ that $(\alpha_1, \alpha_2)^t \in \Theta$. This proves $\Lambda \setminus \{\mathbf{0}\} \subset \mathcal{Z}(\hat{\mu}_{M,D})$.

Next, we prove that $(\Lambda - \Lambda) \setminus \{\mathbf{0}\} \subset \mathcal{Z}(\hat{\mu}_{M,D})$. For any $\boldsymbol{\lambda_1} \neq \boldsymbol{\lambda_2} \in \Lambda \setminus \{\mathbf{0}\}$, we can write $\boldsymbol{\lambda_1} = p^{3u\ell_1}(1,2)^t/3$ and $\boldsymbol{\lambda_2} = p^{3u\ell_2}(1,2)^t/3$ for two positive integers $\ell_1 > \ell_2$. Similar to (3.4), there exists $(\alpha_1, \alpha_2)^t \in \Theta$ such that

$$\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2} = |\beta|^{3ur\ell_2} \begin{pmatrix} \frac{(p^{3u(\ell_1-\ell_2)}-1)q^{3u\ell_2}}{3} \\ \frac{2\cdot(p^{3u(\ell_1-\ell_2)}-1)q^{3u\ell_2}}{3} \end{pmatrix} = (p^{3u(\ell_1-\ell_2)} - 1)M^{*3ur\ell_2} \begin{pmatrix} \frac{\alpha_1}{3} \\ \frac{\alpha_2}{3} \end{pmatrix}.$$

It is easy to see that $(p^{3u(\ell_1-\ell_2)} - 1)(\alpha_1, \alpha_2)^t$ also belongs to Θ, then $\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2} \in M^{*3ur\ell_2}\frac{\Theta}{3} \subset \mathcal{Z}(\hat{\mu}_{M,D})$, which implies that $(\Lambda - \Lambda) \setminus \{\mathbf{0}\} \subset \mathcal{Z}(\hat{\mu}_{M,D})$. By (2.2), E_Λ is an infinite orthogonal set in $L^2(\mu_{M,D})$.

The proof of Theorem 1.1 is complete. □

Now we are ready to prove Theorem 1.2. Under the assumptions of Theorem 1.2, without loss of generality, we assume that $\beta = (p/q)^{\frac{1}{r}}$ for some $p, q, r \in \mathbb{N}$ with $\gcd(p, q) = 1$. Observe that $\mathcal{C} \neq \kappa\beta$ for any $\kappa \in \mathbb{Q}$ is equivalent to $\mathcal{C} = \kappa\beta$ for some $\kappa \notin \mathbb{Q}$, we always assume that $\mathcal{C} = \kappa\beta$ for $\kappa \in \mathbb{R}$ in Theorem 1.2.

Proof of Theorem 1.2. (i) We prove the conclusion by contradiction. Suppose there exists an orthogonal set E_Λ such that $\#\Lambda \geq 4$. Without loss of generality, we assume that $\#\Lambda = 4$, and let $\Lambda = \{\mathbf{0}, \boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \boldsymbol{\lambda_3}\}$. For different subscripts $i, j \in \{1, 2, 3\}$, it follows from (2.2) that $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \boldsymbol{\lambda_3}, \boldsymbol{\lambda_i} - \boldsymbol{\lambda_j} \in \mathcal{Z}(\hat{\mu}_{M,D}) = \mathcal{Z}_{\Theta_1} \cup \mathcal{Z}_{\Theta_2}$. Let $\mathcal{C} = \kappa\beta$ for $\kappa \notin \mathbb{Q}$, and by (2.5), we can write

$$\boldsymbol{\lambda_i} = \beta^{j_i} \begin{pmatrix} \frac{\alpha_{i1}}{3} \\ \frac{\kappa j_i \alpha_{i1} + \alpha_{i2}}{3} \end{pmatrix} \qquad (i = 1, 2, 3)$$

for some positive integers $j_1 \geq j_2 \geq j_3$ and $(\alpha_{i1}, \alpha_{i2})^t \in \Theta$. According to the orthogonality of Λ and Lemma 2.1, we have $j_1 \equiv j_2 \equiv j_3 \pmod r$, which means that there exists $0 \leq s \leq r-1$ such that $j_i = \ell_i r + s$ $(i = 1, 2, 3)$. Obviously, $\ell_1 \geq \ell_2 \geq \ell_3 \geq 0$. We will divide our proof into the following two cases.

Case I. $p, q \notin 3\mathbb{Z}$. By the pigeonhole principle and $p, q, \alpha_{i1} \notin 3\mathbb{Z}$ $(i = 1, 2, 3)$, it is obvious that at least two of the $\alpha_{11}p^{\ell_1}, \alpha_{21}p^{\ell_2}q^{\ell_1-\ell_2}, \alpha_{31}p^{\ell_3}q^{\ell_1-\ell_3}$ are in the set $3\mathbb{Z}+1$ or $3\mathbb{Z}+2$. Without loss of generality, we assume that $\alpha_{11}p^{\ell_1}, \alpha_{21}p^{\ell_2}q^{\ell_1-\ell_2} \in 3\mathbb{Z}+1$. Since $\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2} \in \mathcal{Z}_{\Theta_1} \cup \mathcal{Z}_{\Theta_2}$, there exist $j_4 \in \mathbb{N}$ and $(\alpha_{41}, \alpha_{42})^t \in \Theta$ such that

$$\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2} = \beta^{j_4} \begin{pmatrix} \frac{\alpha_{41}}{3} \\ \frac{\kappa j_4 \alpha_{41} + \alpha_{42}}{3} \end{pmatrix}.$$

Using Lemma 2.1, we have $j_1 \equiv j_2 \equiv j_4 \pmod r$. Let $j_4 = \ell_4 r + s$, then the equation $\alpha_{11}\beta^{j_1} - \alpha_{21}\beta^{j_2} = \alpha_{41}\beta^{j_4}$ implies that

$$\alpha_{11}\left(\frac{p}{q}\right)^{\ell_1} - \alpha_{21}\left(\frac{p}{q}\right)^{\ell_2} = \alpha_{41}\left(\frac{p}{q}\right)^{\ell_4}.$$

Multiplying $q^{\ell_1+\ell_4}$ on both sides of the above equation, one can easily get that

$$q^{\ell_4}(\alpha_{11}p^{\ell_1} - \alpha_{21}p^{\ell_2}q^{\ell_1-\ell_2}) = \alpha_{41}p^{\ell_4}q^{\ell_1}.$$

This together with $p, q \notin 3\mathbb{Z}$ and $\alpha_{11}p^{\ell_1}, \alpha_{21}p^{\ell_2}q^{\ell_1-\ell_2} \in 3\mathbb{Z}+1$ yields that $\alpha_{41} \in 3\mathbb{Z}$, which is a contradiction.

Case II. $p \in 3\mathbb{Z}, q \notin 3\mathbb{Z}$ or $p \notin 3\mathbb{Z}, q \in 3\mathbb{Z}$. By the pigeonhole principle, at least two of the $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2}, \boldsymbol{\lambda_3}$ are in the set $\mathcal{Z}_{\Theta_1}$ or $\mathcal{Z}_{\Theta_2}$. Without loss of generality, we assume that $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2} \in \mathcal{Z}_{\Theta_1}$, then $\alpha_{11}, \alpha_{21} \in 3\mathbb{Z}+1$. In view of $\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2} \in \mathcal{Z}_{\Theta_1} \cup \mathcal{Z}_{\Theta_2}$, there exist $j_5 \in \mathbb{N}$ and $(\alpha_{51}, \alpha_{52})^t \in \Theta$ such that

$$\boldsymbol{\lambda_1} - \boldsymbol{\lambda_2} = \beta^{j_5} \begin{pmatrix} \frac{\alpha_{51}}{3} \\ \frac{\kappa j_5 \alpha_{51} + \alpha_{52}}{3} \end{pmatrix}.$$

Therefore, we have

$$\alpha_{11}\beta^{j_1} - \alpha_{21}\beta^{j_2} = \alpha_{51}\beta^{j_5} \tag{3.5}$$

and

$$(\kappa j_1 \alpha_{11} + \alpha_{12})\beta^{j_1} - (\kappa j_2 \alpha_{21} + \alpha_{22})\beta^{j_2} = (\kappa j_5 \alpha_{51} + \alpha_{52})\beta^{j_5}. \tag{3.6}$$

Since one of the p, q is in the set $3\mathbb{Z}$, Lemma 2.1 implies that $j_1 \equiv j_2 \equiv j_5 \pmod r$ and at least two of the j_1, j_2, j_5 are equal. Note that $\alpha_{11} - \alpha_{21} \in 3\mathbb{Z}$ and $\alpha_{51} \notin 3\mathbb{Z}$, it follows from (3.5) that j_1, j_2, j_5 are not all equal. If $j_1 = j_2$, we have $j_1 \neq j_5$. As $j_1 \equiv j_5 \pmod r$, there exists $\ell \in \mathbb{Z} \setminus \{0\}$ such that $j_1 - j_5 = \ell r$. Combining with $\beta = (p/q)^{\frac{1}{r}}$, (3.5) and (3.6), we conclude that

$$\kappa = \frac{(\alpha_{22} - \alpha_{12})p^{\ell} + \alpha_{52}q^{\ell}}{\ell r \alpha_{51} q^{\ell}} \in \mathbb{Q}.$$

This yields a contradiction by $\kappa \notin \mathbb{Q}$. Similarly, if $j_1 = j_5$ or $j_2 = j_5$, we can derive the same contradiction.

Hence there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.

We now show that the number 3 is the best. Fix $\ell \in \mathbb{N}$, let

$$\Lambda = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, M^{*\ell} \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}, M^{*\ell} \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \right\}.$$

It is easy to check that $(\Lambda - \Lambda) \setminus \{\mathbf{0}\} \subset \mathcal{Z}(\hat{\mu}_{M,D})$. By (2.2), E_Λ is an orthogonal set in $L^2(\mu_{M,D})$, and hence the result follows.

(ii) From the proof of Case I in (i), we see that the condition $\kappa \notin \mathbb{Q}$ is not used. Hence (a) holds by (i), and we only need to prove (b).

Since $\kappa \in \mathbb{Q}$ and $q \in 3\mathbb{Z}$, we write $\kappa = v/u$ for some $u \in \mathbb{N}$, $v \in \mathbb{Z}$ with $\gcd(u, v) = 1$, and let $q = 3^s q'$ with $\gcd(q', 3) = 1$. For any positive integers N and $n \leq N$, let

$$\Lambda_{(n,N)} = \left\{ \bigcup_{n=1}^{N} M^{*3urn} \begin{pmatrix} \frac{p^{3u(N-n)} q'^{\,3u(n-1)}}{3} \\ \frac{2p^{3u(N-n)} q'^{\,3u(n-1)}}{3} \end{pmatrix} \right\}.$$

We will show that $E_{\Lambda_{(n,N)}} = \{e^{2\pi i \langle \boldsymbol{\lambda}, \boldsymbol{x} \rangle} : \boldsymbol{\lambda} \in \Lambda_{(n,N)}\}$ is an orthogonal set in $L^2(\mu_{M,D})$. In view of $p, q' \notin 3\mathbb{Z}$, it is easy to obtain that

$$\Lambda_{(n,N)} \setminus \{\mathbf{0}\} \subset \bigcup_{n=1}^{\infty} M^{*3urn} \frac{\Theta}{3} \subset \mathcal{Z}(\hat{\mu}_{M,D}).$$

For any $\boldsymbol{\lambda_1} \neq \boldsymbol{\lambda_2} \in \Lambda_{(n,N)}$, we can let

$$\boldsymbol{\lambda_1} = M^{*3urn} \begin{pmatrix} \frac{p^{3u(N-n)} q'^{\,3u(n-1)}}{3} \\ \frac{2p^{3u(N-n)} q'^{\,3u(n-1)}}{3} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\lambda_2} = M^{*3urm} \begin{pmatrix} \frac{p^{3u(N-m)} q'^{\,3u(m-1)}}{3} \\ \frac{2p^{3u(N-m)} q'^{\,3u(m-1)}}{3} \end{pmatrix},$$

where $n > m$. It follows from (2.3), $q = 3^s q'$ and $p, q' \notin 3\mathbb{Z}$ that

$$\begin{aligned} \boldsymbol{\lambda_1} - \boldsymbol{\lambda_2} &= M^{*3urn} \begin{pmatrix} \frac{p^{3u(N-n)} q'^{\,3u(n-1)}}{3} \\ \frac{2p^{3u(N-n)} q'^{\,3u(n-1)}}{3} \end{pmatrix} - M^{*3urn} \frac{p^{3u(N-n)} q'^{\,3u(n-1)}}{3} \begin{pmatrix} 3^{3u(n-m)s} \\ (3v(m-n)r + 2) \cdot 3^{3u(n-m)s} \end{pmatrix} \\ &= M^{*3urn} \frac{p^{3u(N-n)} q'^{\,3u(n-1)}}{3} \begin{pmatrix} 1 - 3^{3u(n-m)s} \\ 2 - (3v(m-n)r + 2) \cdot 3^{3u(n-m)s} \end{pmatrix} \\ &\in M^{*3urn} \frac{\Theta}{3} \\ &\subset \mathcal{Z}(\hat{\mu}_{M,D}). \end{aligned}$$

This implies that $(\Lambda_{(n,N)} - \Lambda_{(n,N)}) \setminus \{\mathbf{0}\} \subset \mathcal{Z}(\hat{\mu}_{M,D})$. According to (2.2), $E_{\Lambda_{(n,N)}}$ is an orthogonal set in $L^2(\mu_{M,D})$. By the arbitrary of N, the assertion follows.

This completes the proof of Theorem 1.2. $\square$

4. Concluding remarks

In the present section, we will give some remarks and open questions related to our main results. We use the following example to illustrate Theorems 1.1 and 1.2.

Example 4.1. Let $\mu_{M,D}$, (M, D) be defined by (1.1) and (1.2) respectively, and let

$$M_1 = \begin{bmatrix} \sqrt{3} & \frac{\sqrt{3}}{2} \\ 0 & \sqrt{3} \end{bmatrix}, \quad M_2 = \begin{bmatrix} \frac{2\sqrt{3}}{3} & \sqrt{3} \\ 0 & \frac{2\sqrt{3}}{3} \end{bmatrix}, \quad M_3 = \begin{bmatrix} \frac{2\sqrt{3}}{3} & 2 \\ 0 & \frac{2\sqrt{3}}{3} \end{bmatrix} \text{ and } M_4 = \begin{bmatrix} \sqrt{5} & \frac{\sqrt{5}}{2} \\ 0 & \sqrt{5} \end{bmatrix}.$$

Then $L^2(\mu_{M_1,D})$ admits an infinite orthogonal set of exponential functions, and there are any number of orthogonal exponential functions in $L^2(\mu_{M_2,D})$. Moreover, there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M_3,D})$ and $L^2(\mu_{M_4,D})$, and the number 3 is the best.

For the affine pair (M, D) given by (1.2), by Remark 1.3, it is of interest to consider the following question.

(Q1): If β is not the r-th root of a rational, but it is an algebraic number, what is the maximal cardinality of the orthogonal exponential functions in $L^2(\mu_{M,D})$?

Another interesting problem is the completeness of infinite orthogonal exponential functions in $L^2(\mu_{M,D})$. The following question is now naturally raised.

(Q2): For the self-affine measure $\mu_{M,D}$ corresponding to (1.2), what is the sufficient and necessary condition for $\mu_{M,D}$ to be a spectral measure?

It is worth noting that the special case $\mathcal{C} = 0$ in (1.2) has been studied by Deng and Lau in [11], they showed that $\mu_{M,D}$ is a spectral measure if and only if $|\beta| = 3p$ for some $p \in \mathbb{N}$. In fact, if $|\beta| = 3p$ and $\mathcal{C} = \kappa\beta$ for some $p \in \mathbb{N}$ with $p\kappa \in \mathbb{Z}$, it is easy to prove that $\mu_{M,D}$ is a spectral measure, which is inspired by [13, Theorem 1.3]. On the other hand, if $\mu_{M,D}$ is a spectral measure, by using the similar method used in the proof of [8, Proposition 3.1], we can obtain that β is a rational. Based on these analyses, the following conjecture may be a reasonable conjecture to this end.

Conjecture 4.2. *Let $\mu_{M,D}$, (M, D) be defined by (1.1) and (1.2), respectively. Then $\mu_{M,D}$ is a spectral measure if and only if $|\beta| = 3p$ and $\mathcal{C} = \kappa\beta$ for some $p \in \mathbb{N}$ with $p\kappa \in \mathbb{Z}$.*

Acknowledgments

The authors would like to thank the referee for his/her many valuable suggestions.

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