



Existence and large time behavior to the nematic liquid crystal equations in Besov-Morrey spaces



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ABSTRACT

In this paper, we establish the uniquely existence of the global mild solution to the nematic liquid crystal equations in Besov-Morrey spaces. Some self-similarity and large time behavior of the global mild solution are also investigated.

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1. Introduction

Liquid crystal describes a state of matter in which the molecules may be oriented like a crystal. There are three main types of liquid crystals, namely, nematic, smectic and cholesteric. What of frequent occurrence is the nematic type in which the molecules don't present any positional order but organize in long-range orientational order.

In this paper, we study the following incompressible flow of nematic liquid crystals in \mathbb{R}^3 :

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla P = -\nabla \cdot (\nabla d \odot \nabla d), & (x, t) \in \mathbb{R}^3 \times (0, +\infty), \\ \partial_t d + (u \cdot \nabla)d = \Delta d + |\nabla d|^2 d, & (x, t) \in \mathbb{R}^3 \times (0, +\infty), \\ \nabla \cdot u = 0, & (x, t) \in \mathbb{R}^3 \times (0, +\infty), \\ (u, d)|_{t=0} = (u_0, d_0), & x \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $u(x, t) : \mathbb{R}^3 \times (0, +\infty) \rightarrow \mathbb{R}^3$ is the unknown velocity field of the flow, $P(x, t) : \mathbb{R}^3 \times (0, +\infty) \rightarrow \mathbb{R}$ is a scalar pressure, $d(x, t) : \mathbb{R}^3 \times (0, +\infty) \rightarrow \mathbb{S}^2$ is the unknown (averaged) macroscopic/continuum molecule orientation of the nematic liquid crystal flow, where \mathbb{S}^2 is the unit sphere in \mathbb{R}^3 . u_0 is a given initial velocity with $\nabla \cdot u_0 = 0$ in distribution sense, and $d_0 : \mathbb{R}^3 \rightarrow \mathbb{S}^2$ is a given initial liquid crystal orientation field and satisfies $\lim_{|x| \rightarrow \infty} d_0(x) = \underline{d}_0$ with the constant unit vector $\underline{d}_0 \in \mathbb{S}^2$. The notation $\nabla d \odot \nabla d$ denotes the 3×3

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matrix whose (i, j) -th entry is given by $\partial_i d \cdot \partial_j d$ ($1 \leq i, j \leq 3$). Note that we have set the viscosity constants to be 1 for simplicity.

System (1.1) couples the forced Navier-Stokes equation with the transported flow of harmonic maps to S^2 . It has been simplified. The original one was formulated by Ericksen and Leslie in 1960s (see [5,21]) and is one of the most successful models for the nematic liquid crystals.

Lin [23] and Lin-Liu [25–27] initiated the rigorous mathematical analysis of (1.1) and considered the Ginzburg-Landau approximation of it after replacing $|\nabla d|^2 d$ by $\frac{1}{\epsilon^2} (1 - |d|^2) d$ ($\epsilon > 0$). They proved the existence of global weak solutions and their partial regularities.

In 2D, Lin-Lin-Wang [24] established the existence of a global weak solution that is smooth away from at most finitely many times for the original system (1.1) (see also Hong [9], Hong-Xin [10], Hong-Li-Xin [11], Huang-Lin-Wang [13], Li-Lei-Zhang [20], Wang-Wang [40] for relevant results in dimension two, and Liu-Zhang [33] and Ma-Gong-Li [34]).

In 3D, Wen-Ding [41] proved the uniquely existence of local strong solutions. Huang-Wang [14] established a blow-up criterion of strong solutions. The well-posedness for initial data (u_0, d_0) with small $BMO^{-1} \times BMO$ -norm and with small $L^3_{uloc}(\mathbb{R}^3)$ -norm was verified by Wang [39] and Hineman-Wang [8], respectively. Under the assumption that the initial director field $d_0(\Omega) \subset S^2_+$, Lin-Wang [29] established the existence of global weak solutions.

For the issue of large time behavior, Liu-Xu [32] obtained an optimal decay rates for $\|(u, \nabla d)\|_{H^m(\mathbb{R}^3)}$ provided that $(u_0, d_0) \in H^m(\mathbb{R}^3) \times H^{m+1}(\mathbb{R}^3, S^2)$ ($m \geq 3$) has sufficiently small $\|(u_0, \nabla d_0)\|_{L^2(\mathbb{R}^3)}$ -norm, where the smallness depends on norms of higher order derivatives of initial data. Under the assumption that $\|u_0\|_{H^1(\mathbb{R}^3)} + \|d - e_3\|_{H^2(\mathbb{R}^3)}$ is sufficiently small, Dai-Qing-Schonbek [3] and Dai-Schonbek [4] established an optimal decay rates in $H^m(\mathbb{R}^3)$. Very recently, Huang-Wang-Wen [15] consider system (1.1) in \mathbb{R}^3_+ and established some time decay estimates under the condition that $(u_0, d_0) \in L^3_\sigma(\mathbb{R}^3_+) \times \dot{W}^{1,3}(\mathbb{R}^3_+, S^2)$ has small $\|(u_0, \nabla d_0)\|_{L^3(\mathbb{R}^3_+)}$ norm, which improves the conditions on the initial data given by [3,4,32].

Also, there are some results concerning the issue of finite time singularities. For instance, Huang-Lin-Liu-Wang [12] considered the initial-boundary value problem of equation (1.1) in dimension three. They constructed two examples of finite time singularity with the first one belonging to the class of axisymmetric solutions and the second one for generic initial data having small energy and nontrivial d_0 . Lai-Lin-Wang-Wei-Zhou [19] considered the initial-boundary value problem in a bounded smooth domain in dimension two. They developed a new inner-outer gluing method and constructed solutions blowing up exactly at k prescribed distinct points in the domain as $t \rightarrow T$, a finite time. They also gave a precise description of the blow-up.

For more results on the nematic liquid crystal equations, we can refer to [22,25,28,30,31,43].

This paper aims to treat system (1.1) in a new setting. We consider the framework of Besov-Morrey spaces $\mathbf{N}^{-\beta}_{r,\lambda,q}$ which contain strongly singular functions and measures supported in either points (Diracs), filaments, or surfaces (see e.g. [[6], Remark 3.3] for more details). Besov-Morrey spaces have been studied in a large number of literatures and found wide applications in analysis and partial differential equations; see, e.g., [2,7,17,35,42,45–47].

Our motivation of this paper is due to Almeida-Precioso [1] and Yang-Fu-Sun [44]. Almeida-Precioso [1] obtained the global well-posedness and asymptotic behavior for a semilinear heat-wave type equation in Besov-Morrey spaces. Yang-Fu-Sun [44] established the existence and large time behavior of global mild solution to the coupled chemotaxis-fluid equations in Besov-Morrey spaces. Their results are closely related to the scaling property of the corresponding equations and the indexes of the solution spaces they obtained are critical.

Recall that system (1.1) also has a scaling property and is invariant under the following transformation

$$(u_\lambda(x, t), P_\lambda(x, t), d_\lambda(x, t)) := (\lambda u(\lambda x, \lambda^2 t), \lambda^2 P(\lambda x, \lambda^2 t), d(\lambda x, \lambda^2 t)). \quad (1.2)$$

We say a function space is the initial critical space for system (1.1) if the associated norm is invariant under the transformation $(u_0, d_0) \rightarrow (u_{0\lambda}, d_{0\lambda}) := (\lambda u_0(\lambda x), d_0(\lambda x))$ for all $\lambda > 0$.

This fact leads us to consider system (1.1) in some critical spaces and we found the method in [44] can be applied to (1.1) to some degree.

In order to state our results, we first exhibit the following range of the indexes and the solution spaces.

Throughout this paper, we fix $N = 3$. Let

$$\begin{cases} q_j, r_j > N - \lambda, 0 \leq \lambda < N, 1 < r_j \leq q_j < \infty, \\ \frac{1}{q_1} + \frac{1}{q_2} < \frac{2}{N-\lambda}, \\ \frac{2}{q_2} - \frac{1}{q_1} < \frac{1}{N-\lambda}, \end{cases} \quad (1.3)$$

and

$$\begin{cases} \alpha_j = 1 - \frac{N-\lambda}{q_j}, \\ \beta_j = 1 - \frac{N-\lambda}{r_j}, \end{cases} \quad (1.4)$$

where $j = 1, 2$.

Let $M_{q,\lambda}$ and $\dot{\mathbf{N}}_{r,\lambda,q}^\beta$ be the Morrey and Besov-Morrey spaces, respectively. For the precise definition, we can refer to Section 2.

Let $\Theta := \mathbb{X} \times \mathbb{Y}$ with the usual product norm

$$\|(u, d)\|_\Theta := \|u\|_{\mathbb{X}} + \|d\|_{\mathbb{Y}}.$$

For initial data, we choose the following space

$$\mathbb{E} = \dot{\mathbf{N}}_{r_1,\lambda,\infty}^{-\beta_1} \times \dot{\mathbf{N}}_{r_2,\lambda,\infty}^{-\beta_2}.$$

For a Banach space \mathbb{F} , let $BC_*([0, \infty), \mathbb{F})$ be the Banach space of all maps $\pi : [0, \infty) \rightarrow \mathbb{F}$ such that $\pi(t)$ is bounded and continuous for $t > 0$ with the respect to the norm topology of \mathbb{F} and continuous at $t = 0$ with respect to the weakly-star topology of \mathbb{F} .

For the solution, we choose the following space

$$\begin{aligned} \mathbb{X} &:= \left\{ u : \nabla \cdot u = 0, \quad u \in BC_*([0, \infty), \dot{\mathbf{N}}_{r_1,\lambda,\infty}^{-\beta_1}), t^{\frac{\alpha_1}{2}} u \in BC_*([0, \infty), M_{q_1,\lambda}) \right\} \\ \mathbb{Y} &:= \left\{ d : \nabla d \in BC_*([0, \infty), \dot{\mathbf{N}}_{r_2,\lambda,\infty}^{-\beta_2}), t^{\frac{\alpha_2}{2}} \nabla d \in BC_*([0, \infty), M_{q_2,\lambda}), d \in BC_*([0, \infty), L^\infty) \right\} \end{aligned}$$

and

$$\begin{aligned} \|u\|_{\mathbb{X}} &:= \sup_{t>0} \|u(t)\|_{\dot{\mathbf{N}}_{r_1,\lambda,\infty}^{-\beta_1}} + \sup_{t>0} \left(t^{\frac{\alpha_1}{2}} \|u(t)\|_{M_{q_1,\lambda}} \right) \\ \|d\|_{\mathbb{Y}} &:= \sup_{t>0} \|\nabla d(t)\|_{\dot{\mathbf{N}}_{r_2,\lambda,\infty}^{-\beta_2}} + \sup_{t>0} \left(t^{\frac{\alpha_2}{2}} \|\nabla d(t)\|_{M_{q_2,\lambda}} \right) + \sup_{t>0} \|d(t)\|_{L^\infty}. \end{aligned}$$

For each $\epsilon_0 > 0$, we say $u \in \mathbb{X}_{\epsilon_0}$, if $\|u\|_{\mathbb{X}} \leq C\epsilon_0$. We denote \mathbb{Y}_{ϵ_0} , \mathbb{Z}_{ϵ_0} and Θ_{ϵ_0} simply as \mathbb{X}_{ϵ_0} , respectively. Our first result is the uniquely existence of the global mild solution to system (1.1).

Theorem 1.1. Suppose that there hold (1.3) and (1.4). There exists a sufficiently small $\epsilon_0 > 0$ such that if

$$\|(u_0, \nabla d_0)\|_{\mathbb{E}} + \|d_0 - \underline{d}_0\|_{L^\infty} \leq \epsilon_0, \quad (1.5)$$

then there exists a unique global solution to system (1.1) such that $(u, d - \underline{d}_0) \in \Theta_{\epsilon_0}$.

Remark 1.1. We would like to point out a small difficulty in the proof of Theorem 1.1. The small difficulty comes from the term $|\nabla d|^2 d$. This term leads us to include the L^∞ -norm of $d - \underline{d}_0$ to $\|d\|_{\mathbb{Y}}$. In fact, in the proof of the uniform bound part (Lemma 3.1), there's no need to consider the L^∞ -norm of $d - \underline{d}_0$, since $|d - \underline{d}_0| \leq 2$. However, in the proof of the contraction part (Lemma 3.2), we encounter the term $|\nabla \tilde{c}|^2 c^*$, where $c^* = \tilde{c} - \bar{c}$ denotes the difference of two solutions in the approximation sequence. This term denies us to repeat the process of the proof of the uniform bound part to prove the contraction part if L^∞ -norm is not considered. This is also a difference between the proof of Theorem 1.1 in this paper and the proof of Theorem 1.1 in [44], where L^∞ -norm must be considered in the proof of the uniform bound part.

Since we work in Besov-Morrey spaces with critical indexes, we have the following existence result on forward self-similar solutions to system (1.1).

Corollary 1.1. *Let all conditions in Theorem 1.1 hold. If the initial data (u_0, d_0) satisfy*

$$u_0(x) = \lambda u_0(\lambda x), \quad d_0(x) = d_0(\lambda x)$$

for all $x \in \mathbb{R}^N$ and $\lambda > 0$. Then the global solution (u, d) of system (1.1) given by Theorem 1.1 satisfy

$$u(t, x) = \lambda u(\lambda^2 t, \lambda x), \quad d(t, x) = d(\lambda^2 t, \lambda x).$$

We also prove an asymptotic behavior result of the global mild solution obtained in Theorem 1.1 as $t \rightarrow \infty$.

Theorem 1.2. *Let the assumptions in Theorem 1.1 hold, and let (\bar{u}, \bar{d}) and (\tilde{u}, \tilde{d}) be two global solutions for system (1.1) given by Theorem 1.1 corresponding to initial data (\bar{u}_0, \bar{d}_0) and $(\tilde{u}_0, \tilde{d}_0)$, respectively, where*

$$\|(\bar{u}_0, \nabla \bar{d}_0)\|_{\mathbb{E}} + \|\bar{d}_0 - \underline{d}_0\|_{L^\infty} \leq \epsilon_0$$

and

$$\|(\tilde{u}_0, \nabla \tilde{d}_0)\|_{\mathbb{E}} + \|\tilde{d}_0 - \underline{d}_0\|_{L^\infty} \leq \epsilon_0.$$

Then we conclude that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left(t^{\frac{\alpha_1}{2}} \|e^{t\Delta} (\bar{u}_0 - \tilde{u}_0)\|_{q_1, \lambda} + t^{\frac{\alpha_2}{2}} \|\nabla e^{t\Delta} (\bar{d}_0 - \tilde{d}_0)\|_{q_2, \lambda} \right) \\ & + \lim_{t \rightarrow \infty} \left(\|e^{t\Delta} (\bar{u}_0 - \tilde{u}_0, \nabla \bar{d}_0 - \nabla \tilde{d}_0)\|_{\mathbb{E}} + \|e^{t\Delta} (\bar{d}_0 - \tilde{d}_0)\|_{L^\infty} \right) = 0 \end{aligned} \quad (1.6)$$

if and only if

$$\begin{aligned} & \lim_{t \rightarrow \infty} \left(t^{\frac{\alpha_1}{2}} \|(\bar{u} - \tilde{u})\|_{q_1, \lambda} + t^{\frac{\alpha_2}{2}} \|\nabla (\bar{d} - \tilde{d})\|_{q_2, \lambda} \right) \\ & + \lim_{t \rightarrow \infty} \left(\|(\bar{u} - \tilde{u}, \nabla \bar{d} - \nabla \tilde{d})\|_{\mathbb{E}} + \|\bar{d} - \tilde{d}\|_{L^\infty} \right) = 0. \end{aligned} \quad (1.7)$$

The remaining of this paper is organized as follows. In section 2, we review some basic properties of Morrey and Besov-Morrey space. In section 3, we prove Theorem 1.1. In section 4, we prove Theorem 1.2.

2. Preliminaries

In this section, the basic properties of Morrey and Besov-Morrey space is reviewed for the reader's convenience, more details can be found in [1,16,18,35,37,38].

Let $Q_r(x_0)$ be the open ball in \mathbb{R}^N centered at x_0 and with radius $r > 0$. Given two parameters $1 \leq p < \infty$ and $0 \leq \mu < N$, the Morrey spaces $M_{p,\mu} = M_{p,\mu}(\mathbb{R}^N)$ is defined to be the set of functions $f \in L^p(Q_r(x_0))$ such that

$$\|f\|_{p,\mu} \triangleq \sup_{x_0 \in \mathbb{R}^N} \sup_{r>0} r^{-\mu/p} \|f\|_{L^p(Q_r(x_0))} < \infty \quad (2.1)$$

which is a Banach space endowed with norm (2.1). For $s \in \mathbb{R}$ and $1 \leq p < \infty$, the homogeneous Sobolev-Morrey space $M_{p,\mu}^s = (-\Delta)^{-s/2} M_{p,\mu}$ is the Banach space with norm

$$\|f\|_{M_{p,\mu}^s} = \|(-\Delta)^{s/2} f\|_{p,\mu}.$$

Taking $p = 1$, we have $\|f\|_{L^1(Q_r(x_0))}$ denotes the total variation of f on open ball $Q_r(x_0)$ and $M_{1,\mu}$ stands for space of signed measures. In particular, $M_{1,0} = M$ is the space of finite measures. For $p > 1$, we have $M_{p,0} = L^p$ and $M_{p,0}^s = \dot{H}_p^s$ is the well known Sobolev space. The space L^∞ corresponds to $M_{\infty,\mu}$. Morrey and Sobolev-Morrey spaces present the following scaling

$$\|f(\lambda \cdot)\|_{p,\mu} = \lambda^{-\frac{N-\mu}{p}} \|f\|_{p,\mu}$$

and

$$\|f(\lambda \cdot)\|_{M_{p,\mu}^s} = \lambda^{s-\frac{N-\mu}{p}} \|f\|_{M_{p,\mu}^s},$$

where the exponent $s - \frac{N-\mu}{p}$ is called scaling index and s is called regularity index. We have that

$$(-\Delta)^{l/2} M_{p,\mu}^s = M_{p,\mu}^{s-l}.$$

Morrey spaces contain Lebesgue and weak- L^p , with the same scaling index. Precisely, we have the continuous proper inclusions

$$L^p(\mathbb{R}^N) \not\subset \text{weak-}L^p(\mathbb{R}^N) \not\subset M_{r,\mu}(\mathbb{R}^N),$$

where $r < p$ and $\mu = N(1 - r/p)$ (see e.g. [36]).

Let $\mathcal{S}(\mathbb{R}^N)$ and $\mathcal{S}'(\mathbb{R}^N)$ be the Schwartz space and the tempered distributions, respectively. Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$ be nonnegative radial function such that

$$\text{supp}(\varphi) \subset \left\{ \xi \in \mathbb{R}^N; \frac{1}{2} < |\xi| < 2 \right\}$$

and

$$\sum_{j=-\infty}^{\infty} \varphi_j(\xi) = 1, \text{ for all } \xi \neq 0,$$

where $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. Let $\phi(x) = \mathcal{F}^{-1}(\varphi)(x)$ and $\phi_j(x) = \mathcal{F}^{-1}(\varphi_j)(x) = 2^{jn} \phi(2^j x)$ where \mathcal{F}^{-1} stands for inverse Fourier transform. For $1 \leq q < \infty$, $0 \leq \mu < n$ and $s \in \mathbb{R}$, the homogeneous Besov-Morrey space $\dot{\mathbf{N}}_{q,\mu,r}^s(\mathbb{R}^N)$ ($\dot{\mathbf{N}}_{q,\mu,r}^s$ for short) is defined to be the set of $u \in \mathcal{S}'(\mathbb{R}^N)$, modulo polynomials \mathcal{P} , such that $\mathcal{F}^{-1}\varphi_j(\xi)\mathcal{F}u \in M_{q,\mu}$ for all $j \in \mathbb{Z}$ and

$$\|u\|_{\dot{\mathbf{N}}_{q,\mu,r}^s} = \begin{cases} \left(\sum_{j \in \mathbb{Z}} \left(2^{js} \|\phi_j * u\|_{q,\mu} \right)^r \right)^{\frac{1}{r}} < \infty, & 1 \leq r < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\phi_j * u\|_{q,\mu} < \infty, & r = \infty. \end{cases}$$

The space $\dot{\mathbf{N}}_{q,\mu,r}^s(\mathbb{R}^N)$ is a Banach space and, in particular, $\dot{\mathbf{N}}_{q,0,r}^s = \dot{B}_{q,r}^s$ (case $\mu = 0$) corresponds to the homogeneous Besov space. We have the real-interpolation properties

$$\dot{\mathbf{N}}_{q,\mu,r}^s = (M_{q,\mu}^{s_1}, M_{q,\mu}^{s_2})_{\theta,r} \quad (2.2)$$

and

$$\dot{\mathbf{N}}_{q,\mu,r}^s = (\dot{\mathbf{N}}_{q,\mu,r_1}^{s_1}, \dot{\mathbf{N}}_{q,\mu,r_2}^{s_2})_{\theta,r} \quad (2.3)$$

for all $s_1 \neq s_2, 0 < \theta < 1$ and $s = (1 - \theta)s_1 + \theta s_2$. Here $(X, Y)_{\theta,r}$ stands for the real interpolation space between X and Y constructed via the $K_{\theta,q}$ -method. Recall that $(\cdot, \cdot)_{\theta,r}$ is an exact interpolation functor of exponent θ on the category of normed spaces.

In the next lemmas, we collect basic facts about Morrey spaces and Besov-Morrey spaces (see [1,16,38]).

Lemma 2.1. Suppose that $s_1, s_2 \in \mathbb{R}, 1 \leq p_1, p_2, p_3 < \infty$ and $0 \leq \mu_i < N, i = 1, 2, 3$.

(i)(Inclusion) If $\frac{N-\mu_1}{p_1} = \frac{N-\mu_2}{p_2}$ and $p_2 \leq p_1$, then

$$M_{p_1,\mu_1} \hookrightarrow M_{p_2,\mu_2} \quad \text{and} \quad \dot{\mathbf{N}}_{p_1,\mu_1,1}^0 \hookrightarrow M_{p_1,\mu_1} \hookrightarrow \dot{\mathbf{N}}_{p_1,\mu_1,\infty}^0.$$

(ii)(Sobolev-type embedding) Let $j = 1, 2$ and p_j, s_j be $p_2 \leq p_1, s_1 \leq s_2$ such that $s_2 - \frac{N-\mu_2}{p_2} = s_1 - \frac{N-\mu_1}{p_1}$, then we have

$$M_{p_2,\mu}^{s_2} \hookrightarrow M_{p_1,\mu}^{s_1}, \quad (\mu = \mu_1 = \mu_2)$$

and for every $1 \leq r_2 \leq r_1 \leq \infty$, we have

$$\dot{\mathbf{N}}_{p_2,\mu_2,r_2}^{s_2} \hookrightarrow \dot{\mathbf{N}}_{p_1,\mu_1,r_1}^{s_1} \quad \text{and} \quad \dot{\mathbf{N}}_{p_2,\mu_2,r_2}^{s_2} \hookrightarrow \dot{B}_{\infty,r_2}^{s_2 - \frac{n-\mu_2}{p_2}}.$$

(iii)(Hölder inequality) Let $\frac{1}{p_3} = \frac{1}{p_2} + \frac{1}{p_1}$ and $\frac{\mu_3}{p_3} = \frac{\mu_2}{p_2} + \frac{\mu_1}{p_1}$. If $f_j \in M_{p_j,\mu_j}$ with $j = 1, 2$, then $f_1 f_2 \in M_{p_3,\mu_3}$ and

$$\|f_1 f_2\|_{p_3,\mu_3} \leq \|f_1\|_{p_1,\mu_1} \|f_2\|_{p_2,\mu_2}.$$

Set $\alpha = 1$ in [[1], Lemma 3.1], we have the following decay estimates about the heat semi-group in the Sobolev-Morrey or Besov-Morrey space.

Lemma 2.2. Let $s, \beta \in \mathbb{R}, 1 < p \leq q \leq \infty, 0 \leq \mu < N$, and $(\beta - s) + \frac{N-\mu}{p} - \frac{N-\mu}{q} < 2$ where $\beta \geq s$.

There exists $C > 0$ such that

$$\|e^{t\Delta} f\|_{M_{q,\mu}^\beta} \leq C t^{-\frac{1}{2}(\beta-s)-\frac{1}{2}\left(\frac{N-\mu}{p}-\frac{N-\mu}{q}\right)} \|f\|_{M_{p,\mu}^s}$$

for every $t > 0$ and $f \in M_{p,\mu}^s$.

The following Lemma can be found in [35].

Lemma 2.3. If $1 \leq p, q \leq \infty, s > 0, 0 \leq \lambda < N$, then $f \in \dot{\mathbf{N}}_{p,\lambda,q}^{-2s}$ if and only if

$$\begin{cases} \left[\int_0^\infty \left(t^s \|e^{t\Delta} f\|_{p,\lambda} \right)^q \frac{dt}{t} \right]^{\frac{1}{q}} < \infty, & \text{if } 1 \leq q < \infty, \\ \sup_{t>0} \left(t^s \|e^{t\Delta} f\|_{p,\lambda} \right) < \infty, & \text{if } q = \infty. \end{cases}$$

3. Proof of Theorem 1.1 – global mild solution

The proof of Theorem 1.1 is a consequence of the following Lemmas 3.1 and 3.2. We will prove it by a fixed point argument.

Let $\mathbb{P} \triangleq I - \nabla \Delta^{-1} \operatorname{div}$ be the Leray projection operator. Denote $c \triangleq d - \underline{d}_0$ and $c_0 \triangleq d_0 - \underline{d}_0$. Then we can rewrite system (1.1) as

$$\begin{cases} \partial_t u - \Delta u = -\mathbb{P}[u \cdot \nabla u + \operatorname{div}(\nabla c \odot \nabla c)], \\ \partial_t c - \Delta c = -u \cdot \nabla c + |\nabla c|^2 c + |\nabla c|^2 \underline{d}_0, \\ u|_{t=0} = u_0(x), \quad c|_{t=0} = c_0(x), \end{cases} \quad (3.1)$$

where the initial data satisfying the following far field behavior

$$u_0 \rightarrow 0, \quad c_0 \rightarrow 0 \quad \text{as } |x| \rightarrow \infty. \quad (3.2)$$

By the Duhamel principle, we can express a solution (u, c) of (3.1) and (3.2) in the integral form:

$$\begin{cases} u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}[u \cdot \nabla u + \operatorname{div}(\nabla c \odot \nabla c)](s) ds, \\ c(t) = e^{t\Delta} c_0 + \int_0^t e^{(t-s)\Delta} [-u \cdot \nabla c + |\nabla c|^2 c + |\nabla c|^2 \underline{d}_0](s) ds. \end{cases} \quad (3.3)$$

Define the map

$$(u, c) = \mathbb{T}(\tilde{u}, \tilde{c}) = (\mathbb{T}_1(\tilde{u}, \tilde{c}), \mathbb{T}_2(\tilde{u}, \tilde{c})) \quad (3.4)$$

with

$$\begin{cases} u(t) = \mathbb{T}_1(\tilde{u}, \tilde{c}) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}[\tilde{u} \cdot \nabla \tilde{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla \tilde{c})](s) ds, \\ c(t) = \mathbb{T}_2(\tilde{u}, \tilde{c}) = e^{t\Delta} c_0 + \int_0^t e^{(t-s)\Delta} [-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0](s) ds. \end{cases} \quad (3.5)$$

Then we have

Lemma 3.1. *Given a constant $\epsilon_0 > 0$ small enough, the initial data (u_0, c_0) satisfies (1.5) and $(\tilde{u}, \tilde{c}) \in \Theta_{\epsilon_0}$, then the solution of (3.5) satisfies*

$$(u, c) = \mathbb{T}(\tilde{u}, \tilde{c}) \in \Theta_{\epsilon_0}.$$

Proof. Let's first consider the $\dot{\mathbf{N}}_{r_1, \lambda, \infty}^{-\beta_1}$ -norm of

$$\int_0^t e^{(t-s)\Delta} \mathbb{P}[\tilde{u} \cdot \nabla \tilde{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla \tilde{c})](s) ds.$$

Lemma 2.3 and the boundedness of \mathbb{P} in Morrey spaces lead to

$$\begin{aligned} & \left\| \int_0^t e^{(t-s)\Delta} \mathbb{P}[\tilde{u} \cdot \nabla \tilde{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla \tilde{c})](s) ds \right\|_{\dot{\mathbf{N}}_{r_1, \lambda, \infty}^{-\beta_1}} \\ &= \sup_{s>0} \left[s^{\beta_1/2} \left\| e^{s\Delta} \int_0^t e^{(t-\tau)\Delta} \mathbb{P}[\tilde{u} \cdot \nabla \tilde{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla \tilde{c})](\cdot, \tau) d\tau \right\|_{r_1, \lambda} \right] \end{aligned}$$

$$\leq \int_0^t \sup_{s>0} \left[s^{\beta_1/2} \left\| e^{s\Delta} e^{(t-\tau)\Delta} \mathbb{P}[\tilde{u} \cdot \nabla \tilde{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla \tilde{c})](\cdot, \tau) \right\|_{r_1, \lambda} \right] d\tau. \quad (3.6)$$

For $0 < s \leq t - \tau$, we have by Lemma 2.1 that

$$\begin{aligned} & \sup_{0 < s \leq t - \tau} \left[s^{\beta_1/2} \left\| e^{s\Delta} e^{(t-\tau)\Delta} \mathbb{P}[\tilde{u} \cdot \nabla \tilde{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla \tilde{c})](\cdot, \tau) \right\|_{r_1, \lambda} \right] \\ & \leq (t - \tau)^{\frac{\beta_1}{2}} (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_1} - \frac{1}{r_1} \right)} \|\tilde{u} \otimes \tilde{u}\|_{\frac{q_1}{2}, \lambda} \\ & \quad + (t - \tau)^{\frac{\beta_1}{2}} (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{r_1} \right)} \|\nabla \tilde{c} \odot \nabla \tilde{c}\|_{\frac{q_2}{2}, \lambda} \\ & \leq (t - \tau)^{-\frac{N-\lambda}{q_1}} \|\tilde{u}\|_{q_1, \lambda}^2 + (t - \tau)^{-\frac{N-\lambda}{q_2}} \|\nabla \tilde{c}\|_{q_2, \lambda}^2. \end{aligned} \quad (3.7)$$

For $s > t - \tau$, note that $(t - \tau + s)/2 < s < t - \tau + s$, so we have from Lemma 2.1 that

$$\begin{aligned} & \sup_{s > t - \tau} \left[s^{\beta_1/2} \left\| e^{s\Delta} e^{(t-\tau)\Delta} \mathbb{P}[\tilde{u} \cdot \nabla \tilde{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla \tilde{c})](\cdot, \tau) \right\|_{r_1, \lambda} \right] \\ & \leq \sup_{s > t - \tau} \left[s^{\frac{\beta_1}{2}} (s + t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_1} - \frac{1}{r_1} \right)} \right] \|(\tilde{u} \otimes \tilde{u})(\cdot, \tau)\|_{\frac{q_1}{2}, \lambda} \\ & \quad + \sup_{s > t - \tau} \left[s^{\frac{\beta_1}{2}} (s + t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{r_1} \right)} \right] \|(\nabla \tilde{c} \odot \nabla \tilde{c})(\cdot, \tau)\|_{\frac{q_2}{2}, \lambda} \\ & \leq (t - \tau)^{-\frac{N-\lambda}{q_1}} \sup_{s > t - \tau} \left(1 + \frac{s}{t - \tau} \right)^{-\frac{N-\lambda}{q_1}} \|\tilde{u}(\cdot, \tau)\|_{q_1, \lambda}^2 \\ & \quad + (t - \tau)^{-\frac{N-\lambda}{q_2}} \sup_{s > t - \tau} \left(1 + \frac{s}{t - \tau} \right)^{-\frac{N-\lambda}{q_2}} \|\nabla \tilde{c}(\cdot, \tau)\|_{q_2, \lambda}^2 \\ & \leq (t - \tau)^{-\frac{N-\lambda}{q_1}} \|\tilde{u}(\cdot, \tau)\|_{q_1, \lambda}^2 + (t - \tau)^{-\frac{N-\lambda}{q_2}} \|\nabla \tilde{c}(\cdot, \tau)\|_{q_2, \lambda}^2. \end{aligned} \quad (3.8)$$

Substituting (3.7) and (3.8) into (3.6) and using $q_1 > N - \lambda, q_2 > N - \lambda$ yield

$$\begin{aligned} & \left\| \int_0^t e^{(t-s)\Delta} \mathbb{P}[\tilde{u} \cdot \nabla \tilde{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla \tilde{c})](s) ds \right\|_{\dot{\mathbf{N}}_{r_1, \lambda, \infty}^{-\beta_1}} \\ & \lesssim \left\{ \left[\sup_{t>0} \left(t^{\frac{\alpha_1}{2}} \|\tilde{u}(\cdot, t)\|_{q_1, \lambda} \right) \right]^2 + \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} \|\nabla \tilde{c}(\cdot, t)\|_{q_2, \lambda} \right) \right]^2 \right\} \\ & \quad \times \int_0^t (t - \tau)^{-\frac{N-\lambda}{q_1}} \tau^{-\alpha_1} + (t - \tau)^{-\frac{N-\lambda}{q_2}} \tau^{-\alpha_2} d\tau \\ & \lesssim [B(\alpha_1, 1 - \alpha_1) + B(\alpha_2, 1 - \alpha_2)] (\|\tilde{u}\|_{\mathbf{X}}^2 + \|\tilde{c}\|_{\mathbf{Y}}^2) \\ & \lesssim \|\tilde{u}\|_{\mathbf{X}}^2 + \|\tilde{c}\|_{\mathbf{Y}}^2, \end{aligned} \quad (3.9)$$

where $B(x, y) := \int_0^1 (1 - t)^{x-1} t^{y-1} dt$, for $x > 0, y > 0$.

Next, we calculate the $M_{q_1, \lambda}$ -norm of

$$\int_0^t e^{(t-s)\Delta} \mathbb{P}[\tilde{u} \cdot \nabla \tilde{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla \tilde{c})](s) ds.$$

Using Lemmas 2.1, 2.2 and 2.3 and noting that $2/q_2 - 1/q_1 < 1/(N - \lambda)$, $q_1 > N - \lambda$, we have

$$\begin{aligned}
 & \left\| \int_0^t e^{(t-\tau)\Delta} \mathbb{P}[\tilde{u} \cdot \nabla \tilde{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla \tilde{c})](\cdot, \tau) d\tau \right\|_{q_1, \lambda} \\
 & \lesssim \int_0^t (t-\tau)^{-1/2 - \frac{N-\lambda}{2} \left(\frac{2}{q_1} - \frac{1}{q_1} \right)} \|\tilde{u} \otimes \tilde{u}\|_{q_1/2, \lambda} \\
 & \quad + (t-\tau)^{-1/2 - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_1} \right)} \|\nabla \tilde{c} \odot \nabla \tilde{c}\|_{q_2/2, \lambda} d\tau \\
 & \lesssim \left\{ \left[\sup_{t>0} \left(t^{\frac{\alpha_1}{2}} \|\tilde{u}(\cdot, t)\|_{q_1, \lambda} \right) \right]^2 + \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} \|\nabla \tilde{c}(\cdot, t)\|_{q_2, \lambda} \right) \right]^2 \right\} \\
 & \quad \times \int_0^t (t-\tau)^{-1/2 - \frac{N-\lambda}{2} \left(\frac{2}{q_1} - \frac{1}{q_1} \right)} \tau^{-\alpha_1} + (t-\tau)^{-1/2 - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_1} \right)} \tau^{-\alpha_2} d\tau \\
 & \lesssim t^{-\frac{1}{2} + \frac{N-\lambda}{2q_1}} (\|\tilde{u}\|_{\mathbb{X}}^2 + \|\tilde{c}\|_{\mathbb{Y}}^2). \tag{3.10}
 \end{aligned}$$

Next, we calculate the $\dot{\mathbf{N}}_{r_2, \lambda, \infty}^{-\beta_2}$ -norm of $\nabla \int_0^t e^{(t-\tau)\Delta} [-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0](\tau) d\tau$. By Lemma 2.3,

$$\begin{aligned}
 & \left\| \nabla \int_0^t e^{(t-\tau)\Delta} [-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0](\tau) d\tau \right\|_{\dot{\mathbf{N}}_{r_2, \lambda, \infty}^{-\beta_2}} \\
 & = \sup_{s>0} \left[s^{\beta_2/2} \left\| \nabla e^{s\Delta} \int_0^t e^{(t-\tau)\Delta} (-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0)(\cdot, \tau) d\tau \right\|_{r_2, \lambda} \right] \\
 & \leq \int_0^t \sup_{s>0} \left[s^{\beta_2/2} \left\| e^{s\Delta} \nabla e^{(t-\tau)\Delta} (-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0)(\cdot, \tau) \right\|_{r_2, \lambda} \right] d\tau. \tag{3.11}
 \end{aligned}$$

Employing the fact that $|\tilde{c}| \leq 2$ and $|\underline{d}_0| \leq 1$, we obtain by Lemmas 2.1 and 2.2 that

$$\begin{aligned}
 & \sup_{0 < s \leq t-\tau} \left[s^{\frac{\beta_2}{2}} \left\| e^{s\Delta} \nabla e^{(t-\tau)\Delta} (-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0)(\cdot, \tau) \right\|_{r_2, \lambda} \right] \\
 & \lesssim (t-\tau)^{\frac{\beta_2}{2}} (t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{r_2} \right)} \|\tilde{u}(\cdot, \tau) \nabla \tilde{c}(\cdot, \tau)\|_{\frac{q_1 q_2}{q_1 + q_2}, \lambda} \\
 & \quad + (t-\tau)^{\frac{\beta_2}{2}} (t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{r_2} \right)} \| |\nabla \tilde{c}|^2(\cdot, \tau) \|_{\frac{q_2}{2}, \lambda} \\
 & \lesssim (t-\tau)^{-\frac{N-\lambda}{2q_1} - \frac{N-\lambda}{2q_2}} \|\tilde{u}(\cdot, \tau)\|_{q_1, \lambda} \|\nabla \tilde{c}(\cdot, \tau)\|_{q_2, \lambda} + (t-\tau)^{-\frac{N-\lambda}{q_2}} \|\nabla \tilde{c}(\cdot, \tau)\|_{q_2, \lambda}^2. \tag{3.12}
 \end{aligned}$$

For $s > t - \tau$, note that $(t - \tau + s)/2 < s < t - \tau + s$, one obtains

$$\begin{aligned}
 & \sup_{s>t-\tau} \left[s^{\frac{\beta_2}{2}} \left\| e^{s\Delta} \nabla e^{(t-\tau)\Delta} (-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0)(\cdot, \tau) \right\|_{r_2, \lambda} \right] \\
 & \lesssim s^{\frac{\beta_2}{2}} (s+t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{r_2} \right)} \|\tilde{u}(\cdot, \tau) \nabla \tilde{c}(\cdot, \tau)\|_{\frac{q_1 q_2}{q_1 + q_2}, \lambda} \\
 & \quad + s^{\frac{\beta_2}{2}} (s+t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{r_2} \right)} \| |\nabla \tilde{c}|^2(\cdot, \tau) \|_{\frac{q_2}{2}, \lambda}
 \end{aligned}$$

$$\begin{aligned}
&\lesssim (t-\tau)^{-\frac{N-\lambda}{2q_1}-\frac{N-\lambda}{2q_2}} \sup_{s>t-\tau} \left(1 + \frac{s}{t-\tau}\right)^{-\frac{N-\lambda}{2q_1}-\frac{N-\lambda}{2q_2}} \|\tilde{u}(\cdot, \tau)\|_{q_1, \lambda} \|\nabla \tilde{c}(\cdot, \tau)\|_{q_2, \lambda} \\
&\quad + (t-\tau)^{-\frac{N-\lambda}{q_2}} \sup_{s>t-\tau} \left(1 + \frac{s}{t-\tau}\right)^{-\frac{N-\lambda}{q_2}} \|\nabla \tilde{c}(\cdot, \tau)\|_{q_2, \lambda}^2.
\end{aligned} \tag{3.13}$$

Note that $\frac{1}{q_1} + \frac{1}{q_2} < \frac{2}{N-\lambda}$, $\frac{1}{q_2} < \frac{1}{N-\lambda}$, we thus obtain

$$\begin{aligned}
&\left\| \nabla \int_0^t e^{(t-\tau)\Delta} [-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0] (\tau) d\tau \right\|_{\dot{\mathbf{N}}_{r_2, \lambda, \infty}^{-\beta_2}} \\
&\leq \left\{ \left[\sup_{t>0} \left(t^{\frac{\alpha_1}{2}} \|\tilde{u}(\cdot, t)\|_{q_1, \lambda} \right) \right] \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} \|\nabla \tilde{c}(\cdot, t)\|_{q_2, \lambda} \right) \right] + \left[\sup_{t>0} \left(t^{\alpha_2} \|\nabla \tilde{c}(\cdot, t)\|_{q_2, \lambda}^2 \right) \right] \right\} \\
&\quad \times \int_0^t (t-\tau)^{-\frac{N-\lambda}{2q_1}-\frac{N-\lambda}{2q_2}} \tau^{-\frac{\alpha_1}{2}} \tau^{-\frac{\alpha_2}{2}} + (t-\tau)^{-\frac{N-\lambda}{q_2}} \tau^{-\alpha_2} d\tau \\
&\lesssim \|u\|_{\mathbb{X}} \|\tilde{c}\|_{\mathbb{Y}} + \|\tilde{c}\|_{\mathbb{Y}}^2.
\end{aligned} \tag{3.14}$$

Next, we calculate the $M_{q_2, \lambda}$ -norm of $\nabla \int_0^t e^{(t-\tau)\Delta} [-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0] (\tau) d\tau$.

$$\begin{aligned}
&\left\| \nabla \int_0^t e^{(t-\tau)\Delta} (-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0) (\cdot, \tau) d\tau \right\|_{q_2, \lambda} \\
&\lesssim \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N-\lambda}{2}\left(\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_2}\right)} \|\tilde{u} \nabla \tilde{c}\|_{\frac{q_1 q_2}{q_1+q_2}, \lambda} + (t-\tau)^{-\frac{1}{2}-\frac{N-\lambda}{2}\left(\frac{2}{q_2}-\frac{1}{q_2}\right)} \| |\nabla \tilde{c}|^2 \|_{\frac{q_2}{2}, \lambda} d\tau \\
&\lesssim \left\{ \left[\sup_{t>0} \left(t^{\frac{\alpha_1}{2}} \|\tilde{u}\|_{q_1, \lambda} \right) \right] \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} \|\nabla \tilde{c}\|_{q_2, \lambda} \right) \right] + \left[\sup_{t>0} \left(t^{\alpha_2} \|\nabla \tilde{c}\|_{q_2, \lambda}^2 \right) \right] \right\} \\
&\quad \times \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N-\lambda}{2}\left(\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{q_2}\right)} \tau^{-\frac{\alpha_1}{2}} \tau^{-\frac{\alpha_2}{2}} + (t-\tau)^{-\frac{1}{2}-\frac{N-\lambda}{2}\left(\frac{2}{q_2}-\frac{1}{q_2}\right)} \tau^{-\alpha_2} d\tau \\
&\lesssim t^{-\frac{1}{2}+\frac{N-\lambda}{2q_2}} (\|\tilde{u}\|_{\mathbb{X}} \|\tilde{c}\|_{\mathbb{Y}} + \|\tilde{c}\|_{\mathbb{Y}}^2).
\end{aligned} \tag{3.15}$$

We calculate L^∞ -norm of $\int_0^t e^{(t-\tau)\Delta} [-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0] (\tau) d\tau$. Note that $q_1 > N-\lambda$, $q_2 > N-\lambda$, we get

$$\begin{aligned}
&\left\| \int_0^t e^{(t-\tau)\Delta} [-\tilde{u} \cdot \nabla \tilde{c} + |\nabla \tilde{c}|^2 \tilde{c} + |\nabla \tilde{c}|^2 \underline{d}_0] (\cdot, \tau) d\tau \right\|_{L^\infty} \\
&\lesssim \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N-\lambda}{2q_1}} \|\tilde{u} \otimes \tilde{c}\|_{q_1, \lambda} + (t-\tau)^{-\frac{N-\lambda}{q_2}} \| |\nabla \tilde{c}|^2 \|_{\frac{q_2}{2}, \lambda} d\tau \\
&\lesssim \left\{ \left[\sup_{t>0} \left(t^{\frac{\alpha_1}{2}} \|\tilde{u}(\cdot, t)\|_{q_1, \lambda} \right) \right] \|\tilde{c}(\cdot, t)\|_{L^\infty} + \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} \|\nabla \tilde{c}(\cdot, t)\|_{q_2, \lambda} \right) \right]^2 \right\}
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{N-\lambda}{2q_1}} \tau^{-\frac{\alpha_1}{2}} + (t-\tau)^{-\frac{N-\lambda}{q_2}} \tau^{-\alpha_2} d\tau \\
& \lesssim \|\tilde{u}\|_{\mathbb{X}} \|\tilde{c}\|_{\mathbb{Y}} + \|\tilde{c}\|_{\mathbb{Y}}^2.
\end{aligned} \tag{3.16}$$

Moreover, note that $r_1 > N - \lambda, r_2 > N - \lambda$ by Lemmas 2.1, 2.2 and 2.3, we have

$$\begin{aligned}
\|e^{t\Delta} u_0\|_{\mathbb{X}} &:= \sup_{t>0} \|e^{t\Delta} u_0\|_{\dot{N}_{r_1, \lambda, \infty}^{-\beta_1}} + \sup_{t>0} \left[t^{\frac{\alpha_1}{2}} \|e^{t\Delta} u_0\|_{M_{q_1, \lambda}} \right] \\
&\lesssim \|u_0\|_{\dot{N}_{r_1, \lambda, \infty}^{-\beta_1}} + \|u_0\|_{\dot{N}_{q_1, \lambda, \infty}^{-\alpha_1}} \\
&\lesssim \|u_0\|_{\dot{N}_{r_1, \lambda, \infty}^{-\beta_1}}
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
\|e^{t\Delta} c_0\|_{\mathbb{Y}} &:= \sup_{t>0} \|\nabla e^{t\Delta} c_0\|_{\dot{N}_{r_2, \lambda, \infty}^{-\beta_2}} + \sup_{t>0} \left[t^{\frac{\alpha_2}{2}} \|\nabla e^{t\Delta} c_0\|_{M_{q_2, \lambda}} \right] + \sup_{t>0} \|e^{t\Delta} c_0\|_{L^\infty} \\
&\lesssim \|\nabla c_0\|_{\dot{N}_{r_2, \lambda, \infty}^{-\beta_2}} + \|\nabla c_0\|_{\dot{N}_{q_2, \lambda, \infty}^{-\alpha_2}} + \|c_0\|_{L^\infty} \\
&\lesssim \|\nabla c_0\|_{\dot{N}_{r_2, \lambda, \infty}^{-\beta_2}} + \|c_0\|_{L^\infty}.
\end{aligned} \tag{3.18}$$

Combining (3.9)-(3.18) and (3.5), we obtain

$$\begin{aligned}
\|(u, c)\|_{\Theta} &:= \|\mathbb{T}(\tilde{u}, \tilde{c})\|_{\Theta} \\
&\lesssim \|\tilde{u}\|_{\mathbb{X}}^2 + \|\tilde{c}\|_{\mathbb{Y}}^2 + \|\tilde{u}\|_{\mathbb{X}} \|\tilde{c}\|_{\mathbb{Y}} + \|(u_0, \nabla c_0)\|_{\mathbb{E}} + \|c_0\|_{L^\infty} \\
&\lesssim \epsilon_0,
\end{aligned} \tag{3.19}$$

which implies that

$$(u, c) = \mathbb{T}(\tilde{u}, \tilde{c}) \in \Theta_{\epsilon_0}.$$

We thus complete the proof of Lemma 3.1. \square

To complete the proof of Theorem 1.1, we need the following Lemma.

Lemma 3.2. For $\epsilon_0 > 0$ small enough, let $(\bar{u}, \bar{c}) \in \Theta_{\epsilon_0}$ and $(\tilde{u}, \tilde{c}) \in \Theta_{\epsilon_0}$ with $(\bar{u}, \bar{c})|_{t=0} = (\tilde{u}, \tilde{c})|_{t=0} = (u_0, c_0)$, where (u_0, c_0) satisfies (1.5), then the map $\mathbb{T} = (\mathbb{T}_1, \mathbb{T}_2)$ defined in (3.4) is contractive.

Proof. For simplicity, we write $(u^*, c^*) = (\tilde{u} - \bar{u}, \tilde{c} - \bar{c})$. Then we have

$$|\mathbb{T}_1(\tilde{u}, \tilde{c}) - \mathbb{T}_1(\bar{u}, \bar{c})| = \left| \int_0^t e^{(t-s)\Delta} \mathbb{P}(\tilde{u} \cdot \nabla u^* + u^* \cdot \nabla \bar{u} + \operatorname{div}(\nabla c^* \odot \nabla \bar{c} + \nabla \tilde{c} \odot \nabla c^*))(\cdot, s) ds \right|$$

and

$$\begin{aligned}
& |\mathbb{T}_2(\tilde{u}, \tilde{c}) - \mathbb{T}_2(\bar{u}, \bar{c})| \\
&= \left| \int_0^t e^{(t-s)\Delta} (-\tilde{u} \cdot \nabla c^* - u^* \cdot \nabla \bar{c} + (|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \bar{c}) + |\nabla \bar{c}|^2 c^*)(\cdot, s) ds \right|.
\end{aligned}$$

We first compute the $\dot{\mathbf{N}}_{r_2, \lambda, \infty}^{-\beta_2}$ -norm of

$$\nabla \int_0^t e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\tau) d\tau.$$

Using Lemma 2.3, one has

$$\begin{aligned} & \left\| \nabla \int_0^t e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\tau) d\tau \right\|_{\dot{\mathbf{N}}_{r_2, \lambda, \infty}^{-\beta_2}} \\ &= \sup_{s>0} \left[s^{\beta_2/2} \left\| \nabla e^{s\Delta} \int_0^t e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\cdot, \tau) d\tau \right\|_{r_2, \lambda} \right] \\ &\leq \int_0^t \sup_{s>0} \left[s^{\beta_2/2} \left\| e^{s\Delta} \nabla e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\cdot, \tau) \right\|_{r_2, \lambda} \right] d\tau. \end{aligned} \quad (3.20)$$

Employing the fact that $|\tilde{c} - \underline{d}_0| \leq 3$ and $||\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2| \leq C|\nabla c^*|(|\nabla \tilde{c}| + |\nabla \bar{c}|)$, one obtains by Lemmas 2.1 and 2.2 that

$$\begin{aligned} & \sup_{0 < s \leq t-\tau} \left[s^{\frac{\beta_2}{2}} \left\| e^{s\Delta} \nabla e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\cdot, \tau) \right\|_{r_2, \lambda} \right] \\ &\lesssim (t-\tau)^{\frac{\beta_2}{2}} (t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{r_2} \right)} \| |\nabla c^*| (|\nabla \tilde{c}| + |\nabla \bar{c}|) \|_{\frac{q_2}{2}, \lambda} \\ &\quad + (t-\tau)^{\frac{\beta_2}{2}} (t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{r_2} \right)} \| |\nabla \bar{c}|^2 c^* \|_{\frac{q_2}{2}, \lambda} \\ &\lesssim (t-\tau)^{-\frac{N-\lambda}{q_2}} \| \nabla c^* \|_{q_2, \lambda} \| (\nabla \tilde{c}, \nabla \bar{c}) \|_{q_2, \lambda} + (t-\tau)^{-\frac{N-\lambda}{q_2}} \| \nabla \bar{c}(\cdot, \tau) \|_{q_2, \lambda}^2 \| c^* \|_{L^\infty}. \end{aligned}$$

For $s > t - \tau$, note that $(t - \tau + s)/2 < s < t - \tau + s$, it then holds that

$$\begin{aligned} & \sup_{s>t-\tau} \left[s^{\frac{\beta_2}{2}} \left\| e^{s\Delta} \nabla e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\cdot, \tau) \right\|_{r_2, \lambda} \right] \\ &\lesssim s^{\frac{\beta_2}{2}} (s+t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{r_2} \right)} \| |\nabla c^*| (|\nabla \tilde{c}| + |\nabla \bar{c}|) \|_{\frac{q_2}{2}, \lambda} \\ &\quad + s^{\frac{\beta_2}{2}} (s+t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{r_2} \right)} \| |\nabla \bar{c}|^2 c^* \|_{\frac{q_2}{2}, \lambda} \\ &\lesssim (t-\tau)^{-\frac{N-\lambda}{q_2}} \sup_{s>t-\tau} \left(1 + \frac{s}{t-\tau} \right)^{-\frac{N-\lambda}{q_2}} \| \nabla c^* \|_{q_2, \lambda} \| (\nabla \tilde{c}, \nabla \bar{c}) \|_{q_2, \lambda} \\ &\quad + (t-\tau)^{-\frac{N-\lambda}{q_2}} \sup_{s>t-\tau} \left(1 + \frac{s}{t-\tau} \right)^{-\frac{N-\lambda}{q_2}} \| \nabla \bar{c}(\cdot, \tau) \|_{q_2, \lambda}^2 \| c^* \|_{L^\infty}. \end{aligned}$$

Since $\frac{1}{q_2} < \frac{1}{N-\lambda}$, we thus obtain

$$\left\| \nabla \int_0^t e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\tau) d\tau \right\|_{\dot{\mathbf{N}}_{r_2, \lambda, \infty}^{-\beta_2}}$$

$$\begin{aligned}
&\lesssim \left\{ \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, t)\|_{q_2, \lambda} \right) \right] \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} (\|\nabla \tilde{c}(\cdot, t)\|_{q_2, \lambda} + \|\nabla \bar{c}(\cdot, t)\|_{q_2, \lambda}) \right) \right] \right. \\
&\quad \left. + \left[\sup_{t>0} (t^{\alpha_2} \|\nabla \bar{c}(\cdot, t)\|_{q_2, \lambda}^2) \right] \|c^*\|_{L^\infty} \right\} \times \int_0^t (t-\tau)^{-\frac{N-\lambda}{q_2}} \tau^{-\alpha_2} d\tau \\
&\lesssim \|c^*\|_{\mathbb{Y}} (\|\tilde{c}\|_{\mathbb{Y}} + \|\bar{c}\|_{\mathbb{Y}} + \|\bar{c}\|_{\mathbb{Y}}^2).
\end{aligned}$$

Next, we calculate the $M_{q_2, \lambda}$ -norm of $\nabla \int_0^t e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\tau) d\tau$.

$$\begin{aligned}
&\left\| \nabla \int_0^t e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\cdot, \tau) d\tau \right\|_{q_2, \lambda} \\
&\lesssim \int_0^t (t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_2} \right)} \|\nabla c^*\|_{q_2, \lambda} \|(\nabla \tilde{c}, \nabla \bar{c})\|_{q_2, \lambda} \\
&\quad + (t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_2} \right)} \| |\nabla \bar{c}|^2 c^* \|_{\frac{q_2}{2}, \lambda} d\tau \\
&\lesssim \left\{ \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} \|\nabla c^*\|_{q_2, \lambda} \right) \right] \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} (\|\nabla \tilde{c}\|_{q_2, \lambda} + \|\nabla \bar{c}\|_{q_2, \lambda}) \right) \right] \right. \\
&\quad \left. + \left[\sup_{t>0} (t^{\alpha_2} \|\nabla \bar{c}\|_{q_2, \lambda}^2) \right] \|c^*\|_{L^\infty} \right\} \times \int_0^t (t-\tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_2} \right)} \tau^{-\alpha_2} d\tau \\
&\lesssim t^{-\frac{1}{2} + \frac{N-\lambda}{2q_2}} \|c^*\|_{\mathbb{Y}} (\|\tilde{c}\|_{\mathbb{Y}} + \|\bar{c}\|_{\mathbb{Y}} + \|\bar{c}\|_{\mathbb{Y}}^2).
\end{aligned}$$

We calculate L^∞ -norm of $\int_0^t e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\tau) d\tau$. Note that $q_2 > N - \lambda$, so there holds

$$\begin{aligned}
&\left\| \int_0^t e^{(t-\tau)\Delta} [(|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*] (\cdot, \tau) d\tau \right\|_{L^\infty} \\
&\lesssim \int_0^t (t-\tau)^{-\frac{N-\lambda}{q_2}} \|\nabla c^*\|_{q_2, \lambda} \|(\nabla \tilde{c}, \nabla \bar{c})\|_{q_2, \lambda} + (t-\tau)^{-\frac{N-\lambda}{q_2}} \|\nabla \bar{c}\|_{q_2, \lambda}^2 \|c^*\|_{L^\infty} d\tau \\
&\lesssim \left\{ \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, t)\|_{q_2, \lambda} \right) \right] \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} (\|\nabla \tilde{c}\|_{q_2, \lambda} + \|\nabla \bar{c}\|_{q_2, \lambda}) \right) \right] \right. \\
&\quad \left. + \left[\sup_{t>0} \left(t^{\frac{\alpha_2}{2}} \|\nabla \bar{c}(\cdot, t)\|_{q_2, \lambda} \right)^2 \|c^*\|_{L^\infty} \right] \right\} \times \int_0^t (t-\tau)^{-\frac{N-\lambda}{q_2}} \tau^{-\alpha_2} d\tau \\
&\lesssim \|c^*\|_{\mathbb{Y}} (\|\tilde{c}\|_{\mathbb{Y}} + \|\bar{c}\|_{\mathbb{Y}} + \|\bar{c}\|_{\mathbb{Y}}^2).
\end{aligned}$$

For the estimates of the remaining part of $\|\mathbb{T}_1(\tilde{u}, \tilde{c}) - \mathbb{T}_1(\bar{u}, \bar{c})\|_{\mathbb{X}}$ and $\|\mathbb{T}_2(\tilde{u}, \tilde{c}) - \mathbb{T}_2(\bar{u}, \bar{c})\|_{\mathbb{Y}}$, we can repeat the proof of the corresponding part as in Lemma 3.1, and thus we conclude that

$$\begin{aligned}
&\|\mathbb{T}_1(\tilde{u}, \tilde{c}) - \mathbb{T}_1(\bar{u}, \bar{c})\|_{\mathbb{X}} \\
&\lesssim \|u^*\|_{\mathbb{X}} (\|\tilde{u}\|_{\mathbb{X}} + \|\bar{u}\|_{\mathbb{X}}) + \|c^*\|_{\mathbb{Y}} (\|\tilde{c}\|_{\mathbb{Y}} + \|\bar{c}\|_{\mathbb{Y}}) \\
&\lesssim C_1 \epsilon_0 (\|u^*\|_{\mathbb{X}} + \|c^*\|_{\mathbb{Y}})
\end{aligned}$$

and

$$\begin{aligned} & \|\mathbb{T}_2(\tilde{u}, \tilde{c}) - \mathbb{T}_2(\bar{u}, \bar{c})\|_{\mathbb{Y}} \\ & \lesssim (\|u^*\|_{\mathbb{X}} + \|c^*\|_{\mathbb{Y}})(\|\tilde{u}\|_{\mathbb{X}} + \|\bar{u}\|_{\mathbb{X}} + \|\tilde{c}\|_{\mathbb{Y}} + \|\bar{c}\|_{\mathbb{Y}} + \|\bar{c}\|_{\mathbb{Y}}^2) \\ & \lesssim C_2 \epsilon_0 (\|u^*\|_{\mathbb{X}} + \|c^*\|_{\mathbb{Y}}). \end{aligned}$$

Choosing $\epsilon_0 > 0$ small enough so that $(C_1 + C_2)\epsilon_0 \leq \frac{1}{2}$, we can then prove Lemma 3.2. \square

4. Proof of Theorem 1.2 – large time behavior

In this section we prove Theorem 1.2.

The proof of Theorem 1.2 is a consequence of the following Lemma 4.1.

Let (\bar{u}, \bar{c}) and (\tilde{u}, \tilde{c}) , respectively, be the solutions of (3.5) constructed in Theorem 1.1 corresponding to the initial data (\bar{u}_0, \bar{c}_0) and $(\tilde{u}_0, \tilde{c}_0)$, respectively. According to Theorem 1.1, there exists a constant C_0 such that

$$\|(\bar{u}, \bar{c})\|_{\Theta} \leq C_0 \epsilon_0, \quad \|(\tilde{u}, \tilde{c})\|_{\Theta} \leq C_0 \epsilon_0. \quad (4.1)$$

Let $(u^*, c^*) = (\tilde{u} - \bar{u}, \tilde{c} - \bar{c})$, then we have

$$\begin{aligned} \tilde{u} - \bar{u} &= e^{t\Delta}(\tilde{u}_0 - \bar{u}_0) \\ &+ \int_0^t e^{(t-s)\Delta} \mathbb{P}(\tilde{u} \cdot \nabla u^* + u^* \cdot \nabla \bar{u} + \operatorname{div}(\nabla c^* \odot \nabla \bar{c} + \nabla \tilde{c} \odot \nabla c^*))(\cdot, s) ds, \\ \tilde{c} - \bar{c} &= e^{t\Delta}(\tilde{c}_0 - \bar{c}_0) \\ &+ \int_0^t e^{(t-s)\Delta} (-\tilde{u} \cdot \nabla c^* - u^* \cdot \nabla \bar{c} + (|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \bar{c}) + |\nabla \bar{c}|^2 c^*)(\cdot, s) ds. \end{aligned}$$

Next, we introduce two auxiliary functions

$$\begin{aligned} h(t) &= t^{\frac{\alpha_1}{2}} \|e^{t\Delta}(\tilde{u}_0 - \bar{u}_0)\|_{q_1, \lambda} + t^{\frac{\alpha_2}{2}} \|\nabla e^{t\Delta}(\tilde{c}_0 - \bar{c}_0)\|_{q_2, \lambda} \\ &+ \|e^{t\Delta}(\tilde{u}_0 - \bar{u}_0, \nabla \tilde{c}_0 - \nabla \bar{c}_0)\|_{\mathbb{E}} + \|e^{t\Delta}(\tilde{c}_0 - \bar{c}_0)\|_{L^\infty} \end{aligned}$$

and

$$l(t) = t^{\frac{\alpha_1}{2}} \|\tilde{u} - \bar{u}\|_{q_1, \lambda} + t^{\frac{\alpha_2}{2}} \|\nabla(\tilde{c} - \bar{c})\|_{q_2, \lambda} + \|(\tilde{u} - \bar{u}, \nabla \tilde{c} - \nabla \bar{c})\|_{\mathbb{E}} + \|\tilde{c} - \bar{c}\|_{L^\infty}.$$

Lemma 4.1. *There holds*

$$\lim_{t \rightarrow \infty} h(t) = 0 \quad (4.2)$$

$$\Updownarrow$$

$$\lim_{t \rightarrow \infty} l(t) = 0. \quad (4.3)$$

Proof. We just prove “(4.2) \implies (4.3)”, the proof of the opposite direction is similar.

First, invoking Lemma 2.3 and the boundedness of \mathbb{P} in Morrey space, one finds

$$\begin{aligned}
 & \|(\bar{u} - \tilde{u}, \nabla \bar{c} - \nabla \tilde{c})\|_{\dot{\mathbf{N}}_{r_1, \lambda, \infty}^{-\beta_1} \times \dot{\mathbf{N}}_{r_2, \lambda, \infty}^{-\beta_2}} \\
 &= \| (e^{t\Delta} \bar{u}_0 - e^{t\Delta} \tilde{u}_0, \nabla e^{t\Delta} \bar{c}_0 - \nabla e^{t\Delta} \tilde{c}_0) \|_{\dot{\mathbf{N}}_{r_1, \lambda, \infty}^{-\beta_1} \times \dot{\mathbf{N}}_{r_2, \lambda, \infty}^{-\beta_2}} \\
 &+ \int_0^t \sup_{s>0} \left[s^{\beta_1/2} \| e^{s\Delta} e^{(t-\tau)\Delta} \mathbb{P} (\tilde{u} \cdot \nabla u^* + u^* \cdot \nabla \bar{u} \right. \\
 &\quad \left. + \operatorname{div}(\nabla c^* \odot \nabla \bar{c} + \nabla \tilde{c} \odot \nabla c^*))(\cdot, \tau) \|_{r_1, \lambda} \right] d\tau \\
 &+ \int_0^t \sup_{s>0} \left[s^{\beta_2/2} \| e^{s\Delta} \nabla e^{(t-\tau)\Delta} (\tilde{u} \cdot \nabla c^* + u^* \cdot \nabla \bar{c} \right. \\
 &\quad \left. + (|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*)(\cdot, \tau) \|_{r_2, \lambda} \right] d\tau \\
 &= \| (e^{t\Delta} \bar{u}_0 - e^{t\Delta} \tilde{u}_0, \nabla e^{t\Delta} \bar{c}_0 - \nabla e^{t\Delta} \tilde{c}_0) \|_{\mathbb{E}} + I + II.
 \end{aligned} \tag{4.4}$$

Employing Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
 & \sup_{s>0} \left[s^{\frac{\beta_1}{2}} \| e^{s\Delta} e^{(t-\tau)\Delta} \mathbb{P} (\tilde{u} \cdot \nabla u^* + u^* \cdot \nabla \bar{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla c^* + \nabla c^* \odot \nabla \bar{c}))(\cdot, \tau) \|_{r_1, \lambda} \right] \\
 &\leq (t-\tau)^{-\frac{N-\lambda}{q_1}} \| u^*(\cdot, \tau) \|_{q_1, \lambda} \| (\tilde{u}(\cdot, \tau), \bar{u}(\cdot, \tau)) \|_{q_1, \lambda} \\
 &\quad + (t-\tau)^{-\frac{N-\lambda}{q_2}} \| \nabla c^*(\cdot, \tau) \|_{q_2, \lambda} \| (\nabla \tilde{c}(\cdot, \tau), \nabla \bar{c}(\cdot, \tau)) \|_{q_2, \lambda}
 \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
 & \sup_{s>0} \left[s^{\frac{\beta_2}{2}} \| e^{s\Delta} \nabla e^{(t-\tau)\Delta} (\tilde{u} \cdot \nabla c^* + u^* \cdot \nabla \bar{c} + (|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^*)(\cdot, \tau) \|_{r_2, \lambda} \right] \\
 &\leq (t-\tau)^{-\frac{N-\lambda}{2q_1} - \frac{N-\lambda}{2q_2}} \left(\| \tilde{u}(\cdot, \tau) \|_{q_1, \lambda} \| \nabla c^*(\cdot, \tau) \|_{q_2, \lambda} + \| u^*(\cdot, \tau) \|_{q_1, \lambda} \| \nabla \bar{c} \|_{q_2, \lambda} \right) \\
 &\quad + (t-\tau)^{-\frac{N-\lambda}{q_2}} \| \nabla c^*(\cdot, \tau) \|_{q_2, \lambda} (\| \nabla \bar{c} \|_{q_2, \lambda} + \| \nabla \tilde{c} \|_{q_2, \lambda}).
 \end{aligned} \tag{4.6}$$

Let $0 < \delta < 1$, using (4.1), we estimate I as follows

$$\begin{aligned}
 I &\lesssim \left(\int_0^{\delta t} + \int_{\delta t}^t \right) \left[(t-\tau)^{-\frac{N-\lambda}{q_1}} \| u^* \|_{q_1, \lambda} \| (\tilde{u}, \bar{u}) \|_{q_1, \lambda} \right. \\
 &\quad \left. + (t-\tau)^{-\frac{N-\lambda}{q_2}} \| \nabla c^* \|_{q_2, \lambda} \| (\nabla \tilde{c}, \nabla \bar{c}) \|_{q_2, \lambda} \right] d\tau \\
 &\lesssim \| (\bar{u}, \tilde{u}) \|_{\mathbb{X}} \int_0^{\delta t} (t-\tau)^{-\frac{N-\lambda}{q_1}} \tau^{-\alpha_1} \left(\tau^{\frac{\alpha_1}{2}} \| u^*(\cdot, \tau) \|_{q_1, \lambda} \right) d\tau \\
 &\quad + \| (\bar{c}, \tilde{c}) \|_{\mathbb{Y}} \int_0^{\delta t} (t-\tau)^{-\frac{N-\lambda}{q_2}} \tau^{-\alpha_2} \left(\tau^{\frac{\alpha_2}{2}} \| \nabla c^*(\cdot, \tau) \|_{q_2, \lambda} \right) d\tau \\
 &\quad + \| (\bar{u}, \tilde{u}) \|_{\mathbb{X}} \left[\sup_{\delta t \leq \tau \leq t} \left(\tau^{\frac{\alpha_1}{2}} \| u^*(\cdot, \tau) \|_{q_1, \lambda} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
& + \|(\bar{c}, \tilde{c})\|_{\mathbb{X}} \left[\sup_{\delta t \leq \tau \leq t} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) \right] \\
& \lesssim \epsilon_0 \int_0^{\delta t} (t - \tau)^{-\frac{N-\lambda}{q_1}} \tau^{-\alpha_1} \left(\tau^{\frac{\alpha_1}{2}} \|u^*(\cdot, \tau)\|_{q_1, \lambda} \right) d\tau \\
& \quad + \epsilon_0 \int_0^{\delta t} (t - \tau)^{-\frac{N-\lambda}{q_2}} \tau^{-\alpha_2} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) d\tau \\
& \quad + \epsilon_0 \left\{ \sup_{\delta t \leq \tau \leq t} \left[\tau^{\frac{\alpha_1}{2}} \|u^*(\cdot, \tau)\|_{q_1, \lambda} \right] + \sup_{\delta t \leq \tau \leq t} \left[\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right] \right\}. \tag{4.7}
\end{aligned}$$

For II , we use the same argument as above to get

$$\begin{aligned}
II & \lesssim \epsilon_0 \int_0^{\delta t} (t - \tau)^{-\frac{N-\lambda}{2q_1} - \frac{N-\lambda}{2q_2}} \tau^{-\frac{\alpha_1 + \alpha_2}{2}} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} + \tau^{\frac{\alpha_1}{2}} \|u^*(\cdot, \tau)\|_{q_1, \lambda} \right) d\tau \\
& \quad + \epsilon_0 \int_0^{\delta t} (t - \tau)^{-\frac{N-\lambda}{q_2}} \tau^{-\alpha_2} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) d\tau \\
& \quad + \epsilon_0 \left[\sup_{\delta t \leq \tau \leq t} \left(\tau^{\alpha_1/2} \|u^*(\cdot, \tau)\|_{q_1, \lambda} \right) + \sup_{\delta t \leq \tau \leq t} \left(\tau^{\alpha_2/2} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) \right]. \tag{4.8}
\end{aligned}$$

From (4.7)-(4.8), we get

$$\begin{aligned}
& \|(\bar{u} - \tilde{u}, \nabla \bar{c} - \nabla \tilde{c})\|_{\dot{\mathbf{N}}_{r_1, \lambda, \infty}^{-\beta_1} \times \dot{\mathbf{N}}_{r_2, \lambda, \infty}^{-\beta_2}} \\
& = \|(e^{t\Delta} \bar{u}_0 - e^{t\Delta} \tilde{u}_0, \nabla e^{t\Delta} \bar{c}_0 - \nabla e^{t\Delta} \tilde{c}_0)\|_{\mathbb{E}} + I + II \\
& \lesssim \epsilon_0 \int_0^{\delta} (1 - s)^{-\frac{N-\lambda}{q_1}} s^{-\alpha_1} \left[(ts)^{\alpha_1/2} \|u^*(ts)\|_{q_1, \lambda} \right] ds \\
& \quad + \epsilon_0 \int_0^{\delta} (1 - s)^{-\frac{N-\lambda}{q_2}} s^{-\alpha_2} \left[(ts)^{\alpha_2/2} \|\nabla c^*(ts)\|_{q_2, \lambda} \right] ds \\
& \quad + \epsilon_0 \int_0^{\delta} (1 - s)^{-\frac{N-\lambda}{2q_1} - \frac{N-\lambda}{2q_2}} s^{-\frac{\alpha_1 + \alpha_2}{2}} \left[(ts)^{\frac{\alpha_2}{2}} \|\nabla c^*(ts)\|_{q_2, \lambda} + (ts)^{\frac{\alpha_1}{2}} \|u^*(ts)\|_{q_1, \lambda} \right] ds \\
& \quad + \epsilon_0 \left[\sup_{\delta t \leq \tau \leq t} \left(\tau^{\alpha_1/2} \|u^*(\cdot, \tau)\|_{q_1, \lambda} \right) + \sup_{\delta t \leq \tau \leq t} \left(\tau^{\alpha_2/2} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) \right]. \tag{4.9}
\end{aligned}$$

Next, we calculate $M_{q_1, \lambda} \times M_{q_2, \lambda} \times L^\infty$ norm of $(\bar{u} - \tilde{u}, \nabla \bar{c} - \nabla \tilde{c}, \bar{c} - \tilde{c})$.

$$\begin{aligned}
& \|(\bar{u} - \tilde{u}, \nabla \bar{c} - \nabla \tilde{c}, \bar{c} - \tilde{c})\|_{M_{q_1, \lambda} \times M_{q_2, \lambda} \times L^\infty} \\
& \lesssim \|(e^{t\Delta} \bar{u}_0 - e^{t\Delta} \tilde{u}_0, \nabla e^{t\Delta} \bar{c}_0 - \nabla e^{t\Delta} \tilde{c}_0, e^{t\Delta} \bar{c}_0 - e^{t\Delta} \tilde{c}_0)\|_{M_{q_1, \lambda} \times M_{q_2, \lambda} \times L^\infty} \\
& \quad + J_1 + J_2 + J_3,
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \int_0^t \left\| e^{(t-\tau)\Delta} \mathbb{P} \left(\tilde{u} \cdot \nabla u^* + u^* \cdot \nabla \bar{u} + \operatorname{div}(\nabla \tilde{c} \odot \nabla c^* + \nabla c^* \odot \nabla \bar{c}) \right) (\cdot, \tau) \right\|_{q_1, \lambda} d\tau, \\
J_2 &= \int_0^t \left\| e^{(t-\tau)\Delta} \mathbb{P} \left(\tilde{u} \cdot \nabla c^* + u^* \cdot \nabla \bar{c} + (|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^* \right) (\cdot, \tau) \right\|_{q_2, \lambda} d\tau, \\
J_3 &= \int_0^t \left\| e^{(t-\tau)\Delta} \mathbb{P} \left(\tilde{u} \cdot \nabla c^* + u^* \cdot \nabla \bar{c} + (|\nabla \tilde{c}|^2 - |\nabla \bar{c}|^2)(\tilde{c} - \underline{d}_0) + |\nabla \bar{c}|^2 c^* \right) (\cdot, \tau) \right\|_{L^\infty} d\tau.
\end{aligned}$$

Let $0 < \delta < 1$, we estimate J_1 as follows

$$\begin{aligned}
J_1 &\lesssim \left(\int_0^{\delta t} + \int_{\delta t}^t \right) (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_1} - \frac{1}{q_1} \right)} \left\| \tilde{u}(\cdot, \tau) \otimes u^*(\cdot, \tau) + u^*(\cdot, \tau) \otimes \bar{u}(\cdot, \tau) \right\|_{\frac{q_1}{2}, \lambda} \\
&\quad + (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_1} \right)} \left\| \nabla \tilde{c}(\cdot, \tau) \otimes \nabla c^*(\cdot, \tau) + \nabla c^*(\cdot, \tau) \otimes \nabla \bar{c}(\cdot, \tau) \right\|_{\frac{q_2}{2}, \lambda} d\tau \\
&\lesssim \epsilon_0 \int_0^{\delta t} (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_1} - \frac{1}{q_1} \right)} \tau^{-\alpha_1} \left(\tau^{\frac{\alpha_1}{2}} \|u^*(\cdot, \tau)\|_{q_1, \lambda} \right) \\
&\quad + (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_1} \right)} \tau^{-\alpha_2} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) d\tau \\
&\quad + \epsilon_0 t^{-1/2 + \frac{N-\lambda}{2q_1}} \left[\sup_{\delta t \leq \tau \leq t} \left(\tau^{\frac{\alpha_1}{2}} \|u^*(\cdot, \tau)\|_{q_1, \lambda} \right) + \sup_{\delta t \leq \tau \leq t} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) \right]. \tag{4.10}
\end{aligned}$$

Similarly, we estimate J_2 as follows:

$$\begin{aligned}
J_2 &\lesssim \left(\int_0^{\delta t} + \int_{\delta t}^t \right) (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_2} \right)} \left\| (\tilde{u} \cdot \nabla c^* + u^* \cdot \nabla \bar{c}) (\cdot, \tau) \right\|_{\frac{q_1 q_2}{q_1 + q_2}, \lambda} \\
&\quad + (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_2} \right)} \left\| (\nabla \tilde{c} \cdot \nabla c^* + \nabla c^* \cdot \nabla \bar{c}) (\cdot, \tau) \right\|_{\frac{q_2}{2}, \lambda} d\tau \\
&\lesssim \epsilon_0 \int_0^{\delta t} (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_2} \right)} \tau^{-\frac{\alpha_1 + \alpha_2}{2}} \left(\tau^{\frac{\alpha_1}{2}} \|u^*\|_{q_1, \lambda} + \tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) \\
&\quad + (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_2} \right)} \tau^{-\alpha_2} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) d\tau \\
&\quad + \epsilon_0 t^{-\frac{1}{2} + \frac{N-\lambda}{2q_2}} \left[\sup_{\delta t \leq \tau \leq t} \left(\tau^{\frac{\alpha_1}{2}} \|u^*(\cdot, \tau)\|_{q_1, \lambda} \right) + \sup_{\delta t \leq \tau \leq t} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) \right].
\end{aligned}$$

For J_3 , it holds that

$$\begin{aligned}
J_3 &\lesssim \epsilon_0 \int_0^{\delta t} (t - \tau)^{-\frac{1}{2} - \frac{N-\lambda}{2q_1}} \tau^{-\frac{\alpha_1}{2}} \left(\tau^{\frac{\alpha_1}{2}} \|u^*(\cdot, \tau)\|_{q_1, \lambda} + \|c^*(\cdot, \tau)\|_{L^\infty} \right) d\tau \\
&\quad + \epsilon_0 \int_0^{\delta t} (t - \tau)^{-\frac{N-\lambda}{q_2}} \tau^{-\alpha_2} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} + \|c^*(\cdot, \tau)\|_{L^\infty} \right) d\tau
\end{aligned}$$

$$\begin{aligned}
& + \epsilon_0 \left[\sup_{\delta t \leq \tau \leq t} \left(\tau^{\frac{\alpha_1}{2}} \|u^*(\cdot, \tau)\|_{q_1, \lambda} \right) \right. \\
& \left. + \sup_{\delta t \leq \tau \leq t} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_2, \lambda} \right) + \sup_{\delta t \leq \tau \leq t} \|c^*(\cdot, \tau)\|_{L^\infty} \right].
\end{aligned}$$

Let

$$\begin{aligned}
\Omega = & \epsilon_0 \int_0^\delta (1-s)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_1} - \frac{1}{q_1} \right)} s^{-\alpha_1} \left((ts)^{\frac{\alpha_1}{2}} \|u^*(\cdot, ts)\|_{q_1, \lambda} \right) \\
& + (1-s)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_1} \right)} s^{-\alpha_2} \left((ts)^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, ts)\|_{q_2, \lambda} \right) ds \\
& + \epsilon_0 \int_0^\delta (1-s)^{-\frac{1}{2} - \frac{N-\lambda}{2q_1}} s^{-\frac{\alpha_1 + \alpha_2}{2}} \left((ts)^{\frac{\alpha_1}{2}} \|u^*(ts)\|_{q_1, \lambda} + (ts)^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, ts)\|_{q_2, \lambda} \right) \\
& + (1-s)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_2} \right)} s^{-\alpha_2} \left((ts)^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, ts)\|_{q_2, \lambda} \right) ds \\
& + \epsilon_0 \int_0^\delta (1-s)^{-\frac{1}{2} - \frac{N-\lambda}{2q_1}} s^{-\frac{\alpha_1}{2}} \left((ts)^{\frac{\alpha_1}{2}} \|u^*(\cdot, ts)\|_{q_1, \lambda} + \|c^*(\cdot, ts)\|_{L^\infty} \right) ds \\
& + \epsilon_0 \int_0^\delta (1-s)^{-\frac{N-\lambda}{q_2}} s^{-\alpha_2} \left((ts)^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, ts)\|_{q_2, \lambda} + \|c^*(\cdot, ts)\|_{L^\infty} \right) ds.
\end{aligned}$$

Then we conclude that

$$\begin{aligned}
& \left\| (t^{\frac{\alpha_1}{2}} \bar{u} - t^{\frac{\alpha_1}{2}} \tilde{u}, t^{\frac{\alpha_2}{2}} \nabla \bar{c} - t^{\frac{\alpha_2}{2}} \nabla \tilde{c}, \bar{c} - \tilde{c}) \right\|_{M_{q_1, \lambda} \times M_{q_2, \lambda} \times L^\infty} \\
& \lesssim \left\| (t^{\frac{\alpha_1}{2}} e^{t\Delta} \bar{u}_0 - t^{\frac{\alpha_1}{2}} e^{t\Delta} \tilde{u}_0, t^{\frac{\alpha_2}{2}} \nabla e^{t\Delta} \bar{c}_0 - t^{\frac{\alpha_2}{2}} \nabla e^{t\Delta} \tilde{c}_0, \bar{c}_0 - \tilde{c}_0) \right\|_{M_{q_1, \lambda} \times M_{q_2, \lambda} \times L^\infty} \\
& + \epsilon_0 \left[\sup_{\delta t \leq \tau \leq t} \left(\tau^{\frac{\alpha_1}{2}} \|u^*(\cdot, \tau)\|_{q_1, \lambda} \right) + \sup_{\delta t \leq \tau \leq t} \left(\tau^{\frac{\alpha_2}{2}} \|\nabla c^*(\cdot, \tau)\|_{q_3, \lambda} \right) \right. \\
& \left. + \sup_{\delta t \leq \tau \leq t} \|c^*(\cdot, \tau)\|_{L^\infty} \right] + \Omega.
\end{aligned} \tag{4.11}$$

From (3.17)-(3.18) and condition (4.2), we have

$$h(t) \in L^\infty([0, \infty)), \quad \lim_{t \rightarrow \infty} h(t) = 0. \tag{4.12}$$

Let

$$M = \limsup_{t \rightarrow \infty} l(t) = \lim_{k \in \mathbb{Z}, k \rightarrow \infty} \sup_{t \geq k} l(t),$$

then it suffices to prove $M = 0$. (4.1) implies that M is non-negative and finite. Hence combining (4.9) and (4.11), then using the Lebesgue dominated convergence theorem and (4.12), it finds

$$M \leq C_1 \epsilon_0 (1 + F(\delta)) M, \tag{4.13}$$

where $F(\delta)$ is defined by

$$\begin{aligned}
F(\delta) = & \epsilon_0 \int_0^\delta (1-s)^{-\frac{N-\lambda}{q_1}} s^{-\alpha_1} ds + \epsilon_0 \int_0^\delta (1-s)^{-\frac{N-\lambda}{q_2}} s^{-\alpha_2} ds \\
& + \epsilon_0 \int_0^\delta (1-s)^{-\frac{N-\lambda}{2q_1} - \frac{N-\lambda}{2q_2}} s^{-\frac{\alpha_1+\alpha_2}{2}} ds \\
& + \epsilon_0 \int_0^\delta (1-s)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_1} - \frac{1}{q_1} \right)} s^{-\alpha_1} + (1-s)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_2} \right)} s^{-\alpha_2} ds \\
& + \epsilon_0 \int_0^\delta (1-s)^{-\frac{1}{2} - \frac{N-\lambda}{2q_1}} s^{-\frac{\alpha_1+\alpha_2}{2}} + (1-s)^{-\frac{1}{2} - \frac{N-\lambda}{2} \left(\frac{2}{q_2} - \frac{1}{q_2} \right)} s^{-\alpha_2} ds \\
& + \epsilon_0 \int_0^\delta (1-s)^{-\frac{1}{2} - \frac{N-\lambda}{2q_1}} s^{-\frac{\alpha_1}{2}} ds + \epsilon_0 \int_0^\delta (1-s)^{-\frac{N-\lambda}{q_2}} s^{-\alpha_2} ds
\end{aligned}$$

with

$$\lim_{\delta \rightarrow 0} F(\delta) = 0.$$

Hence, choosing ϵ_0 and δ small enough and using (4.11), we deduce $M = 0$. \square

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